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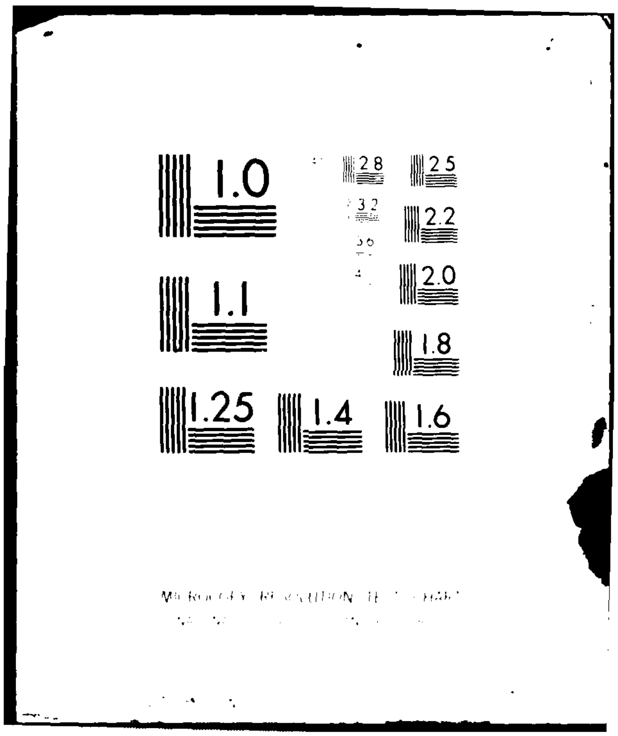
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August 1980

FINITE ELEMENTS FOR FLUID DYNAMICS

Nima Geffen

Principal Investigator

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Tel-Aviv University

Ramat-Aviv, Israel

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The first two parts of this report wind up a few questions in the mathematical formulation of vector fields governed by conservation and rotationality laws, with explicit application to fluid dynamic fields, possibly with shock waves. The points treated have a strong bearing on computational schemes and the stability of numerical calculations and the results provide a priori information on the way to select the appropriate set of equations, the right functional, and the most promising approximation space for finite element discretizations. The last assertion is then tested for the tricomi equation in a nonuniformly

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) elliptic domain. A mixed Tricomi problem is discretized by an alternative collocation scheme which proves to be accurate and stable as demonstrated on a few test-cases. The collocation finite difference schemes have proven superior thus to the finite elements ones for this case. They are, however, more specialized and natural for linear problems and simple geometries. The variationally based finite elements, on the other hand, hold a better promise for complex geometries, and for an accurate treatment of shocks.

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Scientific Report

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Table of Contents:

Forward

The first two parts of this report wind up a few questions in the mathematical formulation of vector fields governed by conservation and rotationality laws, with explicit application to fluidynamic fields, possibly with shock waves. The points treated have a strong bearing on computational schemes and the stability of numerical calculations and the results provide a-priori information on the way to select the appropriate set of equations, the right functional and the most promising approximation space for a finite element descretizations. The last assertion is then tested for the tricomi equation in a non-uniformly elliptic domain.

A mixed Tricomi problem is descretized by an alternative collocation scheme which proves to be accurate and stable as demonstrated on a few test-cases. The collocation finite difference schemes have superior to the finite elements ones for this case, so far. They are, however, more specialized and natural for linear problems and simple geometries. The variationally based finite elements, on the other hand, hold a better promise for complex geometries, and for an accurate treatment of shocks.

I. A Note on how to select from too many equations the right ones to solve a given problem.

Nima Geffen

Abstract

The field laws for a physical continuum are often described by too many first order partial differential equations for the number of required field quantities. The question described and a simple way to resolve it is given for the so-called conservative and non-conservative representations of continuum mechanics .

General Equations

Continuum mechanics, electrodynamics and other physical theories can be modelled by various specialization of the following equations:

(1) $\nabla \circ \underline{A} = G$

(2) $\nabla \times \underline{u} = \underline{W}$ (n equations)

for:

$\underline{x} = x_i \quad i = 1,\ldots,m$ independent variables

$\underline{u} = u_j \quad j = 1,\ldots,n$ dependent variables

$\underline{A}\,(x_i,u_j) = A^{(k)} \quad k = 1,\ldots,m$

and:

$G(x_i,u_j),\ \underline{W}(x_i,u_j) = W^{(j)}$

where the source function G is arbitrary, but the vorticity $\underline{W}$ has to satisfy a compatibility condition:

(3) $\nabla \circ \underline{W} = 0$

The system (1) (2) includes (n+1) first-order partial differential equations for the n unknown u_j. The overdeterminacy is apparent only, due to the fact that the n rotationality conditions are not independent (note eq. (3)) because any (n-1) statements imply the one left as will be shown explicitly in the following.

Examples

1. Maxwell's equations for the electromagnetic field are [1]

e1) $\nabla \cdot \underline{H} = 0$ e3) $\nabla \times \underline{E} = -\frac{1}{c} \underline{H}_t$

e2) $\nabla \cdot \underline{E} = 4\pi\rho$ e4) $\nabla \times \underline{H} = \frac{1}{c} \underline{E}_t + \frac{4\pi}{c} \underline{j}$

Equations e1) - e4) are not independent: for smooth fields (twice differentiable) eq. 3. implies $(\nabla \cdot \underline{H})_t = 0$ and eq. e1) holds for all times if it holds for $t = t_0$, thus it is an initial condition (at most).

The electric charge density and currents (ρ and $\underline{j}$ respectively) cannot be prescribed arbitrarily, since eq. e2 and e4) imply a constraint on their source terms ρ and $\underline{j}$:

e5) $\rho_t + \frac{1}{c} \nabla \cdot \underline{j} = 0$ (continuity of charge)

II. Steady-state Aerodynamics

for which:

$$\underline{x} = (x_1, x_2, x_3) = (x, y, z) \text{ - space coordinates}$$

$$\underline{u} = (u_1, u_2, u_3) = (u, v, w) \text{ - velocity components}$$

$$\underline{A} = \rho \underline{u}$$

and: $$\rho = \rho(\underline{u}^2) \quad \text{density}$$

$$G = 0, \quad \underline{W} = \underline{W}(\underline{x})$$

where: $\underline{W} = 0$ for irrotational flow, but changes across curved shocks. Spelled out in Cartesian coordinates we get:

e6) $$(a^2-u^2)u_x + (a^2-v^2)v_y + (a^2-w^2)w_z - 2uww_x - 2uvv_y - 2vww_z = 0, \tag{6}$$

$$\text{(i)} \quad w_x - u_z = W^{(1)}$$

$$\text{(ii)} \quad w_y - v_z = W^{(2)}$$

$$\text{(iii)} \quad u_y - v_x = W^{(3)}$$

i.e. 4 equations for the 3 unknown functions (u,v,w).

Methods of Solution and Descretization

The system equations (1), (2) even when simplified for specific physical field, is multidimensional and coupled, which renders it complicated to analyse and inconvenient to solve. Auxiliary functions (e.g. scalar and vector potentials) have been taylored to simplify and clarify the mathematical picture and render it elegant, intelligible and solvable.

For irrotational fields $\underline{u}$, i.e. $\underline{W} \equiv 0$ eq. (1) reduce to one second order equation for the scalar potential ϕ, defined by: $\underline{u} = \nabla\phi$, so that eq. (2) is satisfied identically.

For Maxwells equations the standard analysis is done via the scalar and vector potentials: ϕ and $\underline{A}$ respectively, (e.g. [1]), where:

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \nabla\phi, \quad \underline{H} = \nabla \times \underline{A}.$$

Written in terms of $(\phi, \underline{A})$ Maxwell's system reduce to 4 equations for the 4 components.

The resulting equations are higher order, and admit a wider family of solutions, all equivalent under gauge transformations:

$$\phi' = \phi - \frac{1}{c} f_t(\underline{x}, t)$$

$$\underline{A}' = \underline{A} + \nabla f$$

or in 4 dimensional notation:

$$(\underline{A},-\phi)' = (\underline{A},-\phi) + [\nabla f,-\frac{1}{c} f_t]$$

The potential formulation for the electromagnetic field is endowed with a beautiful structure, lucid transformation properties and striking accessible information content (e.g. decoupling into $2^{\underline{nd}}$ order wave equations for each of the components) unfolding the wealth of electromagnetic waves ref. [1].

Another suggestion, to equalise the number of equations and unknowns is to add a new dependent variable:

$$v = \nabla\cdot\underline{u}$$

and using the relation:

$$\nabla x \nabla x \underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$$

replace eq. (2) by its rotor:

(4) $\nabla^2\underline{u} = \nabla v - \nabla x \underline{W}$,

which, with eq. (1) gives (n+1) equations for the (n+1) unknowns ($\underline{u}$,v).

The formulation above has been suggested by M. Mock for computational purposes, with a staggered mesh for (u,v) (to avoid decoupling of the descretized equations for $\underline{u}$ and v) to solve boundary value problems.

Direct field formulation

The auxiliary function is (e.g. potentials) formulations invariably raise the order of the equations to be solved; the first-order system becomes second-order. This requires a higher degree of smoothness for the solution function and may be a draw-back for numerical analysis and calculations. Thus, although many large computer simulations are based on 'potential' formulations (e.g. ref [2]) a direct solution of the first order system has been found beneficial [3]. Sometimes essential [4], especially for 'initial' rather than boundary value problems.

The question is how to choose the 'right' (n-1) rotationality conditions that will give with the continuity eq. (1), the 'right' (nxn) system for a stable descretization for a marching scheme to solve the first order system, to obtain directly the n field components (so-called primitive variables). A simple treatment for irrotational, 3-dimensional fields (worked out for the problem in [4]) is given in [5]. It is extended here for the general case (eq. (1), (2)), where the field may have sources and be rotational (e.g. an electromagnetic field with moving charges, flow behind a curved shock, motion of reacting gases.)

Choice of Equations for 3-Dimensions

Spelled-out for 3 dependent variables and 3 independent ones (e.g. Cartesian space coordinates) we get:

$$\underline{x} = (x_1, x_2, x_3) = (x, y, z)$$

$$\underline{u} = (u_1, u_2, u_3) = (u, v, w)\ (x, y, z)$$

$$\underline{A} = (A^{(1)}, A^{(2)}, A^{(3)})\ (u, v, w;\ x, y, z)$$

and the system of equations is:

(1) $A_x^{(1)} + A_y^{(2)} + A_z^{(3)} = G$

$$w_y - v_z = W^{(1)}$$

(2) $u_z - w_x = W^{(2)}$

$$v_x - u_y = W^{(3)}$$

i.e. 4 equations for the three unkonwon functions (u,v,w), a redundant system for a well defined field.

In addition, a compatibility on $\underline{W}$ requires:

(3) $W_x^{(1)} + W_y^{(2)} + W_z^{(3)} = 0$

via which 2 components of $\underline{W}$ determines the dependence of the third on its corresponding coordinate, e.g. when $W^{(2)}$, $W^{(3)}$ are given, the following must hold for $W^{(1)}$:

$$W_x^{(1)} = -(W_y^{(2)} + W_z^{(3)})$$

$$W^{(1)} = -\int_{x_0}^{x} (W_y^{(2)} + W_z^{(3)})\,(\xi, y, z)\,d\xi + W^{(1)}(x_0, y, z)$$

e.g. i) $w_y - v_z - W^{(1)}$

ii) $u_z - w_x = 0$

iii) $v_x - u_y = 0$

(3) $W_x^{(1)} = 0$

or: $W^{(1)}(x,y,z) = W^{(1)}(y,z) + C$

$$= W^{(1)}(x = x_0, y, z)$$

Thus $W^{(1)}$ cannot be a function of x and along x-lines retains its value at one point: $x = x_0$. The non-zero component of the vorticity is constant along lines parallel to the corresponding coordinate.

Irrotational Fields

In this case $\underline{W} \equiv 0$, eq. (3) is satisfied automatically and equations (2) become:

i) $w_y - v_z = 0$

ii) $u_z - w_x = 0$

iii) $v_x - u_y = 0$

for $(u,v,w)(x,y,z)$ unique and sufficiently smooth, (e.g. twice differentiable in each of the independent variables) we get:

ii) $\longrightarrow 0 = u_{zy} - w_{xy} = (u_y)_z - (w_y)_x = (v_x)_z - (w_y)_x = (v_z - w_y)_x$

$\partial/\partial x$ (ii) $(u,w) \in C^2$ (i) $v \in C^2$

$$v_z - w_y = c(y,z)$$

For $\underline{u}$ irrotational at any $x = x_0$,

$$(v_z - w_y)(x_0, y, z) = 0 \Rightarrow C(y,z) \equiv 0 \qquad \forall (y,z)$$

and i), ii) $\Longrightarrow$ (iii) which can be considered and "initial condition" at most, (e.g. $x_0 \to -\infty$, and a uniform field there). In exactly the same manner:

i) iii) $\longrightarrow [u_z - w_x]_y = 0$

$$u_z - w_y = F(x,z)$$

and irrotationality at any $y = y_0$ plane implies $F \equiv 0$ and eq. (ii) and:

i) ii) $\longrightarrow [v_x - u_y]_z = 0$

and irrotationality at any $z = z_0$ implies iii).

Thus any of the 3 irrotationality conditions implies the third, for unique smooth, irrotational fields everywhere. For stable numerical algorithms however, one has to choose set that carries the appropriate "boundary information" along the marching coordinate; if the integration is to be carried out along the x_i direction, the $i^{\underline{th}}$ component of the iirrotationality equations has to be omitted [4], [5].

For the general irrotational Case:

(i) $w_y - v_z = W^{(1)}$

(ii) $u_z - w_x = W^{(2)}$

(iii) $v_x - u_y = W^{(3)}$

(3) $W^{(1)}_x + W^{(2)}_y + W^{(3)}_z = 0.$

Differentiating and substituting we get:

$\frac{\partial}{\partial x}$ i) $w_{yx} - v_{\;x} = W^{(1)}_x$

$= (w_x)_y - v_{zx} =$

$(u_z)_y - W^{(2)}_y - v_{zx} = W^{(1)}_x$

(ii)

$(u_y - v_x)_z = W^{(1)}_x + W^{(2)}_y = -W^{(3)}_z$

(iii)

$v_x - u_y = W^{(3)} + F(x,y)$

$[(v_x - u_y) - W^{(3)}](x,y,z_0) = F(x,y,z_0)$

Thus i) ii) $\Longrightarrow$ iii) provided $v_x - u_y$ is given at $z = z_0$.

In the same manner each two of the equations determine the third, for which initial conditions have to be given for uniqueness.

Algebraic treatment

Equations (1), (2) can be written in a matrix form (used widely for numerical analysis and descretizations):

(5)

$$\begin{pmatrix} A_u^{(1)} & A_v^{(1)} & A_w^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} A_u^{(2)} & A_v^{(2)} & A_w^{(2)} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_y + \begin{pmatrix} A_u^{(3)} & A_v^{(3)} & A_w^{(3)} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_z$$

$$= \begin{pmatrix} G - \sum\limits_{i=1}^{3} \dfrac{\partial A^{(i)}}{\partial x_i} \\ W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix}$$

or in short:

$$C^{(x)} \underline{u}_x + C^{(y)} \underline{u}_y + C^{(z)} \underline{u}_2 = \underline{F}$$

with the corresponding (4x3) matrices C and 4 forcing functions F.

The system (5) is again redundant, and one of the last 3 equations has to be deleted to render the (4x3) coefficient matrices C into (3x3) matrices $\tilde{C}$. The choice is directed by the marching "time like" direction, for which the $\tilde{C}^{(i)}$ matrix has to be invertible, hence nonsingular, hence with no row (or column) of zeros. This automatically rules out one equation: when (u,v,w) are given at $x = x_0$ (e.g. $x_0 \longrightarrow -\infty$ for steady flow about an obstacle), the first irrotationality condition has to omitted. A marching procedure along the x axis requires the inversion of the non-redundant matrix $\tilde{C}(x)$ such that:

$$\underline{u}_x = -[\tilde{C}^{(x)}]^{-1}\, \tilde{C}^{(y)}\underline{u}_y + (\tilde{C}^{(z)}u_z) + [\tilde{C}^{(x)}]^{-1}\underline{F}$$

In the same manner, integration schemes along the y and z directions require the omission of (5i0 and (5iii), respectively, as a necessary condition (not always sufficient!) for a well defined, stable scheme, (as is seen in [4]).

Final Remarks

The question and system treated are elementary and so general, the analysis and answer so simple, to be judged trivial, if not for the fact that the question does come up occasionally (e.g. [4]), and the answer not always immediate. Essentially the same problem has been treated recently (and has come to our attention while writing this note), in a completely different context for different reasons and aims in [6]. It also seems to be related to other recent approaches [7] and to variational formulation and analysis of the system at hand with related physical applications [8].

Work on relaxing the smoothness requirement, re-formulation and analysis for non-smooth fields (e.g. flows with shocks) is in progress.

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II. ALTERNATIVE VARIATIONAL FORMULATIONS FOR NONLINEAR VECTOR SYSTEMS

Nima Geffen

Abstract

Variational formulations for vector fields described by a system of partial differential equations of whatever type, possibly nonlinear, and with initial-boundary conditions are given. Smoothness properties of suitable approximation spaces are viewed and the effect of coercive , natural boundary conditions discussed briefly. Examples are drawn from electromagnetic field theory and fluidynamics.

Differential equations

Consider the following conservation and rotationality conditions, for a vector field $\underline{u}(\underline{x})$:

(1) $\nabla \cdot \underline{V} = G$

(2) $\nabla \times \underline{u} = \underline{W}$

(3) $\underline{u}(\underline{x}) = f(\underline{x}) \quad \underline{x} \in \partial\Omega$:

where:

$\underline{u} = \underline{u}_i$ is the vector of dependent variables $i = 1,\dots,m$

$\underline{x} = x_j$ is the vector of independent variables $j = 1,\dots,n$

$G = G(\underline{x})$ is a given function

$\underline{W} = W_j(\underline{x})$ is the vorticity

$\underline{V} = V_j(\underline{x},\underline{u})$

The vorticity $\underline{W}$ has to obey a compatibility condition:

(4) $\nabla \cdot \underline{W} = 0$

The system of equations (1), (2) is quite general, it can be linear and nonlinear, elliptic, hyperbolic or mixed with smooth or non smooth solutions. The independent variables x_i may designate space and time coordinates and different kinds of initial and/or boundary conditions may be appropriate for different problems. Higher order equations may be put into this form, and examples of applications include the description of electromagnetic fields, the theory of elasticity, fluidynamics, and plasmadynamics, including flows with shocks.

Variational Formulations

A variational formulation of the field $\underline{u}$ satisfying (1), (2), (3) is a functional $J(\underline{v})$ defined on Ω whose stationary value is obtained for $\underline{v} = \underline{u}$:

$$\delta J(\underline{v}) = 0 \quad \Leftarrow \quad \underline{v} = \underline{u}$$

For well posed problems, for which $\underline{u}(\underline{x})$ is unique

$$\delta J(\underline{v}) = 0 \quad \Leftrightarrow \quad \underline{v} = \underline{u}$$

Variational formulations are scalar, short, additive (the functionals for complex systems are direct sums of their simpler parts), invariant under appropriate classes of transformations and are often convenient for theoretical analysis and for numerical simulations, e.g. by the finite elements method. Integrals can easily be descritized and approximated, and the smoothness requirements on the functions are less stringent than for the corresponding differential system. This last point is most important from the numerical view point in addition to a better rationale for the treatment of shocks.

The case G = 0 is described in [1]. The functional $J(\underline{v},\underline{x})$:

$$J = \int_{\Omega} L(\underline{v},\underline{x}) + \underline{\lambda}.(\nabla x \underline{v} - \underline{W}) + \int_{\partial\Omega_m} \underline{\lambda} x \underline{v} \cdot d\underline{\sigma} \quad \text{is stationary:}$$

$\delta J(\underline{v}) = 0$ at $\underline{v} = \underline{u}$

provided that: $\nabla_{\underline{u}} \times \underline{v} = 0.$

$$\underline{V} = \nabla_{\underline{u}} L$$

resulting in:

$$\underline{V} = -\nabla \times \underline{\lambda}$$

The variation is done on all $\underline{v}$ for which J is defined and $\underline{v}$ satisfying coercive boundary conditions on the boundary $\partial\Omega_i = \partial\Omega - \partial\Omega_m$ or for which $\underline{\lambda} \parallel \underline{v}$ or $\underline{v} \parallel \underline{d\sigma}$ on $\partial\Omega$.

For the non-sourceless (or sourceful) case; the following variational statements hold:

Th. 1

The functional:

$$(5)\quad J(\underline{v}) = \int_{\Omega} [L - \underline{g}\cdot\underline{v} + \underline{\lambda}\cdot(\nabla \times \underline{v} - \underline{W})]\cdot d\underline{x} + \int_{\partial\Omega_m} \underline{\lambda} \times \underline{v}\cdot\underline{d\sigma}$$

is stationary for $\underline{v} = \underline{u}$ satisfying (1), (2), (3) provided that

$$(6)\quad \nabla_{\underline{u}} \times \underline{V} = 0$$

where:

$$(7)\quad \nabla\cdot\underline{g} = G \quad \text{or} \quad \underline{g} = \nabla^{-1}G$$

$$(8)\quad \underline{V} = \nabla_{\underline{u}} L$$

(9) $\underline{V} = \nabla \times \underline{\lambda} + \underline{g}$

$\underline{v}$ is allowed to vary over all functions for which (5) is defined and finite and which satisfy the coercive B.C. (3). The proof is straight forward and follows the details in [1] exactly.

Corollary

The functional:

$$(10) \quad J(\underline{v}) = \int_{\Omega} (L-\underline{g}\cdot v + \underline{\lambda}\cdot\underline{W})dx + \int_{\partial\Omega_m} \underline{\lambda} x \underline{v}\cdot\underline{d\sigma} \qquad (3)$$

is stationary for $\underline{v} = \underline{u}$ satisfying (1), (3) when the variation is taken over all fields satisfying eq. (2) and the initial/boundary conditions (3): i.e.

$$\delta J(\underline{v}) = 0 \iff \underline{v} = \underline{u} : \begin{array}{l} \nabla\cdot\underline{V} = G \\ \nabla x\underline{V} = \underline{W} \\ \underline{v}_j = \underline{g}_j(\underline{x}) \quad \underline{x}\in\partial\Omega_i \end{array}$$

The restricted variational statement (10) follows from the statement in (5) by inspection.

An alternative formulation is obtained by integrating the 2^{nd} term by parts, using the vector identity:

$$\nabla\cdot(\underline{\lambda}\times\underline{v}) = \underline{v}\cdot\nabla\times\underline{\lambda} - \underline{\lambda}\cdot\nabla\times\underline{v}$$

substituting in (5):

$$\underline{\lambda}\cdot\nabla\times\underline{v} = \underline{v}\cdot\nabla\times\underline{\lambda} - \nabla\cdot(\underline{\lambda}\times\underline{v})$$

Th. 2

The functional

$$(11) \quad J(\underline{v}) = \int (L-\underline{g}\cdot\underline{v} + \underline{v}\cdot\nabla\times\underline{\lambda} - \underline{\lambda}\cdot\underline{W})\underline{dx}$$

is stationary for

$$\underline{v} = \underline{u} \quad \text{satisfying (1), (2).}$$

In the variational formulation (11) is made over all $\underline{v}$ in Ω which render all terms integrable. The lagrange multiplier $\underline{\lambda}$ is required to have integrable first derivatives, which appear explicitly in J.

The surface term drops out, and the solution $\underline{v} = \underline{u}$ satisfies the natural boundary conditions: $\underline{\lambda} \parallel \underline{u}$ or $\underline{u} \parallel \underline{d\sigma}$ on $\partial\Omega$

Smoothness requirements

In the variational formulation (5) $\underline{v}$ is required to be at least once differentiable (for the $2^{\underline{nd}}$ term to be defined) which holds also for (10) (where eq. (2) has to be satisfied). The lagrange multiplier $\underline{\lambda}$ in this formulation can be just integrable, e.g. a step function. The variational statement (11), on the other hand, does not involve derivatives of $\underline{v}$ (hence admit integrability - only there) but requires $\underline{\lambda}$ to differentiable at least once. Formulations (5), (10) include a surface term and involve spaces satisfying appropriate boundary conditions; in statement (11) the surface term has dropped out, and the solution $\underline{v} = \underline{u}$ making it stationary satisfies a natural boundary condition. Both smoothness requirements and behavior on the boundary has bearings on approximation spaces used for numerical calculations, and on the stability of numerical schemes. Simple examples are described in the following part of this report. [2].

Examples:

1. Steady electromagnetic field

Maxwells' equations for a steady state can be written as [3]:

i) $\nabla \cdot \underline{E} = \rho$ ii) $\nabla x \underline{E} = 0$

iii) $\nabla \cdot \underline{B} = 0$ iv) $\nabla x \underline{B} = \underline{j}$

A variational statement for i) ii) is:

$$(5') \quad J(\underline{E}) = \int_{\Omega} (E^2/2 - \underline{g} \cdot \underline{E}) + \underline{\lambda}^{(E)} (\nabla x \underline{E}) + \int_{\partial \Omega_i} \underline{\lambda}^{(E)} x \underline{E} \cdot \underline{d\sigma}$$

for $\underline{E}$ arbitrary, or:

$$(10') \quad J(\underline{E}) = \int_{\Omega} (E^2/2 - \underline{g} \cdot \underline{E}) \quad \text{for } \underline{E} \text{ irrotational.}$$

where: $\underline{g}$ is any solution to

$$\nabla \cdot \underline{g} = \rho \quad \text{or:} \quad \underline{g} = \nabla^{-1} \rho$$

The variational statement (11) for i) ii) becomes:

$$(11') \quad J(\underline{E}) = \int_{\Omega} [E^2/2 - \underline{g} \cdot \underline{E}) - \underline{E} \cdot (\nabla x \underline{\lambda}^{E})]$$

The corresponding variational formulations for the magnetic equations iii) iv) are:

$$(5'')\quad J(\underline{B}) = \int B^2/2 + \underline{\lambda}^{(B)}(\nabla x\underline{B}-\underline{j}) + \int_{\partial\Omega_m} \underline{\lambda}^{(B)} x\underline{B}\cdot \underline{d\sigma}$$

$$(10'')\quad J(\underline{B}) = \int B^2/2 - \underline{\lambda}^{B}\cdot j + \int_{\partial\Omega_m} \underline{\lambda}^{(B)} x\underline{B}\cdot \underline{d\sigma}$$

for $\underline{B}$ satisfying eq, iv)

and

$$(11'')\quad J(\underline{B}) = \int_{\Omega} [E^2/2 - \underline{B}\cdot(\nabla x\underline{\lambda}^{(B)})]$$

the functionals for the combined field are obtained by simply adding the ones for the 'separated' system:

$$(5''')\quad J(\underline{B},\underline{E}) = \int_{\Omega}\Big[\frac{E^2-B^2}{2} - \underline{g}\cdot E + \underline{\lambda}^{B}\cdot\underline{j} + \underline{\lambda}^{(E)}(\nabla x\underline{E}) - \underline{\lambda}^{(B)}(\nabla x\underline{B})\Big] + \int_{\partial\Omega_m^E} \underline{\lambda}^{E} x\underline{E}\cdot\underline{d\sigma} - \int_{\partial\Omega_m^B} \underline{\lambda}^{B} x\underline{B}\cdot\underline{d\sigma}$$

$$(10''')\quad J(\underline{B},\underline{E}) = \int \frac{E^2-B^2}{2} - \underline{g}\cdot\underline{E} + \underline{\lambda}^{B}\cdot j + \int_{\partial\Omega}$$

for irrotational $\underline{E}$ and $\underline{B}$ satisfying iv).

The statement (10") can be reduced to the one used for the scalar and vector potentials for the irrotational electric and solenoidal magnetic fields, (e.g. [3] pg. 366 eq. (11-65).

Finally (11)' and (11") combine to give:

(11"'):

$$J(\underline{E},\underline{B}) = \int \frac{E^2-B^2}{2} - \underline{g}\cdot\underline{E} - \underline{E}\cdot\nabla\times\underline{\lambda}^E + \underline{B}\cdot\nabla\times\underline{\lambda}^B$$

2. Steady fluidynamics

The differential equations are:

$$\nabla\cdot(\rho q) = 0 \qquad \rho = \rho(q^2)$$

$$\nabla x\cdot \underline{q} = w$$

and the corresponding Lagrangian L and functionals are:

$$(9) \qquad \rho\underline{u} = \frac{\partial L}{\partial \underline{u}} \quad \text{or:} \quad \rho u_i = \frac{\partial L}{\partial u_i}$$

$$J(\underline{u}) = \int_{\Omega} [L + \underline{\lambda}\cdot(\nabla x\underline{u} - \underline{w})] + \int_{\partial\Omega} \underline{\lambda} x \underline{u}\cdot d\sigma$$

(10) or:

$$J(u) = \int_{\Omega} L - \underline{\lambda} \cdot \underline{w} + \int_{\partial\Omega} \underline{\lambda} \times \underline{u}\cdot \underline{d\sigma}$$

or:

$$(11) \quad J(\underline{u}) = \int_{\Omega} L - \underline{\lambda}\cdot\underline{w} + \underline{u}\cdot\nabla x\underline{\lambda}$$

for the appropriate function spaces, with the required smoothness properties, and constraints in the region and/or on the boundary.

The results (9) have been described and analyzed in [1a], [1b] where applications to specific flows are given; the formulation

(10) of a slight modification used in electromagnetics and other field theories but not generally useful for computations due to the practical difficulty in actually constructing a wide enough family of $\underline{v}$ fields that satisfy the rotationality condition (2)). The formulation (11) is new and can be readily specialized to particular fields; it offers an interesting alternative from the theoretical point of view but holds rather doubtful promise computationally (stability problems?).

Concluding Remarks

A preliminary report on alternative variational formulations is given for general vector fields governed by systems of partial differential equations, possibly nonlinear specifying their sources (eq. (1), e.g. conservation of mass for $G = 0$) and vorticity (eq. (2), e.g. $\underline{w} = 0$ for irrotational fields). This framework is pregnant with information and connections with other variational approaches and with related mathematical and computational questions. It also involves the questions of redundancy symmetry and the appropriate way to describe these fields for the continuous and non continuous cases, which may (and most often will!) occur for nonlinear systems, e.g. flows with shocks.

The last question as well as the elaboration on the other points are in the works now, to be reported at a later date.

Simple computational examples, where alternative formulations and trial functions are used for the Laplace and Tricomi problems are reported in the following chapter.

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III. ON DIFFERENT MIXED FINITE ELEMENT APPROXIMATIONS FOR SYMMETRIC ELLIPTIC SYSTEMS

Sara Yaniv

Abstract

Different finite-dimensional spaces are tried for mixed finite element approximations for a functional which has a saddle point at the solution of elliptic symmetric linear system, with a Lagrange multiplier. Calculations are carried out for 2 first order equations, $(u,v)(x,y)$ and boundary conditions, e.g. Laplace and Tricomi's problems. Brezzis' convergence condition is found hard to verify rigorously, even for descretizations that seem to work well. Preliminary analysis is tried and experiments conducted for bilinear variations on rectangles for all components and for bilinear (u,v) and piece-wise constant (λ) trial functions.

1. Introduction and variational formulation

Consider the equation

$$(1a)\qquad A_x + B_y = f(x,y) \qquad (A,B)(x,y,\phi_x,\phi_y)$$

in a domain Ω with given boundary conditions:

$$(1b)\qquad \phi(x,y)\Big|_{\partial\Omega} = 0$$

Assuming

$$u = \phi_x$$

$$v = \phi_y$$

and $$A_v = B_u$$

then the operator $F(x,y,u,v) = A_x + B_y$ is a potential [1, p. 35].

The variational formulation for the problem is: find $(u,v;\lambda)$ so that

$$(2)\qquad J(u,v;\lambda) = \int_\Omega\int [L(x,y,u,v) + \lambda(u_y - v_x)]dxdy$$

$$- 2\int_\Omega\int F(x,y)udxdy$$

$$F(x,y) = \int^x f(\xi,y)d\xi$$

is stationary, for all functions $(u,v) \in VxV$ and $\lambda \in W$, where every $(\tilde{u},\tilde{v}) \in V \times V$ satisfying boundary conditions:

$$\tilde{u}dx + \tilde{v}dy = 0.$$

$L(x,y,u,v)$ is the Lagrangian of the problem, i.e.

$$L_u = A$$

$$L_v = B$$

and $\lambda(x,y)$ is a Lagrange multiplier which is the stream function of the problem [1, page 39].

The variation of $J(u,v;\lambda)$ for fixed λ gives the following weak formulation:

$$(3) \quad \delta J(u,v;\lambda) = \iint_\Omega [L_u \delta u + L_v \delta v + \lambda(\delta u_y - \delta v_x)]dxdy = 0$$

Adding the equation

$$u_y - v_x = 0$$

in weak formulation:

$$\iint_\Omega q(u_y - v_x)dxdy = 0 \quad \text{for} \quad \forall q \in W,$$

we get a saddle-point problem:

$$(4a) \quad \iint_\Omega [L_u\, u_1 + L_v\, v_1 + \lambda(\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x})]dxdy = 0$$

$$\forall\ (u_1, v_1) \in V \times V$$

and

(4b) $$\int_{\Omega}\int q\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)dxdy = 0 \qquad \forall q \in W.$$

The functions u, v and λ which satisfy (1a) and (1b) are the solution of (4a) and (4b).

For linear problems:

$$L_u\, u + L_v\, v = A\, u + B\, v \equiv L_{uu}\, u^2 + 2L_{uv}\, u\, v + $$

$$+ L_{vv}\, v^2$$

and for an elliptic equation it is positive definite form.

Hence

$$V \times V = \{(u,v)/\int_{\Omega}\int [L_{uu} \cdot u^2 + 2L_{uv} \cdot uv + L_{vv} \cdot v^2 + $$

$$+ (u_y - v_x)^2]dxdy < \infty \;,\; udx + vdy\Big|_{\partial\Omega} = 0\}$$

$$\|(u,v)\|^2_{V \times V} = \int_{\Omega}\int [L_{uu}u^2 + 2L_{uv}uv + L_{vv}v^2 + (u_y - v_x)^2]dxdy$$

and, since λ depends on an arbitrary constant:

$$W = \{q/\int_{\Omega}\int q^2 dxdy < \infty \;,\; q(x_0, y_0) = 0\}$$

$$(x_0, y_0) \in \Omega.$$

This formulation gives a saddle-point problem [2], which is determined as follows:

Given $f \in V'$, $g \in W'$, find $u \in V$, $\psi \in W$ such that:

$$
(5) \qquad \begin{aligned} a(u,v) + b(v,\psi) &= f(v) \qquad && \forall v \in V \\ b(u,\phi) &= g(\phi) && \forall \phi \in W \end{aligned}
$$

Brezzi's theorem states the following conditions for existence and uniqueness of the solution:

Suppose a and b are bounded and

$$
(6a) \qquad \inf_{\substack{u \in z \\ \|u\|_V = 1}} \; \sup_{\substack{v \in z \\ \|v\|_V = 1}} |a(u,v)| \geq \gamma > 0
$$

$$
z \equiv \{u \in V / b(u,\phi) = 0 \quad \forall \phi \in W\}
$$

$$
(6b) \qquad \sup_{v \in V} \frac{|b(v,\psi)|}{\|v\|_V} \geq k \; \|\psi\|_W \qquad \forall \psi \in W, \quad k > 0
$$

then the solution of (5) is unique

and

$$
\|u\|_V + \|\psi\|_W \leq C(\|f\|_{V'} + \|g\|_{W'}) .
$$

It is easy to verify that problem (4a), (4b) satisfies (6a), (6b), (6c), hence has a unique solution which is the only solution of (1a), (1b).

2. Approximation

In order to approximate the solution of problem (4a), (4b) using the saddle-point weak formulation or the saddle-point variational principle, we use finite-dimensional spaces $V_h \subset V$, $W_h \subset W$ satisfying:

$$\text{(7a)} \quad \inf_{\substack{u \in Z_h \\ \|u\|_V = 1}} \ \sup_{\substack{v \in Z_h \\ \|v\|_V = 1}} |a(u,v)| \geq \tau \geq 0 \ , \quad \tau \text{ independent of } h$$

$$\text{(7b)} \quad \sup_{v \in Z_h} |a(u,v)| > 0 \quad \forall \quad 0 \neq u \in Z_h$$

$$Z_h \equiv \{u \in V_h ; \ b(u,\phi) = 0 \quad \forall \phi \in W_h\}$$

$$\text{(7c)} \quad \sup_{v \in V_h} \frac{|b(v,\psi)|}{\|v\|_V} \geq \ell \ \|\psi\|_W$$

$$\forall \psi \in W_h$$

$$\ell > 0, \text{ independent of } h.$$

The approximated problem is:

$$\text{(8)} \quad \begin{aligned} a(u_h,v) + b(v,\psi_h) &= f(v) \qquad && \forall v \in V_h \\ b(u_h,\phi) &= g(\phi) && \forall \phi \in W_h \ . \end{aligned}$$

Under hypothesis (7a), (7b), (7c) problem (8) has a unique solution [2] and:

$$\|u - u_h\|_V + \|\psi - \psi_h\|_W \leq c \inf_{\substack{\chi \in V_h \\ \delta \in W_h}} (\|u-\chi\|_V + \|\psi-\delta\|_W$$

3. Examples

We used the variational principle (2) to solve, approximately, the Laplace equation and the Tricomi equation in an elliptic domain.

(i) The Dirichlet problem for the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0$$

$$\phi\Big|_{\partial\Omega} = f(x,y)$$

The variational functional is:

$$J(u,v;\lambda) = \int_\Omega\!\!\int [u^2 + v^2 + \lambda(u_y - v_x)]dxdy \tag{9}$$

for $(u,v) \in V\times\tilde{V}$, $\lambda \in W$

$$V\times V = \{(u,v)/\iint (u^2+v^2+(u_y-v_x)^2]dxdy < \infty,\ udx + vdy\Big|_{\partial\Omega} = df\}$$

$$W = \{q\,|\int_\Omega\!\!\int q^2 dxdy < \infty\ ,\ q(x_0,y_0) = 0\}$$

for this problem conditions (6a), (6b), (6c) are fulfilled, hence the problem has a unique solution.

Let Ω be the rectangle:

$$\Omega = \{(x,y)/\ -1 \leq x \leq 1,\ -1 \leq y \leq 0\}.$$

We have divided Ω into rectangles of size h and chosen different finite dimensional spaces for the trial functions

a) The first attempt was to use the bilinear trial functions for (u,v) and λ, for these spaces Brezzi's conditions are not, necessarily satisfied; this is concluded from the instability of the numerical solution, in some of the cases tried,

b) We tried to take other finite dimensional spaces so that (7a), (7b), (7c) will be satisfied. Using the same finite-element discretization of Ω as in a), approximating u and v by bilinear trial functions and piece-wise constant functions for λ (intuitively, this may help to get rid of the 4 constants and leave us with the only arbitrary constant of the problem which is $q(x_0,y_0) = 0$).

For this approximation the solution converged to the analytic solution;

$u - e^x \sin y$, $v = e^x \cos y$, $\lambda = -2e^x \cos y + 2/e^{-1}$ as shown in the following table:

	u	v	λ
L_2-error $h = 0.25$	0.0.415	0.053	0.184
L_2-error $h = 0.125$	0.0163	0.0194	0.097
L_2-error $h = 0.0625$			
numerical rate of convergence	1.35	1.45	0.92

(The rates of convergence for all of the problems we solved were the same).

In [3, page 75-77] the authors show that for the Stokes equation (for a mixed variational formulation) quadratic trial functions for u and v and piece-wise constant function for λ for triangular elements are permitted, and then conditions (7a), (7b), (7c) are satisfied.

c) We tried to solve the variational principle: find $(u,v;\lambda)$ so that

$$J(u,v;\lambda) = \int_{\Omega}\int [u^2 + v^2 - \lambda_y u + \lambda_x v]dxdy +$$

$$+ \int_{\partial\Omega} \lambda(f_x dx + f_y dy)$$

is stationary, for:

$$(u,v) \in VxV \quad , \quad \lambda \in W$$

where

$$VxV = \{(u,v)/\int_{\Omega}\int (u^2 + v^2)dxdy < \infty\}$$

$$W = \{q/\int_{\Omega}\int (q_y^2 + q_x^2)dxdy < \infty \ , \ q(x_0,y_0) = 0\},$$

as an equivalent problem for the Laplace equation.

For this formulation we used the same element discretization, but took piece-wise constant trial functions for u and v and bilinear trial functions for λ. The numerical solutions was unstable, we received 2 independent solutions for λ and the solution for u and v did not converge.

ii) The Dirichlet problem for the Tricomi equation in an elliptic domain

$$y\phi_{xx} - \phi_{yy} = 0$$

$$\phi\Big|_{\partial\Omega} = f(x,y)$$

$$\Omega = \{(x,y)/\ -1 \leq x \leq 1,\ -1 \leq y \leq 0\}$$

The variational functional is:

(10) $$J(u,v;\lambda) = \int_\Omega\int (yu^2 - v^2 + \lambda(u_y - v_x)]dxdy$$

for $(u,v) \in VxV$, $\lambda \in W$

$$VxV = \{(u,v)/\int_\Omega\int[yu^2 - v^2 + (u_y - v_x)^2]dxdy < \infty,$$

$$udx + vdy\Big|_{\partial\Omega} = df\}$$

$$W = \{q|\ \int_\Omega(q^2dxdy < \infty\ ,\ \ q(x_0,y_0) = 0\}$$

conditions (6a), (6b), (6c) are satisfied, hence there exist a unique solution.

We followed the numerical procedures a) and b) used for the Laplace equation. The approximated solutions behave the same [1, page 73].

For procedure a) the solution for λ was unstable depending on 4 arbitrary additive constants while for u and v the approximation was good.

For Procedure b) the approximated solution was stable and converging for u, v and λ as shown in the following table:

	u	v	λ
L_2-error h = 0.25	0.0628	0.0962	0.333
L_2-error h = 0.125	0.0257	0.0382	0.175
L_2-error h = 0.0625			
numerical rate of convergence	1.3	1.3	0.93

The analytic solution in this case is:

$$u = \sinh x\left(y + \sum_{1}^{\infty} \frac{y^{3n+1}}{(3n+1)3n(3n-2)(3n-3)\ldots 4.3}\right)$$

$$v = \cosh x\left(1 + \sum_{1}^{\infty} \frac{y^{3n}}{3n(3n-2)(3n-3)\ldots 4.3}\right)$$

$$\lambda = 2\sinh x\left(1 + \sum_{1}^{\infty} \frac{y^{3n}}{3n(3n-2)(3n-3)\ldots 4.3}\right)$$

The rate of convergence is the same for all the cases solved.

4. Remarks

Equation (1a) is equivalent to the first order system:

$$
\begin{aligned}
L_u - \lambda_y &= 0 \\
L_v + \lambda_x &= 0 \qquad (11) \\
u_y - v_x &= 0
\end{aligned}
$$

where

$$L_u = 2u \quad \text{for the Laplace equation}$$

$$L_v = 2v$$

and

$$L_u = 2yu \quad \text{for the Tricomi equation}$$

$$L_v = -2v$$

It is obvious that the solution for λ has more derivatives than u and v.

The trial functions used in procedures a) and b) do not satisfy this feature while those of c) satisfy it. But the only procedure which gave a converging solution is b).

The only conditions which insure convergence of the approximated solutions are those of Brezzi (7a), (7b), (7c), (which are quite difficult to show in the finite-dimensional problem).

Summary and Concluding Remarks

A summary of results of finite elements calculations in the non-uniformly elliptic parts are given in [[illegible]]. A rectangular re-gion is descretized into rectangles and a bilinear variation was assumed for both field variables (u,v) and the Lagrange multiplier λ appearing in the functional:

$$(12) \quad J(u,v;\lambda) = \iint_{\Omega} [L + \lambda(u_y - v_x)]\, dxdy$$

to be made stationary at the solution functions $(u,v;\lambda)$.

The values of λ at one corner ($i = 0$, $j = 0$ say) deter-mine its values at even points only and the set of equations [illegible] the odd points consist of a separate system, leading to [illegible] ac-ceptable behavior of the overall solution (a phenomenon [illegible] other computational contexts). To couple the equations [illegible] points the values of λ at the 4 points of the corner [illegible] had to be predetermined (e.g. by a Taylor's expansion about the (0,0) point and the relation to the values of u, v in the [illegible] rectangle). The procedure is not considered satisfactory [illegible] it yields acceptable results. On the hyperbolic region [illegible] intermittency causing unstability has been observed.

Following a series of lectures by J. Osborn [5], [illegible] variation was retained for (u,v) but the test space [illegible] replaced by piece-wise constants on rectangles. No decoupling occurs and the calculations are completely stable and [illegible].

The price is paid in a lower accuracy, but the scheme is nevertheless considered feasible.

The functional:

$$J(u,v;\lambda) = \iint_{\Omega} [L + u\lambda_y - v\lambda_x]dxdy + \int_{\partial\Omega} \lambda(f_x dx + f_y dy)$$

obtained from (12) by integrating the 2^{nd} term by parts, is also stationary at the solution to the Laplace system. The same descritization and the corresponding trial spaces yield unstable scheme, This is somewhat unexpected, because the relative differentiability required for (u,v) and λ in (11) fits better the analytic relation than the one in b). The stability analysis for this case (via Brezzi's coercivity condition for the approximation spaces) is difficult, and we have not been able to carry it to a successful conclusion.

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IV. FINITE DIFFERENCE APPROXIMATIONS FOR THE SOLUTION OF THE TRICOMI EQUATION IN A MIXED ELLIPTIC-HYPERBOLIC REGION

by

David Levin

Abstract

Difference schemes for the solution of the Tricomi problem are derived. In both the elliptic and hyperbolic domains the schemes are constructed so to be exact for several polynomial solutions of the Tricomi equation. The method presented is adaptable to non-linear equations and to non-standard meshes. A high accuracy is demonstrated for several Tricomi boundary conditions.

1. Introduction

The Tricomi equation

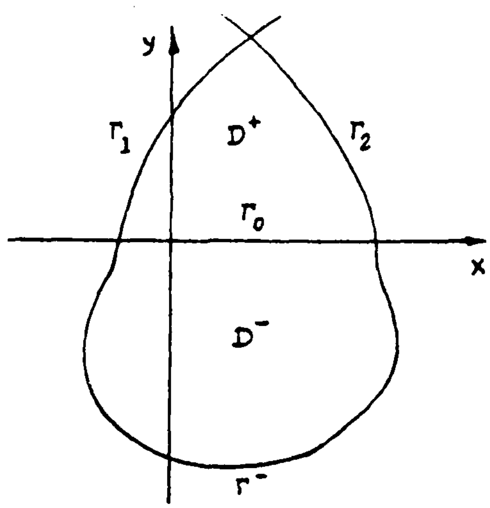
$$y\psi_{xx} - \psi_{yy} = 0 \tag{1.1}$$

is elliptic for $y < 0$ and hyperbolic for $y > 0$ with characteristics defined by:

$$\begin{cases} \xi = x + \frac{2}{3}y^{3/2} \\ \eta = x - \frac{2}{3}y^{3/2} \end{cases} \tag{1.2}$$

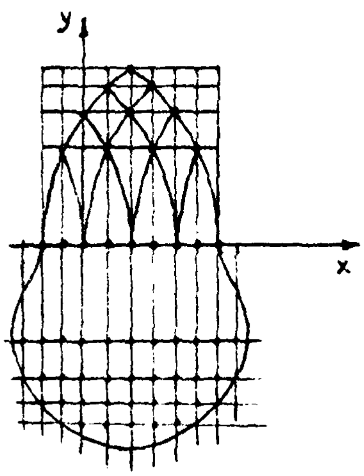
Tricomi [6] showed that (1.1) has a unique solution in a domain $D = D^+ \cup D^-$ bounded by a simple arc Γ^- in the elliptic domain with endpoints on the x-axis at $x = A$ and $x = B$ and by the two characteristics Γ_1 and Γ_2 through these endpoints (figure 1.1).

Figure 1.1



Problem (1.1) is known to be well posed in D^- with Dirichlet boundary conditions, i.e., ϕ given on Γ^- and on $\Gamma_0 \equiv \{y = 0, A \leq x \leq B\}$. In D^+ problem (1.1) is well posed with either Cauchy conditions - ϕ and ϕ_y given on Γ_0, or Goursat conditions - ϕ given on Γ_0 and on either Γ_1 or Γ_2. For solving the Tricomi problem in D one should match schemes in D^+ with schemes in D^-, e.g., Filipov [1] used a five-point formula in D^- and a four-point formula in D^+ matched by a difference equation corresponding to $\phi_{yy} = 0$ along the "parabolic line" Γ_0, on a mesh as in figure 2.1

Figure 1.2



Filipov also proved that his scheme is regular and converging. The change in the structure of the mesh across the parabolic line makes it difficult to construct higher order schemes of this kind in the usual manner.

Another strategy for solving the Tricomi problem is that of Vincenti and Wagoner [7] who reduced the problem to a pure elliptic problem. However, the boundary conditions on Γ_0, which are obtained by projecting the given conditions on Γ_1, are quite complicated and cause great numerical difficulties. Several authors, [2], [5] have used expansions in terms of certain particular solutions of (1.1), and this method proved to be quite effective for cases of very smooth boundary conditions.

The method presented in this work combines somehow the motifs of the above three strategies; a local expansion in terms of particular solutions of the Tricomi equation are used to produce high order difference schemes for the Tricomi problem. Using these schemes the problem is reduced to an elliptic problem in D^- with certain boundary conditions on $y = 0$.

The method used here for producing the difference scheme has recently been found to be useful for solving both Cauchy and Goursat problems in D^+. It is the versatility of this method which enables us to obtain specially structured difference schemes of high order for the desired matching along the parabolic line.

In section 4 we describe some numerical experiments with the suggested procedure, exhibiting a global $O(h^3)$ accuracy.

2. The expansion method for producing difference schemes

Let $z_i = (x_i, y_i)$ $i = 1,\dots,N+1$ be $N+1$ adjacent mesh points in a grid covering the domain D. We look for a difference approximation for the Tricomi equation (1.1) based upon $\{z_i\}_{i=1}^{N+1}$. The usual way of obtaining difference schemes is by expanding the approximation $u(x,y)$ in power series, around z_{N+1} for instance, then form a linear combination of $u(z_1),\dots,u(z_{N+1})$, using (1.1), to get the desired degree of accuracy. This process can be simplified by using an expansion in terms of the polynomial solutions of (1.1). These can be found by expressing $\phi(x,y)$ as a double power series expansion around (0,0) and comparing the expansions of $y\phi_{xx}$ with that of ϕ_{yy}. The result is that all the polynomial solutions of the Tricomi equation can be written as

$$\begin{cases} P_{2M+1}(x,y) = \sum_{i=0}^{[\frac{M}{2}]} c_i x^{M-2i} y^{3i} \\ P_{2M+2}(x,y) = \sum_{i=0}^{[\frac{M}{2}]} d_i x^{M-2i} y^{3i+1} \end{cases} \qquad M = 0,1,2,\dots \qquad (2.1)$$

where $c_0 = d_0 = 1$ and

$$\begin{cases} c_{i+1} = \dfrac{(M-2i)(M-2i-1)}{(3i+3)(3i+2)}\, c_i \\ \\ d_{i+1} = \dfrac{(M-2i)(M-2i-1)}{(3i+4)(3i+3)}\, d_i \end{cases} \qquad i = 1,\dots,[\tfrac{M}{2}]. \tag{2.2}$$

LEMMA.

Let $u(x,y)$ be an analytic solution of (1.1) in a neighbourhood S of (x_0,y_0) then

$$u(x,y) = \sum_{j=1}^{2M+1} a_j P_j(x,y) + o((\max(h,k)^M) \quad \forall\ (x,y) \in S \tag{2.3}$$

where $h = |x-x_0|$ and $k = |y-y_0|$. Also, for $y_0 = 0$ and $k \leq h^{2/3}$

$$\begin{aligned} u(x,y) &= \sum_{j=1}^{2M+1} b_j P_j(x,y) + o(h^M) \\ &= \sum_{j=1}^{2M+2} b_j P_j(x,y) + o(h^{M+2/3}). \end{aligned} \tag{2.4}$$

The proof is straightforward.

Using the above lemma it is clear that if we find a scheme which is accurate for $\{P_j\}_{j=1}^{N}$, i.e.

$$\sum_{i=1}^{N} \alpha_i P_j(x_i,y_i) = P_j(x_{N+1},y_{N+1}),\ j = 1,2,\dots,N, \tag{2.5}$$

then this scheme has a truncation error of order $[\frac{N}{2}]'$ Therefore we simply use (2.5) as the defining equations for our schemes. This method of obtaining difference schemes is convenient to use for any given distribution of mesh points, provided that the system (2.5) has a solution. The method has recently been used successfully for solving the Cauchy and Goursat problems in D^+ [3]. However, the schemes presented in [3] are not in a suitable form to be used for solving the mixed problem in D since they cannot be matched nicely with schemes in the elliptic domain D^-. In order to obtain suitable schemes we consider the "discrete Cauchy problem" in D^+, i.e. solving (1.1) with the two-level conditions

$$\begin{cases} \phi(x,0) = f_1(x) \\ & A \leq x \leq B \\ \phi(x,-\delta) = f_2(x) \end{cases} \qquad (2.6)$$

This problem is not well posed in D^+ in the usual sense. However, if we recall that ϕ should be a solution of the mixed problem in D, ϕ should be in C^1 in a neighbourhood of $y = 0$ for any $A < x < B$. Therefore, $\phi_y(x,0^+) = \phi_y(x,0^-)$ and as $\delta \to 0$ the problem (2.6) turns to be a Cauchy problem in D^+. For a fixed δ we should also give the additional boundary conditions:

$$\begin{cases} \phi(A,y) = g_1(y) \\ & -\delta < y < 0 \\ \phi(B,y) = g_2(y) \end{cases} \qquad (2.7)$$

to make the problem well posed in $A \leq x \leq B$ $-\delta \leq y \leq 0$, and hence also in D^+.

For a small δ it is expected that the domain of influence of the boundary conditions (2.6) is similar to that of the Cauchy conditions. Therefore, we consider a mesh defined by characteristic lines in D^+ as in [1] and [3]. Hence, we look for a scheme of the form

$$\sum_{i=1}^{M} [\alpha_i \phi(x_i, y_m) + \alpha_{M+i} \phi(x_i, y_{m-1})] = \phi(x_0, y_{m+1}) \qquad (2.8)$$

$m = 0, 1, \ldots, 2\tilde{N}-1$, where

$$y_{-1} = -\delta \qquad \delta \leq (\tfrac{3}{4}h)^{\frac{2}{3}},$$

$$y_0 = 0,$$

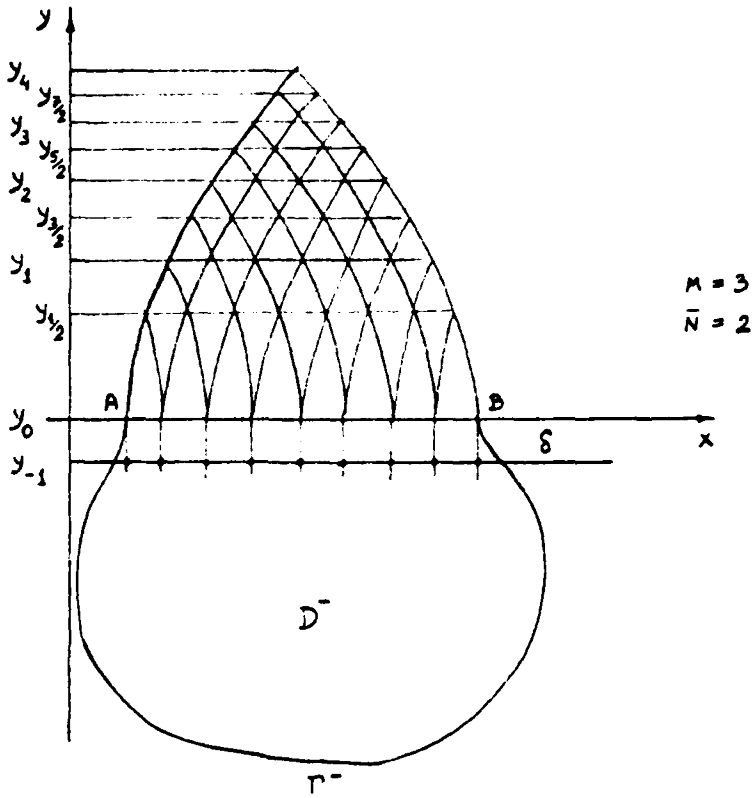
$$y_{m+1} = (\tfrac{3}{4}h + y_m^{\frac{3}{2}})^{\frac{2}{3}} \qquad m = 0, 1, \ldots, 2\tilde{N}-1$$

and

$$x_i = x_0 + h(\tfrac{i-1}{M-1} - \tfrac{1}{2}) \quad i = 1, 2, \ldots, M$$

where $h = \frac{B-A}{2\tilde{N}}$ as in figure 2.1.

Figure 1.1



The coefficients $\{\alpha_i\}_{i=1}^{2M}$ in (2.8) are determined by (2.[illegible]) with N = 2M, i.e. by the system of 2M linear equations

$$\sum_{i=1}^{M} [\alpha_i P_j(x_i, y_m) + \alpha_{M+i} P_j(x_i, y_{m-1})] = P_j(x_0, y_{m+1}) \quad (2.9)$$
$$j = 1, 2, \ldots, N$$

Given the "discrete Cauchy conditions" (2.6) and the scheme (2.8) for m = 1 (assuming that (2.9) has a solution) one can get an approximation for the level $y = y_1$ and so on.

to move upward to $y_{2\bar{N}}$ at the top of D^+. Before proceeding to the mixed problem we present some numerical results with the above scheme for the "discrete Cauchy problem"; we examined the case $M = 3$, i.e. the scheme is built to be exact for $P_1,\ldots,P_6$. However, it so turns out that the system (2.9) is singular. Therefore, we could make the scheme accurate for P_7 and P_8 as well, and thus, by (2.3) and (2.4) the resulting scheme is of order $o(h^{3+2/3})$ near $y = 0$ and of order $o((y_{m+1}-y_m)^3)$ for $m \geq 1$. A global $O(h^{3+2/3})$ accuracy has been detected in a series of numerical experiments. An example of this is shown in the following table where an analytic solution of the Tricomi equation,

$$\phi(x,y) = \cosh x\left(y + \sum_{n=1}^{\infty} \frac{y^{3n+1}}{(3n+1)3n\cdot(3n-2)(3n-3)\ldots4\cdot3}\right) \qquad (2.10)$$

is taken as a test function. The table shows the maximum absolute value of the error obtained by using the suggested scheme with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, where the "discrete Cauchy conditions (2.6) are given with $\delta = \frac{h}{2}$.

Table 2.1

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
max|error|	.67E-4	.57E-5	.39E-6	25E-7

3. The scheme for the mixed problem

We now consider the Tricomi problem in the mixed domain $D = D^+ \cup D^-$ with boundary conditions on Γ^- and on Γ_1. Assuming that Γ^- is rectangular we can use a rectangular mesh in D^- and 5 or 9 point schemes around any internal mesh point in D^-. The scheme (2.8) for $m = 0$ can be regarded as difference equation for the mesh points on $y = 0$. However, when we move on upwards we find that we still miss some difference equations to stand for the unknowns at the mesh points on the free boundary Γ_2. Therefore, some additional schemes should be introduced.

Let us denote by φ_0 the vector of the values of the approximation at the mesh points on $y = 0$ and by φ_{-1} the vector of the values on $y = -\delta$, i.e. $\varphi_0 = \{\varphi_{0i}\}_{i=1}^{2\bar{N}(M-1)-1}$ and $\varphi_1 = \{\varphi_{1i}\}_{i=1}^{2\bar{N}(M-1)-1}$ where

$$\begin{cases} \varphi_{0i} = \phi(A + i\frac{h}{M-1}, 0) \\ \\ \varphi_{1i} = \phi(A + i\frac{h}{M-1}, -\delta) \end{cases} \qquad i = 1,\dots,2\bar{N}(M-1)-1. \tag{3.1}$$

Successive use of the schemes (2.8) for $m = 0,1,\dots,2\bar{N}-1$ finally yields a linear relation between the approximation at the mesh points on Γ_1 and φ_0 and φ_{-1} in the form:

$$\phi(A+\frac{m}{2}h, y_m) = S_m^T \varphi_0 + T_0^T \varphi_{-1}$$
$$m = 1,2,\dots,2\bar{N}$$

where S_m and T_m are $(2\bar{N}(M-1)-1)$-dimensional vectors. Since ϕ is given on Γ_1 we can consider (3.2) as $2\bar{N}$ equations relating the unknown coefficients in φ_0 and φ_{-1}. Yet we are short of $2\bar{N}(M-2)-1$ equations to complete the system. To get these equations we use some additional schemes for solving the "discrete Cauchy problem" which result in relations of the form (3.2) between φ_0, φ_{-1} and boundary values at intermediate points on Γ_1. This can be done for any M and with no less accuracy than in (3.2). To demonstrate it we present the treatment for the case M = 3;

For M = 3 we are short of 2N-1 equations and we could have just the right number of equations if we could get relations of the form (3.2) for the boundary values $\phi(x_{m+\frac{1}{2}}, y_{m+\frac{1}{2}})$ where

$$\begin{cases} x_{m+\frac{1}{2}} = A + \frac{m+\frac{1}{2}}{2}h \\ y_{m+\frac{1}{2}} = \left(\frac{3}{8}h + y_m^{\frac{3}{2}}\right)^{\frac{2}{3}} \end{cases} \qquad m = 1,\dots,2\bar{N}-1 \qquad (3.3)$$

Such relations can be obtained by using additional 4th order schemes of the form

$$\sum_{i=1}^{4} [a_i \phi(x_i, y_m) + \alpha_{4+i}\phi(x_i, y_{m-1})] = \phi(x_{m+\frac{3}{2}}, y_{m+\frac{3}{2}}). \quad (3.4)$$

Again the coefficients of these schemes are obtained by systems of equations of the form (2.5) and here we could make the schemes be accurate for $P_1,\ldots,P_8$.

In the end we have the $2\bar{N}(M-1)-1$ relations

$$\phi(A+\tfrac{n}{4}h,\, y_{\frac{n}{2}}) = S^T_{\frac{n}{2}}\varphi_0 + T^T_{\frac{n}{2}}\varphi_{-1} \qquad n = 2,\ldots,4\bar{N}, \quad (3.5)$$

where $S_{\frac{n}{2}}$ and $T_{\frac{n}{2}}$ are given $(2\bar{N}(M-1)-1)$-dimensional vectors. These $4\bar{N}-1$ relations (for $M = 3$) are combined with the ordinary scheme for the internal points in D^- to give a full system of equations for all the unknowns in D^- and on the parabolic line $y = 0$. After solving this system we can use the vectors $S_{\frac{n}{2}}$ and $T_{\frac{n}{2}}$ $n = 2,\ldots,4\bar{N}$ to produce an approximation at all the mesh points in D^+.

4. Numerical experiments

In this section we present some numerical results of applying the new schemes for several Tricomi problems. We used the presented schemes (with $M = 3$) in the hyperbolic domain combined with a simple 5 and 9-point formula in the elliptic domain. The 9-point formula is also obtained by a procedure of the type (2.5), i.e., by demanding that the scheme is accurate for polynomial solutions of the Tricomi equation. In that way we obtain a local $o(\delta^4)$ scheme in D^- which, experimentally, proved to be a global $0(\delta^4)$ where a square mesh of size δ is used in D^-. In order to match the hyperbolic and elliptic schemes we chose $\delta = \frac{h}{2}$.

We consider boundary conditions given on the lines $\Gamma_- = \{x = \pm 1, -1 \le y \le 0\}$ and $\{y = -1, -1 \le x \le 1\}$ in the elliptic domain and on the characteristic line $\Gamma_1 = \{x - \frac{2}{3}y^{\frac{3}{2}} = -1, 0 \le y \le (\frac{3}{2})^{\frac{2}{3}}\}$ in the hyperbolic domain. Such conditions define a unique solution in the domain bounded by the above lines and by the characteristic

$$\Gamma_2 = \{x + \frac{2}{3}y^{\frac{3}{2}} = 1,\ 0 \le y \le (\frac{3}{2})^{\frac{2}{3}}\}.$$

The computational aspects of the method and some additional numerical results are to be described in a separate report [4].

Table 3.1

Maximum error in $D = D^+ \cup D^-$ for the Tricomi problem with the values of the analytic solution (2.10) of the Tricomi equation as boundary values on $\Gamma_1 \cup \Gamma^-$. A scheme with $M = 3$ is used in the hyperbolic domain and 5 and 9-point formula are used in the elliptic domain.

$\Delta x = \Delta y = \frac{h}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
5-point formula	0.47E-2	0.14E-2	0.37E-3
9-point formula	0.44E-3	0.59E-4	0.48E-5

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V. DIFFERENCE SCHEMES FOR THE SOLUTION OF TRICOMI'S EQUATION IN A MIXED REGION

Frieda Loinger

Abstract

The Tricomi problem in a mixed region is solved via a numerical projection of the hyperbolic boundary conditions onto the parabolic line, thus coupling the two regions. Difference schemes, exact for polynomials up to a certain degree, are used; high accuracy is achieved in all the examples computed.

1. Introduction

Tricomi's equation:

$$y\varphi_{xx} - \varphi_{yy} = 0$$

is elliptic for $y < 0$, parabolic for $y = 0$ and hyperbolic for $y > 0$. We look for a solution in a mixed domain $D = D_- \cup D_+$ where D_- is an elliptic rectangle bounded by $x = -1$, $y = -1$, $x = +1$, $y = 0$ and D_+ the hyperbolic curved triangle bounded by $y = 0$ and the two characteristics Γ_1 and Γ_2:

$$\Gamma_1 : \quad -1 = \eta = x - \frac{2}{3}y^{3/2} \qquad -1 \leqslant x \leqslant 0$$

$$\Gamma_2 : \quad 1 = \xi = x + \frac{2}{3}y^{3/2} \qquad 0 \leqslant x \leqslant 1$$

The problem is assume well-posed with appropriate boundary conditions given on $x = -1$, $y = -1$, $x = +1$ and one of the characteristics, say Γ_1.

2. Notation

For the numerical solution of the problem we divide the region into the following elements (see figure 1): rectangles in the elliptic domain along the lines x = const, y = const and isoparametric triangular elements in the hyperbolic domain, with the following enumeration:

j - the no. of the row

$j = -M, -M+1, \ldots, 0, 1, \ldots, N$

$j < 0$ elliptic region

$j = 0$ parabolic line

$j > 0$ hyperbolic region

$\cdot$ is a mesh point

no. of points on the parabolic line : $2N + 1$

n_j - the no. of points in the j-th-row

$x_{i+1} - x_i = \Delta x = 1/N \quad \forall i$ in the grid

$y_{j+1} - y_j = \Delta y = 1/M$ for $j < 0$

$y_{j+1} - y_j = \Delta y_j = (1.5\Delta x(j+1))^{2/3} - (1.5\Delta x \cdot j)^{2/3}$ for $j \geq 0$

mesh points in the elliptic part including the parabolic line:

$$(x_i, y_j) \quad i = -N, -N + 1, \ldots, 0, 1, \ldots, N \; ; \; j = -M, \ldots, 0$$

$$x_i = i\Delta x \quad y_j = j\Delta y$$

mesh point in the hyperbolic part:

$$(x_{i,j}, y_j) \quad j = 1, 2, \ldots, N \; ; \quad i = -N, \ldots, N - 2j$$

$$y_j = (1.5\Delta x j)^{2/3} \; ; \; x_{i,j} = (i+j)\Delta x$$

and the numerical solution at the j-th-row:

$$\varphi^{(j)} = \varphi(x_{i,j}, y_j) \quad i = -N, -N + 1, \ldots, n_j$$

3. Projection of the boundary condition along Γ_1 on the parabolic line

The hyperbolic problem with either Goursat conditions (φ given on Γ_1 and $y = 0$) or Cauchy conditions (φ and φ_y given on $y = 0$) is treated in [1] (D. Levin) using high accuracy difference schemes. The difference scheme for (x_i, y_{j+1}) is based on the values of φ, φ_y at the points $(x_i \pm \Delta x, y_j)$, (x_i, y_j) when Cauchy conditions are given (figure 2).

The coefficients are determined by the demand that the scheme is accurate for 6 polynomial solutions of the Tricomi equation: $1, x, y, xy, 3x^2 + y^3, x^3 + xy^3$. Numerical experiments showed that the schemes obtained are accurate for 8 basic functions:

$$1, x, y, xy, 3x^2 + y^3, x^3 + xy^3, 6x^2y + y^4, 2x^3y + xy^4.$$

We now look for a solution in the mixed region.

An analytic connection between the boundary conditions on Γ_1 and the parabolic line is known:

Bitsadze [2]:

$$\tau(x) = \varphi(x,0) = \frac{1}{2\pi\gamma_1} x^{5/6} \frac{d}{dx} \int_0^x \frac{\psi(t/2)}{t^{2/3}(x-t)^{1/6}} dt + \gamma \int_0^x \frac{\nu(t)}{(x-t)^{1/3}} dt$$

where $\nu(x) = \varphi_y(x,0)$

$\psi(x,0) = \varphi(x,y(x))$ on Γ_1

The connection on the parabolic line enables one to find the solution of Tricomi's equation in the elliptic region and then in the hyperbolic problem can be solved with either as a Goursat or a Cauchy boundary conditions.

This report presents a numerical process for finding a connection between the hyperbolic and elliptic region, without using φ_y (see also [3]). Alternatively, we look for a connection between the values on Γ_1 and $\varphi^{(0)}$, $\varphi^{(-1)}$.

Assuming the elliptic problem already solved, i.e. $\varphi^{(j)} = (\varphi_1^{(j)},\dots,\varphi_{2N+1}^{(j)})$, $j = -M,\dots,0$, known, where $\varphi^{(0)}$ is the solution on the parabolic line. We proceed from row (j) to $(j+1)$ $(j > 1)$ by choosing 6 basic functions as before (in the Cauchy problem), but instead of using values of φ,φ_y on $y = y_j$ as it is done in [1] for the Cauchy problem, we take an additional row y_{j-1} and the difference scheme will be based on the values of φ at the points (figure 3): (x_i,y_{j+1}) ; $(x_i \pm \Delta x,y_j),(x_i,y_j)$; $(x_i \pm \Delta x,y_{j-1}),(x_i,y_{j-1})$;
and we require that the difference scheme is exact for the 6 basic functions. Numerical experiments show that these 7-part differ-

ence scheme using 6 basic functions are exact for 8 polynomials.

Note: if $j = 0$, $y_j = 0$, $y_{j-1} = -\Delta y$, i.e. the first row of the elliptic region is taken.

The difference scheme can be expressed by

$$\sum_{\ell=1}^{3} t_\ell \, \varphi(x_\ell, y_j) + s_\ell \, \varphi(x_\ell, y_{j-1}) = \varphi(x_\ell, y_{j+1})$$

$$x_\ell = x_i + (\ell-1)\Delta x \qquad i = -N, \ldots, n_j$$

The coefficients (t_ℓ, s_ℓ) $(\ell = 1,3)$ do not depend on x, but do depend on y. In matrix form

$$\varphi^{(j+1)} = T^{(j+1)} \, \varphi^{(j)} + S^{(j+1)} \, \varphi^{(j-1)} \qquad j = 0, \ldots, N-1$$

where

$$T^{(j+1)} = \begin{bmatrix} t_1 & t_2 & t_3 & & & 0 \\ & t_1 & t_2 & t_3 & & \\ & & t_1 & t_2 & t_3 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \qquad j = 0, \ldots, N-1$$

$$T^{(j+1)} \text{ of order : } n_{j+1} \times n_j$$

$$S^{(j+1)} = \begin{bmatrix} 0 & S_1 & S_2 & S_3 & & & \\ 0 & 0 & S_1 & S_2 & S_3 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

$j = 1,\ldots,N-1$

$S^{(j+1)}$ of order $n_{j+1} \times n_{j-1}$

$$S^{(1)} = \begin{bmatrix} S_2 & S_2 & S_3 & & & 0 \\ & S_1 & S_2 & S_3 & & \\ & & S_1 & S_2 & S_3 & \\ 0 & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

Suppose we found $B^{(j)}$ *and* $A^{(j)}$, *such that*

$$\varphi^{(j)} = B^{(j)}\,\varphi^{(0)} + A^{(j)}\,\varphi^{(-1)}$$

where

$$B^{(j)} : n_j \times (2N+1)$$

$$A^{(j)} : n_j \times (2N+1)$$

$$B^{(1)} = T^{(1)} \qquad A^{(1)} = S^{(1)}$$

$$B^{(0)} = I \qquad A^{(0)} = 0$$

then:

$$\varphi^{(j+1)} = T^{(j+1)}\varphi^{(j)} + S^{(j+1)}\varphi^{(j-1)}$$

$$= T^{(j+1)}(B^{(j)}\varphi^{(0)}+A^{(j)}\varphi^{(-1)})+S^{(j+1)}(B^{(j-1)}\varphi^{(0)}+A^{(j-1)}\varphi^{(-1)})$$

$$= B^{(j+1)}\varphi^{(0)}+A^{(j+1)}\varphi^{(-1)}$$

where:

$$B^{(j+1)} = T^{(j+1)}B^{(j)} + S^{(j+1)}B^{(j-1)}$$

$$A^{(j+1)} = T^{(j+1)}A^{(j)} + S^{(j+1)}A^{(j-1)}$$

$\varphi^{(j)}(1) = g_j$ are the boundary conditions along Γ_1.

We get only N equations for the 2N-1 unknowns: $\varphi_2^{(0)},\dots,\varphi_{2N}^{(0)}$.

$j = 1,\dots,N$ [illegible] $+ A^{(j)}(1)\varphi^{(-1)}$

where [illegible] are the first rows [illegible], respectively.

To get N-1 additional equations we take N-1 intermediate points on Γ_1 (marked by x in figures [illegible]) and build a 9-point difference scheme for them.

The difference scheme has the form:

$$\sum_{\ell=1}^{4} [\text{illegible}]\,\varphi(x_\ell,y_j) + [\text{illegible}]\,\varphi(x_\ell,y_{j-1}) = \varphi(x_{-N,j+1/2},y_{j+1/2})$$

where:

$$x_\ell = x_{-N,j+1/2} + (\ell - \tfrac{5}{2})\Delta x$$

The coefficients of the scheme are obtained by the demand that it is accurate for 8-polynomial lead. Since the 7-point formula is accurate for 8 polynomials the order of accuracy is not changed. The 9-point scheme can be expressed as:

$$\varphi^{(j+3/2)} = \hat{T}^{(j+3/2)}\ \varphi^{(j)} + \hat{S}^{(j+3/2)}\ \varphi^{(j-1)}$$

where:

$$\hat{T}^{(j+3/2)} : n_{j+3/2} \times n_j$$

$$\hat{S}^{(j+3/2)} : n_{j+3/2} \times n_{j-1}$$

$\varphi^{(j)}$ and $\varphi^{(j-1)}$ can be substituted from the equation obtained by the 7-point formula:

$$\varphi^{(j)} = B^{(j)}\ \varphi^{(0)} + A^{(j)}\ \varphi^{(-1)}$$

$$\Rightarrow \varphi^{(j+3/2)} = \hat{T}^{(j+3/2)}(B^{(j)}\varphi^{(0)} + A^{(j)}\varphi^{(-1)}) + \hat{S}^{(j+3/2)}(B^{(j-1)}\varphi^{(0)}$$

$$+A^{(j-1)}\varphi^{(-1)})$$

$$= \hat{B}^{(j+3/2)}\varphi^{(0)} + \hat{A}^{(j+3/2)}\varphi^{(-1)}$$

where:

$$\hat{B}^{(j+3/2)} = \hat{T}^{(j+3/2)}B^{(j)} + \hat{S}^{(j+3/2)}B^{(j-1)}$$

$$\hat{A}^{(j+3/2)} = \hat{T}^{(j+3/2)}A^{(j)} + \hat{S}^{(j+3/2)}A^{(j-1)}$$

We now have 2N-1 equations for the 2N-1 unknowns on the parabolic line:

$$g_j = \varphi^{(j)}(1) = B^{(j)}(1)\varphi^{(0)} + A^{(j)}(1)\varphi^{(-1)} \qquad j = 1,\dots,N$$

$$g_{j+\frac{1}{2}} = \varphi^{(j+1/2)}(1) = \hat{B}^{(j+1/2)}(1)\varphi^{(0)} + \hat{A}^{j+1/2}(1)\varphi^{(-1)} \qquad j = 1,\dots,N-1$$

where $B^{(j)}(1)$, $A^{(j)}(1)$, $\hat{B}^{(j+1/2)}(1)$, $\hat{A}^{(j+1/2)}(1)$

are the first rows of $B^{(j)}$, $A^{(j)}$, $\hat{B}^{(j+1/2)}$, $\hat{A}^{(j+1/2)}$ respectively. The projection on the parabolic line can be written in matrix form:

$$\boxed{B\ \varphi^{(0)} + A\ \varphi^{(-1)} = g}$$

The projection matrices A and B, of order $(2N+1)x(2N+1)$ are:

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ B^{(1)}(1) & & & \\ \hat{B}^{(3/2)} & & & \\ B^{(2)}(1) & & & \\ \hat{B}^{(5/2)}(1) & & & \\ \vdots & & & \\ B^{(N-1)}(1) & & & \\ \hat{B}^{(N-\frac{1}{2})}(1) & & & \\ B^{N}(1) & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & \cdots & 0 \\ A^{(1)}(1) & & \\ \hat{A}^{(3/2)}(1) & & \\ A^{(2)}(1) & & \\ \hat{A}^{(5/2)}(1) & & \\ \vdots & & \\ A^{(N-1)}(1) & & \\ \hat{A}^{(N-1/2)}(1) & & \\ A^{(N)}(1) & & \\ 0 & \cdots & 0 \end{bmatrix}$$

4. Solution of the elliptic problem

In the elliptic region we use either a 5-point (I) or a 9-point difference scheme.

I)
$$\Delta y^2\ \varphi_{yy} \approx \varphi(x,y+\Delta y) - 2\varphi(x,y) + \varphi(x,y-\Delta y)$$

$$= \varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}$$

$$\Delta x^2\ \varphi_{xx} \approx \varphi(x+\Delta x,y) - 2\varphi(x,y) + \varphi(x-\Delta x,y)$$

$$= \varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}$$

The 5-point difference scheme for the Tricomi equation $y\varphi_{xx}-\varphi_{yy}=0$ for the point (i,j) is:

$$y(\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}) - (\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}) = 0$$

where:

$$y = j\Delta y \qquad j = -M + 1,\ -M + 2, \ldots, 1$$

This scheme is accurate for: $1, x, y, xy, 3x^2 + y^3$

II) We build a 9-point difference scheme by taking the first 8 polynomials which solve Tricomi's equation, i.e.:

$$1, x, y, xy, 3x^2 + y^3, x^3 + xy^3, 6x^2y + y^4, 2x^3y + xy^4$$

The demand that the difference scheme based on the points: $(x_i \pm \Delta x, y_j \pm \Delta y)$, $(x_i \pm \Delta x, y_j)$, $(x_i, y_j \pm \Delta y)$ and (x_i, y_j) be accurate for those functions leads to a dependent system of equations. Therefore one polynomial has to be exchanged with one with a higher degree.

An independent set of functions are:

$$1, y, xy, 3x^2 + y^3, x^3 + xy^3, 6x^2y + y^4, 2x^3y + xy^4, 15x^4 + 30x^2y^3 + 2y^6$$

The scheme is accurate for $P = x$ as well, hence it is accurate for 9 functions.

Both types of the difference schemes give a system of equations which is triangular in blocks:

$$\begin{bmatrix} B^{(-M)} & C^{(-M)} & & & & \\ A^{(-M+1)} & B^{(-M+1)} & C^{(-M+1)} & & & \\ & A^{(-M+2)} & B^{(-M+2)} & C^{(-M+2)} & & \\ & \ddots & \ddots & \ddots & & \\ & & & A^{(-1)} & B^{(-1)} & C^{(-1)} \\ & & & & A^{(0)} & B^{(0)} \end{bmatrix} \cdot \begin{bmatrix} \varphi^{(-M)} \\ \varphi^{(-M+1)} \\ \varphi^{(-M+2)} \\ \vdots \\ \varphi^{(-1)} \\ \varphi^{(0)} \end{bmatrix} = \begin{bmatrix} f^{(-M)} \\ f^{(-M+1)} \\ f^{(-M+2)} \\ \vdots \\ f^{(-1)} \\ f^{(0)} \end{bmatrix}$$

with all the blocks having the same size: (2N+1) x (2N+1)

$B^{(-M)} = I$

$C^{(-M)} = 0$

$A^{(j)}$, $B^{(j)}$, $C^{(j)}$, $-M+1 \leq j \leq 1$, are matrices of either the 5-point or 9-point difference schemes including identity equations for the boundary points $(-1, j\Delta y)$, $(+1, j\Delta y)$. $A^{(0)}$ and $B^{(0)}$ are the projection matrices (section 3), $f^{(-M)}$ is the boundary condition on $y = -1$, and $f^{(j)}(1)$ and $f^{(j)}(2N+1)$ are the boundary conditions on $x = -1$ and $x = +1$, accordingly.

$$\left.\begin{aligned} f^{(j)}(1) &= \varphi(-1, j\Delta y) \\ f^{(j)}(2N+1) &= \varphi(+1, j\Delta y) \\ f^{j}(i) &= 0, \quad 2 \leq i \leq 2N \end{aligned}\right\} -M + 1 \leq j \leq 1$$

$f^{(0)} = g$ - vector of solution on Γ_1.

5. Numerical results

The mixed problem is solved in 3 steps:

a) Computation of the projection matrices $A^{(0)}$ and $B^{(0)}$.

b) Solution of the elliptic problem with $A^{(0)}\varphi^{(-1)} + B^{(0)}\varphi^{(0)}=g$ as boundary condition on $y = 0$.

c) Solution of the hyperbolic problem using the results of $\varphi^{(0)}$ and $\varphi^{(-1)}$ from (b), and proceeding with the 7-point formula:

To see the influence of the projection condition on $y = 0$ the elliptic problem is also solved with exact boundary conditions on $y=0$, i.e. $A^{(0)} = 0$, $B^{(0)} = I$ and $f^{(0)} = \varphi(x,0)$ is substituted in the system (d); the hyperbolic problem is also solved with exact values for $\varphi^{(0)}$ and $\varphi^{(-1)}$. (e)

The folloiwng 3 examples are considered:

i) $\varphi_1(x,y) = 15x^4 + 30x^2y^3 + 2y^6$

Note: For this function the 9-point formula in the elliptic region is exact.

ii) $\varphi_2(x,y) = -21x^4y - 21x^2y^4 - y^7$

iii) $\varphi_3(x,y) = \cosh x(y + \sum_{n=1}^{\infty} \frac{y^{n+1}}{(3n+1)3n(3n-2)(3n-3)\ldots 4.3})$

	$\Delta y = \Delta x = \frac{1}{4}$		$\Delta y = \Delta x = \frac{1}{8}$		$\Delta y = \Delta x = \frac{1}{16}$	
	max abs. error in $y \leq 0$	max abs. error in $y > 0$	max abs. error in $y \leq 0$	max abs. error in $y > 0$	max abs. error in $y \leq 0$	max abs. error in $y > 0$
5-point formula in the elliptic region and projection on $y=0$ (a,b,c)	0.411	0.360	0.106	0.883×10^{-1}	0.267×10^{-1}	0.226×10^{-1}
9-point formula in the elliptic region and projection on $y=0$ (a,b,c)	0.676×10^{-1}	0.110	0.555×10^{-2}	0.113×10^{-1}	0.407×10^{-3}	0.786×10^{-3}
5-point formula in the elliptic region and exact values on $y = 0$ (d)	0.242		0.646×10^{-1}		0.162×10^{-1}	
9-point formula in the elliptic region and exact values on $y = 0$ (d)	0		0		0	
hyperbolic problem with exact values for $\varphi^{(0)}$, $\varphi^{(-1)}$ (e)		0.646×10^{-1}		0.538×10^{-2}		0.476×10^{-3}

Table 1:

i) $\varphi_1(x,y) = 15x^4 + 30x^2y^3 + 2y^6$

	$\Delta y = \Delta x = \frac{1}{4}$		$\Delta y = \Delta x = \frac{1}{8}$		$\Delta y = \Delta x = \frac{1}{16}$	
	max abs.error in $y \leq 0$	max abs.error in $y > 0$	max abs.error in $y \leq 0$	max abs.error in $y > 0$	max abs.error in $y \leq 0$	max abs.error in $y > 0$
5-point formula in the elliptic region and projection on y=0 (a,b,c)	0.211	0.111	0.563×10^{-1}	0.296×10^{-1}	0.142×10^{-1}	0.862×10^{-2}
9-point formula in the elliptic region and projtion on y = 0 (a,b,c)	0.729×10^{-1}	0.106	0.629×10^{-2}	0.135×10^{-1}	0.381×10^{-3}	0.108×10^{-2}
5-point formula in the elliptic region and exact values on y = 0 (d)	0.175		0.466×10^{-1}		0.118×10^{-1}	
9-point formula in the elliptic region and exact values on y = 0 (d)	0.290×10^{-2}		0.189×10^{-3}		0.120×10^{-4}	
hyperbolic problem with exact values for $\varphi^{(0)}$, $\varphi^{(-1)}$ (e)		0.149×10^{-1}		0.122×10^{-2}		0.826×10^{-4}

Table 2:

ii) $\varphi_2(x,y) = -(21x^4y + 21x^2y^4 + y^7)$

	$\Delta y = \Delta x = \frac{1}{4}$		$\Delta y = \Delta x = \frac{1}{8}$		$\Delta y = \Delta x = \frac{1}{16}$	
	max abs.error in $y \leq 0$	max abs.error in $y > 0$	max abs.error in $y \leq 0$	max abs.error in $y > 0$	max abs.error in $y \leq 0$	max abs.error in $y > 0$
5-point formula in the elliptic region and projection on $y = 0$ (a,b,c)	0.473×10^{-2}	0.394×10^{-2}	0.137×10^{-2}	0.125×10^{-2}	0.366×10^{-3}	0.339×10^{-3}
9-point formula in the elliptic region and projection on $y = 0$ (a,b,c)	0.301×10^{-3}	0.437×10^{-3}	0.267×10^{-4}	0.586×10^{-4}	0.163×10^{-5}	0.483×10^{-5}
5-point formula in the elliptic region and exact values on $y = 0$ (d)	0.188×10^{-2}		0.472×10^{-3}		0.120×10^{-3}	
9-point formula in the elliptic region and exact values on $y = 0$ (d)	0.118×10^{-4}		0.751×10^{-6}		0.479×10^{-7}	
hyperbolic problem with exact values for $\varphi^{(0)}, \varphi^{(-1)}$ (e)		0.675×10^{-4}		0.566×10^{-5}		0.390×10^{-6}

Table 3:

iii) $\varphi_3(x,y) = \cos hx(y + \sum_{n=1}^{\infty} \frac{y^{3n+1}}{(3n+1)3n(3n-2)(3n-3)\ldots 4\cdot 3})$

Conclusions

The order of convergence of the 5-point formula alone is $O(\Delta x^2)$ and the order of convergence of the 9-point formula alone is $O(\Delta x^4)$ (see results (d), table 1-3). The results of (e) show that the order of convergence in the hyperbolic region is $O(\Delta x^3)$, approximately:

$$O(\Delta x^{3+2/3}) \approx O(\Delta x^3 \cdot \Delta y_j) \quad (j>0)$$

(estimate of error, see [3]).

The schemes used in the hyperbolic region are more accurate than the 5-point formula and less accurate than the 9-point formula, used in the elliptic region. Therefore, when the mixed problem is solved with the projection, the maximal absolute error is obtained in $y \leq 0$ in case of the 5-point scheme and the order of convergence is $O(\Delta x^2)$ in the whole region, and the maximal absolute error is achieved in $y>0$ when we use the 9-point scheme in the elliptic region and the order of convergence is $O(\Delta x^3 \Delta y_j)$ $(j>0)$

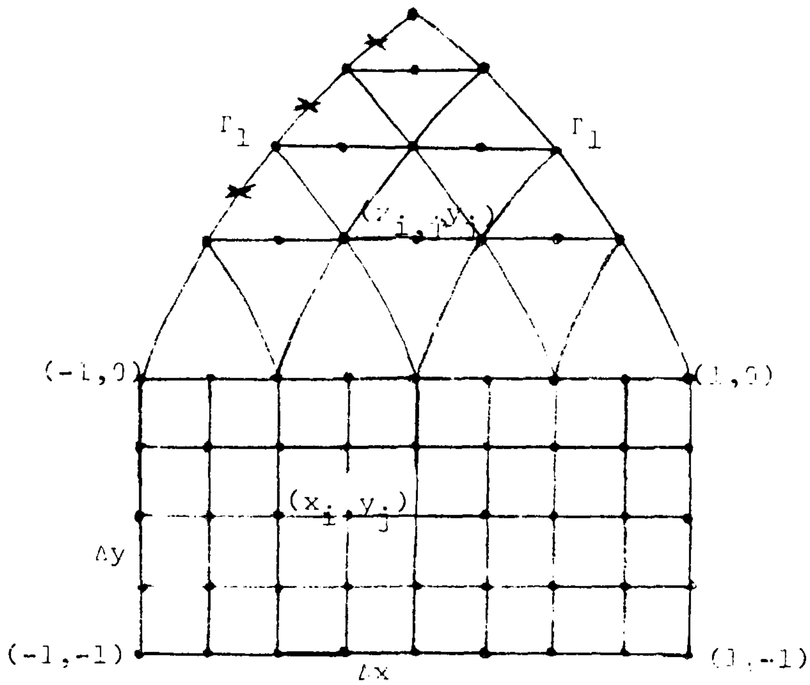

Figure 1

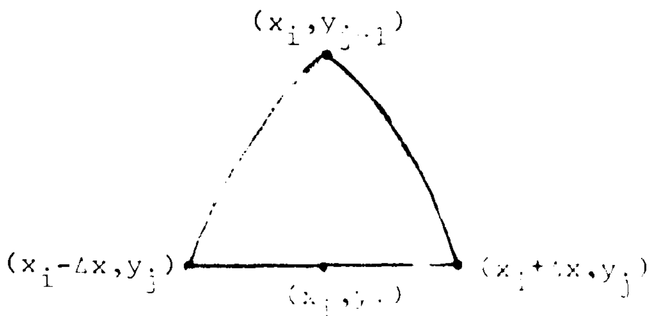

Figure 2

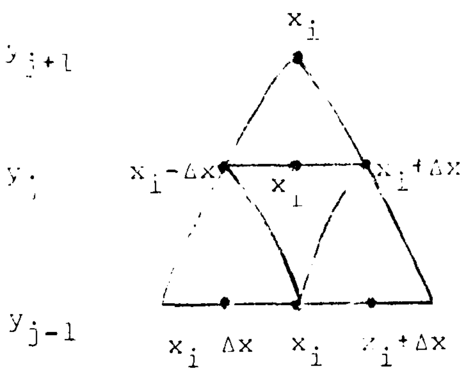

Figure 3

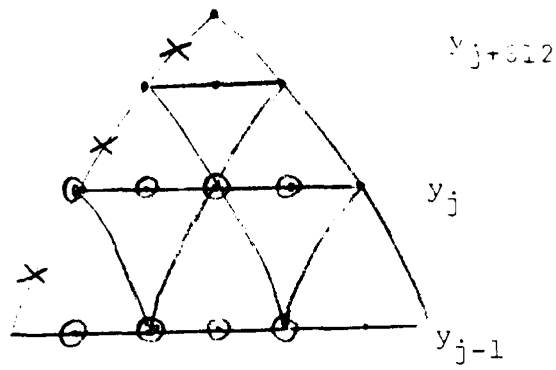

Figure 4

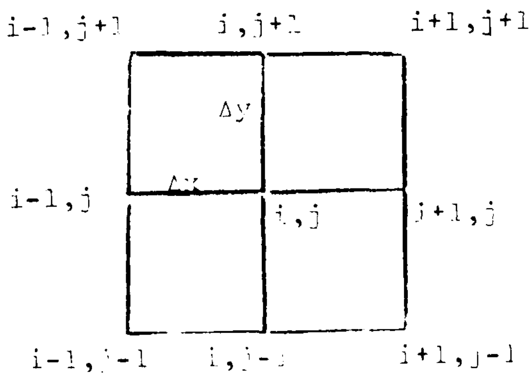

Figure 5

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[2] Bitsadze, A.V., Equations of the mixed type, The MacMillan Company, New York (1964), pp. 76.

[3] Levin, D., Finite difference approximations for the solution of the Tricomi equation in a mixed elliptic-hyperbolic region, 1980.

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