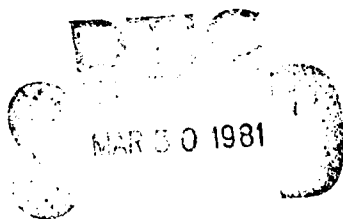LEVEL

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GOODNESS OF FIT TESTS FOR THE WEIBULL AND THE EXTREME
VALUE DISTRIBUTION WITH ESTIMATED PARAMETERS

by

Mahesh Chandra
Nozer D. Singpurwalla
Michael A. Stephens

Serial-T-410
20 November 1979

The George Washington University
School of Engineering and Applied Science
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NONE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER T-410	2. GOVT ACCESSION NO. *AD-A096 981*	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) GOODNESS OF FIT TESTS FOR THE WEIBULL AND THE EXTREME VALUE DISTRIBUTION WITH ESTIMATED PARAMETERS		5. TYPE OF REPORT & PERIOD COVERED SCIENTIFIC
		6. PERFORMING ORG. REPORT NUMBER T-410
7. AUTHOR(s) MAHESH CHANDRA NOZER D. SINGPURWALLA MICHAEL A. STEPHENS		8. CONTRACT OR GRANT NUMBER(s) N00014-77-C-0263 NRC-04-78- N00014-76-C-0475 239 DAAG29-77-C-0031
9. PERFORMING ORGANIZATION NAME AND ADDRESS THE GEORGE WASHINGTON UNIVERSITY INSTITUTE FOR MANAGEMENT SCIENCE & ENGINEERING WASHINGTON, DC 20052		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS NUCLEAR REGULATORY COMMISSION WASHINGTON, DC 20555		12. REPORT DATE 20 November 1979
		13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) NONE
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION OF THIS REPORT IS UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

GOODNESS OF FIT TESTS
WEIBULL DISTRIBUTION
EXTREME VALUE DISTRIBUTION

KOLMOGOROV-SMIRNOV STATISTICS
SIMULATING THE MAXIMUM OF BROWNIAN
MOTION

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Mahesh Chandra*
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THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
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Program in Logistics

ON GOODNESS OF FIT TESTS FOR THE WEIBULL AND THE
EXTREME VALUE DISTRIBUTION WITH ESTIMATED PARAMETERS

by

Mahesh Chandra
Nozer D. Singpurwalla
Michael A. Stephens

1. Introduction

The two parameter Weibull distribution has found many applications in the biological, engineering, and the hydrological sciences. For instance it has been used by Doll (1971), to describe the observed age distribution of many human cancers. Its use for describing failures of electrical and mechanical components is well documented in the engineering literature, and in a comprehensive study, Benson (1968) discusses its use for analyzing flood data.

In this paper, we address ourselves to the problem of testing the null hypothesis H_0: that a given random sample belongs to a Weibull or an extreme value distribution with unknown parameters. The test statistics will be EDF statistics, i.e., those based on the empirical distribution function, and we present tables of critical values for testing H_0.

A foundation for developing the tables of critical values is the recent theory by Durbin (1973) on the weak convergence of an "empirical" stochastic process. This stochastic process is based on the empirical distribution function and estimates of the unknown parameters. The statistics that we discuss can be represented as well-behaved functionals of this empirical process.

2. Preliminaries

The two-parameter Weibull distribution is given by

$$P(T \leq t) = 1 - \exp\left(-\left(\frac{t}{\delta}\right)^{\beta}\right), \quad t \geq 0 \qquad (2.1)$$

$$= 0 \qquad \qquad , \text{ otherwise;}$$

the scale parameter δ and the shape parameter β are both assumed to be positive.

If we make the transformation $X = -\ln T$, where T has the distribution (2.1), then X is said to have the extreme value distribution.

$$P(X \leq x) = F(x) = \exp\left\{-\exp-\left(\frac{X-a}{b}\right)\right\}, \quad -\infty < x < \infty, \qquad (2.2)$$

where $a = -\ln\delta$ and $b = \frac{1}{\beta}$ are the location and the scale parameters, respectively.

The tests that we discuss in this paper are for the extreme value distribution. To make a test of fit for the Weibull distribution we shall take the negative of the natural logarithms of the supposed Weibull data. Thus, we wish to consider the case of testing whether the distribution of a random sample X_1, X_2, ..., X_n, say F, is an extreme value distribution with unknown location and scale parameters a and b , respectively. Specifically, we wish to test the null hypothesis

$$H_0: \quad F(x) = G(x)$$

for all x and for some (a,b) , where $G(\cdot)$ is the distribution $F(\cdot)$ given by (2.2).

When a and b are specified, the H_0 is said to be "simple," and the test reduces to testing the hypothesis that the independent random variables

$$Z_i = G\left(\frac{X_i - a}{b}\right) = \exp\left[-\exp\left(-\left(\frac{X_i - a}{b}\right)\right)\right] \quad , \quad 1 \leq i \leq n$$

have a common uniform $(0,1)$ distribution. The Kolmogorov-Smirnov test is based on the statistic

$$\sqrt{n} \sup_{0 \leq t \leq 1} |G_n(t) - t| \tag{2.3}$$

where

$$G_n(t) = \frac{1}{n} \sum_{i=1}^{n} I\left(G\left(\frac{X_i - a}{b}\right) \leq t\right), \quad 0 \leq t \leq 1, \text{ and} \tag{2.4}$$

where $I(E)$ denotes the indicator of the event E. Under the null hypothesis, the "empirical" stochastic process

$$W_n(t) = \sqrt{n} \, (G_n(t) - t), \qquad 0 \leq t \leq 1 \tag{2.5}$$

satisfies

$$W_n(t) \xrightarrow{d} W^0 \quad \text{in } \mathcal{D}[0,1] , \tag{2.6}$$

where $\xrightarrow{d}$ denotes convergence in distribution, and W^0 the Gaussian process determined by $E(W^0(t)) = 0$, and

$E(W^0(s)W^0(t)) = \min(s,t) - st$, $0 \le s$, $t \le 1$. $\mathscr{D}[0,1]$ denotes the space of functions on $[0,1]$ which are right continuous and have left-hand limits.

When H_0 is composite, an analogous test statistic and a convergence theorem are obtained; these are discussed below.

3. Asymptotic Results When H_0 is Composite

When a and b are not specified H_0 is composite, and we shall use $(\hat{a}_n, \hat{b}_n)$, the maximum likelihood estimators of (a,b). Following Stephens (1977) we call this situation Case 3. (Cases 1 and 2 refer to the less important cases in which only a is unknown or only b is unknown.)

Let

$$Y_{n,1} = \frac{X_i - \hat{a}_n}{\hat{b}_n}, \qquad\qquad 1 \le i \le n,$$

and define

$$H_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(G(Y_{n,i}) \le t), \quad 0 \le t \le 1 \qquad (3.1)$$

and

$$Y_n(t) = \sqrt{n}\,(H_n(t) - t), \qquad\qquad 0 \le t \le 1 . \qquad (3.2)$$

Then from a theorem of Durbin (1973) and the appropriate regularity conditions, the empirical process $\{Y_n(t);\ 0 \le t \le 1\}$ is such that

$$Y_n \xrightarrow{d} Y^0 \quad \text{in} \quad \mathscr{D}[0,1] ,$$

where Y^0 is a Gaussian process determined by

$$E(Y^0(t)) = 0 \qquad\qquad , 0 \le t \le 1$$

- 4 -

and

$$E(Y^0(s)Y^0(t)) = \min(s,t) - st - 1.108(slogs)(tlogt) \tag{3.3}$$

$$+ .257(slogs)(tlogtlog(-logt))$$

$$+ .257(slogslog(-logs)(tlogt))$$

$$- .60793(slogslog(-logs)tlogtlog(-logt)), \quad 0 \leq s, t \leq 1 .$$

The above has also been shown by Stephens (1977). The statistics of interest in connection with H_0 are:

(i) the one-sided Kolmogorov-Smirnov statistics

$$D^+ = \sup_{0 \leq t \leq 1} Y_n(t) \quad , \tag{3.4}$$

$$D^- = -\inf_{0 \leq t \leq 1} Y_n(t) \quad , \tag{3.5}$$

(ii) the Kolmogorov-Smirnov statistic

$$D = \max (D^+, D^-) \quad , \tag{3.6}$$

(iii) the Kuiper statistic

$$V = D^+ + D^- , \tag{3.7}$$

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(iv) the Cramer-Von Mises statistics

$$W^2 = \int_0^1 Y_n^2(t)dt \quad , \tag{3.8}$$

(v) the Watson statistic

$$U^2 = \int_0^1 Y_n^2(t)dt - \left[\int_0^1 Y_n(t)dt\right]^2 \quad . \tag{3.9}$$

and the Anderson-Darling statistic

$$A^2 = \int_0^1 \frac{Y_n^2(t)}{t(1-t)} dt \quad . \tag{3.10}$$

As a consequence of the continuous mapping theorem, the limit laws of D^+, D^-, D, V, W^2, U^2, and A^2 under H_0, are given by the laws of the random variables $\sup_{0 \leq t \leq 1} Y^0(t)$, $-\inf_{-0 \leq t \leq 1} Y^0(t)$, $\max(D^+, D^-)$, $\int(Y^0(t)^2 dt$,

$$\left[\int_0^1 (Y^0(t)dt)^2 - \left(\int_0^1 Y^0(t)dt\right)^2\right], \text{ and } \lim_{\varepsilon \to 0} \int_1^{1-\varepsilon} \frac{(Y^0(t))^2}{t(1-t)} dt \quad ,$$

respectively.

4. Sampling Distributions of the Test Statistics

Stephens (1977) has found from theoretical work the cumulants of the limiting distributions of W^2, U^2, and A^2, and has used these to approximate the distributions. The pth quantiles of these limiting distributions, together with a modification for these variables when the sample size is finite are given by Stephens (1977, Table 1).

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The sampling distributions of D^+, D^-, D , and V have also been obtained by Stephens (in some unpublished work), using Monte Carlo methods, for samples of size 10, 20, and 50. Using these finite sample results, Stephens uses an extrapolation of the quantiles for finite n , to obtain the corresponding asymptotic quantiles. The smoothed Monte Carlo points are given under Case 3, in Table 4.0. Also given in Table 4.0 are the quantiles of the distributions of D^+, D^-, D , and V when only the scale parameter is unknown (Case 1), and when only the location parameter is unknown (Case 2). The points for Case 2 were obtained by Monte Carlo methods similar to those for Case 3, but for Case 1, we include some exact points given by Durbin (1975).

Durbin's paper is concerned with testing for exponentiality but, with some rearrangement, his points apply for Case 1. This is because, in Case 1, b in the extreme value distribution is known; then if we make the transformation $y = \exp(-X/b)$, it is easily shown that the distribution of y is the exponential distribution $F(y) = 1-\exp(-Ay)$, $y > 0$, where A is $\exp(a/b)$. Thus the problem reduces to testing that y has the exponential distribution with A unknown (since a is unknown). Durbin has found the exact points for $\sqrt{n}D^+$, $\sqrt{n}D^-$ and $\sqrt{n}D$ for this situation, and has given extensive tables for $n \leq 100$. Durbin's exact points have been used where possible in Table 4.0, Case 1. Because the transformation $y = \exp(-X/b)$ is monotonically decreasing, the D^+ calculated directly from the X-values will equal D^- calculated from the y-values, and D^-(X-values) will equal D^+ (y-values); therefore the values in Table 1, Case 1, for $\sqrt{n}D^+$ are Durbin's values for $\sqrt{n}D^-$ and vice versa. Values of $\sqrt{n}V$ in Table 4.0, Case 1 are obtained from Monte Carlo methods; asymptotic values for $\sqrt{n}D^+$, $\sqrt{n}D^-$ and $\sqrt{n}D$ are obtained by extrapolating Durbin's exact points, bearing in mind that $\sqrt{n}D^+$ and $\sqrt{n}D^-$ should have the same asymptotic distributions.

- 7 -

The sampling distributions of D^+, D^-, D, and V have also been obtained by Stephens (in some unpublished work), using Monte Carlo methods, for samples of size 10, 20, and 50. Using these finite sample results, Stephens uses an extrapolation of the quantiles for finite n, to obtain the corresponding asymptotic quantiles. The smoothed Monte Carlo points are given under Case 3, in Table 4.0. Also given in Table 4.0 are the quantiles of the distributions of D^+, D^-, D, and V when only the scale parameter is unknown (Case 1), and when only the location parameter is unknown (Case 2). The points for Case 2 were obtained by Monte Carlo methods similar to those for Case 3, but for Case 1, we include some exact points given by Durbin (1975).

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- 7 -

A second method of obtaining the asymptotic sampling distributions of the variables (3.4) through (3.10) is suggested in Wood (1978), and involves a direct simulation of the Gaussian process Y^0. Specifically, the process Y^0 is approximated by its finite dimensional distribution, corresponding to an evaluation of the process at k equally spaced points in the unit interval. Ten thousand multivariate normal random vectors with the covariance matrix given by (3.3) were generated using the extended precision version of a program from the IMS Library. The empirical distributions of the supremum, the infimum, and the difference between the supremum and the infimum of the resulting multivariate normal vectors were then tabulated, thus approximating the limit laws of $\sqrt{n}\ D^+$, $\sqrt{n}\ D^-$, $\sqrt{n}\ D$, and $\sqrt{n}\ V$. The limit laws of W^2, U^2, and A^2 were approximated by using numerical integration techniques. For this, we used subroutine QSF from the IBM Scientific Subroutine Package. In order to obtain the quantiles of the true approximating limiting distribution, (i.e., for for $k = \infty$), extrapolations from finite values of k must be performed; how this is done is explained in Section 4.1.

Since Stephens (1977) has already obtained the quantiles of the limiting distributions of W^2, U^2, and A^2 using theoretical methods, the main purpose served by simulating the process V^0 is to obtain the quantiles of the limiting distributions of $\sqrt{n}\ D^+$, $\sqrt{n}\ D^-$, $\sqrt{n}\ D$, and $\sqrt{n}\ V$ by this alternative method. The Kolmogorov-Smirnov statistics D^+, D^-, and D are known in similar goodness-of-fit situations to have relatively low power. However, they are commonly used in practice and D^+ and D^- are very useful for one-sided tests; thus a comparison of the two methods of obtaining quantiles, both involving extrapolation, would be valuable. We now proceed to this comparison.

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4.1 Results of the Monte Carlo Simulation and Extrapolations

In Tables 4.1 through 4.4, we give the quantiles of the limiting distributions of the test statistics (3.4) through (3.10), obtained from statistics 10,000 replications of the process Y^0 , and using k = 29, 59, 89 and 119 equally spaced points.

In order to obtain the quantiles of the distributions when k is infinite, we shall plot the kth quantile versus $\frac{1}{\sqrt{k}}$, for each of the test statistics and extrapolate to zero. The quantiles considered are for p = 0.75, 0.90, 0.95, 0.975 and 0.99.

For example, in Figure 4.1 we show a plot of the 0.75th quantile versus $\frac{1}{\sqrt{k}}$, for k = 29, 59, 89 and 119, for the test statistic D^+ . The dotted line is our linear extrapolation to obtain the 0.75th quantile of the true limiting distribution (i.e., when k = ∞). In Figure 4.2, we show the plot for the 0.90th quantile. Also shown on the vertical axis of Figure 4.2 by an asterisk, is the 0.90th quantile of the asymptotic distribution of D^+ obtained by Stephens, and given in Table 4.0. In Figures (4.3) through (4.5), we show analogous plots for p = 0.95, 0.975, and 0.99. Similar plots, for the other statistics D^-, D, V, W^2, U^2, and A^2 , are given in Figures (4.6) through (4.23). The asterisks on the vertical axis of Figures (4.17) through (4.23) represent the appropriate quantiles of the limiting distributions of W^2, U^2, and A^2 obtained by Stephens (1977, Table 1). Since these have been obtained theoretically, they provide us with a benchmark for assessing the accuracy of the simulations of the asymptotic process, and also give us some guidelines for extrapolations.

Examination of Figures (4.1) through (4.23) suggests the following comments:

(a) The plots of the pth quantile versus $\frac{1}{\sqrt{k}}$, for
all the variables considered here, are approximately
linear for small values of p , say p = 0.75 , but
tend to curve down for the larger values of p ,
especially for p = 0.99. Figure (4.1) is an
example of the former and Figure 4.12 is an example
of the latter. We should expect the curve to be
monotonic in $\frac{1}{\sqrt{k}}$, and the curving down
suggests that, as k becomes larger, the
accuracy of the simulation of Y^0 may
become suspect.

(b) The linearly extrapolated values shown in the
plots are, in most instances, larger than the
corresponding values obtained by Stephens (i.e.,
those indicated by the asterisks). However,
parabolic extrapolation, also shown in the plots,
gives asymptotic values much closer to the
asterisks; since the asterisks for W^2, U^2, and
A^2 can be regarded as quite accurate, it appears
that parabolic extrapolation is to be preferred
to linear extrapolation.

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4.2 Simulating the Maximum of a Brownian Motion Process

In comment (a) above, we have noted that it appears that the direct method of simulating the asymptotic process by its discrete analogue evaluated at k points may lead to inaccuracies when k becomes large. In order to investigate this possibility further, it was decided to simulate the following Brownian motion process, for which the distribution of the maximum is well known. The process $\{\tilde{W}(t); 0 \leq t \leq 1\}$ has mean 0 and covariance.

$$\{E \tilde{W}(s)\tilde{W}(t)\} = \min (s,t) . \tag{4.1}$$

In order to approximate the process, we obtain $\tilde{W}(t)$ at $k + 1$ equally spaced points on the interval (0,1), i.e., we evaluate $\{\tilde{W}(j)/k); j = 0, 1, \ldots k\}$ by generating multivariate normal vectors as described above. Siegmund(1978) has shown that the distribution of the maximum of this discretized process can be approximated by

$$P\{ \max_{0 \leq j \leq k} \tilde{W}(j/k) \geq x\} = 2\{1 - \Phi (x + \frac{.583}{\sqrt{k}})\} \tag{4.2}$$

where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2} dt$, Siegmund's result is exact when k is infinite.

In Table 4.5 we show the quantiles of the distribution of $M_k = \max_j \tilde{W}(j/k) , 0 \leq j \leq k $, for $k = 20, 30, 50, 60,$ and 90, obtained by simulation, using 10,000 replications, and these are compared with the results given by Siegmund's approximation (4.2). In Figure 4.24 these results are shown graphically. The results of this simulation

indicate that it would be difficult to use the simulated points
alone to obtain the correct intercept on the y-axis; the turning-
down effect is again present when k becomes large. Also, it
appears as though either linear or parabolic extrapolation would
give reasonable results, the variability in the Monte Carlo points
making it difficult to distinguish between these methods.

5. Quantiles of Limiting Distributions of EDF Statistics

In view of comment (b) above, supported by the above results,
it seems reasonable to extrapolate parabolically to obtain asymptotic
percentage points in Figures 4.1 to 4.16 for statistics D^+, D^-, D ,
and V. The results are given in Table 5.1, for D^+, D^-, D , and V;
the values given by Stephens' Monte Carlo method are included for com-
parison. It is clear that there is negligible difference between the
values, in terms of the percentage level, so that the mean of the two
estimates (or of all four for D^+ and D^- which should have the same
quantiles) might be taken as a reasonable compromise till more accur-
ate methods of finding true values are available. Table 5.2 lists
the quantiles of the asymptotic distributions of all the EDF statistics,
using this compromise estimate for D^+, D^-, D, and V , and Stephens'
theoretically calculated values for W^2, U^2, and A^2.

6. Further Remarks

(a) The above extensive study was motivated by the desire to
compare two methods of obtaining asymptotic quantiles of test statistics
which are functionals of a process which is asymptotically Gaussian.

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By a comparison of the two methods, even though each one is based on Monte Carlo simulations followed by extrapolation, points are obtained which we feel will be accurate for practical purposes.

(b) The second method, of directly approximating the asymptotic process by simulating a discretized version at k points, is a naturally appealing one. However, indications are that it is very difficult to preserve accuracy as k becomes large. We have to be cautious, since we do not always know what to expect of the calculated quantity (in our case, values of functionals of the process) as k becomes larger. However, we had one good indication, given by Siegmund's approximation for the quantiles of the maximum of the Brownian motion process, which suggests that these quantiles should vary monotonically in k . This was not the case for the simulated results; and although we must remember that these are subject to sampling variations, the evidence overall in these studies suggests that increasing k will not necessarily give better asymptotic results, probably because the handling of k multivariate normal vectors produces inaccuracies.

ACKNOWLEDGEMENT

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Figure 4.1

Plot of the .75th Quantile of D^+ vs $\dfrac{1}{\sqrt{k}}$

Linear Extrapolation for D^+ at $k = \infty$

Parabolic Extrapolation for D^+ at $k = \infty$

$\dfrac{1}{\sqrt{119}}$

$\dfrac{1}{\sqrt{89}}$

$\dfrac{1}{\sqrt{59}}$

$(D^+ ; .750)$

$\dfrac{1}{\sqrt{29}}$

Values of the .75th Quantile of the Distribution of D^+

Values of $\dfrac{1}{\sqrt{k}}$
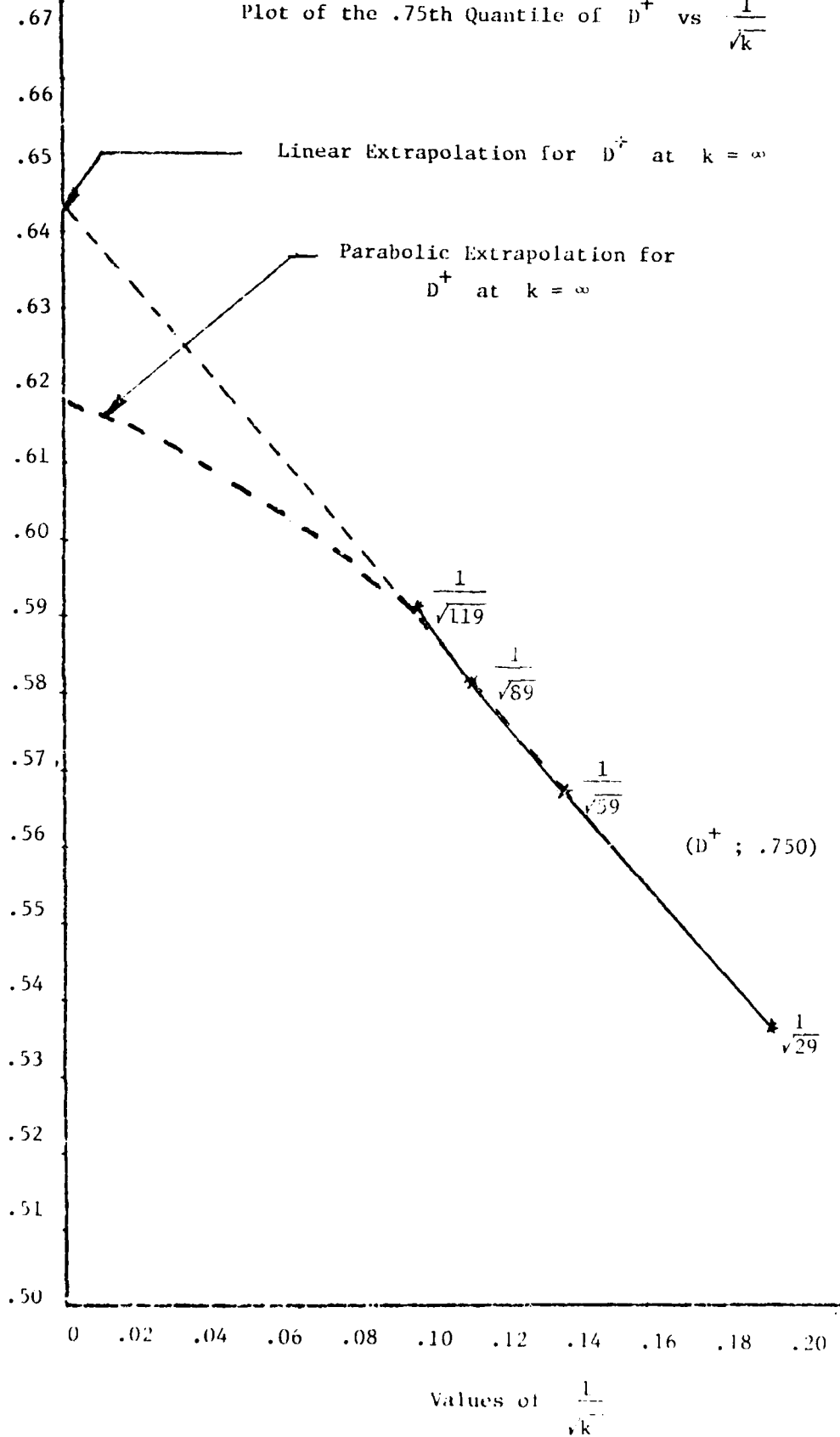
Figure 4.2

Plot of the .90th Quantile of D^+ vs $\frac{1}{\sqrt{k}}$

Linear Extrapolation for D^+ at $\kappa = \infty$

Parabolic Extrapolation for D^+ at $k = \infty$

Value Given by Stephens' Simulation

$(D^+ ; .900)$

Values of the .90th Quantile of the Distribution of D^+

Values of $\frac{1}{\sqrt{k}}$

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Figure 4.3

Plot of the .95th Quantile of D_n^+ vs $\frac{1}{\sqrt{k}}$

(Explanatory Comments on Figure 4.2)

Values of the .950th Quantile of the Distribution of D^+

$(D_n^+ ; .950)$

Values of $\frac{1}{\sqrt{k}}$
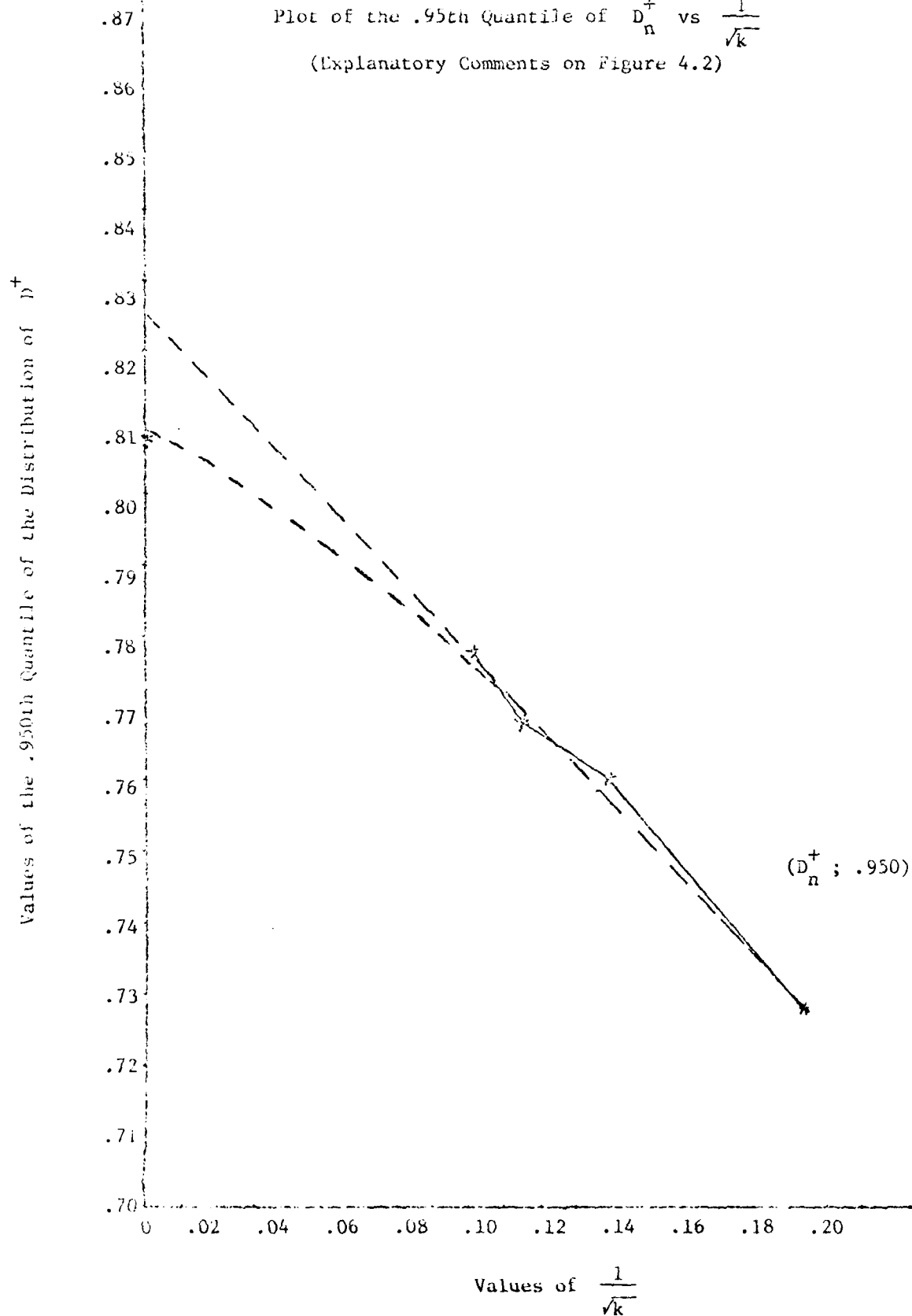
Figure 4.4

Plot of the .975th Quantile of D^+ vs $\frac{1}{\sqrt{k}}$

(Explanatory Comments on Figure 4.2)

$(D^+ ; .975)$

Values of $\frac{1}{\sqrt{k}}$
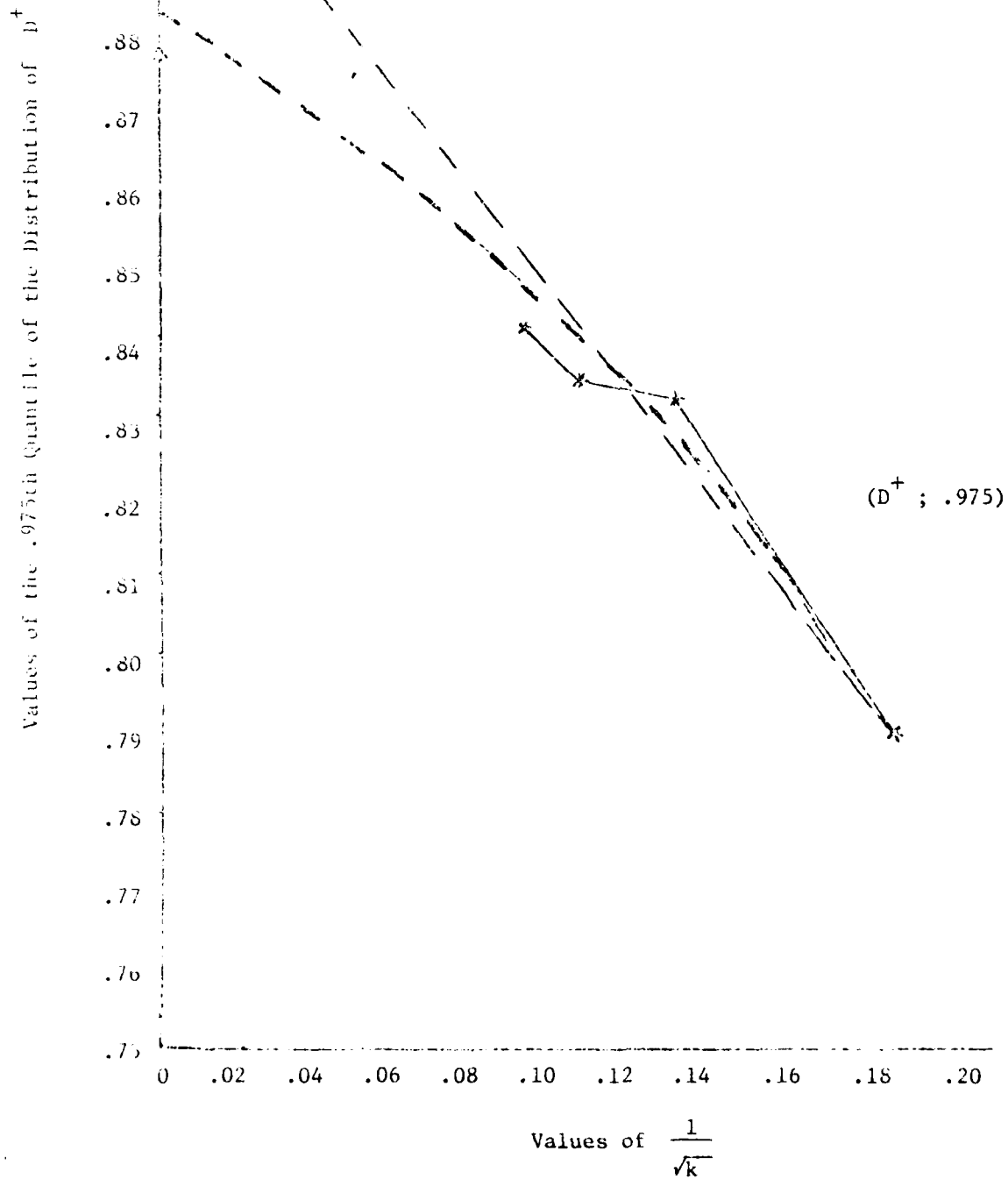
Figure 4.5

Plot of the .990th Quantile of D^+ vs $\frac{1}{\sqrt{k}}$

Values of $\frac{1}{\sqrt{k}}$

$(D^+ ; .990)$

0 .02   .04   .06   .08   .10   .12   .14   .16   .18   .20
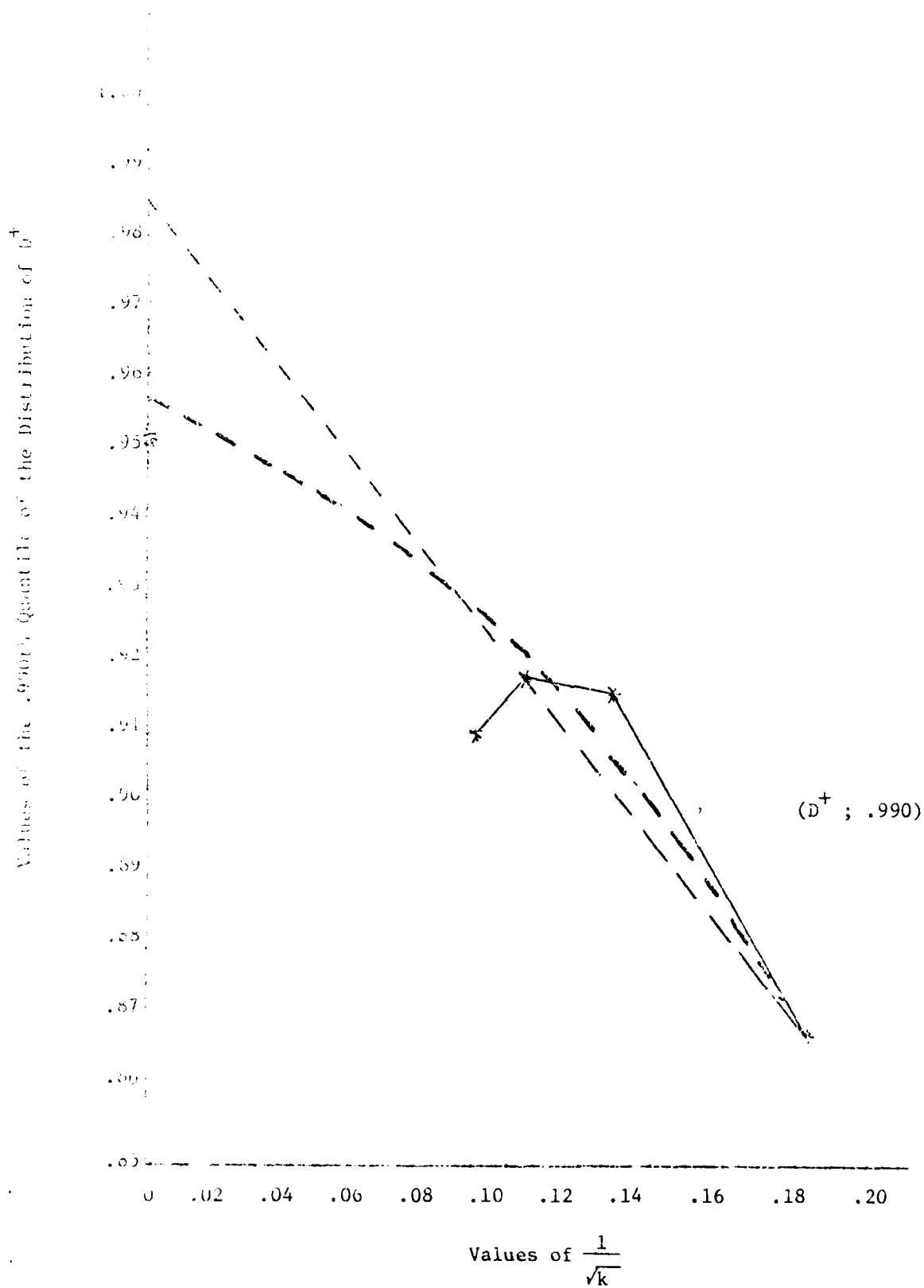
Values of $\frac{1}{\sqrt{k}}$

Figure 4.6

Plot of the .750th Quantile of D^+ vs $\frac{1}{\sqrt{k}}$

$(D^-\ ;\ .750)$
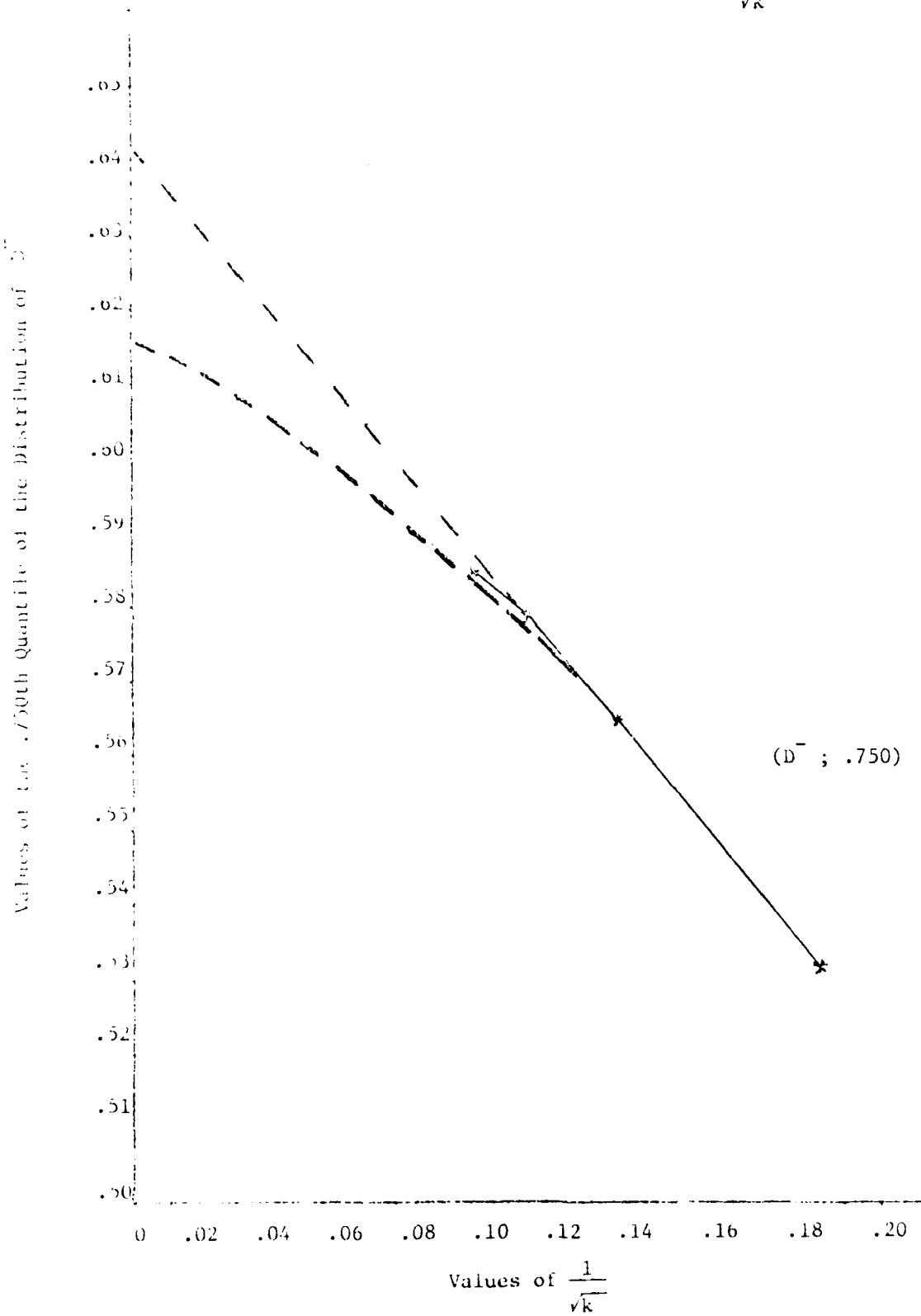
Values of $\frac{1}{\sqrt{k}}$

Figure 4.7

Plot of the .900th Quantile of D^- vs $\frac{1}{\sqrt{k}}$

$(D^- \; ; \; .900)$
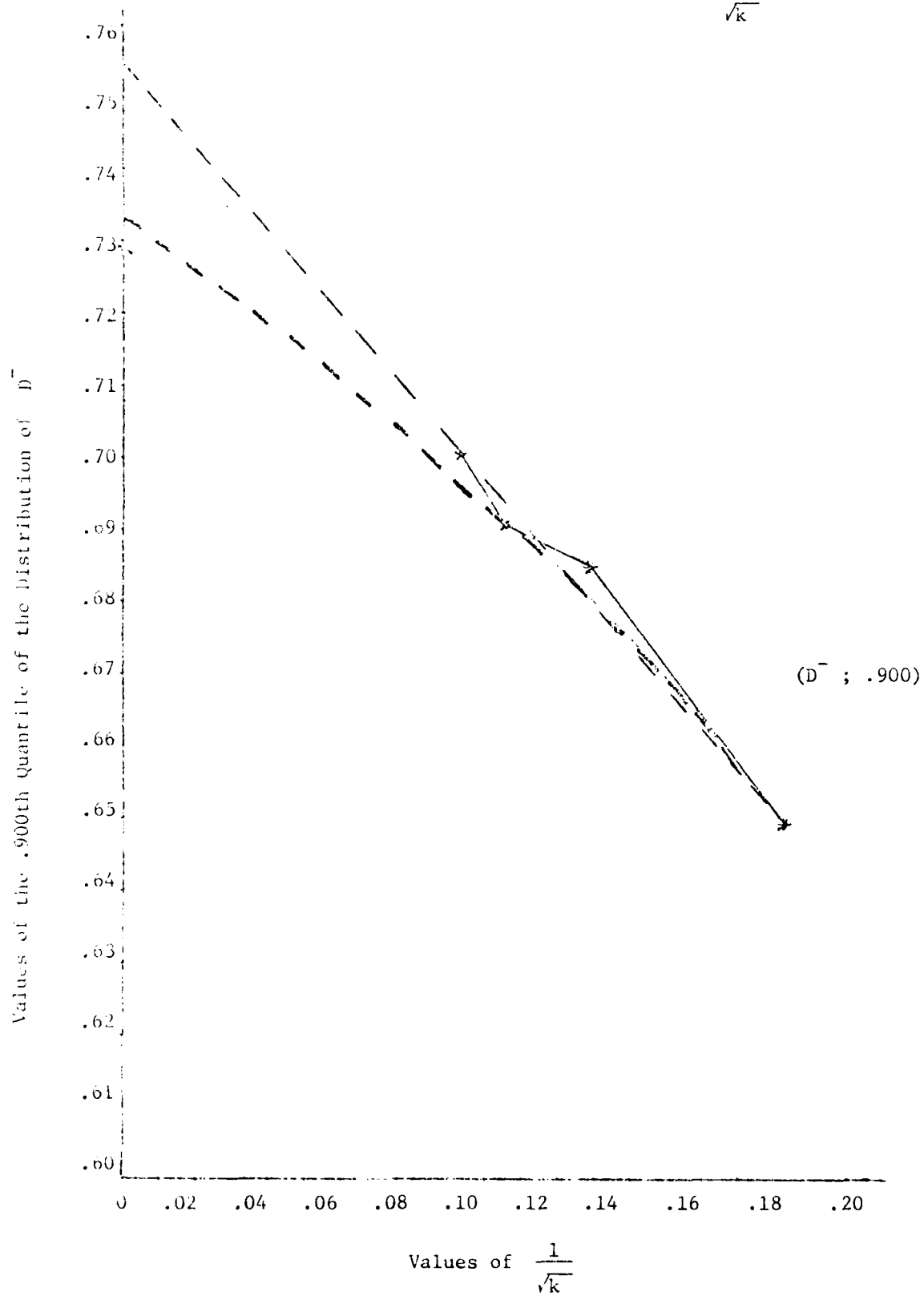
Values of $\frac{1}{\sqrt{k}}$

Figure 4.8

Plot of D^- vs $\frac{1}{\sqrt{k}}$

$(D^- ; .975)$

$(D_n^- ; .950)$

Values of the Distribution of D^-
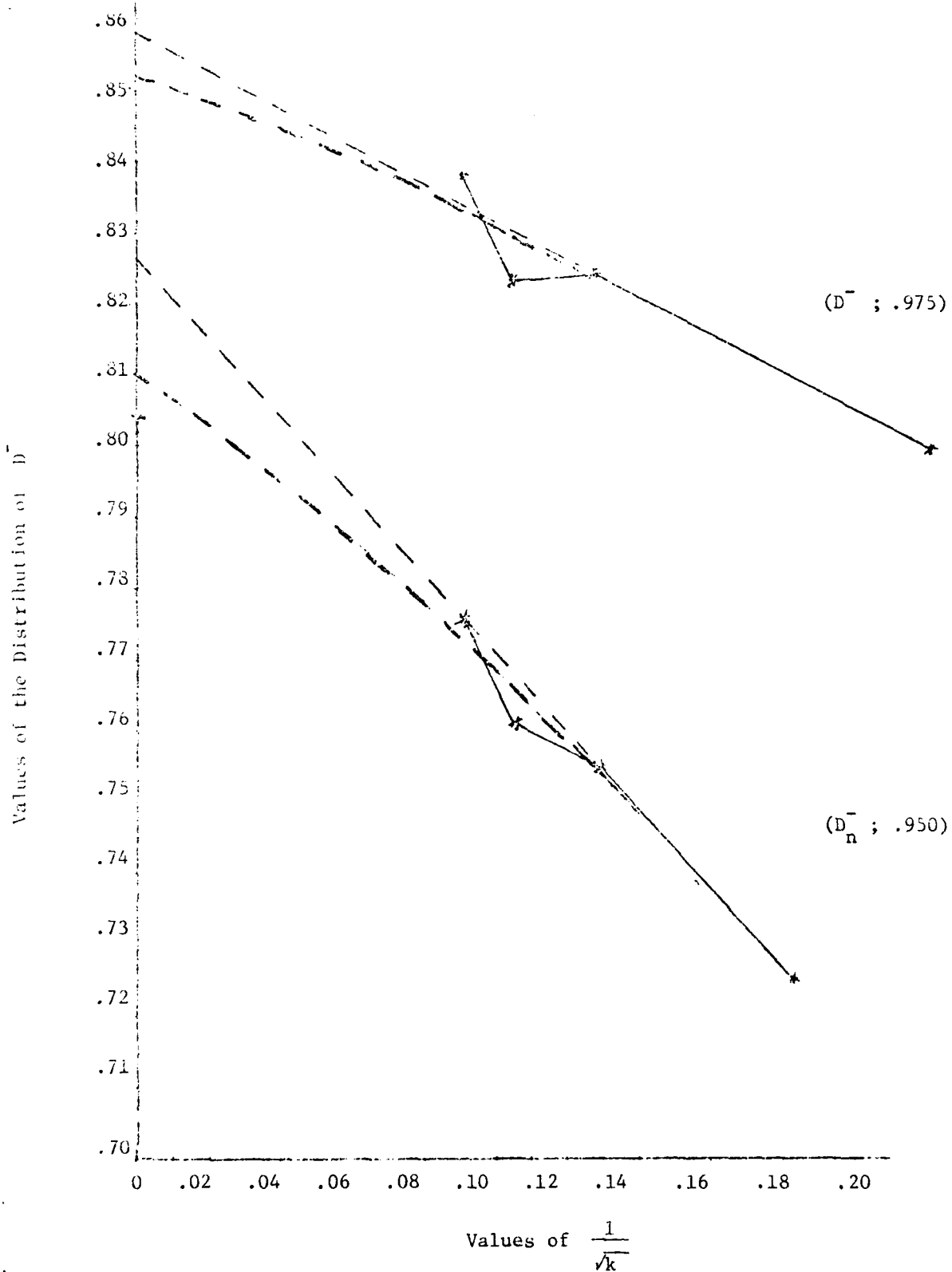
Values of $\frac{1}{\sqrt{k}}$

Figure 4.9

Plot of the .990th Quantile of D^- vs $\frac{1}{\sqrt{k}}$

$(D_n^- \; ; \; .990)$
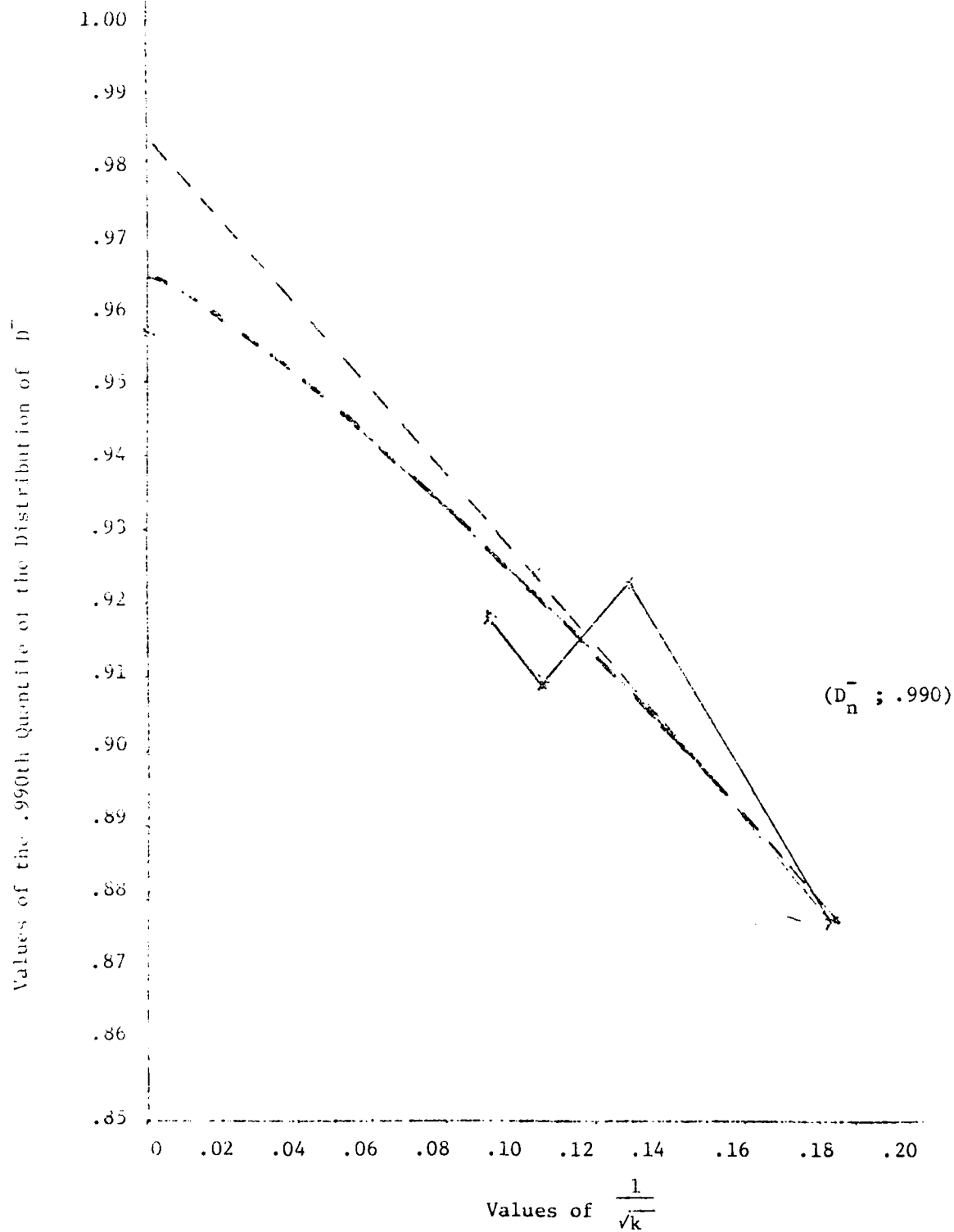
Values of $\frac{1}{\sqrt{k}}$

Figure 4.10

Plot of the .750th Quantile of D vs $\frac{1}{\sqrt{k}}$

$(D \; ; \; .750)$

Values of $\frac{1}{\sqrt{k}}$

Figure 4.11

Plot of D vs $\frac{1}{\sqrt{k}}$

T-410

(D ; .950)

(D ; .900)

Values of the Distribution of D

Values of $\frac{1}{\sqrt{k}}$

- 24 -
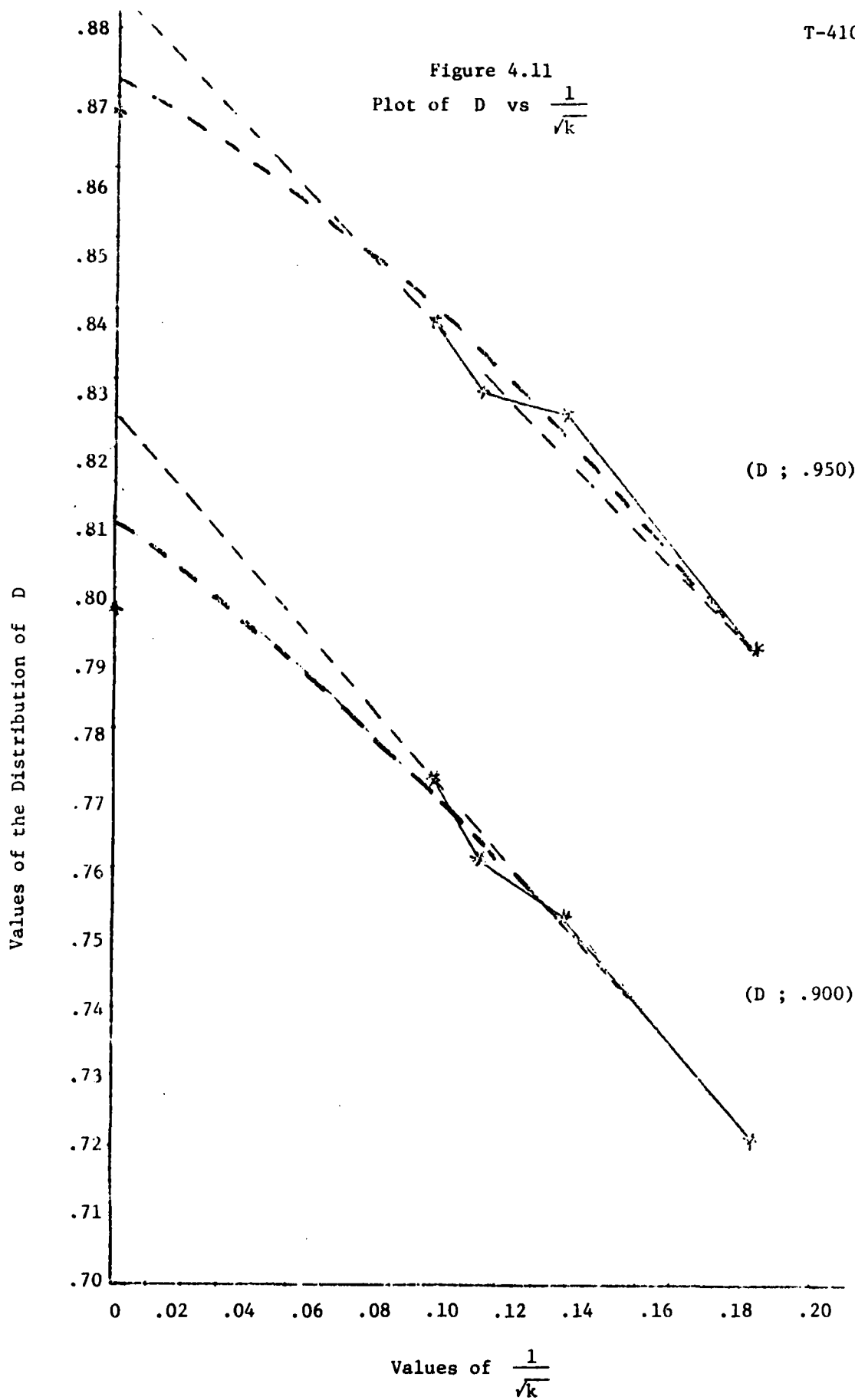
Figure 4.12

Plot of D vs $\frac{1}{\sqrt{k}}$

$(D \; ; \; 990)$

$(D \; ; \; .978)$

Values of $\frac{1}{\sqrt{k}}$

Values of the Distribution of D
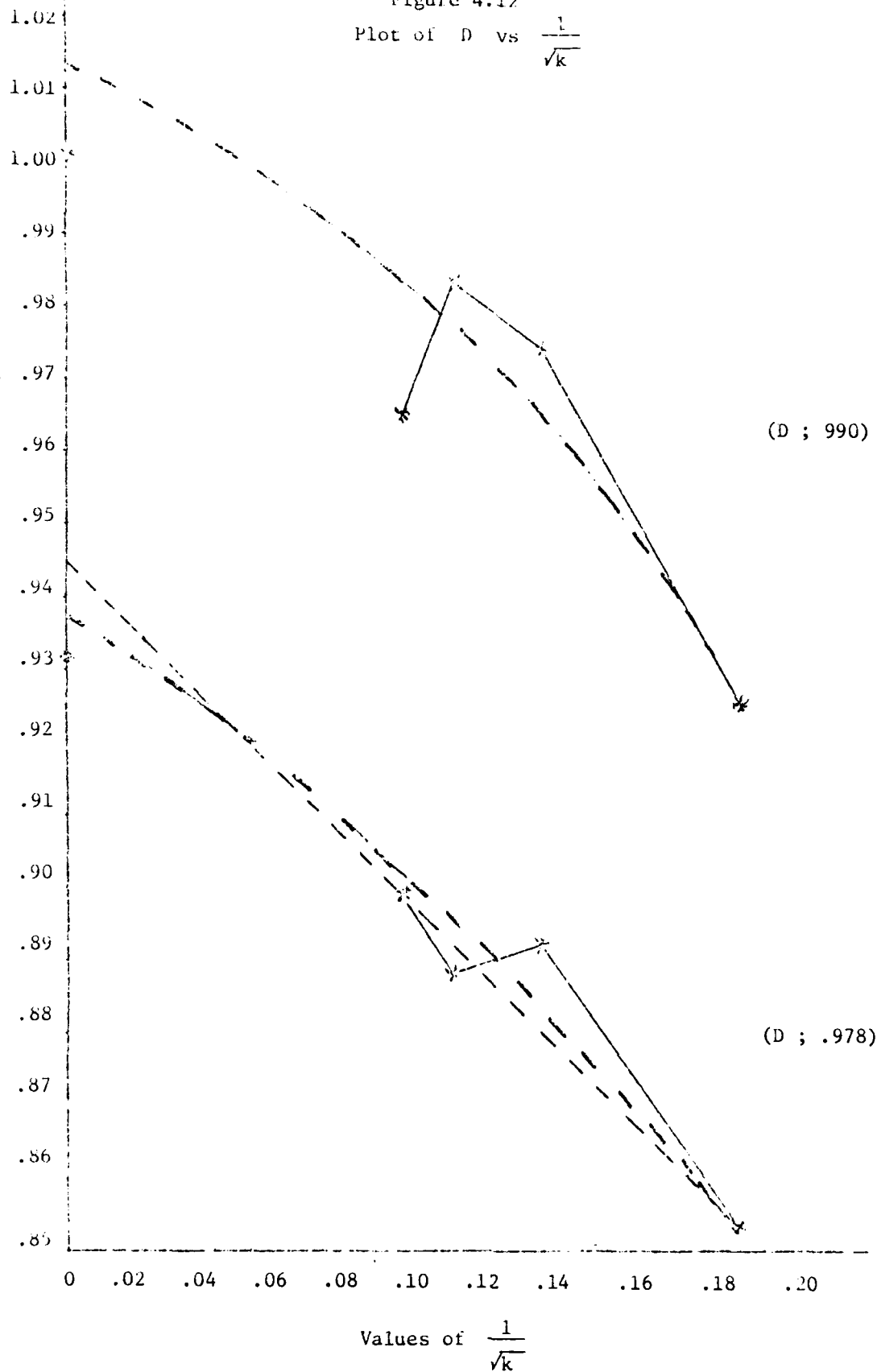
Figure 4.13
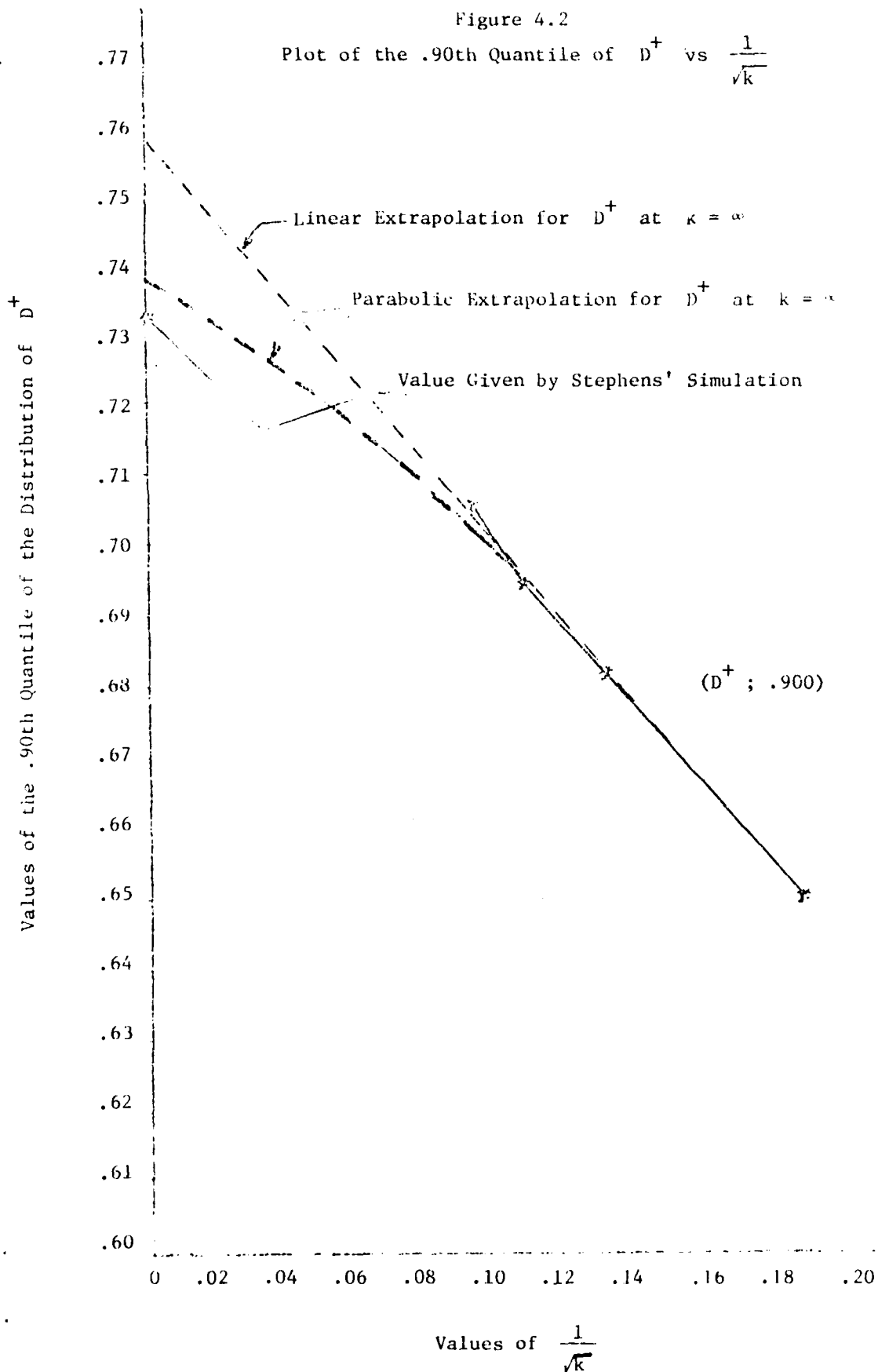
Plot of the .750th Quantile of V vs $\frac{1}{\sqrt{k}}$

Values of the .750th Quantile of the Distribution of V

(V ; .750)
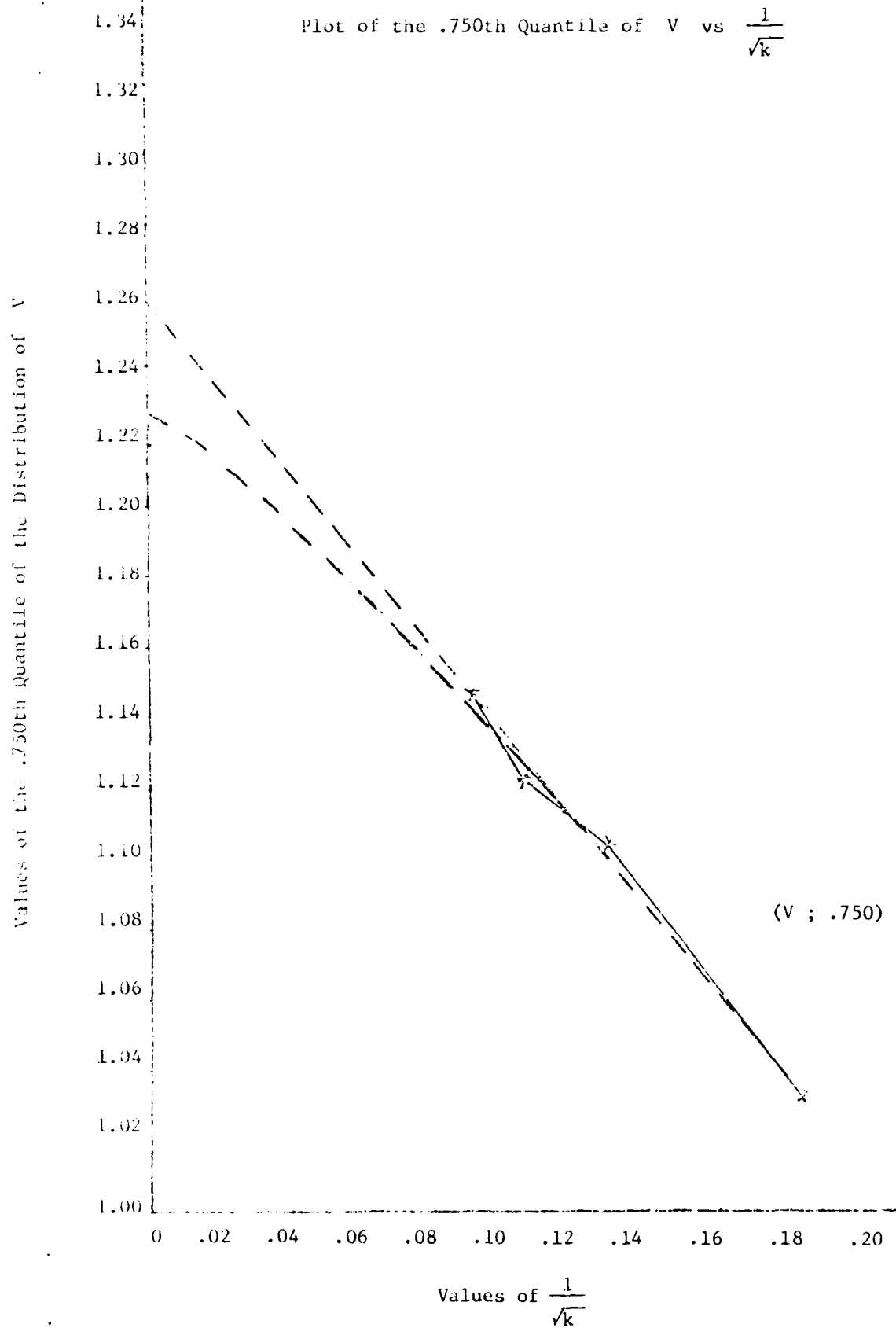
Values of $\frac{1}{\sqrt{k}}$

Figure 4.14

Plot of the .900th Quantile of V vs $\frac{1}{\sqrt{k}}$

Values of the .900th Quantile of the Distribution of V

$(V \; ; \; .900)$

Values of $\frac{1}{\sqrt{k}}$

Figure 4.1?

Plot of V vs $\frac{1}{\sqrt{k}}$

(V ; .975)

(V ; .950)
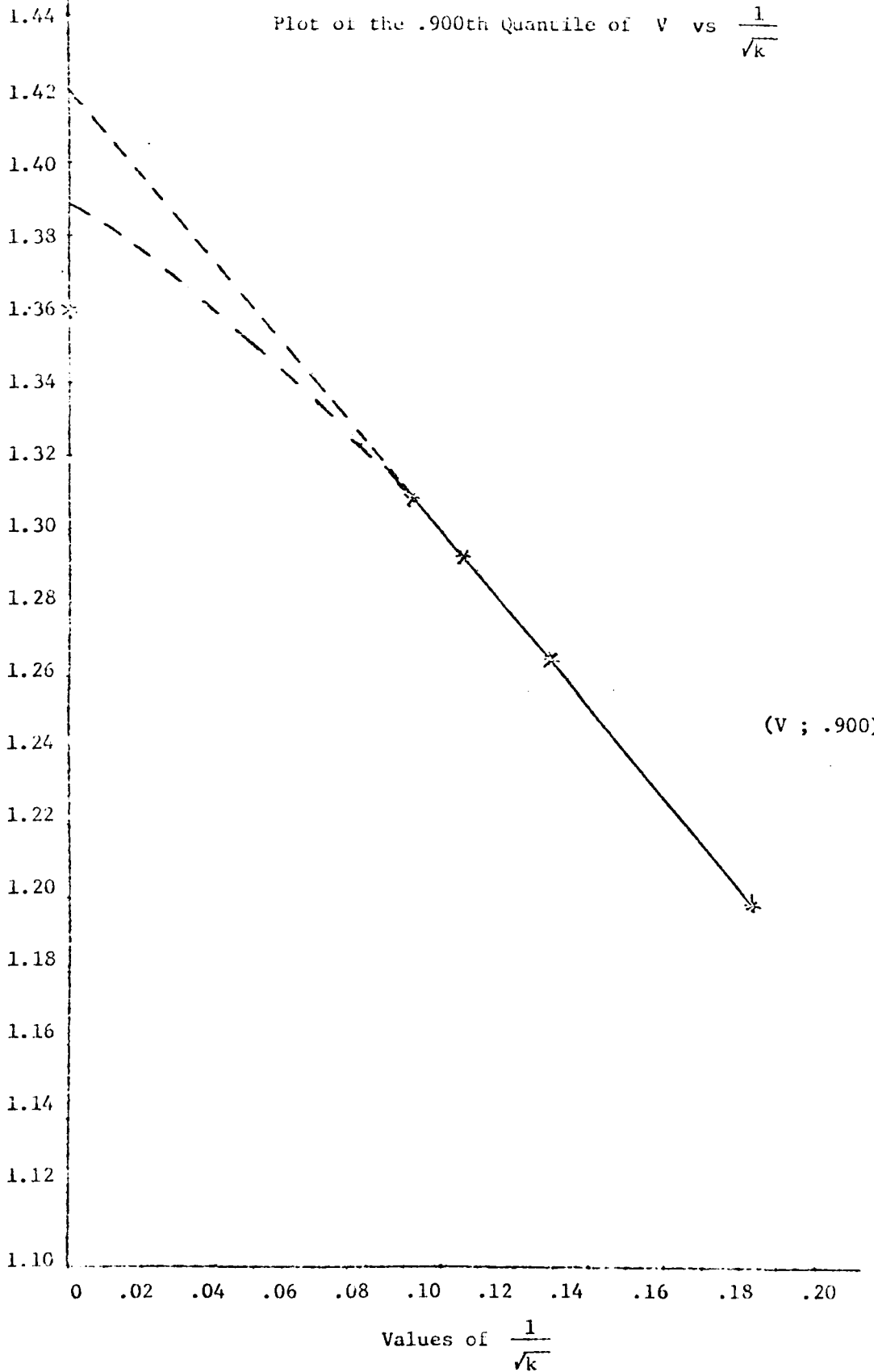
Values of the Distribution of V
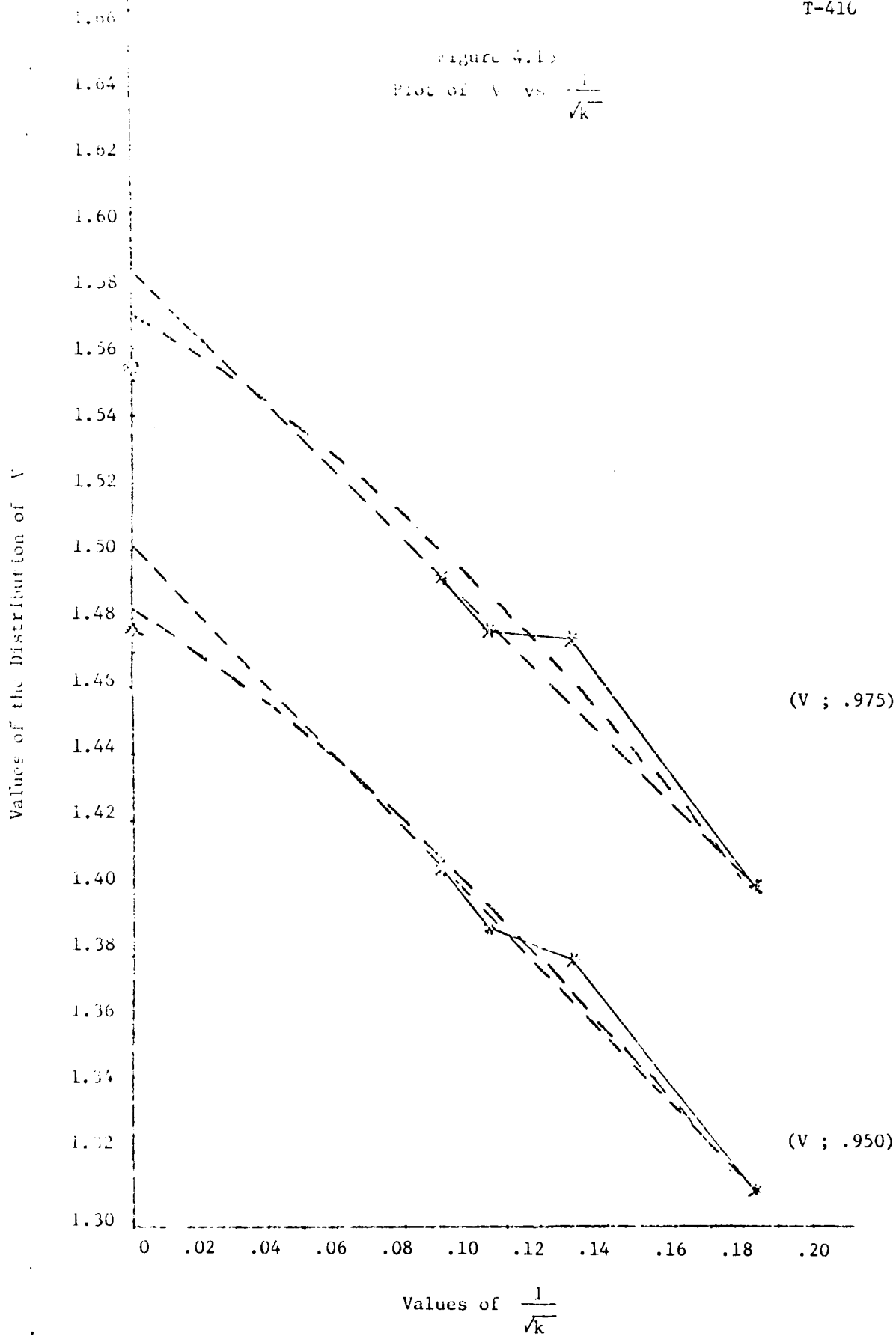
Values of $\frac{1}{\sqrt{k}}$

Figure 4.16

Plot of the .990th Quantile of V vs $\frac{1}{\sqrt{k}}$

(V ; .990)

Values of $\frac{1}{\sqrt{k}}$
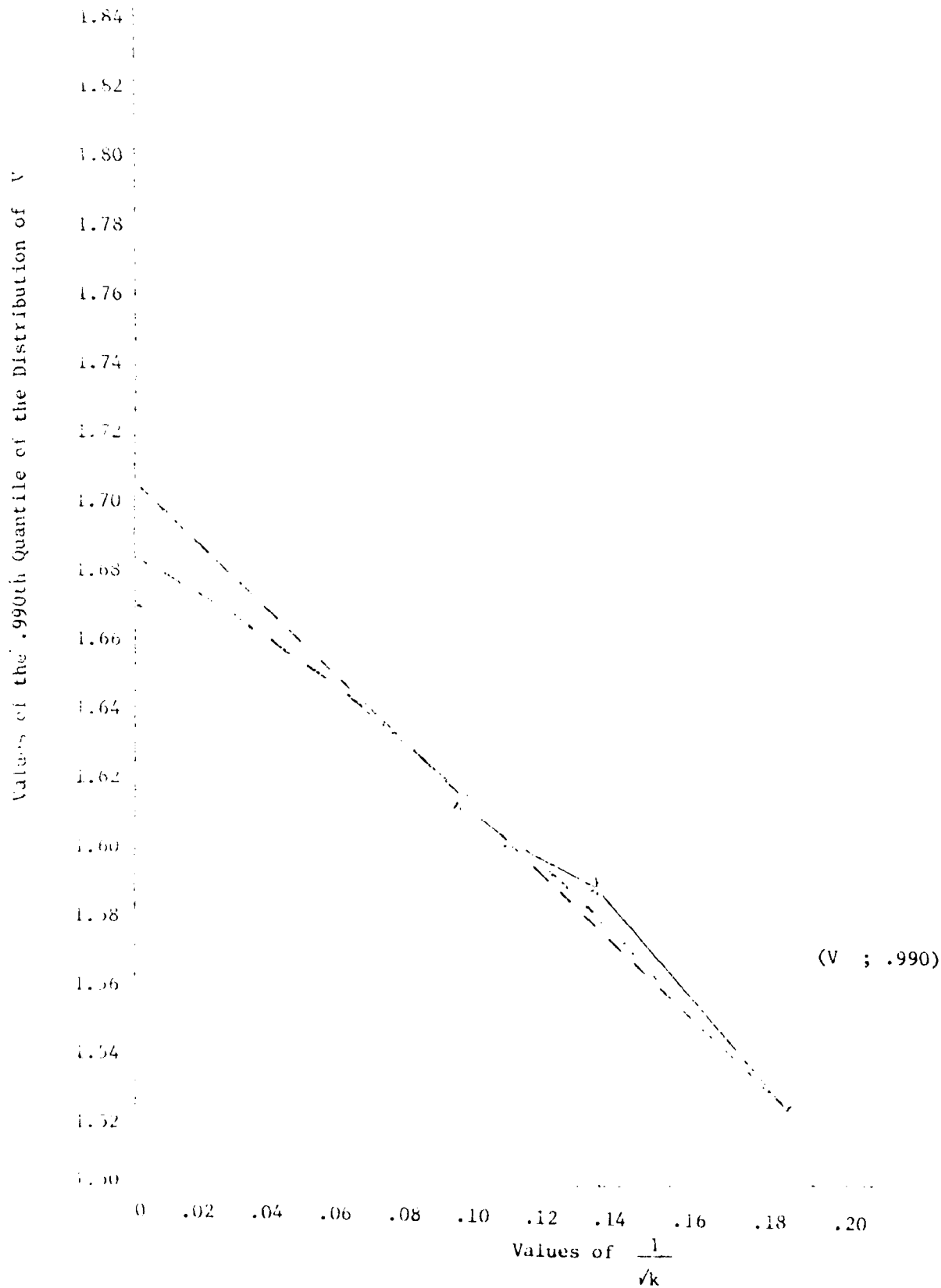
Figure 4.16

Figure 4.17

Plot of W^2 and U^2 vs $\frac{1}{\sqrt{k}}$

$(W^2 ; .950)$

$(U^2 ; .950)$

$(W^2 ; .900)$

$(U^2 ; .900)$

$(W^2 ; .750)$

$(U^2 ; .750)$

Values of the Distributions of W^2 and U^2

0   .02   .04   .06   .08   .10   .12   .14   .16   .18   .20

Values of $\frac{1}{\sqrt{k}}$

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Figure 4.15

Plot of W^2 and U^2 vs $\frac{1}{\sqrt{k}}$

T-410

$(W^2 \; ; \; .990)$

$(U^2 \; ; \; .990)$

$(W^2 \; ; \; .975)$

$(U^2 \; ; \; .975)$

Values of $\frac{1}{\sqrt{k}}$

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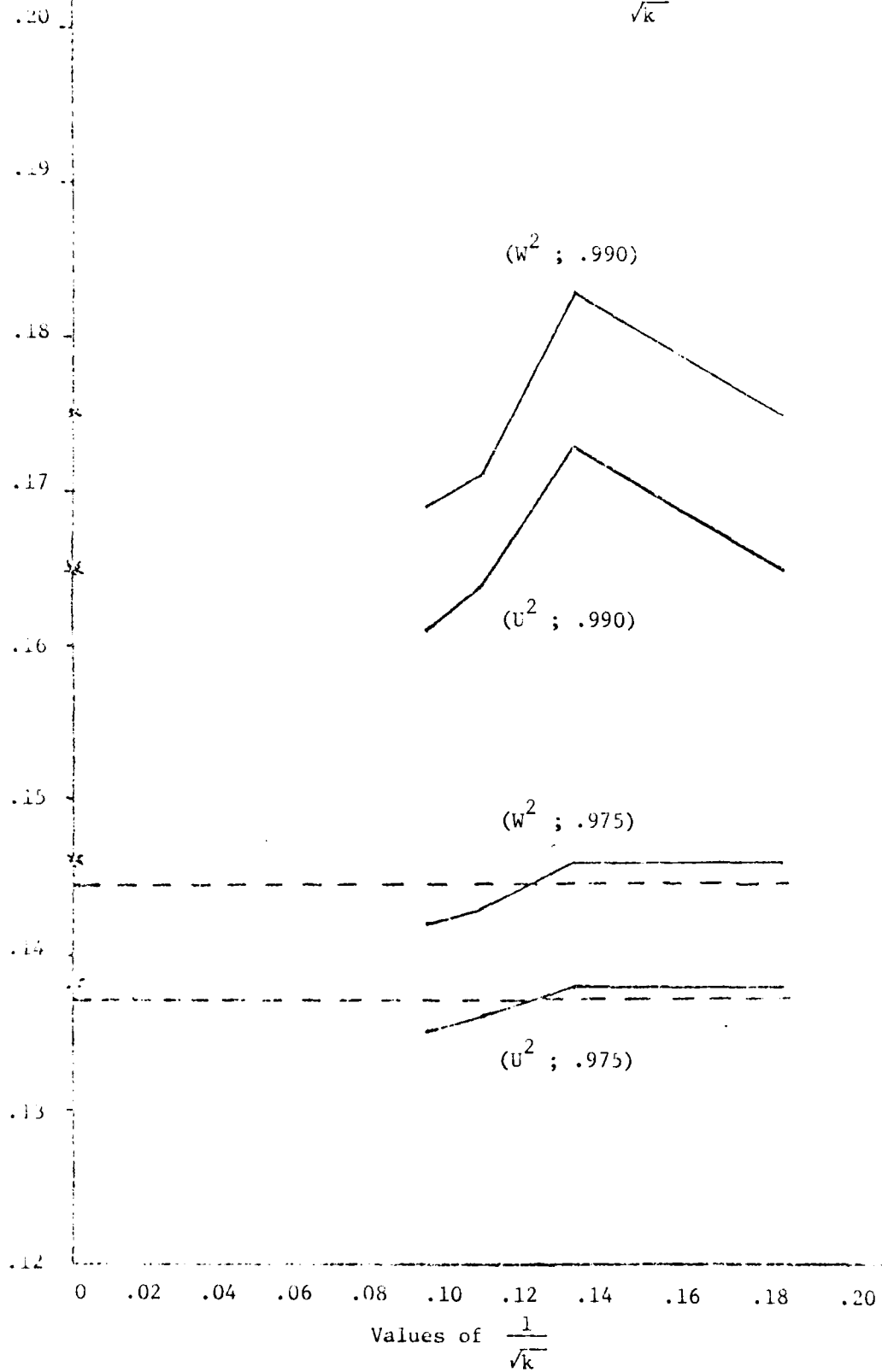
Figure 4.19

Plot of the .750th Quantile of A^2 vs $\frac{1}{\sqrt{k}}$

$(A^2 ; .750)$

Values of $\frac{1}{\sqrt{k}}$

Figure 4.16

Plot of the .900th Quantile of A^2 vs $\frac{1}{\sqrt{k}}$

Values of the .900th Quantile of the Distribution of A^2

$(A^2 ; .900)$

Values of $\frac{1}{\sqrt{k}}$

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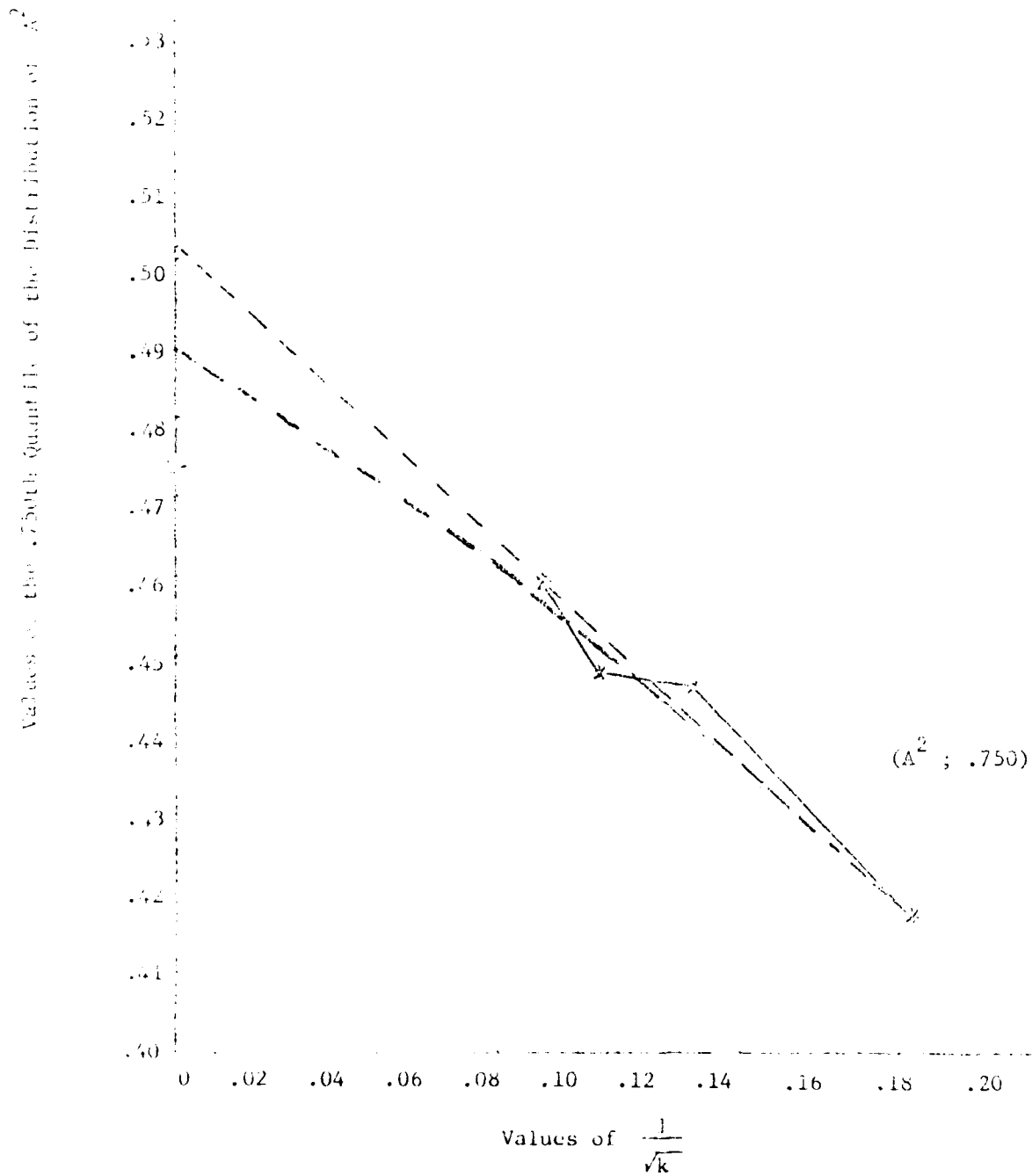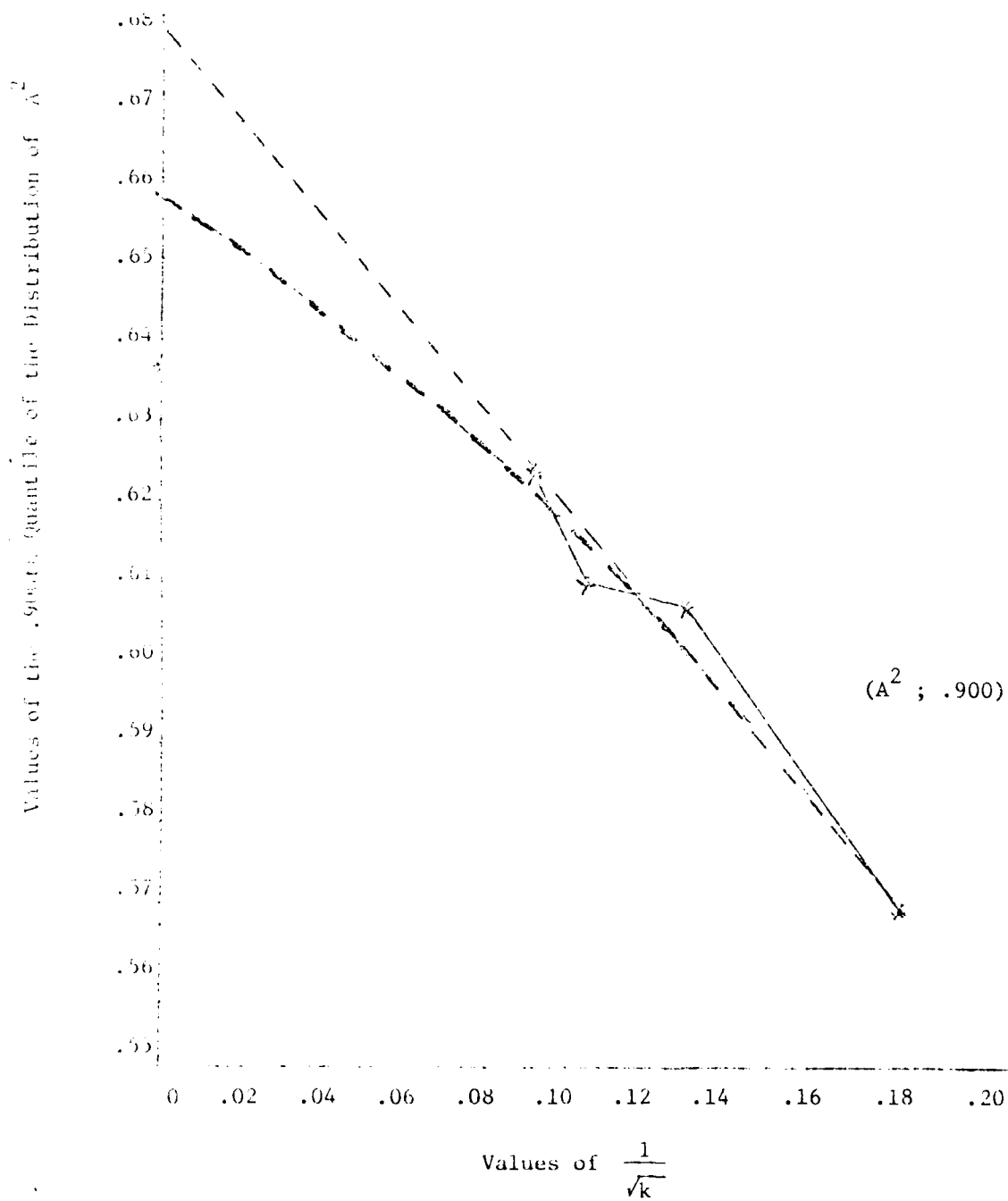
Figure 4.21

Plot of the .950th quantile of A^2 vs $\frac{1}{\sqrt{k}}$

$(A^2 ; .950)$

Values of $\frac{1}{\sqrt{k}}$

Figure 4.22

Plot of the .975th Quantile of A^2 vs $\frac{1}{\sqrt{k}}$

$(A^2 ; .975)$

Values of $\frac{1}{\sqrt{k}}$
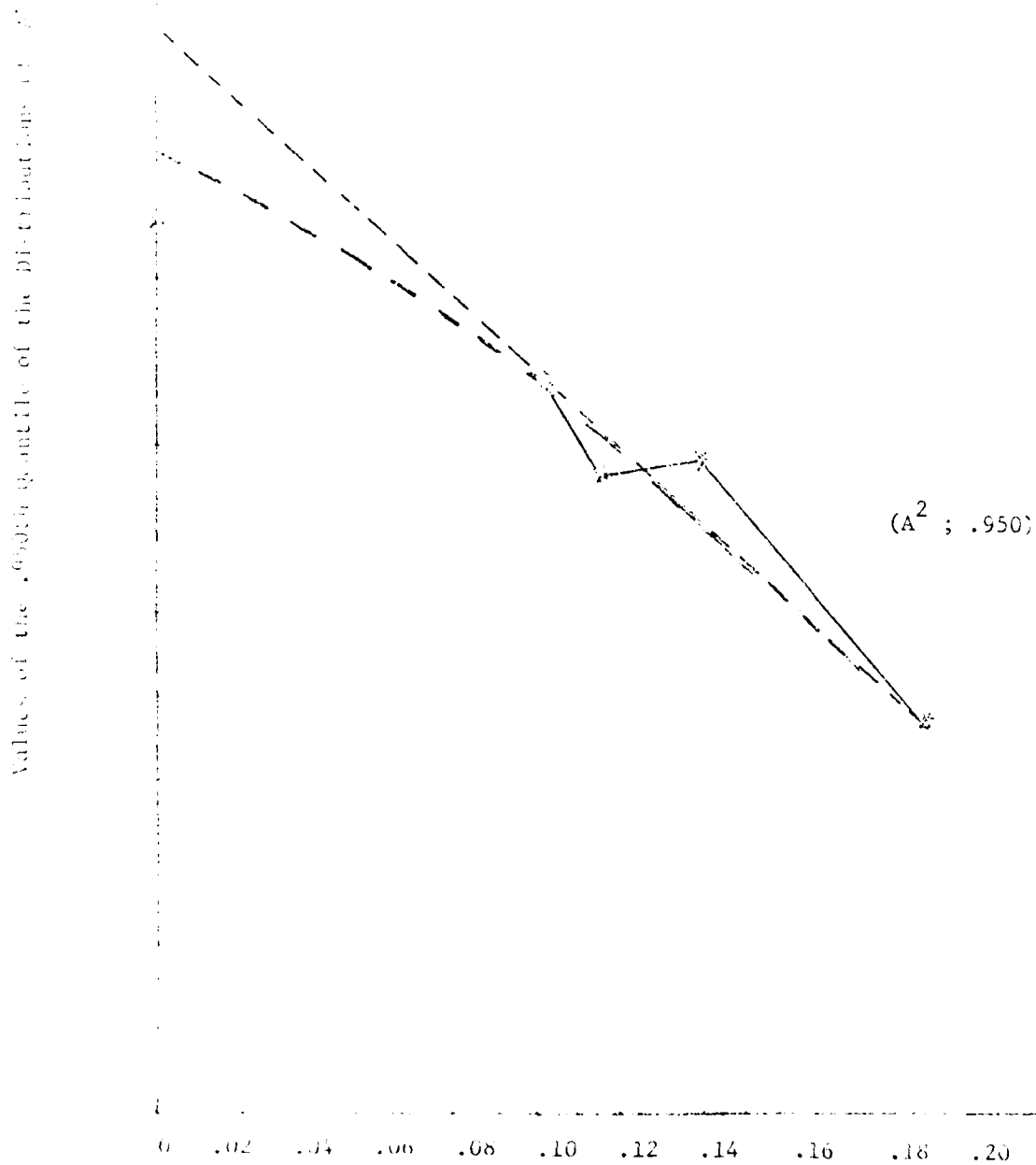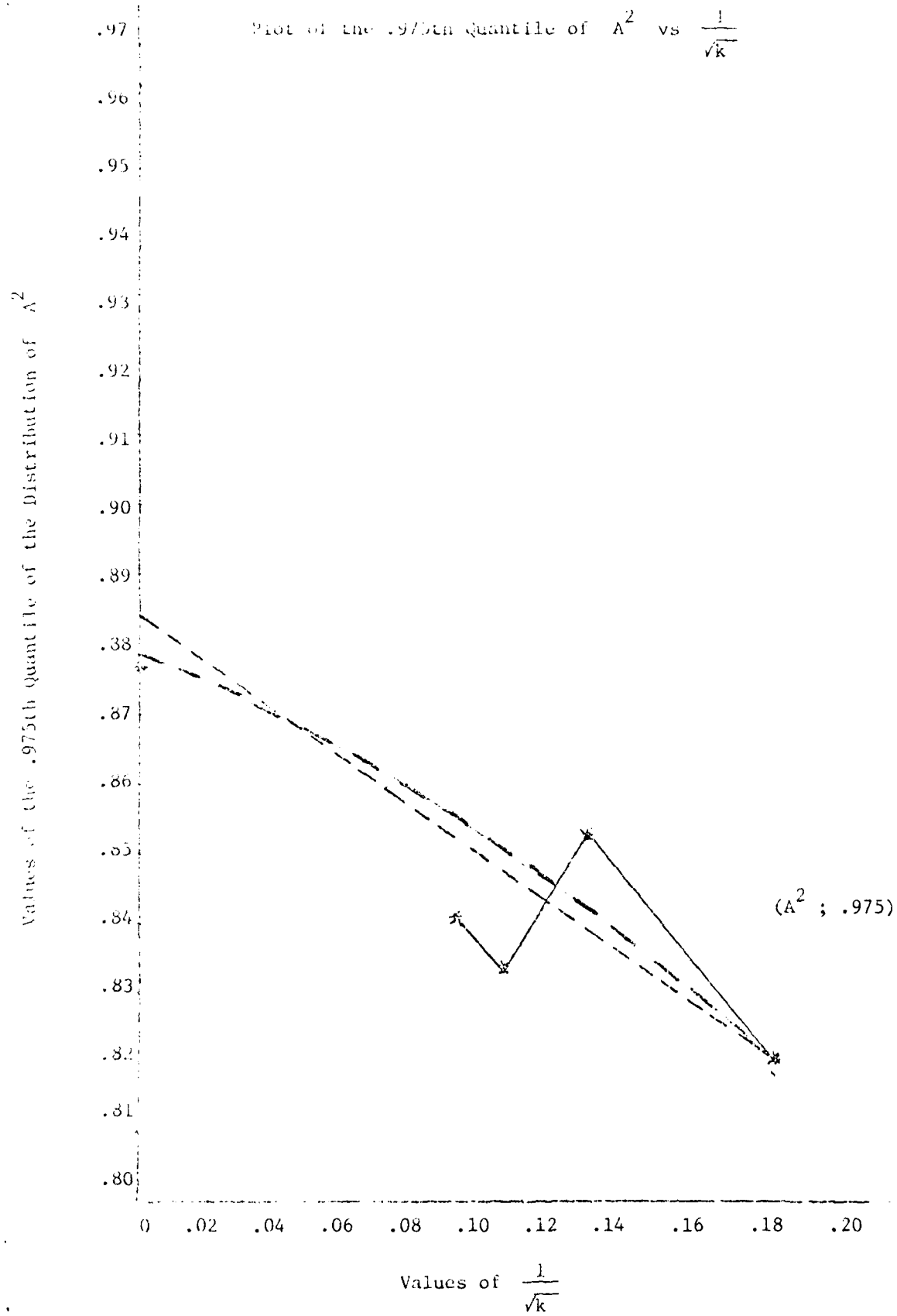
Figure 4.23

Plot of the .990th Quantile of Λ^2 vs $\dfrac{1}{\sqrt{\kappa}}$

$(A^2 \; ; \; .990)$

Values of $\dfrac{1}{\sqrt{k}}$
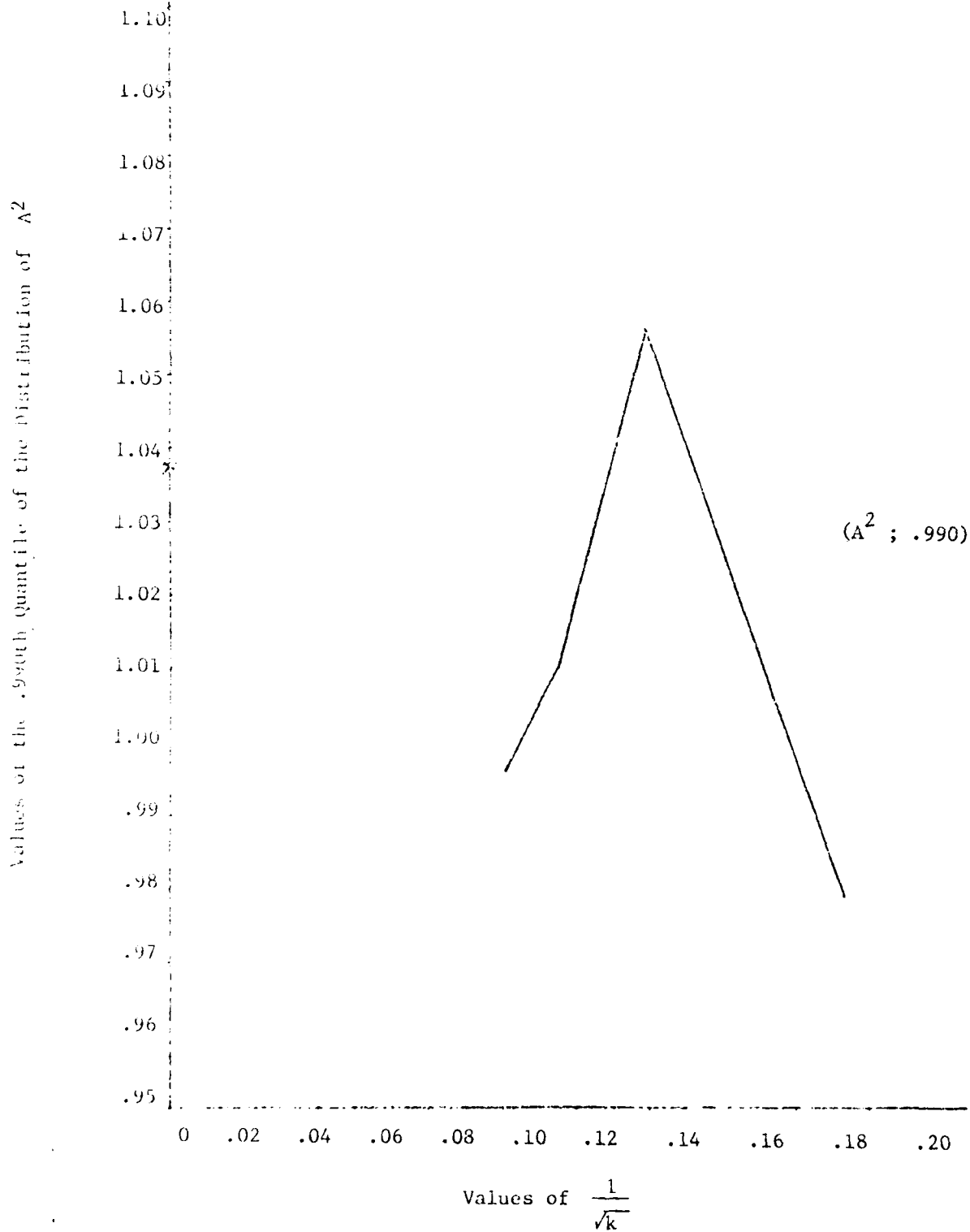
Figure 4.24

Graph of the Quantiles of the Maximum of a
Brownian Motion Process from Monte Carlo
Simulation and Siegmund's Approximation

p = .90

Monte Carlo

Siegmund's
Approximation.

p = .95
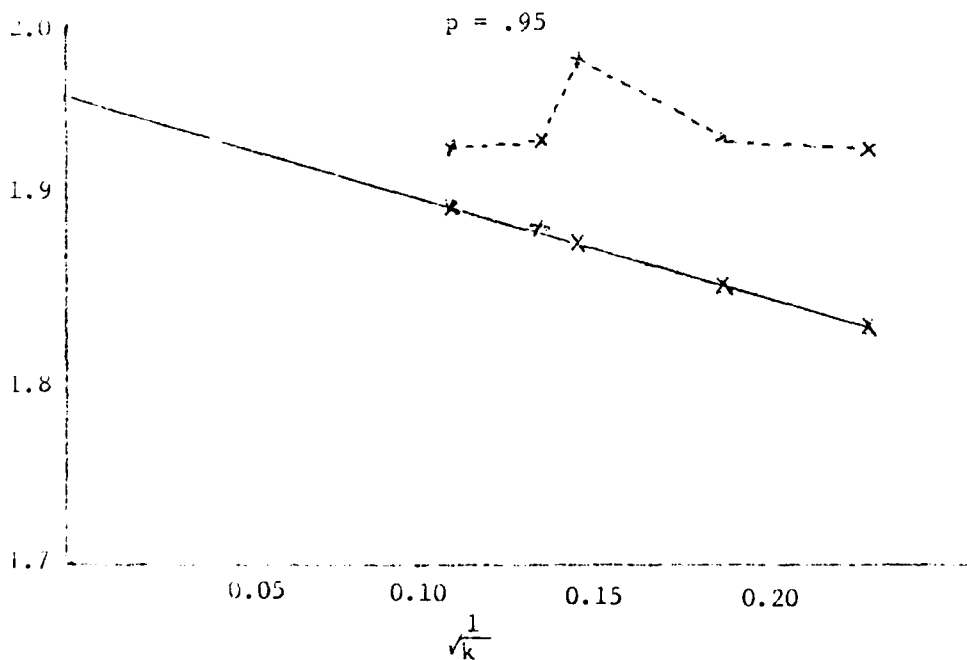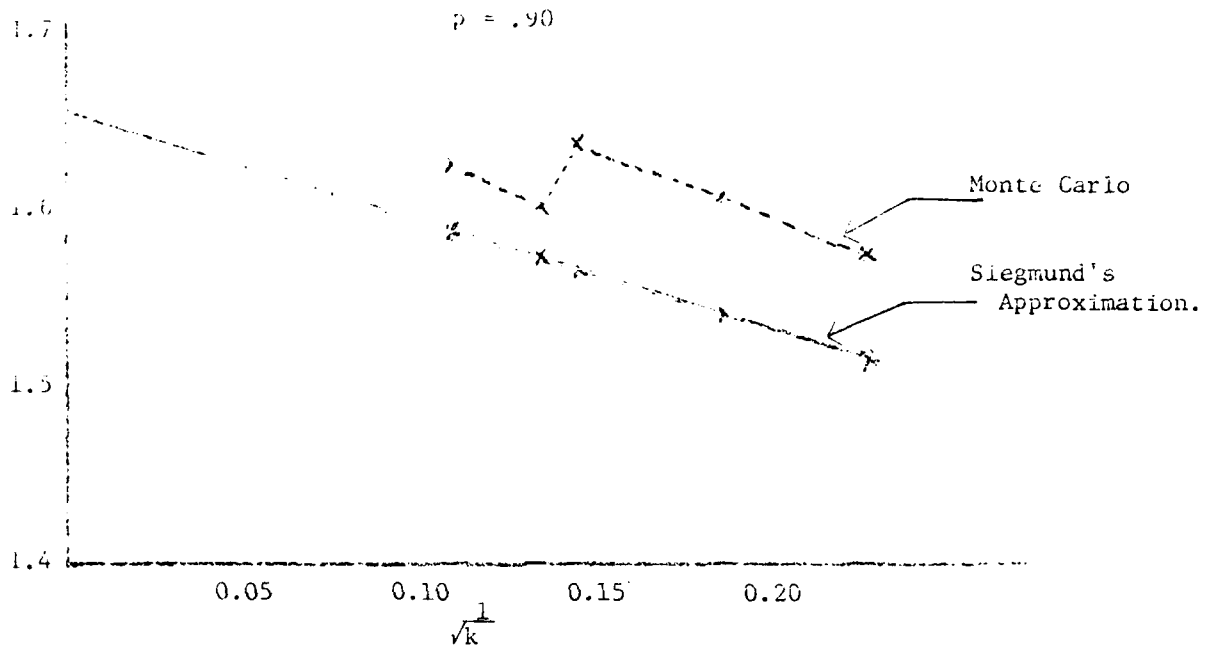
Figure 4.24 (Continued)

p = .975

p = .990

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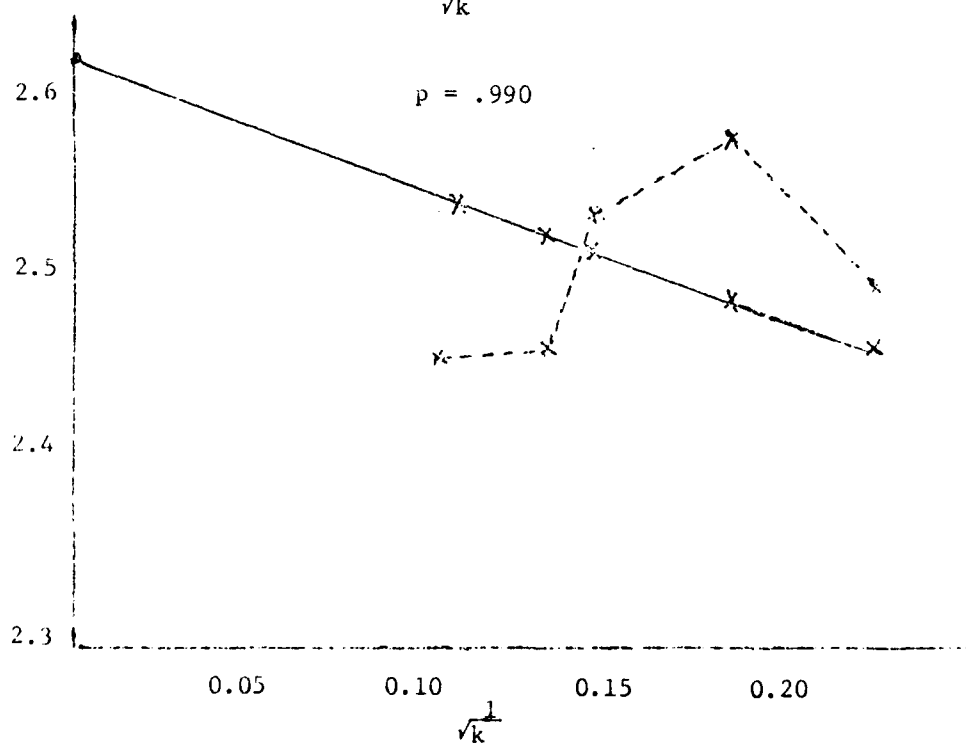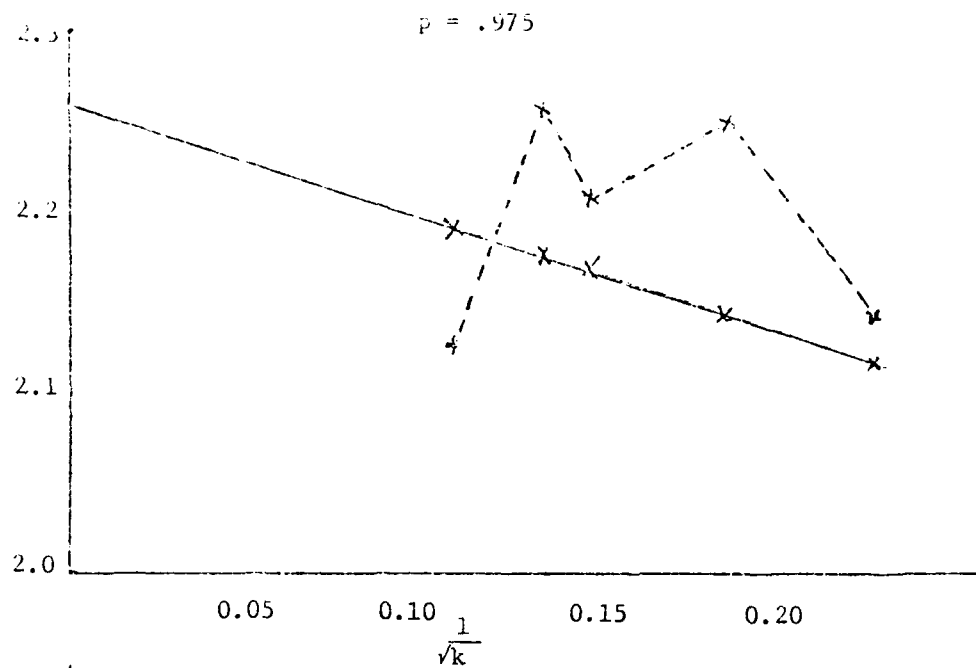
Table 4.0

Upper Tail Quantiles for EDF Statistics $\sqrt{n}\ D^+$, $\sqrt{n}\ D^-$
$\sqrt{n}\ D$, $\sqrt{n}\ V$. (For explanation of Cases 1, 2 and 3, and the
Monte Carlo technique Used see Section 4)

Statistic	n\p	.90	.95	.975	.99
$\sqrt{n}\ D^+$	10	.872	.969	1.061	1.152
Case 1	20	.878	.979	1.068	1.176
	50	.882	.987	1.070	1.193
	∞	.886	.994	1.104	1.207
$\sqrt{n}\ D^+$	10	.988	1.135	1.273	1.419
Case 2	20	1.003	1.152	1.282	1.432
	50	1.012	1.168	1.287	1.439
	∞	1.019	1.174	1.289	1.444
$\sqrt{n}\ D^+$	10	.685	.755	.842	.897
Case 3	20	.710	.780	.859	.926
	50	.727	.796	.870	.940
	∞	.732	.808	.876	.951
$\sqrt{n}\ D^-$	10	.773	.883	.987	1.103
Case 1	20	.810	.921	1.013	1.142
	50	.840	.950	1.031	1.171
	∞	.886	.994	1.104	1.207
$\sqrt{n}\ D^-$	10	1.012	1.162	1.275	1.409
Case 2	20	1.006	1.150	1.280	1.432
	50	1.001	1.142	1.290	1.448
	∞	1.019	1.17	1.296	1.456
$\sqrt{n}\ D^-$	10	.700	.766	.814	.892
Case 3	10	.715	.785	.843	.926
	50	.724	.796	.860	.944
	∞	.73	.81	.87	.96

Table 4.0 (Continued)

Statistic	n\p	.90	.95	.975	.99
$\sqrt{n}$ D	10	.934	1.026	1.113	1.206
Case 1	20	.954	1.049	1.134	1.239
	50	.970	1.067	1.148	1.263
	∞	.990	1.086	1.200	1.300
$\sqrt{n}$ D	10	1.141	1.270	1.390	1.520
Case 2	20	1.152	1.281	1.403	1.525
	50	1.157	1.286	1.411	1.528
	∞	1.161	1.290	1.417	1.530
$\sqrt{n}$ D	10	.760	.819	.880	.944
Case 3	20	.779	.843	.907	.973
	50	.790	.856	.922	.988
	∞	.797	.868	.932	1.001
$\sqrt{n}$ V	10	1.428	1.547	1.650	1.772
Case 1	20	1.460	1.575	1.685	1.813
	50	1.480	1.593	1.716	1.838
	∞	1.53	1.65	1.77	1.91
$\sqrt{n}$ V	10	1.388	1.493	1.596	1.715
Case 2	20	1.424	1.538	1.641	1.763
	50	1.445	1.564	1.667	1.793
	∞	1.459	1.584	1.686	1.812
$\sqrt{n}$ V	10	1.287	1.381	1.459	1.535
Case 3	20	1.323	1.428	1.509	1.600
	50	1.344	1.453	1.538	1.639
	∞	1.360	1.471	1.558	1.664

Table 4.1

Asymptotic Quantiles* (Estimates) of EDF Test
Statistics for the Extreme Value Distribution
With Both Parameters Estimated

P	pth Quantile						
	D^+	D^-	D	V	W^2	U^2	A^2
0.010	0.198	0.197	0.264	0.489	0.015	0.014	0.098
0.025	0.224	0.226	0.299	0.537	0.018	0.017	0.116
0.050	0.250	0.251	0.326	0.581	0.021	0.020	0.134
0.100	0.280	0.281	0.358	0.637	0.025	0.024	0.158
0.250	0.344	0.343	0.421	0.739	0.035	0.032	0.213
0.500	0.429	0.427	0.505	0.871	0.049	0.047	0.295
0.750	0.536	0.532	0.611	1.032	0.072	0.068	0.417
0.900	0.650	0.649	0.721	1.199	0.100	0.095	0.569
0.950	0.728	0.725	0.791	1.310	0.123	0.117	0.696
0.975	0.790	0.794	0.853	1.401	0.146	0.138	0.820
0.990	0.868	0.877	0.925	1.521	0.175	0.165	0.979

*Based on direct simulation of the asymptotic process,
using k = 30 intervals and 10,000 replicates

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Table 4.2

Asymptotic Quantiles* (Estimates) of EDF Test
Statistics for the Extreme Value Distribution
With Both Parameters Estimated

p	pth Quantile						
	D^+	D^-	D	V	$\hat{W}^2$	U^2	A^2
0.010	0.230	0.232	0.300	0.554	0.016	0.016	0.113
0.025	0.258	0.256	0.327	0.596	0.019	0.018	0.132
0.050	0.281	0.279	0.352	0.636	0.022	0.021	0.150
0.100	0.311	0.311	0.387	0.694	0.026	0.025	0.173
0.250	0.373	0.375	0.452	0.799	0.035	0.034	0.229
0.500	0.459	0.461	0.541	0.941	0.050	0.048	0.316
0.750	0.567	0.565	0.644	1.103	0.073	0.070	0.446
0.900	0.681	0.685	0.753	1.265	0.103	0.097	0.607
0.950	0.760	0.755	0.825	1.379	0.124	0.118	0.728
0.975	0.832	0.824	0.892	1.474	0.146	0.138	0.853
0.990	0.916	0.923	0.974	1.584	0.183	0.173	1.056

*Based on direct simulation of the asymptotic process,
using k = 60 intervals and 10,000 replicates

Table 4.3

Asymptotic Quantiles* (Estimates) of EDF Test
Statistics for the Extreme Value Distribution
With Both Parameters Estimated

p	pth Quantile						
	D^+	D^-	D	V	W^2	U^2	A^2
0.010	0.248	0.244	0.317	0.579	0.016	0.015	0.115
0.025	0.269	0.270	0.342	0.626	0.019	0.018	0.134
0.050	0.292	0.295	0.368	0.669	0.022	0.021	0.152
0.100	0.323	0.325	0.400	0.722	0.026	0.025	0.179
0.250	0.387	0.387	0.463	0.824	0.035	0.034	0.234
0.500	0.472	0.470	0.551	0.961	0.049	0.047	0.320
0.750	0.581	0.579	0.654	1.122	0.073	0.069	0.448
0.900	0.694	0.691	0.761	1.293	0.101	0.096	0.610
0.950	0.768	0.751	0.828	1.388	0.123	0.117	0.726
0.975	0.834	0.823	0.888	1.476	0.143	0.136	0.833
0.990	0.918	0.909	0.983	1.597	0.171	0.164	1.010

*Based on direct simulation of the asymptotic process,
using $k = 90$ intervals and 10,000 replicates

Table 4.4

Asymptotic Quantiles* (Estimates) of EDF Test
Statistics for the Extreme Value Distribution
With Both Parameters Estimated

P	pth Quantile						
	D^+	D^-	D	V	W^2	U^2	A^2
0.010	0.254	0.254	0.328	0.598	0.017	0.016	0.123
0.025	0.281	0.278	0.352	0.638	0.019	0.019	0.140
0.050	0.304	0.303	0.378	0.683	0.022	0.021	0.158
0.100	0.338	0.334	0.407	0.739	0.033	0.031	0.183
0.250	0.397	0.395	0.472	0.844	0.035	0.034	0.241
0.500	0.484	0.480	0.559	0.983	0.051	0.048	0.329
0.750	0.591	0.585	0.666	1.146	0.074	0.070	0.459
0.900	0.705	0.702	0.773	1.309	0.102	0.098	0.624
0.950	0.778	0.775	0.838	1.406	0.122	0.116	0.738
0.975	0.841	0.838	0.899	1.492	0.142	0.135	0.841
0.990	0.910	0.918	0.965	1.608	0.169	0.161	0.996

*Based on direct simulation of the asymptotic process,
using k = 120 intervals and 10,000 replicates

Table 4.5

Quantiles of the Distribution of $\hat{M}_k$, the Maximum of
a Brownian Motion Process Discretized at k Points

k	p	pth Quantile by Simulation	pth Quantile by Approximation (4.2)
20	.90	1.5725	1.5146
	.95	1.9318	1.8296
	.975	2.1466	2.1196
	.990	2.5053	2.4696
30	.90	1.6049	1.5386
	.95	1.9362	1.8536
	.975	2.2495	2.1436
	.990	2.5815	2.4936
50	.90	1.6376	1.5626
	.95	1.9798	1.8776
	.975	2.2085	2.1676
	.990	2.5407	2.5176
60	.90	1.5997	1.5697
	.95	1.9416	1.8847
	.975	2.2583	2.1747
	.990	2.4642	2.5250
90	.90	1.6239	1.5835
	.95	1.8913	1.8985
	.975	2.1240	2.1885
	.990	2.4625	2.5385

Table 5.1

Asymptotic Quantiles of EDF Test Statistics $\sqrt{n}\,D^+$, $\sqrt{n}\,D^-$, $\sqrt{n}\,D$, and $\sqrt{n}\,V$ for the Extreme Value Distribution With Both Parameters Estimated: Comparison of Two Methods

	pth Quantile							
	$\sqrt{n}\,D^+$		$\sqrt{n}\,D^-$		$\sqrt{n}\,D$		$\sqrt{n}\,V$	
p	Parabolic Extrapol.	Stephens*	Parabolic Extrapol.	Stephens*	Parabolic Extrapol.	Stephens*	Parabolic Extrapol.	Stephens*
.75	.618	-	.616	-	.704	-	1.224	-
.90	.736	.732	.733	.729	.808	.797	1.384	1.340
.95	.810	.808	.810	.804	.872	.868	1.452	1.431
.975	.882	.876	.858	.873	.945	.932	1.54.	1.555
.990	.956	.951	.964	.957	1.013	1.001	1.678	1.664

*Quantiles obtained by Stephens (unpublished) via a direct simulation and extrapolation.

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Table 5.2

Asymptotic Quantiles of EDF Test Statistics
For the Extreme Value Distribution
With Both Parameters Estimated

P	pth Quantile						
	$\sqrt{n}\ D^+$	$\sqrt{n}\ D^-$	$\sqrt{n}\ D$	$\sqrt{n}\ V$	W^{2*}	U^{2*}	A^{2*}
.75	.62	.62	.70	1.22	.126	.073	.474
.90	.73	.73	.80	1.37	.142	.102	.637
.95	.81	.81	.87	1.48	.150	.124	.757
.975	.87	.87	.94	1.56	.158	.146	.877
.990	.96	.96	1.01	1.67	.170	.175	1.038

*From Stephens (1977, Table 1, Case 3)

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