ASYMPTOTIC POWER OF EDF STATISTICS FOR EXPONENTIALLY AGAINST WEIBULL AND GAMMA ALTERNATIVES

By

MICHAEL A. STEPHENS

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
In recent papers, asymptotic distributions of EDF statistics, i.e. goodness-of-fit statistics based on the empirical distribution function, have been found by expanding a related function in a series of orthogonal functions. Suppose an ordered sample $y_1 \leq y_2 \leq \cdots \leq y_n$ is given, and we wish to test $H_0$: the parent population is the (continuous) distribution $F(y;\theta)$, where $\theta$ is a vector of parameters. The transformation $x_i = F(y_i;\theta)$, if $\theta$ is known, gives a set $x_i$ of ordered uniform variates. Let $F_n(x)$ be the EDF of the $x$-values, and let $y_n(x) = \sqrt{n} \{ F_n(x) - x \}$. The test statistics to be considered are $W_n^2$, $U_n^2$, and $A_n^2$, of the Cramer-von Mises type, and can be defined in terms of $y_n(x)$:

$$W_n^2 = \int_0^1 \{ y_n(x) \}^2 \, dx,$$

$$U_n^2 = \int_0^1 \{ y_n(x) - \bar{y} \}^2 \, dx,$$

$$A_n^2 = \int_0^1 \{ y_n(x) \}^2 \, w(x) \, dx,$$

where

$$\bar{y} = \int_0^1 y_n(x) \, dx \quad \text{and} \quad w(x) = 1/(x-x^2).$$

As $n \to \infty$, $y_n(x)$ approaches a Gaussian process $y(x)$, and the asymptotic distributions of the statistics are found from the integrals above with $y(x)$ replacing $y_n(x)$. This is done by first expanding the appropriate integrand as a series of orthogonal functions. The method was given by Watson (1967), and was developed in this connection by Durbin and Knott.
(1972), by Stephens (1974), and most recently by Durbin, Knott and Taylor (1975). We shall illustrate the procedure with $W_n^2$. Suppose $y_n(x)$ is expanded as a Fourier series:

$$y_n(x) = \sum_{j=1}^{\infty} b_j \sin \pi j x, \quad 0 < x < 1;$$

then

$$b_j = \frac{\sqrt{2}}{\pi} z_{nj}/j^n$$

where

$$z_{nj} = \sqrt{\frac{2}{n}} \sum_{i=1}^{n} \cos j \pi x_i.$$ 

Also

$$W_n^2 = \sum_{j=1}^{\infty} \frac{z_{nj}^2}{j^2 \pi^2}.$$ 

Durbin and Knott refer to $z_{nj}$ as components. Stephens (1974) expanded $y_n(x)$ as a Fourier series with both sine and cosine terms to deal with $U_n^2$; for $A_n^2$, one expands $y_n(x)(w(x))^{1/2}$ in associated Legendre functions. This approach to asymptotic theory of these statistics is a different approach from that taken by Anderson and Darling (1952). In the latter paper it was shown that if $W$ is a random variable with the asymptotic distribution of $W_n^2$, it was possible to express $W^2$ as

$$W^2 = \sum_{j=1}^{\infty} \frac{z_{1j}^2}{j^2 \pi^2}.$$
where \( z_j \) are i.i.d. standard normal variables; the newer method gives a finite-n form of (3), and one can see the approach of the \( z_{nj} \) to the \( z_j \).

In Durbin, Knott and Taylor (1975; henceforth referred to as DKT) the above technique is extended to the case where one or more components of \( \theta \) are unknown and must be estimated from the data. Again the new approach offers an elegant and interesting alternative to the theory of unknown-parameter situations, begun by Darling (1955) and developed by Sukhatme (1972) and Stephens (1976). A modern treatment, particularly of the behavior of \( y_n(x) \), is in Durbin (1973). DKT examined the distributions of \( W^2 \), \( U^2 \) and \( A^2 \) (absence of a subscript indicates asymptotic distributions) for tests for normality, and also gave some asymptotic power results for certain alternatives tending, as \( n \to \infty \), to the null hypothesis. Similar theory was also given for tests of exponentiality, i.e. that \( y \) has distribution \( F(y) = 1 - \exp(-\theta y) \), \( y \geq 0 \), but for \( W^2 \) and \( U^2 \) only. In this paper we examine \( A^2 \) for this situation, and compare asymptotic powers of \( W^2 \), \( U^2 \) and \( A^2 \) against Weibull and Gamma alternatives. In repeating some of the DKT calculations, we have found that the coefficients given to calculate components, both for normal and exponential tests, are not correct; apparently the wrong matrix has been printed, in DKT Tables 1 and 6.

In order to save space, we follow as closely as possible the DKT notation, and simplify it for the special case of interest; the reader is referred to the DKT paper for theoretical details.
2. Components of $\hat{W}_n^2$, when parameters must be estimated.

We continue by illustrating the general theory with $\hat{W}^2$. Let unknown components of $\mathbf{B}$ be estimated by maximum likelihood, and let $\hat{\mathbf{B}}$ be the $\mathbf{B}$ so calculated; write $\mathbf{2}_1 = F(y; \hat{\mathbf{B}})$, and suppose $\hat{Y}_n(x), \hat{Y}(x), \hat{W}_n^2, \hat{W}_n^2$ refer to the quantities defined in Section 1, but with $\hat{\theta}$ replacing $\theta$. Then

$$(4) \quad \hat{W}_n^2 = \int_0^1 (\hat{Y}_n(x))^2 \, dx \quad \text{and} \quad \hat{W}_n^2 = \int (\hat{Y}(x))^2 \, dx.$$ 

DKT show that

$$(5) \quad \hat{W}_n^2 = \sum_{j=1}^{\infty} \mu_j \hat{z}_{nj}^2$$

and asymptotically

$$(6) \quad \hat{W}_n^2 = \sum_{j=1}^{\infty} \mu_j \hat{z}_j^2$$

Where the $\mu_j$ and the components $\hat{z}_{nj}$, which are asymptotically $\hat{z}_j$, are to be found. The $\hat{z}_j$ are obtained first, as follows. Define $\lambda_j = (j\pi)^{-2}$ and $\hat{z}_j(x) = \sqrt{2} \sin(j\pi x)$, $j = 1, 2, \ldots$; the $\lambda_j$ are the weights of the $\hat{z}_{nj}^2$ in representation (2). Define vectors $\zeta_n$, and $\zeta$ with j-th components $\zeta_{nj}$ and $\zeta_j$:

$$(7) \quad \zeta_{nj} = \lambda_j \int_0^1 \varphi_n(x) \, \hat{z}_j(x) \, dx.$$
and

\[
\zeta_j = \frac{1}{2} \int_0^1 \varphi(x) I_j(x) \, dx.
\]

The \( \zeta_j \) will be found as a linear combination of the \( \zeta_j \), and then \( \zeta_{nj} \) will be defined as the same linear combination of the \( \zeta_{nj} \).

Suppose, for our purposes, that in the distribution \( F(y;\theta) \), density \( f(y;\theta) \), the vector \( \theta \) contains only two components, so that \( \theta' = (\theta_1, \theta_2) \); \( \theta_1 \) is known, and on \( H_0 \) is \( \theta_{10} \), and \( \theta_2 \) is to be estimated by maximum likelihood, with estimate \( \hat{\theta}_2 \). Let \( \theta_0' = (\theta_{10}, \theta_2) \) and \( \hat{\theta}_0' = (\theta_{10}, \hat{\theta}_2) \). Let \( F, f \) be \( F(y;\theta) \) and \( f(y;\theta) \) respectively; define

\[
\begin{align*}
\varrho_1(x) &= \left( \frac{\partial F}{\partial \theta_1} \right)_{\theta=\theta_0} \\
\varrho_2(x) &= \left( \frac{\partial F}{\partial \theta_2} \right)_{\theta=\theta_0}
\end{align*}
\]

with \( x = F(y;\theta_0) \) making \( \varrho_1 \) and \( \varrho_2 \) functions of \( x \); also let

\[
\begin{align*}
I_1 &= -E\left( \frac{\partial^2 \ln F}{\partial \theta_1^2} \right)_{\theta=\theta_0} \\
I_{12} &= E\left( \frac{\partial \ln F}{\partial \theta_1} \cdot \frac{\partial \ln F}{\partial \theta_2} \right)_{\theta=\theta_0}
\end{align*}
\]

Suppose vector \( \delta \) has components \( \delta_j \), for \( j=1, 2, \ldots \)

\[
\delta_j = \frac{1}{2} \int_0^1 \varphi(x) \xi_j(x) \, dx
\]

It can be shown that \( I = \Delta' \Delta = \sum_{j=1}^\infty \delta_j^2 \). Then, from DKT, equation 5.1, adapted for only one unknown parameter,

\[
E(\xi_1') = T - \Delta_1' / I = M
\]
where $\mathbb{T}$ is the identity matrix of infinite order. Let $\lambda^r$ be the diagonal matrix with diagonal elements $\lambda_1^r, \lambda_2^r, ...$ where

$\lambda_1 \geq \lambda_2 \geq ...$, and let $\mathbb{N}$ be the diagonal matrix which diagonalises

$\frac{1}{\lambda^2} M \lambda^2$, i.e.

(13) \[
\frac{1}{\lambda^2} M \lambda^2 = \mathbb{N} \mu \mathbb{N}'.
\]

where $\mu$ is a diagonal matrix with elements $\mu_1 \geq \mu_2 \geq ...$ on the diagonal. Finally, define

(14) \[
B = \frac{1}{\mu^2} \mathbb{N}^{-\frac{1}{2}} \lambda^\frac{1}{2}.
\]

and

(15) \[
\hat{z} = B \xi; \quad \hat{z}_n = B \xi_n.
\]

These are the vectors whose components appear in (5) and (6).

Note that $E(zz') = BM'B = T$, using (12) for the first equality and (13) and (14) for the second. Recall that $\hat{z}_i = F(y_i; \theta)$; DKT show that

$$\xi_{nj} = \sqrt{\frac{2}{n}} \sum_{i=1}^{n} \cos \pi j \hat{z}_i$$

so that, using this and (15), $\hat{z}_n$ can be calculated from a sample (theoretically infinite) of $y$-values. DKT advocate using the components $\hat{z}_{nj}$ as test statistics for $H_0$; the above results show they are asymptotically $N(0,1)$ and independent.
Power calculations

For asymptotic power calculations we suppose the known parameter \( \theta_1 \) in \( \theta \) takes on values \( \theta_1 = \theta_{10} + \gamma / \sqrt{n} \), where \( \gamma \) is a constant, as \( n \to \infty \). Thus in the limit \( \theta_1 = \theta_{10} \), the value on \( H_0 \). As \( n \to \infty \), each component \( \delta_j \) will be independently distributed \( N(\gamma \delta_j, 1) \) where \( \delta_j \) is found below, so that asymptotic power can be found for individual components and for the entire statistic \( \mathbb{H}^2 \). To obtain \( \delta_j \) we define

\[
\delta(x) = g_1(x) - I_{21} g_2(x)/I.
\]

The asymptotic process \( \gamma(x) \) is Gaussian with mean \( \gamma \delta(x) \) and covariance

\[
\mathbb{E}[\gamma(s), \gamma(t)] = \min(s, t) - st - g_2(s) g_2(t)/I;
\]

this follows from much more general results of Durbin (1973) quoted in DKT. Define \( b_j = \int_0^1 \ell_j(x) \delta(x) dx \), and let \( b_j \) be the \( j \)-th component of vector \( b \); \( \delta_j \) will then be the \( j \)-th component of \( \delta \) given by (DKT, equation 6.4)

\[
(16) \quad \delta = \frac{1}{2} N' b
\]

and, on \( H_1 \), \( \mathbb{E}(\delta) = \gamma \delta \). Let the length of \( \delta \),

\[
i.e. \left( \sum_{j=1}^{\infty} \delta_j^2 \right)^{1/2}, \text{ be } |\delta|.
\]
3. Practical Calculations: Comments

(a) In the calculations above we work in principle with infinite matrices, beginning with $\tilde{M}$. Obviously in practice we take $\tilde{M}$ to be a square matrix of order $q$, say.

DKT suggest (bottom of P. 224) that this be obtained using $\tilde{\Delta}$ of length $q$, say $\tilde{\Delta}_q$, so that $I$ is replaced by $I_q = \sum_{j=1}^{q} \tilde{\epsilon}_j^2$, and $\tilde{\Sigma}$ by $\tilde{\Sigma}_q$, an identity matrix of order $q$. Let the resulting matrix $\tilde{\Sigma}_q - \tilde{\Delta}_q \tilde{\Delta}_q' / \tilde{\Sigma}_q$ be $\tilde{M}_q$; it may be used to continue the calculations, using the $q \times q$ top left hand corners of $\tilde{\lambda}$ and $\tilde{\mu}$, and finally gives a $q \times q$ matrix $B_q$ from (14). Let the first $q$ components of $\tilde{\xi}$ be a vector $\tilde{\xi}_q$, then a $q$-vector $\hat{\tilde{\xi}}_q$ is given by $\hat{\tilde{\xi}}_q = B_q \tilde{\xi}_q$ and in practice this vector gives $q$ components, used, say, for testing purposes.

(b') We can however find $I$ exactly from (10), and use it to give, say, $M^*_q = \tilde{\Sigma}_q - \tilde{\Delta}_q \tilde{\Delta}_q' / \tilde{\Sigma}_q$, and ultimately $B^*_q$ and $\hat{\tilde{\xi}}^*_q$ following the same steps as for $B_q$ and $\hat{\tilde{\xi}}_q$. It seems to us preferable to do this, since then $M^*_q$ is the correct asymptotic covariance matrix of the first $q$ components of $\tilde{\xi}$, whereas $M_q$ is only an approximation. Also the resulting $\hat{\tilde{\xi}}^*_q$ has asymptotic covariance matrix $\tilde{\Sigma}_q$ (the identity matrix), since

$$E(\hat{\tilde{\xi}}^*_q \tilde{\xi}_q^*) = B^*_q E(\tilde{\xi}_q \tilde{\xi}_q'^\prime) B^*_q' = B^*_q M^*_q B^*_q' = \tilde{\Sigma}_q;$$

the asymptotic covariance matrix of $\hat{\tilde{\xi}}^*_q$ is slightly different from $\tilde{\Sigma}_q$.

(c) Ultimately we must extrapolate, as $q \to \infty$, to obtain values of $\mu_j$ which are used to obtain percentage points of $\tilde{\nu}^2$; we used $q = 10, 12, 20$, and 40 and the extrapolated $\mu_j$ were checked against eigenvalues obtained from the Darling approach; these are given in
Stephens (1976). DKT and Stephens, by different methods, use the $\mu_j$ and (6) to give percentage points for $W^2$; the two sets match very well for all practical purposes.

(d) As stated earlier, DKT suggest the use of $\hat{Z}_{nj}$ as test statistics; they give coefficients in their Tables 1 and 6, from which these components, for $i=1,2,...,10$, are to be calculated. Unfortunately the values in the tables appear to be wrong. They should be the first rows of matrix $B$ (modified slightly to deal with the finiteness of $Z$ in practical calculations) and DKT discuss this on p. 228; however, they appear to have given the rows of matrix $N'$ and not those of $B$. Thus the components as given will not be independent nor have asymptotic variances $1$, on $H_0$; and on the alternative, their means will not be $\gamma_{i}^j$.

(e) Any orthogonal transformation, say $H$, applied to the $\hat{Z}$ of (15) will give a new set of components $\hat{\pi} = H\hat{Z}$ with asymptotic covariance matrix equal to the identity matrix, on both $H_0$ and $H_1$. If we choose the top row of $H$ to be the unit vector $u' = \hat{\pi}/|\hat{\pi}|$, then $E(\hat{\pi}_1)$ becomes $\gamma u' \hat{\pi} = \gamma |\hat{\pi}|$; since $u$ is parallel to $\hat{\pi}$, this is the maximum mean attainable for given $\gamma$, by a single component. It can be used to calculate the maximum power attainable by a single component, for a given alternative; the alternative enters in the calculations of $\hat{e}(x)$ and hence of $\hat{e}_j$. We shall show later that this power is almost equal to that achieved by the Likelihood Ratio test, for Gamma and Wiebull alternatives. Thus the best single component is almost as effective as the L.R. test.
(f) Components have great mathematical interest, and are especially useful in power calculations, as we see later. We are not however, enthusiastic for their practical use as test statistics. They are very tedious to calculate, and the calculations take one far away from the original data, so that the final values are difficult to interpret; these are factors which will not appeal to the practical user. Further, although they will have occasionally better power than the entire statistic, e.g. $\chi^2$, for some alternatives, this is probably not true in general. Stephens (1974) illustrates this point in connection with tests where $\theta$ is completely specified.

We now give power calculations for both Wiebull and Gamma alternatives.
4. **Power against the Weibull alternative.**

For the Weibull alternative we have

$$F(y; \theta) = 1 - \exp\left(\frac{-(y/\theta_2)^{\theta_1}}{e^{l}}\right)$$

where

$$\theta_1 = 1 + \gamma \sqrt{n} \quad (i.e. \quad \theta_{10} = 1).$$

Then (DKT, section 9)

$$g_1(x) = -(1-x)\ln(1-x)\ln(-\ln(1-x)); \quad g_2(x) = (1-x)\ln(1-x);$$

and it may be shown that

$$I = 1, \quad I_{21} = \tilde{\gamma} - 1$$

where \(\tilde{\gamma}\) is Euler's constant 0.577216. (DKT, Equation (9.4), gives I incorrectly). These have been used to give values of \(\hat{\theta}\) for

\(q = 10, 12, 20, 40\) (as before, \(q\) is the order of magnitude in the matrix calculations) and these were extrapolated to give \(\hat{\theta}\) for

\(q \to \infty\). The first four components of \(\hat{\theta}\) are given in Table 1. For individual components, the asymptotic power can be calculated since

\(z_j\) has a \(N(\hat{\gamma}^j, 1)\) distribution. DKT choose \(\gamma\) to be the values required to give power equal to 0.5 and 0.95 for the appropriate L.R. test of \(\theta_1\). This test is based on the M.L. estimator \(\hat{\theta}_1\) of \(\theta_1\) for which, asymptotically
\[ \sqrt{n}(\hat{\phi}_1 - 1) \] is distributed \( N(0, 0.608) \).

(Johnson and Kotz, 1970, Vol. 1, p. 254). Then for 0.5 power \( \gamma = 1.283 \) and for 0.95 power \( \gamma = 2.563 \) for one-sided tests; for two sided tests \( \gamma = 1.528 \) and 2.811. These are the values corresponding to the results in the columns of DKT, Table 8. We have repeated some of these calculations and the results are included in Table 2.

For the complete statistic \( \hat{W}^2 \) we have the asymptotic distribution represented by

\[ \hat{W}^2 = \sum_{j=1}^{\infty} \mu_j \hat{z}_j^2 \]

where the

\[ \hat{z}_j = N(\gamma \hat{\phi}_j, 1). \]

The asymptotic distribution can be calculated by a method due to Imhof, adapted for the case of an infinite set of components by Durbin and Knott (1972). Alternatively, we can approximate it by finding the cumulants of \( \hat{W}^2 \) and fitting a curve of the form \( a + bx_2 \), where \( a, b, p \) are chosen so that the first three cumulants match those of \( \hat{W}^2 \).

This was done, e.g. in Stephens (1974), for power studies where all parameters were known, and a graph was given for power against \( \gamma \).

Note that cumulants \( K_j \) of \( \hat{W}^2 \) on \( H_1 \) are

\[ K_1 = E(\hat{W}^2) = \sum \mu_j + \gamma^2 \sum \mu_j \hat{\phi}_j^2 = \sum \mu_j \gamma^2 \sum \hat{c}_j^2 \]
where \( c = \{c_1, c_2, \ldots \} \)' = \( N \)' in (16), and all sums are for \( j=1,2,\ldots,\infty \); also,

\[
K_2 = 2 \sum \mu_j + 4\gamma^2 \sum c_j^2 \mu_j
\]

and

\[
K_3 = 8 \sum \mu_j^3 + 24\gamma^2 \sum c_j^2 \mu_j^2
\]

so that for the cumulants it is not strictly necessary to know the \( \delta_j \), but only the \( c_j \) and \( \mu_j \).

We have done power calculations by both methods, and the results agreed very closely; a graph of power against \( \gamma^2 \) is given in Figure 1.

Calculations for \( U^2 \) and for \( A^2 \).

The calculations of \( \mu_j \) and \( \delta_j \) for \( U^2 \) are based on the asymptotic behaviour of \( \hat{Y}_n(x) - \bar{y} \) where \( \bar{y} = \int_0^1 \hat{Y}_n(x) dx \), and those for \( A^2 \) on \( \hat{Y}_n(x)/(x(1-x))^{1/2} \). The appropriate changes in the theory are given in DKT, Section 4 and we will not repeat them here. The statistic is still expressed in the form (6) with \( \hat{W}^2 \) replaced by \( \hat{U}^2 \) or \( \hat{A}^2 \), and asymptotically the components \( \hat{z}_j \) have \( N(\gamma \delta_j, 1) \) distributions; the values of \( \delta_j \) are given in Table 1. We have also used the \( \mu_j \) to give null-distribution points by Imhof's method as adapted by Durbin and Knott; the upper tail points at level \( \alpha \) are then the following:

\[
\alpha(\%): \quad 10 \quad 5 \quad 2.5 \quad 1
\]

\%

point: \quad 1.061 \quad 1.321 \quad 1.590 \quad 1.947

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5. Power against the Gamma alternative.

For the Gamma distribution, we have

\[ F(y; \theta) = \frac{1}{\Gamma(m)} \frac{1}{(\alpha)^m} \int_0^y e^{-z/\alpha} z^{m-1} dz \]

where, to simplify notation, we have let \( \theta_1 = m \) and \( \theta_2 = \alpha \). On \( H_0 \), \( m = 1 \), and on \( H_1 \), \( m = l\gamma \sqrt{n} \). We then have, with \( x = F(y; \theta) \)

\[ g_1(x) = \frac{1}{\Gamma(m)} \int_0^y e^{-z} z^{m-1} \ln z \, dz - \frac{\Gamma'(m)}{\Gamma(m)^2} \int_0^y e^{-z} z^{m-1} \, dz \]

and \( g_2(x) = (1-x)\ln(1-x) \) as before.

I becomes \( m/\theta^2 \), which, for \( m = 1, \theta = 1 \) is 1;

\[ \int_0^\infty ye^{-y} \ln y \, dy - \int_0^\infty e^{-y} \ln y \, dy \]

= \( \Gamma'(2) - \Gamma'(1) = (1-\gamma)(-\gamma) = 1 \)

Thus \( \hat{e}(x) = g_1(x) - g_2(x) \) and for \( \hat{e}_j \) we need integrals

\[ J_1 = \int_0^1 g_1(x) \ell_j(x) \, dx \quad \text{and} \quad J_2 = \int_0^1 g_2(x) \ell_j(x) \, dx \]

The second integral is already known from the calculations needed in Section 4; the first gives \( J_1 = J_{11} - J_{12} \) where

\[ J_{11} = \frac{1}{\Gamma(m)} \int_0^1 \int_0^y e^{-z} z^{m-1} \ln z \, dz \ell_j(x) \, dx \]

with \( y = -\ln(1-x') \). The integral is calculated by reversing the order.
of integration. Let $s = 1 - e^{-z}$; when $m = 1$ we have

$$J_{11} = -\int_0^1 \ln(-\ln(1-s)) \frac{\sqrt{s}}{\pi j} (\cos \pi j - \cos \pi js) ds$$

The original integral has been reduced to a single integral, and although some limit problems remain at the end points, it can be evaluated numerically. The integral $J_{12}$ is

$$J_{12} = \Gamma'(1) \int_0^1 \int_0^y e^{-z}dz \ell_j(x)dx$$

$$= \Gamma'(1) \int_0^1 x \ell_j(x)dx$$

$$= \Gamma'(1) \sqrt{2} \pi C$$

where $C = -1$ if $j$ is even and 1 if $j$ is odd. Using these results we have found the $\ell_j$ for the Gamma alternative, and thus the asymptotic power of components and the entire $\mathcal{W}^2$ can be evaluated.

For $U^2$ and $A^2$ the integrals are evaluated on the same lines as those above, though for $A^2$, where $\ell_j(x)$ consists of Ferrar associated Legendre functions, the algebra is more cumbersome. These details will be omitted. For comparison purposes we need the best test against the $\Gamma$-alternative. This is based on the $\gamma, L$ estimate $\hat{m}$ of $m$, which asymptotically for $m = 1$ has distribution $N(1, \sigma^2)$ where $1/\sigma^2 = n(\psi'(1) - 1) = 0.645n$; $\psi'(x)$ is the digamma function (Johnson and Kotz, 1970, vol. 1, p. 188). Thus for the best one-sided 5% test, we reject if $\hat{m} > 1.645\sigma$; on the
alternative the power is

\[ Pr(\hat{m} - 1 > 1.645\sigma) = P((\hat{m} - (1 + \gamma / \sqrt{n})) > 1.645\sigma - \gamma / \sqrt{n}) \]

\[ = P((\hat{m} - (1 + \gamma / \sqrt{n})/\sigma) > 1.645 - \gamma / (\sigma \sqrt{n})) . \]

The left hand side of the inequality is a \( N(0,1) \) variable; thus for 50\% power we have \( \gamma = 1.645\sigma \sqrt{n} = 1.645/(0.645)^{1/2} = 2.060; \) for 95\% power \( 1.645-\gamma/(\sigma \sqrt{n}) = -1.65, \) giving \( \gamma = 4.119. \) For two-tailed tests, \( \gamma = 2.441 \) and 4.489 for 50\% and 95\% power respectively. Calculations on the lines of those for the Weibull alternative give power graphs against \( \gamma^2, \) shown in Figure 2.

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References


Table 1

Values of $\hat{\delta}_j$ and of $|\hat{\delta}|$, for the test of exponentiality vs. Weibull and Gamma alternatives

| Statistic | $j$: | 1     | 2     | 3     | 4     | $|\hat{\delta}|$ |
|-----------|------|-------|-------|-------|-------|------------------|
| $\hat{\nu}^2$ (Weibull) | :     | 1.1322 | 0.1062 | 0.4176 | 0.0693 | 1.2798          |
| $\hat{\nu}^2$ (Weibull) | :     | 0.9632 | 0.0645 | 0.5075 | 0.0555 | 1.2591          |
| $\hat{\Lambda}^2$ (Weibull) | :     | 1.2050 | 0.0087 | 0.3479 | 0.0018 | 1.2823          |
| $\hat{\nu}^2$ (Gamma) | :     | 0.6466 | 0.1917 | 0.2608 | 0.1237 | 0.7953          |
| $\hat{\nu}^2$ (Gamma) | :     | 0.5836 | 0.0901 | 0.3211 | 0.0442 | 0.7866          |
| $\hat{\Lambda}^2$ (Gamma) | :     | 0.7256 | 0.1610 | 0.2238 | 0.0801 | 0.7978          |
Table 2

Asymptotic powers of components, and of the entire statistics $t^2$, $U^2$, and $A^2$ in tests for exponentiality against Weibull alternatives

<table>
<thead>
<tr>
<th>Component</th>
<th>One-Sided Test</th>
<th>Two-sided Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^2$</td>
<td>1.645 6.580</td>
<td>2.335 7.901</td>
</tr>
<tr>
<td>Best test power:</td>
<td>.50 .95</td>
<td>.50 .95</td>
</tr>
<tr>
<td>$y^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_1$</td>
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<td>.409  .889</td>
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<td>.053 .060</td>
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<td>$z_3$</td>
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<td>.098  .217</td>
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<td>$z_4$</td>
<td>.060  .071</td>
<td>.051  .054</td>
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<tr>
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<td>\delta</td>
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<td>.313  .773</td>
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<td>.051 .054</td>
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<td>.058  .067</td>
<td>.051  .053</td>
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<td>.453  .923</td>
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<td>.050 .050</td>
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<td>.050  .051</td>
<td>.050  .050</td>
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<tr>
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<td>$</td>
<td>\delta</td>
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<td>0.403  0.890</td>
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<td>.294  .744</td>
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<td>$A^2$</td>
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<td>.438  .917</td>
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Table 3

Asymptotic powers of components, and of the entire statistics $\hat{\chi}^2$, $\hat{\chi}^2$ and $\hat{\chi}^2$ in tests for exponentiality against Gamma alternatives.

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<th>Component</th>
<th>One-Sided Test</th>
<th>Two-Sided Test</th>
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<td>.853</td>
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Figure 2: Graphs of Power $P$ Against $\chi^2$ for the Gamma Distribution Alternatives.
**Title:** Asymptotic Power of EDF Statistics for Exponentiality Against Weibull and Gamma Alternatives.

**Authors:** Michael A. Stephens

**Performing Organization:** Department of Statistics, Stanford University, Stanford, CA 94305

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**Please see reverse side.**
ASYMPTOTIC POWER OF EDF STATISTICS FOR EXPONENTIALITY AGAINST WEIBULL AND GAMMA ALTERNATIVES

Asymptotic distributions of EDF statistics — goodness-of-fit statistics based on the empirical distribution function — can be obtained by expanding a related function in a series of orthogonal functions. This method was given by Watson and developed especially in this connection by Durbin and Knott and by the author. An earlier approach by Anderson and Darling is different and the newer methods provide a sharper view of the approach to infinity. Similar approaches were employed by Durbin, Knott and Taylor to test for normality and also gave asymptotic power results for certain alternatives and in some cases for tests of exponentiality. In this paper, asymptotic powers of some EDF statistics for testing exponentiality against Weibull and Gamma alternatives are developed.
DATE
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