SOME RESULTS ON THE OVERALL RELIABILITY OF UNDIRECTED GRAPHS (U)

FEB 81
A SATYANARAYANA, M K CHANG, Z S KHALIL
N00014-75-C-0781

UNCLASSIFIED
ORC-81-2
MICROCOPY RESOLUTION TEST CHART

1.0  1.25  2.2
1.1  2.0  1.8
1.25 1.4  1.6
SOME RESULTS ON THE OVERALL RELIABILITY
OF UNDIRECTED GRAPHS

by

A. Satyanarayana and Mark K. Chang
Operations Research Center
University of California, Berkeley

and

Zohel S. Khalil
Department of Mathematics
Concordia University
Montreal, Canada

This research has been partially supported by the National Science Foundation under Grant PFR-7822265, the Office of Naval Research under Contract N00014-75-C-0781 with the University of California and the Natural Sciences and Engineering Research Council, Grant No. A9095, Canada. Reproduction in whole or in part is permitted for any purpose of the United States Government.
SOME RESULTS ON THE OVERALL RELIABILITY OF UNDIRECTED GRAPHS

A. Satyanarayana, Mark K. Chang, Zohal S. Khalil

PERFORMING ORGANIZATION NAME AND ADDRESS
Operations Research Center
University of California
Berkeley, California 94720

CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Department of the Navy
Arlington, Virginia 22217

 DISTRIBUTION STATEMENT (of this report)
Approved for public release; distribution unlimited.

SUPPLEMENTARY NOTES
(SEE TITLE PAGE)

KEY WORDS (Enter as many as appropriate and identify by block number)
Overall Reliability
Undirected Graphs
Biconnected Graph

ABSTRACT (Enter as many as appropriate and identify by block number)
(SEE ABSTRACT)
ABSTRACT

A probabilistic graph consists of vertices and links that fail with some known probabilities. For such a graph, overall reliability is the probability that there exists communication between all vertex-pairs. In this paper, some useful results are presented to simplify the overall reliability computation of an undirected graph when the failure events of the links are statistically independent.
Some Results on the Overall Reliability of Undirected Graphs

by

A. Satyanarayana, M. Chang and Z. Khalil

1. Introduction

A probabilistic graph consists of a set of vertices and links that fail with some known probability distribution. In reliability analysis of such a graph, one of the problems of considerable interest is the overall reliability. For a probabilistic graph, overall reliability is the probability that there exists communication between all vertex-pairs.

Wing and Demetriou [1] studied this problem by considering all possible vertex-pairs in the graph but the approach is not tractable for large graphs. Fu [2] provided approximate solutions using the well known techniques of electrical network analysis. Bell and Van Slyke [3] proposed a method of enumerating modified cutsets by backtracking. Satyanarayana [4], Satyanarayana and Hagstrom [5] made topological observations and proposed subgraph enumeration algorithms to compute the overall reliability. However all these methods have worst case computation time that grows exponentially in the size of the graph. Furthermore, this problem is believed to possess no algorithms with a polynomial time bound [6]. Therefore, it is essential to explore the use of graph partitioning techniques for efficient computation of the overall reliability. Graph partitioning may involve splitting the graph into reduced graphs, each analyzed separately and then the results combined. When the computational complexity grows more than linearly in the size of the graph, substantial savings may be achieved using reduced graphs instead of the given large graph.

In this paper, we present some results concerning the overall reliability of an undirected
graph in which the failure events of links are assumed to be statistically independent. Further ramifications of these results, the algorithmic implementation and computational complexity will be discussed elsewhere [7]. Suppose $G$ is an undirected graph and $m$ is a vertex of degree $k > 1$ in $G$. In section 2, we present a scheme for decomposing $G$ into $\left( \frac{k}{2} \right) + 1$ reduced graphs, and express the reliability of $G$ in terms of these reduced graphs. In section 3, we present two simple results which are useful in expressing the reliability of a biconnected graph in terms of the reliabilities of its subgraphs.

2. Overall Reliability of an Undirected Graph

One of the earliest decomposition techniques is that of factoring the graph with respect to a particular link [8,9]. For a given undirected graph $G$, two reduced graphs $G-e$ and $G_e$ are obtained by deleting and contracting respectively a link $e$ in $G$. Denote by $R(G)$ the overall reliability of $G$, and $Pr(e)$ by $p_e$.

Property 1:

$$R(G) = p_e R(G_e) + (1-p_e) R(G-e).$$

Consider the example graph $G$ of Fig. 1. Using Property 1, the overall reliability

$$R(G) = p_1(p_1 + p_2 - p_1p_2) + (1-p_1)p_1p_2).$$

For a complete solution of $R(G)$ when $G$ is larger, Property 1 may be repeatedly applied to the reduced graphs. Because of the binary structure of this procedure, the computation tree grows exponentially in the size of $G$. Simple techniques for reducing the size of the computation tree are therefore important. Besides the obvious savings obtained by not factoring on essential or irrelevant links, replacement of parallel links by a single link with the appropriate probability is another simple reduction [9]. Bell [10] and Johnson [11] independently have shown that the complexity of an algorithm based on repeated application of Property 1 and performing parallel reductions is of order $(n-1)!$ for a complete graph on $n$ vertices. This algorithm in [10, 11] is perhaps the best known for computing overall reliability.

In what follows, we present a decomposition scheme in which the graph is factored with
FIGURE 1

Example Graphs $G$, $G - e$ and $G_{e_3}$
respect to a particular vertex. This scheme results in a better computational bound than in [10,11]. In section 3 we introduce another decomposition technique to be applied on a biconnected graph.

Let \( m \) be a vertex of degree \( k > 1 \) in \( G \) such that \( e_1, e_2, \ldots, e_k \) are the links incident on \( m \). Also, without loss of generality, let \( K = \{1, 2, \ldots, k\} \) be the other end vertices of links \( e_1, e_2, \ldots, e_k \). Denote by \( G - m \) the subgraph obtained by deleting vertex \( m \) and its incident links from \( G \). Let \( X \subseteq K \) and \( G_X \) be the reduced graph obtained by coalescing vertices of \( X \) in \( G - m \). We have the following lemma:

Lemma 1:

\[
R(G) = \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \prod_{j=1}^{k} (1 - p_i) R(G - m) + \sum_{X \subseteq K} \prod_{i \in X} p_i \prod_{i \notin X} (1 - p_i) R(G_X),
\]

where \( p_j = Pr(e_j) \).

Proof: By deleting and contracting links \( e_1, e_2, \ldots, e_k \) successively from vertex \( m \) in \( G \) and applying Property 1 repeatedly on the resulting graphs, we can easily show that

\[
R(G) = \sum_{X \subseteq K} \prod_{i \in X} p_i \prod_{i \notin X} (1 - p_i) R(G_X).
\]

Since \( G_X = G - m \) for \( |X| = 1 \), we have the lemma. QED.

To illustrate Lemma 1, consider the example \( G \) of Fig.2. We factor \( G \) on vertex 4. Vertices 1, 2 and 3 are adjacent to 4. \( G - m \) and all possible graphs obtained from \( G - m \) by coalescing 1, 2, 3 in all possible combinations taking two vertices and three vertices at a time are shown in Fig.2. By Lemma 1 the overall reliability of \( G \) can be written as

\[
R(G) = p_1(1-p_2)(1-p_3)+p_2(1-p_1)(1-p_3)+p_3(1-p_1)(1-p_2)R(G - m) + p_1p_2(1-p_3)R(G_{1,2,3})+p_1p_3(1-p_2)R(G_{1,3,4})+p_2p_3(1-p_1)R(G_{2,3,4})
\]

Clearly, for a given vertex \( m \) of degree \( k \), the number of possible nonempty subsets \( X \) is \( 2^k - 1 \). This is the number of reduced graphs obtained if \( G \) and the reduced graphs are
FIGURE 2
Illustration for Lemma 1
successively factored w.r.t. links $e_1, e_2, ..., e_k$ using Property 1 until none of them contain these links. Among the $2^k-1$ graphs, $k-1$ of them correspond to $G-m$. Lemma 1 recognizes this fact and the number of reduced graphs obtained by factoring $G$ on $m$ is $2^k - k$. For computing the overall reliability of $G$, application of Lemma 1 thus provides a significant computational advantage over Property 1. In fact, we can do even better when we realize that the $2^k - k - 1$ terms in the second sum correspond to the partition of the event that at least two links out of $\{e_1, e_2, ..., e_k\}$ are functional. By the following theorem, we show that the number of reduced graphs need to be considered is $\left(\frac{k}{2}\right) + 1$.

**Theorem 1**: The overall reliability of $G$,

$$R(G) = \left(\sum_{i=1}^{k} \prod_{j=1}^{i-1} (1-p_j)\right) R(G-m) + \sum_{i<j} \prod_{l \neq i, j} (1-p_l) R(\hat{G}_{i,j}),$$

where $\hat{G}_{i,j}$ is the graph obtained from $G$ by contracting links $e_i$ and $e_j$ ($i < j$) and deleting links $e_q$ ($i \neq q \neq j$).

**Proof**: Let $\xi_1$ and $\xi_2$ be the events that out of $\{e_1, e_2, ..., e_k\}$, exactly one link functions and at least two links function, respectively. Then,

$$R(G) = \Pr\{\text{all vertices in } G \text{ are connected } \cap \xi_1\} + \Pr\{\text{all vertices in } G \text{ are connected } \cap \xi_2\}.$$

We know $\Pr(\xi_1) = \sum_{i=1}^{k} \prod_{j=1}^{i-1} (1-p_j)$. For $\xi_2$ to occur, one or more of the pairs $\{e_i, e_j\}$, $i < j$, must be functional. Applying the formula $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap \bar{A})$ on the lexicographic ordering of these link pairs, we obtain a partition of $\xi_2$:

$$\Pr(\xi_2) = \sum_{i<j} \prod_{l \neq i, j} (1-p_l).$$

Conditioning on this partition and that for $\xi_1$, we have

$$R(G) = \sum_{i=1}^{k} \prod_{j=1}^{i-1} (1-p_j) \Pr\{\text{all vertices in } G \text{ are connected} \mid e_i \text{ functions but not } e_j, j \neq i\}$$

$$+ \sum_{i<j} \prod_{l \neq i, j} (1-p_l) \Pr\{\text{all vertices in } G \text{ are connected} \mid e_i \text{ and } e_j \text{ function but not } e_q, i \neq q < j\}.$$
We now illustrate Theorem 1 using the graph of Fig.2. Reduced graphs \((G-m)\), \(G_{1}^{11,31}\), \(G_{2}^{11,31}\) and \(G_{3}^{11,31}\) obtained by factoring \(G\) on vertex 4 are shown in Fig.3. By Theorem 1 we have

\[
R(G) = p_{1}(1-p_{2})(1-p_{3}) + p_{2}(1-p_{1})(1-p_{3}) + p_{3}(1-p_{1})(1-p_{2}) R(G-m) + p_{1}p_{2} R(G_{1}^{11,31}) + p_{1}p_{3}(1-p_{2}) R(G_{2}^{11,31}) + p_{2}p_{3}(1-p_{1}) R(G_{3}^{11,31})
\]

\[
= p_{1}p_{2} (p_{1}(1-p_{2})(1-p_{3}) + p_{2}(1-p_{1})(1-p_{3}) + p_{3}(1-p_{1})(1-p_{2}))
\]

\[+ (p_{1} + p_{2} - p_{1}p_{2})p_{1}p_{2} + (p_{3} + p_{2} - p_{2}p_{3})p_{1}p_{2}(1-p_{2}) + p_{1}p_{2}p_{3}(1-p_{1}).
\]

### 3. Overall Reliability of a Biconnected Graph

For a connected graph \(G\), a vertex \(a\) is a cut-vertex if removing \(a\) splits \(G\) into two or more parts. Suppose \(G_{1}, G_{2}, \ldots, G_{k}\) are the subgraphs of \(G\) which partition the links of \(G\) and furthermore, have a cut-vertex of \(G\) in common among them. The overall reliability of \(G\) can then be easily expressed as a product of the overall reliabilities of \(G_{1}, G_{2}, \ldots, G_{k}\).

\(G\) is biconnected if it contains no cut-vertices, but there exists a pair of vertices whose deletion disconnects \(G\). To emphasize the fact that deletion of a pair of vertices disconnects \(G\), it is sometimes referred to as a strictly biconnected graph. Suppose \(G_{1}\) and \(G_{2}\) are the two subgraphs of \(G\) such that \(G_{1} \cup G_{2} = G\) and \(G_{1} \cap G_{2}\) contains only two vertices \(i\) and \(j\). Let \(G_{i}\) and \(G_{j}\) denote the graphs obtained by coalescing \(i\) and \(j\) in \(G_{1}\) and \(G_{2}\) respectively. We have the following theorem, which may be viewed as a generalization of Property 1.

**Theorem 2:** The overall reliability of \(G\),

\[
R(G) = R(G_{1})R(G_{2}) + [R(G_{1}) - R(G_{1})]R(G_{2}).
\]

**Proof:** The proof is by induction on the number of links in \(G_{2}\).

Suppose \(G_{2}\) has exactly one link \(e\). Using Property 1 on \(e\) we have

\[
R(G) = p_{e}R(G_{1}) + (1-p_{e})R(G_{1}).
\]

Since \(R(G_{1}) = p_{e}\) and \(R(G_{2}) = 1\), we have

\[
R(G) = R(G_{1})R(G_{2}) + R(G_{2})R(G_{1}) - R(G_{1})R(G_{2}).
\]

and the theorem is true.
FIGURE 3
Illustration for Theorem 1
Suppose the theorem is true for any \( G_2 \) with \( k \) links. Consider a graph \( G \) such that its \( G_2 \) has \( k+1 \) links. Let \( e \) be a link in \( G_2 \) and \( G_e \) be the graph obtained by contracting \( e \) in \( G \). By Property 1,

\[
R(G) = p_e R(G_e) + (1-p_e) R(G-e).
\]

Note that deletion of \( i \) and \( j \) still disconnects \( G_e \) and of course \( G-e \). Let \( G_e \) be partitioned as \( G_1 \) and \( G_2 \), w.r.t. vertices \( i \) and \( j \), where \( G_2 \) is identical to the graph obtained from \( G_2 \) by contracting \( e \). \( G_2 \) has \( k \) links and hence the theorem is true for \( G_e \) and we have

\[
R(G_2) = R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2),
\]

where \( \hat{G}_2 \) is the graph obtained from \( G_2 \) by coalescing \( i \) and \( j \). Similarly, \( G-e \) can be partitioned w.r.t. \( i \) and \( j \) and let the subgraphs be \( G_1 \) and \( G_2-e \). Here again \( G_2-e \) has \( k \) links and hence we have

\[
R(G-e) = R(G_1)R(\hat{G}_2) + R(G_2-e)R(\hat{G}_1) - R(G_1)R(G_2-e),
\]

where \( \hat{G}_2 \) is the graph obtained from \( G_2-e \) by coalescing \( i \) and \( j \). Therefore,

\[
R(G) = p_e [R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2)] + (1-p_e) [R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2)]
\]

\[
= R(G_1)[p_e R(\hat{G}_2) + (1-p_e) R(\hat{G}_1)] + R(\hat{G}_1)[p_e R(G_2) + (1-p_e) R(G_2-e)] - R(G_1)[p_e R(G_2) + (1-p_e) R(G_2-e)].
\]

By Property 1, the above reduces to

\[
R(G) = R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2).
\]

By induction on \( k \), the theorem is true for all \( G \). QED.

Theorem 2 is illustrated using the example graph of Fig.2. \( G_1, \hat{G}_1, G_2 \) and \( \hat{G}_2 \) are shown in Fig.4. By Theorem 2,

\[
R(G) = (p_1p_3 + p_3p_4 + p_2p_4 - 2p_1p_3p_4)(p_1 + p_3 - p_1p_3) + p_3p_5(p_1 + p_4 - p_1p_4)
\]

\[
- (p_1p_3 + p_3p_4 + p_2p_4 - 2p_1p_3p_4)p_3p_5.
\]

We have seen that by means of Theorem 2, the overall reliability of \( G = G_1 \cup G_2 \) can be expressed in terms of the overall reliability of its two subgraphs and their coalesced graphs (\( \hat{G}_1 \) and \( \hat{G}_2 \)). In addition to the property that deletion of a pair of vertices \( i \) and \( j \) disconnects \( G \).
FIGURE 4
Illustration for Theorem 2
suppose $G_1 \cap G_2$ contains a link $e$ between $i$ and $j$. $R(G)$ can then be expressed in terms of $G_1$, $G_2$, $G_1-e$ and $G_2-e$ using the following theorem.

Theorem 3:

$$R(G) = \frac{1}{p_r}[R(G_1)R(G_2) - (1-p_r)R(G_1-e)R(G_2-e)].$$

Proof: Let $G_1$, and $G_2$, be the graphs obtained by contracting $e$ in $G_1$ and $G_2$ respectively. Since $G_1 \cup (G_2-e) = G$ and $G_1 \cap (G_2-e) = \{i, j\}$, the conditions of Theorem 2 are satisfied and we have

$$R(G) = R(G_1)R(G_2) + R(G_2-e)[R(G_1) - R(G)] R(G_2-e).$$

By Property 1, $R(G_1) = p_r R(G_1) + (1-p_r) R(G_1-e)$, which implies

$$R(G_1) = \frac{1}{p_r}[R(G_1) - (1-p_r)R(G_1-e)]. \quad p_r \neq 0.$$ 

Similarly for $R(G_2)$. Substituting the above for $R(G_1)$ and $R(G_2)$ in the expression for $R(G)$ yields the theorem. QED.

Theorem 3, which expresses $R(G)$ in terms of $G_1$, $G_2$, and their corresponding open subgraphs $G_1-e$, $G_2-e$, is illustrated by Fig.5, where

$$R(G) = \frac{1}{p_2}[(p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4)(p_2p_3 + p_2p_5 + p_3p_5 - 2p_2p_3p_5)} - (1-p_2)(p_1p_4)(p_1p_2)].$$
$G: \quad e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5$

$G_1: \quad e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad R(G_1) = p_1 p_2 + p_1 p_4 + p_2 p_4 - 2p_1 p_2 p_4$

$G_2: \quad e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad R(G_2) = p_2 p_3 + p_2 p_5 + p_3 p_5 - 2p_2 p_3 p_5$

$G_1 - e_2: \quad e_1 \quad e_4 \quad R(G_1 - e_2) = p_1 p_4$

$G_2 - e_2: \quad e_3 \quad e_5 \quad R(G_2 - e_2) = p_3 p_5$

**FIGURE 5**
Illustration for Theorem 3
REFERENCES


