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A COST FUNCTION FOR MILITARY AIRFRAMES

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A COST FUNCTION FOR MILITARY AIRFRAMES

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ABSTRACT

Recent theoretical and empirical work in the areas of learning curves, production rate and cost estimation of airframes has seemed to yield contradictory conclusions.

This paper synthesizes this work. The synthesis yields a model of the acquisition process that captures the interaction between learning and both endogenous and exogenous production rate changes.

This is accomplished by modifying a previous model to include previous production experience and yearly production targets. This permits a production program to be modeled as a series of discrete tasks connected by experience. By this device the impact of an exogenous increase or decrease in deliveries can be modeled. Likewise, the impact of stretching a lot out over a longer period of time can be modeled by this procedure.

The model is also expanded to include the impact of several restrictions on production. Finally, plans for estimating the cost function and illustrating its use in program management are discussed.

INTRODUCTION

Due to cost overruns, Congressional concern, and a continuing need for better planning estimates, it is imperative that new techniques be developed and old techniques refined to obtain better cost estimates for major weapon system production and acquisition. Along with these techniques, a better understanding of the factors and forces that determine cost is required. In particular, the sensitivity of program costs to alternative policy decisions must be accurately estimated if we are to meet the challenge of providing wise acquisition policy. Furthermore, the cost impacts of policy decisions must be readily available if they are to have an impact in the dynamic world of systems acquisition.

This research is concerned with estimating the cost impacts of policies that affect the production rate during a program. For convenience we discuss these effects at two stages in the life of the program.

At the outset of a production program, a tentative monthly production schedule for the program is negotiated between the contracting parties. This schedule permits planning for work force buildup, facility and tooling needs, and the ordering of long lead time items. This early situation is referred to as the planning stage.

Although the planned delivery schedule covers the life of the program, formal contractual agreements between the Department of Defense and manufacturers usually cover only one year's delivery requirements. Delivery requirements for subsequent years are funded through the exercise of options or separate contracts as funds are appropriated by the Congress. Over time the situation tends to change. Funding in a particular year may be insufficient to cover the planned production. Or a national emergency or changed mission requirements may argue for changes in production rate. This later situation is referred to as the production stage.

Intuition, economic theory, and recently, empirical studies argue that production rate changes, at either stage in the program, affect program costs. In addition, Gaunt [7] points out that cost penalties for production rate changes are now embodied in some contracts.

The foregoing is generally accepted, but there is substantial disagreement about both the magnitude and the direction of the impact of production rate changes on program costs. Empirical studies of airframe programs in the last five years have documented cases where increases in production rate have been associated with increases, decreases, and no change in the unit cost of production.

BACKGROUND

The theoretical foundations for production rate impacts on costs are as old as the study of economics. Adam Smith's pin factory example [18] is an early statement of the effect. More recently, Asher [3] recognized the potential importance of production rate to aircraft production costs; but he could find little statistical support for
the idea. Since 1956 the idea of combining learning effects and production rate effects in the explanation of aircraft costs has proceeded along two rather separate routes.

In 1959 Alchian [1] provided some theoretical observations concerning the interaction of learning and production rate. His paper was followed by Hirshleifer's 1962 discussion [8]. Preston and Keachie [16], Ol [14] and Rosen [17] also made contributions. All these papers added to the understanding of the process by which learning interacted with production rate to affect cost; but they were conceptual and almost completely data free. Furthermore, for the most part, they generated results that were far too general for statistical estimation.

The second line of development has been mainly empirical. Ever since Alchian's 1959 paper [1], Rand Reports [9, 10, 11, 12] on aircraft cost estimation have attempted to include both volume and production rate as independent variables in their cost estimating relations. In his 1963 paper Alchian [2] reports this attempt as early as 1948. Even though Alchian argues that both variables should be important, the resulting empirical work credits production rate with little, if any, explanatory ability. In fact, a recent study [10] states:

In general, however, we must conclude that for predicting the overall effect of production rate on aircraft cost, generalized estimating equations based on statistical analyses of our sample of military aircraft would be too unreliable to be useful.

The Rand studies have been cross-sectional studies characterized by a few observations on many aircraft programs. More recent work by Womer [23], Smith [20] and his students, Congleton and Kinton [5, 19] has reached the opposite conclusion. The studies under Smith's direction have been time series studies on single airframe programs. Unfortunately, these studies have been almost devoid of economic theory. As a result even though some of the studies indicate that production rate is correlated with costs on a program, our understanding of the process by which this happens is fuzzy at best. Without this knowledge the results cannot be intelligently used for policy guidance.

Recent work has been closing the gap between these two lines of research. Washburn [21] and Womer [23] derive cost relations consistent with economic theory in forms suitable for empirical estimation. This work shows that, in the absence of outside forces, the producer attempting to minimize cost will change the production rate over time. That is, some of the production rate changes in weapon systems' programs do not result from government action. Womer [21] points out that these results refute some previous theoretical work. They also provide a potential explanation for Smith's seemingly contradictory results.

At the same time, the unique data problems of combining variables measured by time periods with others measured by units produced have been examined by Womer [24]. Finally, preliminary work [6, 13] has started on relating Womer's [23] model to the Rand data.

This paper reports on the first stage of a research effort designed to synthesize the existing theoretical and empirical work that relates production rate and learning curves.

Here Womer's model [23] is modified to include resources that cannot be varied during the production program. This permits a more realistic distinction to be drawn between the planning situation before the program begins and the more restricted production situation.

Next the model is applied to the problem of producing to a delivery schedule. This results in the production program being modeled as a series of discrete tasks connected by experience. By this device the impact of an exogenous increase or decrease in deliveries can be modeled. Likewise, the impact of stretching a lot out over a longer period of time can be modeled by this procedure.

Finally, the impact of policy on the model is illustrated by considering the problem subject to a constant workforce constraint.

THE MODEL

The model uses a production function to relate output rate to two classes of inputs. The relative prices of resources within each class are assumed to change. Thus, each class may be represented as a single composite resource. One class is composed of resources whose use rate cannot change during the program. The resources of the other class can be used at changing rates throughout the production program. This simple classification of resources is just the usual distinction between resources which gives rise to fixed and variable costs. The variables of the model are described below:

\[ Q(t) = \int q(t) \, dt \]

\[ c(t) \] = cumulative production experience

\[ \delta \] = a parameter describing learning

\[ v \] = a returns to scale parameter

\[ C \] = discounted program cost

\[ T \] = time horizon for the production program

\[ D \] = discount rate

\[ V \] = volume of output to be produced by \( T \)

\[ p \] = price (in units of the variable resource) of the fixed resource.
The production function relates the quantity of fixed resources, \( k \), the rate of variable resource use, \( x(t) \), and cumulative production experience, \( Q(t) \), to the output rate, \( q(t) \). The production function is assumed to be of the form

\[
q(t) = k^x x^{1/y} Q^y(t) \quad (1)
\]

This functional form embodies two summary characteristics of the production process:
(a) the production function is homogeneous of degree 1/y in the variable resources;
(b) neutral technological change is induced in the production process as a log-linear function of cumulative production experience.

The homogeneity assumption is frequently made in empirical studies of production. Here we also assume that \( y > 1 \) implying decreasing returns to scale. Otherwise, an optimal production program would crowd all production into an arbitrarily short period at the end of the program. This assumption also implies that production rate has no absolute maximum. However, for high values of \( y \), the resource penalties associated with increasing \( q(t) \) may be prohibitive.

The assumption that production experience induces neutral technological change in the production process simplifies the analysis considerably. Otherwise, both the impact of experience on the use of each resource and the relative impact of each resource on output rate must be specified.

The resource prices and the discount rate are assumed to be exogenous constants. Thus discounted program costs in units of the variable resources are:

\[
C = \int_0^T x(t)e^{-r} dt + P_k \quad (2)
\]

So far the model is not much different from the model of [23]. Here, however, we assume that the firm must meet an imposed delivery schedule. That is cumulative production levels, \( Q(t) \), at particular points in time, \( t_1 \), are specified in the contract.

The firm is assumed to minimise the discounted program costs incurred to meet the delivery schedule. The firm's problem is characterised as:

\[
\text{Min } C = \int_0^T x(t)e^{-r} dt + P_k \quad (3)
\]

subject to \( q(t) = k^x x^{1/y} Q^y(t) \)

\[
x(t) \geq 0 \quad Q(t_1) = Q_1 \quad Q(T) = \upsilon \]

\[
k^x \geq 0 \quad Q(0) = 0
\]

All of the interesting results of using this model can be demonstrated with a two point production schedule, i.e. \( t_1 \) and \( T \); so \( t \) is set equal to 1 below.

The solution to this problem is presented for both the production situation and the planning situation.

THE PRODUCTION SITUATION

In the production situation the delivery schedule and \( k \) are fixed. The solution to the optimal control problem at (3) requires that the optimal time path of \( x(t) \) be found over the range \((0, t_1)\) and the range \((t_1, T)\). With a little work the optimal time path is found to be

\[
x(t) = \left[ \frac{\alpha}{(y-1)} \right] (r-1) G_1 (1-\delta) Y_1(1-\delta)
\]  

\[
[\alpha e^{ct}/(y-1)-1]^{-\gamma} e^{ct}(y-1)
\]

when \( 0 < t < t_1 \)

and

\[
x(t) = \left[ \frac{\alpha}{(y-1)} \right] (r-1) G_1 (1-\delta) Y_1(1-\delta)
\]  

\[
-\alpha e^{ct}/(y-1)-1]^{-\gamma} e^{ct}(y-1)
\]

when \( t_1 < t < T \)

Cumulative discounted costs at any point in time are found by substituting (4) into the objective function at (3) and changing the limit of integration from \( T \) to \( t \).

This yields

\[
C(t) = A \left[ \alpha^{-1/y} G_1 \right] (r-1) Y_1(1-\delta) [e^{ct}/(y-1)-1]^{-\gamma} e^{ct}(y-1)-1]
\]

\[
+ P_k
\]

when \( 0 < t < t_1 \)

and

\[
C(t) = A \left[ \alpha^{-1/y} G_1 \right] (r-1) Y_1(1-\delta) [e^{ct}/(y-1)-1]^{-\gamma} + A \left[ \alpha^{-1/y} Y_1 \right] (r-1) Y_1(1-\delta) [e^{ct}/(y-1)-1]^{-\gamma} + P_k
\]

when \( t_1 < t < T \), where \( A = [\alpha/(y-1)]Y_1(1-\delta)^{-\gamma} \)

Figure 1 illustrates the cumulative costs for three different delivery schedules producing 240 aircraft in 40 months.
Figure 1. Cumulative Program Cost as a Function of Time

\[ C = k^{-\alpha} \gamma Q_1 \left( \frac{1-\delta}{1-\gamma} \right) \left( e^\delta t / (1-\gamma) - 1 \right)^{1-\gamma} \]
\[ + k^{-\alpha} \gamma (1-\delta) \left( Q_1 - 1 \right) \left( e^\delta t / (1-\gamma) - e^\delta t / (1-\gamma) \right) \]
\[ + P_k \]

(7)

The optimal value of \( k \) can be found by:
\[ 2k = -\alpha \gamma k^{-\alpha} f(Q_1, t_1, V, T) + P_k = 0 \]
\[ k^{-\alpha} f = P_k / \alpha f(Q_1, t_1, V, T) \]
\[ k = \left( \alpha f(Q_1, t_1, V, T) / P_k \right) ^ {1/(\alpha+1)} \]  

(8)

Figure 2 illustrates \( C \) as a function of \( k \) for the three delivery schedules used previously. Figure 2 reveals two interesting facts. First, there is a unique value of \( k \) that is best for each delivery schedule. Second, the ability to choose \( k \) does not totally remove the cost penalties for imposing delivery schedules.

This planning cost function can be used to determine the appropriate delivery schedule. For example, suppose \( t_1, V, \) and \( T \) are known, Figure 1 shows \( C \) as a function of \( Q_1 \). Using this information, together with information on the benefits of having more or fewer aircraft available at \( t_1 \), the appropriate value of \( Q_1 \) can be chosen.

THE PLANNING SITUATION

In the planning situation \( k \) can be chosen in an optimal way. From (4) and (3), total program costs for any level of \( k \) are found as:

\[ Q_1 = \frac{V(e^{\delta t} / (1-\gamma) - 1)}{\left( e^\delta t / (1-\gamma) - 1 \right)} \]

(6)
Figure 2. Program Cost as a Function of the Quantity of Fixed Resources, k.

Given \( Q \), (8) determines the optimal level of the fixed resources, \( k \); and this in turn is used in (4) to yield the time path for the variable resources, \( X(t) \).

Substituting (8) into (7) yields total program cost in the planning situation as a function of the variables that prescribe the delivery schedule: \( Q \), \( T \), \( V \), and \( T \).

\[
C = B \left[ \frac{\gamma}{(\gamma + 1)} \right] Q \left( 1 - \epsilon \right) \left[ e^{\delta T/(\gamma - 1)} -1 \right] \left( 1 - \gamma \right) + \left( \epsilon^{1-\delta} Q \right) \left( e^{\delta T/(\gamma - 1)} -e^{\delta T/(\gamma - 1)} \left( 1 - \gamma \right) \right] / (\gamma + 1)
\]

where \( B = A^{1/\gamma} / (\gamma + 1) \).

Figure 3. Program Cost as a Function of Delivery Schedule, \( Q \).
Figure 4 shows the relation between average costs of production in the planning situation and the production situation. The lower solid curve (the planning situation) shows the least cost way to produce $V$ aircraft in $T$ months. The dot-dash curve shows the production situation corresponding to $Q_1$ and $k$. It reflects the costs that will be incurred if $V$ is not equal to its planned value. The two curves are not tangent at $V = 240$. This reflects the fact that $Q_1$ was not chosen by the least cost criterion. Nevertheless, $k$ has been chosen so that given $Q_1$ there is no lower cost way to produce 240 aircraft in $T$ months and satisfy the delivery schedule.

Figure 4 illustrates the impact of either crashing or stretching a program in the production situation. Clearly decreasing $V$ results in higher unit costs than planned and substantially higher costs than could have been attained had the correct volume been anticipated. Likewise, crashing the program, increasing $V$ without changing $T$, results in higher costs than would have been available in the planning situation. Increasing $V$ may actually increase unit costs if $V$ is substantially greater than planned.

Finally, Figure 4 sheds some light on Smith's results which show that production rate and unit costs are sometimes positively and sometimes negatively correlated. Decreasing $V$ in the production situation results in an exogenous decrease in production rate and an increase in unit cost. Thus producing a tendency towards negative correlation.

Increasing $V$ in the production situation requires production rate to increase. This too can result in higher unit costs; thus a positive correlation.

Once formed the model can be exercised to analyze the effect of alternative policies on costs and production. The next section provides a sample analysis.

**A CONSTANT WORKFORCE**

Suppose national economic policy argues that fluctuations in the demand for labor in the vicinity of the contractor be minimized. One possible policy is the constant workforce policy:

$$k(t) = l$$  \hspace{1cm} (10)

That is, the quantity of labor used cannot vary during the program.

There are several possible specifications of the relation between labor, other variable resources and the class of variable resources. One tractable specification is based on the Cobb-Douglas production function:

$$q(t) = k Q(t)^{\beta} N(t)^{\varepsilon}$$ \hspace{1cm} (11)

where $0 < \beta$, $\varepsilon < 1$ and $1/\varepsilon = \beta + \varepsilon$

Invoking the constant workforce restrictions:

$$q(t) = l Q(t)^{\beta} N(t)^{\varepsilon}$$ \hspace{1cm} (12)
Since neither \( k \) nor \( l \) can be varied during the program they can be joined to form the composite resource \( Z \), yielding

\[
q(t) = 2^{t^3}N_f(t)Q^l(t)
\]  

(13)

Writing the production function in this form we see that the constant workforce problem is just like the variable workforce problem except that \( \alpha + \beta \) plays the role of \( a \), \( \epsilon \) plays the role of \( 1/\gamma \), \( Z \) plays the role of \( k \) and \( M \) plays the role of \( n \).

The impacts of the constant workforce restrictions are to raise costs and to make the production situation even more restrictive. Now the cost penalties for picking the wrong level of \( V \) are even higher. This is illustrated in Figure 5. Here, with \( \beta = \epsilon = 1/4 \), unit costs in production situation are superimposed on Figure 4. If the correct volume is planned, the cost penalties of the constant workforce are minimized. But as \( V \) changes from its planned level, the cost penalties increase.

**SUMMARY AND RESEARCH PLANS**

This paper has expanded an earlier model. The expanded model was seen to deal nicely with the problems of producing to a delivery schedule and it incorporates prior experience on the program. The model also permits the analyst to specify certain policy constraints and trace their implications on program costs. More importantly, the expanded model is seen to contain an explanation for the fact that sometimes production rate has been positively and sometimes negatively correlated with program costs. However, to verify this hypothesis more work needs to be accomplished.

In particular a careful job of estimating the cost function for several airframe programs needs to be done. This requires attention to the kinds of policy constraints in force at various times during the program. Fortunately much of the required data is still available.

In addition to the data sets reported by Smith (20) and Orsini (15) data from OSD reports like "Acceptance Rates and Tooling Capacity for Selected Military Aircraft" (4) and detailed program histories in the ASD Cost Library can be consulted. Interviews with contractors may also be required. While little raw data is expected to be required, the data sets need to be consolidated and transformed to provide consistent observations. The parameters of the model can be estimated for each airframe program. Finally, the estimated model can be programmed and used to provide timely, documented answers to questions about the cost impact of alternative policies.

![Figure 5. The Impact of a Constant Workforce.](image-url)
REFERENCES


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The impacts of an exogenous increase or decrease in deliveries, of stretching a lot out over a longer period of time, and of several restrictions on production can be modeled by this procedure.