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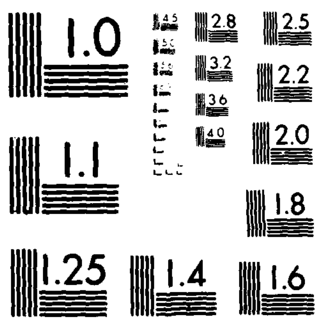

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PARTIAL CHARACTERIZATIONS OF COMPLETELY NONDETERMINISTIC

STOCHASTIC PROCESSES

by

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ABSTRACT

$x(t)$: t < or = 0

H superscript 2

A discrete weakly stationary Gaussian stochastic process $\{x(t)\}$, is completely nondeterministic if no non-trivial set from the SIGMA σ-algebra generated by $\{x(t):t>0\}$ lies in the SIGMA σ-algebra generated by $\{x(t):t\leq 0\}$. ~~In [8]~~ Levinson and McKean essentially showed that a necessary and sufficient condition for complete non-determinism is that the spectrum of the process is given by $|h|^2$ ABSOLUTE VALUE OF H SQUARED where h is an outer function in the Hardy space, H^2, of the unit circle in $\mathbb{C}$ C with the property that $h/\bar{h}$ h/h ba. uniquely determines the outer function h up to an arbitrary constant. In this paper we consider several characterizations of complete non-determinism in terms of the geometry of the unit ball of the Hardy space H^1 H superscript 1 and in terms of Hankel operators, and pose an open problem.

1. INTRODUCTION

In [10] Sarason defines a property of a discrete weakly stationary Gaussian stochastic process, $\{x(t)\}$, which he called complete nondeterminism. This condition is that no set from the future of the process (i.e. the σ-algebra generated by the random variables $x(t)$ for $t>0$) lies in the past (i.e. the σ-algebra generated by $x(t)$ for $t\leq 0$), except for null sets and the complements of null sets. In the spectral representation this condition becomes the following. Let m be the spectral measure of the process and let P denote the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n\leq 0$ where functions are defined on $\mathbb{T}$, the unit circle in $\mathbb{C}$. Let F denote the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n>0$. Then complete nondeterminism is equivalent to the condition that $P\cap F=\{0\}$. It is clear that this condition reflects a certain kind of independence (in a statistical sense) of the past, P, and the future, F.

It is of interest to characterize those measures m on $\mathbb{T}$ which lead to completely nondeterministic (cnd) processes. In [10] a necessary and sufficient condition for complete nondeterminism was stated as the measure m being absolutely continuous with respect to Lebesgue measure, $d\theta$, with $\log\frac{dm}{d\theta}$ integrable. Unfortunately this characterization is incorrect. In [8, p.105] Levinson and McKean essentially describe a partial characterization of cnd processes which we discuss in Section 3. This paper continues an investigation into the problem of characterizing spectral measures of cnd processes.

In Section 2 we examine the relationship between complete nondeterminism and some other familiar kinds of independence of P and F.

In Section 3 we restate the question in several ways which yield partial answers in terms of exposed points of the unit sphere of H^1 and certain Hankel operators.

The complete characterization of complete nondeterminism in terms of the spectral distribution function remains open and seems to be a hard question.

The authors are grateful to D.E. Sarason for some helpful correspondence on the topics of this paper.

2. COMPLETE NONDETERMINISM

A Gaussian process is called deterministic if its past determines the future, i.e., for each $t>0$, $x(t)$ is measurable with respect to the past. This is translated in the spectral representation to the property that $P=L^2(dm)$. A necessary and sufficient condition for this to occur is that $\log\frac{dm}{d\theta}$ be not integrable. Conversely the process is indeterministic if $\log\frac{dm}{d\theta}$ is integrable. A stronger restriction than indeterminism is that the process is purely indeterministic or regular. This is an asymptotic independence condition which, in the spectral representation, is equivalent to $\bigcap_{k=1}^{\infty} F_k=\{0\}$ where F_k is the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n\geq k$. This condition is often referred to by saying that the process has trivial remote future. Results of Szego [11], Kolmogorov [5] and Krein [6] show that $\{x(t)\}$ is regular if and only if m is absolutely continuous with respect to Lebesgue measure and $\log\frac{dm}{d\theta}$ is integrable. First we give an example of a process which is regular but not completely nondeterministic, thereby showing that the characterization in [10] is incorrect. First we establish some notation. L^1(resp.L^2) is the space of integrable (resp. square integrable) functions on $\mathbb{T}$. L^∞ is the space of essentially bounded functions on $\mathbb{T}$. We shall often regard functions in L^1 as extended harmonically into the open unit disc $D=\{z:|z|<1\}$ by means of Poisson's formula. We let H^1 denote those functions in L^1 which have analytic extensions into the disc. We define H^2 and H^∞ similarly. H^2 is a Hilbert space with orthornormal basis $\{z^n : n=0,1,2,\dots\}$. For standard results on the Hardy spaces we refer to [4].

For a regular process we can write $dm = w d\theta = |H| d\theta = |h|^2 d\theta$ where H is an outer function in H^1 and h is an outer function in H^2.

<u>Proposition 1</u>. There is a regular process which is not completely nondeterministic.

<u>Proof</u>. Let $w(e^{i\theta}) = |1+e^{i\theta}|^2 = |1+z|^2$ and put $dm = w d\theta$. Since $\log |1+z|^2 \in L^1$ this process is regular. However $(1+\bar{z})^{-1} \in \mathcal{P} \cap \mathcal{F}$. This follows since $1+z$ is outer. For we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} |1-p_n(1+z)|^2 dz = 0 \quad \text{for some sequence } p_n \text{ of polynomials in } z \, ;$$

$$\text{hence} \quad \int_{\mathbb{T}} |(1+\bar{z})^{-1} - z\, p_n|^2 |1+\bar{z}|^2 \, dz \to 0 \quad \text{as} \quad n\to\infty$$

$$\Rightarrow \int_{\mathbb{T}} |(1+\bar{z})^{-1} - z\, p_n|^2 \, |1+z|^2 \, dz \to 0 \quad \text{as } n\to\infty \, ;$$

i.e. $(1+\bar{z})^{-1} \in \mathcal{F}$.

Similarly $\int_{\mathbb{T}} |(1+\bar{z})^{-1} - \bar{p}_n|^2 \, |1+z|^2 \, dz$

$$= \int_{\mathbb{T}} |1-p_n(1+z)|^2 \, dz \to 0 \quad \text{as} \quad n\to\infty \, ; \text{ i.e. } (1+\bar{z})^{-1} \in \mathcal{P} .$$

We next obtain a simple necessary and sufficient condition for complete nondeterminism. It is straightforward to see that if m is singular with respect to Lebesgue measure then $P \cap F \neq \{0\}$. This, together with earlier comments means that in considering cnd processes we can restrict our attention to regular processes.

We wish to rephrase our question in terms of L^2 rather than $L^2(m)$. We have $dm=|h|^2 d\theta$. Consider the mapping $T:L^2(m) \to L^2$ given by $Tf=hf$. It is easily verified that T is an isometry of $L^2(m)$ onto L^2 . Also T maps F onto $H_0^2 = \{f \epsilon H^2 : f(0) = 0\}$, and T maps P onto $(h/\bar{h})\overline{H^2}$ where $\overline{H^2} = \{\bar{f} : f \epsilon H^2\}$.

<u>Proposition 2</u>. A process is not cnd if and only if $h/\bar{h} = \alpha(F/\bar{F})$ where $F \epsilon H^2$ is outer and α is inner with $\alpha(0)=0$.

<u>Proof</u>. Using the isometry T we see that $P \cap F \neq \{0\}$ if and only if there are non-zero functions g_1, g_2 in H^2 such that $zg_1 = (h/\bar{h})\bar{g}_2$

$$\iff z(g_1/h) = (\bar{g}_2/\bar{h}) \text{ and } z(g_2/h) = (\bar{g}_1/\bar{h})$$

$$\implies z(g_1+g_2)/h = (\overline{g_1+g_2})/\bar{h} \quad .$$

Hence $P \cap F \neq \{0\}$ if and only if there exists a function $G \epsilon H^2$ such that $zG/h = \bar{G}/\bar{h}$. If we use the inner-outer factorization of G then this equality becomes

$$z\phi F/h = \overline{\phi F}/\overline{h} \quad \text{where } \phi \text{ is inner and } F \epsilon H^2 \text{ is outer,}$$

$$\Rightarrow \quad h/\overline{h} = \alpha(F/\overline{F}) \quad \text{and} \quad \alpha(0)=0 \quad .$$

Conversely $h/\overline{h} = \alpha(F/\overline{F})$, $\alpha(0)=0$

$$\Rightarrow \quad (h/\overline{h})\overline{F} = z(\beta F) \quad \text{where} \quad \alpha = z\beta$$

$$\Rightarrow \quad P \cap F \neq \{0\} \text{ by the above.}$$

The same reasoning yields the following result for $k \geq 1$:

$P \cap F_k^- \neq \{0\} \Longleftrightarrow h/\overline{h} = \alpha(F/\overline{F})$ where $F \epsilon H^2$ is outer and α is inner with α having a zero at the origin of order at least k .

Another strictly stronger property than regularity is that of minimality. Introduced by Kolmogorov [5] this property says that a process is minimal if the value of the random variable $x(0)$ cannot be predicted without error from the values of the random variables $\{x(t):t \neq 0\}$. In other words a process is not minimal if it is possible to perfectly interpolate any value of the process from knowledge of the remaining values of the process. Kolmogorov [5] proved that a process is minimal if and only if w^{-1} is in L^1 .

It is immediately of interest to examine the relationship between minimal processes and completely nondeterministic processes.

<u>Proposition 3</u>. If the process $\{x(t)\}$ is minimal then it is completely nondeterministic. On the other hand there exist completely nondeterministic processes which are not minimal.

<u>Proof</u>. Suppose $\{x(t)\}$ is minimal. Then by Kolmogorov's theorem $h^{-1} \in H^2$. Using Proposition 2 we argue by contradiction. For suppose $\{x(t)\}$ is not completely nondeterministic. Then $(h/\overline{h}) = \alpha(f/\overline{f})$ where f is outer and α is inner with $\alpha(0)=0$. This equality implies $\overline{f}/\overline{h} = \alpha(f/h)$. The LHS is in $\overline{H^1}$ and the RHS is in H_0^1 which forces both sides to be zero and thus $f=0$ which is a contradiction. This proves the first statement of the proposition. An example of a process which yields the second statement is given by $w=|1+z|$. In this case $h=(1+z)^{1/2}$ and $h/\overline{h} = z^{1/2}$. By Kolmogorov's criterion this process is not minimal. On the other hand suppose $h/\overline{h} = \alpha(f/\overline{f})$ for f outer, α inner with $\alpha(0)=0$. Then

$$z^{1/2} = \alpha(f/\overline{f}) = z\phi(f/\overline{f}) \quad \text{with } \phi \text{ inner}$$

$$\Rightarrow z^{1/2}\phi f = \overline{f}$$

$$\Rightarrow z(\phi f)^2 = (\overline{f})^2 .$$

The LHS is in H_0^1 and the RHS is in $\overline{H^1}$. Again this forces both sides to be zero and hence $f=0$ which gives a contradiction. Thus the process with $w=|1+z|$ is completely nondeterministic.

Let P_k be the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n \leq k$. A minimal process is one for which the function 1 does not belong to the closed linear span of P_1 and F_1 i.e. $1 \notin P_1 \vee F_1$. There is a similar restatement of the condition of complete nondeterminacy. Let $P_1 + F_1 = \{f \in L^2(m): f = g+h$ with $g \in P_1, h \in F_1\}$.

<u>Proposition 4</u>. A regular process is completely nondeterministic if and only if $1 \notin P_1 + F_1$.

<u>Proof</u>. Assume f is a non-zero element of $P \cap F$. Then, for some $k \geq 1$, $f \in F_k$ but $f \notin F_{k+1}$ (since $\bigcap_{k=1}^{\infty} F_k = \{0\}$). Hence $f = ae^{ik\theta} + f_1$ where $a \neq 0$ and $f_1 \in F_{k+1}$. This implies $e^{ik\theta} = (f-f_1)/a \in P + F_{k+1}$

$$\Rightarrow 1 \in (e^{-ik\theta} P) + F_1 \subseteq P_1 + F_1 .$$

Conversely assume that $1 \in P_1 + F_1$. Then $1 = f_1 + f_2$ with $f_1 \in P$, $f_2 \in F$. Hence $e^{i\theta} f_1 = e^{i\theta} - e^{i\theta} f_2 \in F$. But $e^{i\theta} f_1 \in P$. Hence $e^{i\theta} f_1 \in P \cap F$.

We complete this section by establishing a simple sufficient condition for $P \cap F_n$ to be non-trivial.

<u>Proposition 5</u>. Suppose that $w = |p|^2 w_1$, where p is a trigonometric polynomial of degree n with all its zeros in the closed unit disc, and $w_1 \in L^1$. Then $P \cap F_n \neq \{0\}$.

<u>Proof</u>. We show that $1/\bar{p} \in P \cap F_n$.

(i) $1/\overline{p}\varepsilon P$: without loss of generality we can assume that

$$1/\overline{p} = \prod_{j=1}^{m} (1-\overline{z}/\overline{\zeta}_j)^{-n_j} \qquad |\zeta_j|\leq 1$$

$$= \sum_{j=1}^{m} q_{n_j-1}(\overline{z})(1-\overline{z}/\overline{\zeta}_j)^{-n_j}$$

where q_{n_j-1} is a polynomial.

Now $(1-\overline{z}/\overline{\zeta}_j)^{-n_j}$ can be approximated by polynomials in $\overline{z}$ in $L^2(m)$. In fact

$$\int_{\mathbb{T}} |(1-\overline{z}/\overline{\zeta}_j)^{-n_j} - \{1+\frac{m-1}{m}(\overline{z}/\overline{\zeta}_j)+\frac{m-2}{m}(\overline{z}/\overline{\zeta}_j)^2$$

$$+\ldots+1/m(\overline{z}/\overline{\zeta}_j)^{m-1}\}^{n_j}|^2 \; w(\theta)d\theta$$

$$= \int_{\mathbb{T}} |1-[(1-\overline{z}/\overline{\zeta}_j)\{1+\frac{m-1}{m}(\overline{z}/\overline{\zeta}_j)+\ldots+1/m(\overline{z}/\overline{\zeta}_j)^{m-1}\}]^{n_j}|^2 \; w_2(\theta)d\theta$$

$$\text{where} \quad w_2 = w/(1-z/\zeta_j)^{n_j}$$

$$= \int_{\mathbb{T}} |1-[1-1/m\{\overline{z}/\overline{\zeta}_j+\ldots+(\overline{z}/\overline{\zeta}_j)^m\}]^{n_j}|^2 \; w_2(\theta)d\theta$$

$\to 0$ as $m\to\infty$ by Lebesgue's dominated convergence theorem.

Hence $1/\overline{p} \;\varepsilon\; P$;

(ii) $1/\overline{p} \;\varepsilon\; F_n : 1/\overline{p} = z^n/z^n\overline{p} = z^n/q_n$ where $q_n = z^n\overline{p}$ is also a polynomial of degree n in z. The same construction as in (i) shows that $1/q_n$ can be approximated by polynomials in z in $L^2(m)$. Hence $1/q_n \;\varepsilon\; F_o$ and $1/\overline{p} \;\varepsilon\; F_n$.

Remark. This proposition implies that if we restrict our attention to cnd processes then the strong mixing condition implies the property that P and F_1 be at positive angle; (see [3], [10, p.77] for definitions). For if the angle between P and F_n is converging to $\pi/2$ as $n\to\infty$ then, for some k, P and F_k are at a positive angle which implies by [3] that $w=|p|^2 w_1$ for some trigonometric polynomial p where w_1 is the spectrum of a process for which P and F_1 are at a positive angle. If the process is cnd then Proposition 5 implies that p must have zero degree. In general the strong mixing condition does not imply that P and F_1 are at positive angle (e.g. take $h=1+z$).

3. EXPOSED POINTS OF THE BALL IN H^1 AND HANKEL OPERATORS

It is well known (see [7]) that the extreme points of the unit ball of H^1 are given by the outer functions F in H^1 with $\|F\|_1 = 1$. It is also well known that an H^1 function F of unit norm is not determined by its argument.

In [8, p.205] Levinson and McKean showed that for continuous processes the dimension of $P \cap F_0 = 1$ if and only if $h/\bar{h}$ determines the outer function h up to a constant. In this section we consider this approach which is closely related to the results of Section 2 and consider this characterization in geometrical terms.

In their study of extremum problems in H^1 deLeeuw and Rudin introduced the following sets of H^1 functions indexed by unimodular L^∞ functions. Let $\phi \varepsilon L^\infty$ with $|\phi| = 1$ almost everywhere and define

$$S_\phi = \{F \varepsilon H^1 : \|f\|_1 = 1 , \frac{F}{|F|} = \phi \text{ almost everywhere}\} .$$

Geometrically S_ϕ is the intersection of the ball of H^1 and the hyperplane $\{F \varepsilon H^1 : \int \phi F d\theta = 1\}$ and so S_ϕ is a convex set (which may be empty, in general). When S_ϕ contains exactly one function F, the hyperplane touches the ball of H^1 only at F which means that F is an exposed point of the ball of H^1. (In fact the definition of S_ϕ we have given corresponds to $S_{\bar{\phi}}$ as defined by deLeeuw and Rudin.)

<u>Proposition 6</u>. Let $w=|H|=|h|^2$. Without loss of generality assume that $\int w d\theta=1$. The following statements are equivalent:

(1) $\{x(t)\}$ is completely nondeterministic

(2) $S_{h/\bar{h}}$ contains exactly one function

(3) $h^2=H$ is an exposed point of the unit ball in H^1 .

<u>Proof</u>. Note that $S_{h/\bar{h}}$ always contains h^2 , so that our comments above show the equivalence of (2) and (3). Now suppose that $\{x(t)\}$ is not completely nondeterministic. By Proposition 2 $h/\bar{h} = \alpha(F/\bar{F})$ where α is inner and $\alpha(0)=0$ and $F\varepsilon H^2$ is outer. Hence $\frac{\alpha F^2}{|\alpha F^2|} = \alpha\frac{F^2}{|F|^2} = \alpha\frac{F}{\bar{F}} = h/\bar{h}$.

Thus a positive multiple of αF^2 is in $S_{h/\bar{h}}$. But $a(\alpha F^2) \neq h^2$ for any $a>0$ since α has a zero at the origin. Hence $S_{h/\bar{h}}$ contains more than one function. Conversely suppose $S_{h/\bar{h}}$ contains more than one function. Then, by Theorem 9 of [7] $S_{h/\bar{h}}$ contains a function f with $f(0)=0$. Write $f=bF^2$ where b inner, $b(0)=0$, and $F\varepsilon H^2$ is outer. Now $f\varepsilon S_{h/\bar{h}}$ implies that $h/\bar{h} = bF/\bar{F}$ which, by Proposition 2, shows that $\{x(t)\}$ is not completely nondeterministic.

A similar result is given in the following proposition for $k\geq 1$.

<u>Proposition 7</u>. $P\cap F_k \neq \{0\}$ if and only if there is a function $f\varepsilon S_{h/\bar{h}}$ where f has k zeros (counting multiplicities) in the open unit disc.

Proof. By Proposition 2 $\mathcal{P}\cap F_k \neq \{0\}$ implies that $h/\bar{h} = z^k\phi(F/\bar{F})$ where ϕ is inner and $F \in H^2$ is outer. As in the proof of Proposition 6 it follows that $z^k\phi F^2 \in S_{h/\bar{h}}$. Conversely if $f \in S_{h/\bar{h}}$ and $f(z_1)=f(z_2)=\ldots=f(z_k)=0$ where $z_j \in D$ $(1\leq j\leq k)$ then it is easy to verify that a positive multiple of $g(z) = z^k f(z) \prod_{j=1}^{k}(z-z_j)^{-1}(1-\bar{z}_j z)^{-1}$ is in $S_{h/\bar{h}}$. Factorize g as $g=z^k bF$ where b is inner and $F \in H^2$ is outer. Since $ag \in S_{h/\bar{h}}$ for some $a>0$ it follows that $h/\bar{h}=z^k bF/\bar{F}$ showing that $\mathcal{P}\cap F_k \neq \{0\}$.

Note that Proposition 6 yields the version of the Levinson and McKean result as applied to cnd processes: namely, a process is cnd if and only if $\arg(h/\bar{h})$ is the argument of a unique H^1 function.

Since we have expressed the characterization of completely nondeterministic processes in terms of an extremum problem it is not surprising that there is a version of the problem in terms of the norms of Hankel operators which are closely related to extremum problems on H^1.

Let P be the orthogonal projection of L^2 onto H^2. Recall that the Hankel operator with symbol $\phi \in L^\infty$ is the bounded operator from H^2 to $L^2 \ominus H^2$ defined by

$$H_\phi(f) = (I-P)(\phi f) \qquad (f \in H^2).$$

The norm of H_ϕ is given by $\|H_\phi\| = d(\phi, H^\infty) = \inf_{f\in H^\infty} \|\phi - f\|_\infty$.

It is straightforward to show from first principles that the process $\{x(t)\}$ is completely nondeterministic if and only if $H_{h/\bar{h}}$ attains the norm of 1 (on the unit sphere of H^2). In fact more is true.

In [1] it is essentially shown that H_ϕ attains its norm on the unit sphere on H^2 if and only if $\phi=f+\lambda\psi$ where $f\epsilon H^\infty$, $\lambda>0$ and $|\psi|=1$ a.e on $\mathbb{T}$ with $S_{\bar{\psi}}$ containing more than one function. Also if $\|\phi\|_\infty=1$ then H_ϕ attains the norm 1 if and only if $|\phi|=1$ a.e on $\mathbb{T}$ and $S_{\bar{\phi}}$ contains more than one function [1]. There is another result of this type which does not seem to have appeared in the literature.

<u>Proposition 8</u>. $\|H_\phi\|<\|\phi\|_\infty \Rightarrow \phi = f+\lambda\psi$ where $f\epsilon H^\infty$, $\lambda>0$ and $|\psi|=1$ a.e on $\mathbb{T}$ with $S_{\bar{\psi}}$ containing exactly one function.

<u>Proof</u>. Without loss of generality we assume that $\|\phi\|_\infty=1$. Suppose $\|H_\phi\|<1$. Then by [2] there exists $\psi\epsilon L^\infty$ such that (i) $\phi-\psi\epsilon H^\infty$ and (ii) $\psi=\bar{F}/|F|$ for some $F\epsilon H^1$, $F\neq 0$. Now (i) $\Rightarrow H_\psi=H_\phi$ and so $\|H_\psi\|<1$. So there exists $g\epsilon H^\infty$ such that $\|(\bar{F}/|F|)-g\|_\infty= a<1$ which gives that $|\arg(gF)|<b<\pi/2$. Hence $(gF)^{-1}\epsilon H^1$ (since $gF\neq 0$ on D and if G is analytic on D and $|\arg G|<b<\pi/2$ then $G\epsilon H^p$ for all $p<\pi/2b$) . Thus $g(gF)^{-1}\epsilon H^1$ $\Rightarrow F^{-1}\epsilon H^1$. Now $F/|F|=\bar{\psi}$ so that a positive multiple of F is in $S_{\bar{\psi}}$. Then $F^{-1}\epsilon H^1$ implies that $S_{\bar{\psi}}$ contains one and only one function (if $G\epsilon S_\beta$ and $G^{-1}\epsilon H^1$ then $S_\beta=\{G\}$ - See [7, Theorem 8] and use the fact that positive $H^{1/2}$ functions are constant).

Note however that $S_{\overline{\psi}}$ containing exactly one function does not necessarily imply that $\|H_\psi\| < \|\psi\|_\infty$. For example if $h=(1+z)^{1/2}$, and we take $\psi=\overline{h}/h$ it can be shown that $\|H_\psi\|=1$ but, as we saw in the proof of Proposition 3, $|h|^2$ corresponds to a cnd process so that $S_{h/\overline{h}} = \{h^2\}$.

4. P and F_k : AN OPEN QUESTION

There is an interesting set of results in [9] which describes the relationship between minimal processes and those processes which may not be minimal but, for some fixed k, do not allow perfect interpolation of k "missing" values of the process.

Call a process <u>k-minimal</u> if the k functions $1, e^{i\theta}, \ldots, e^{(k-1)i\theta}$ do not all belong to the closed linear span of P_1 and F_k. The extension of Kolmogorov's result given in [9] is that a process with spectrum w is k-minimal if and only if there exists a polynomial $p(e^{i\theta})$ such that $\int_{\mathbb{T}} \frac{|p(e^{i\theta})|^2}{w(e^{i\theta})} d\theta < \infty$

where the degree of the polynomial p is strictly less than k and we may assume that the zeros of p are all on $\mathbb{T}$. Thus if w is the spectrum of a k-minimal process then $w=|p|^2 w_1$ where w_1 is the spectrum of a minimal process and p is a trigonometric polynomial with degree $<k$ and zeros all on $\mathbb{T}$.

It would be satisfying to have a similar theory relating processes for which $P \cap F_k = \{0\}$ with completely nondeterministic processes. We know that $P \cap F_k \neq \{0\}$ if and only if $h/\bar{h} = \alpha(F/\bar{F})$ where $F \in H^2$ is outer and α is inner with a zero at the origin of multiplicity at least k.

The fact that $p/\bar{p} = \lambda z^k$ for some constant λ with $|\lambda| = 1$ when p is a trigonometric polynomial of degree k, together with our comments above also shows that if a process is k-minimal then $P\ F_k = \{0\}$. We also remark that for a cnd process k-minimal processes are automatically minimal by the same reasoning as in the remark following Proposition 5.

Proposition 9. Suppose $P \cap F_k = \{0\}$ but $P \cap F_{k-1} \neq \{0\}$. Then $h/\bar{h} = \lambda z^k F/\bar{F}$ where F is outer and $|\lambda| = 1$.

Proof. $P \cap F_{k-1} \neq \{0\}$ implies that

$$h/\bar{h} = z^{k-1}\alpha(F/\bar{F}) \quad \text{where} \quad F \in H^2 \text{ is outer,} \quad \alpha \text{ is inner.}$$

Suppose that α is non-constant. Then we can find constants a, b such that $0 \neq aF + b(\alpha F) \in H_0^2$. Then

$$z^{k-1}(aF + b\alpha F)/h = \overline{(\bar{a}\alpha F + \bar{b}F)}/\bar{h} \quad .$$

The LHS ε F_k and the RHS ε P. Hence $P \cap F_k \neq \{0\}$. This contradiction implies that α is a constant λ.

The result we are aiming for is that if $P \cap F_k = \{0\}$ then $w = |p|^2 w_1$ where p is a trigonometric polynomial of degree $<k$ with all its zeros on $\mathbb{T}$ and w_1 is the spectrum of a completely nondeterministic process. (We already know by Proposition 5 that for such a process $P \cap F_j \neq \{0\}$ if j is the degree of p.) Proposition 10 provides a partial answer. First recall that h is a strong outer function [7] if $h/(z-\lambda) \notin H^2$ for all $\lambda \in \mathbb{T}$. For simplicity we will simply look at processes for which $P \cap F_1 \neq \{0\}$ and $P \cap F_2 = \{0\}$.

Proposition 10. The following are equivalent

(i) $w = |p|^2 w_1$ where p is a trigonometric polynomial of degree 1 with its zero on $\mathbb{T}$ and w_1 is the spectrum of a cnd process.

(ii) $P \cap F_2 = \{0\}$ and h is not strong outer.

<u>Proof</u>. Suppose (i). Suppose $h/\bar{h}=z^2\alpha(F/\bar{F})$ for α inner, $F \epsilon H^2$, outer. But $|h|^2=|p|^2|h_1|^2 \Rightarrow h=ph_1 \Rightarrow h/\bar{h} = zh_1/\bar{h}_1$

$$\Rightarrow h_1/\bar{h}_1 = z\alpha(F/\bar{F}) ,$$

contradicting the fact that $|h_1|^2$ corresponds to a cnd process. Trivially (i) $\Rightarrow$ h is not strong outer.

Conversely suppose (i) does not hold, i.e. $h \neq p_\lambda h_1$ for any trigonometric polynomial $p_\lambda = z-\lambda$ ($\lambda \epsilon \mathbb{T}$) and outer function h_1 corresponding to a cnd process. Then either $h/p_\lambda \notin H^2$ for all $\lambda \epsilon \mathbb{T}$ i.e. h is strong outer or $h=p_\lambda h_1$ but h_1 does not correspond to a cnd process, i.e. $h_1/\bar{h}_1 = z\alpha(F/\bar{F})$ for α inner, $F \epsilon H^2$ outer. Thus $h/\bar{h} = z(h_1/\bar{h}_1) = z^2\alpha(F/\bar{F})$, i.e. $P \cap F_2 = \{0\}$.

The missing link that we require leads us to suspect that h strong outer together with $P \cap F_1 \neq \{0\} \Rightarrow P \cap F_2 \neq \{0\}$. In fact we make the following conjecture.

<u>Conjecture 1</u>. h strong outer, $P \cap F_1 \neq \{0\} \Rightarrow P \cap F_k \neq \{0\}$ for all $k \geq 1$.

We finish by translating this conjecture into the language of Section 3. In [7] it was shown that $S_{h/\bar{h}} = \{h^2\}$ implies that h is strong outer. The following conjecture would tell us that if h strong outer does not imply $S_{h/\bar{h}} = \{h^2\}$ then $S_{h/\bar{h}}$ must contain many functions.

<u>Conjecture 2</u>. Suppose S_ϕ contains more than one function, one of which is strong outer. Then S_ϕ contains a function with an inner factor which is not a finite Blaschke product.

If Conjecture 2 is correct then so is Conjecture 1 for the following reason. Suppose h is strong outer and $P \cap F_1 \neq \{0\}$. Then $h^2 \varepsilon S_{h/\bar{h}}$ and so if Conjecture 2 is correct $S_{h/\bar{h}}$ contains a function with an inner factor which is not a finite Blaschke product. [7, Lemma 4.6] shows that this gives a function $g \varepsilon S_{h/\bar{h}}$ with infinitely many zeros in D. Proposition 7 then shows that $P \cap F_k \neq \{0\}$ for all $k \geq 1$. Note that it is easy to construct examples of processes for which $P \cap F_k \neq \{0\}$ for all $k > 1$. In fact by the reasoning above $w = |1+k|^2$ gives such an example if k is an inner function which is not a finite Blaschke product.

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A discrete weakly stationary Gaussian stochastic process $\{x(t)\}$, is completely nondeterministic if no non-trivial set from the σ-algebra generated by $\{x(t):t>0\}$ lies in the σ-algebra generated by $\{x(t):t\leq 0\}$. In [8] Levinson and McKean essentially showed that a necessary and sufficient condition for complete nondeterminism is that the spectrum of the process is given by $|h|^2$ where h is an outer function in the Hardy space, H^2, of the unit circle in $\mathbb{C}$ with the property that $h/\bar{h}$ uniquely

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) determines the outer function h up to an arbitrary constant. In this paper we consider several characterizations of complete non-determinism in terms of the geometry of the unit ball of the Hardy space H^1 and in terms of Hankel operators, and pose an open problem.

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