We present an extension of the analysis of our previously proposed model for auroral kilometric radiation, in which the full three-wave dynamics is considered. The temporal variation of the electrostatic ion cyclotron density fluctuations and the effect of a finite three-wave coupling coefficient, both effects ignored in our steady state theory, are included here. This yields a set of coupled three-wave kinetic equations, which are solved numerically for the evolution of the auroral kilometric radiation. The numerical results tend to confirm all of the predictions of our steady state theory. The growth rate is found to be determined by the resonant growth rate found in our steady state theory, and stages of the growth are found in which...
the radiation grows at twice the resonant growth rate, and sometimes four, eight, etc. times that value. The growth saturates, and this saturation level is found to scale as $\Gamma_b \cdot t^{4/3}$, where $\Gamma_b$ is the rate of transfer of energy from the beam to the beat wave, or quasimode. Saturation levels are found which are adequate to produce the observed radiation levels. It is found that for a sufficiently large growth region, the saturation level is relatively independent of the initial level of density fluctuations.
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A COHERENT NONLINEAR THEORY OF AURORAL KILOMETRIC RADIATION:
II. DYNAMIC INTERACTIONS

I. Introduction

In a recent paper [Grabbe, et al, 1980], a theory of auroral kilometric radiation was proposed, in which electromagnetic noise is amplified by interaction with low frequency coherent quasineutral density fluctuations created by electrostatic ion cyclotron (EIC) waves, in the presence of precipitating auroral electron beams. The result is a three-wave parametric process in which a beat wave is produced that can interact with the beam, much like the theory of Palmadesso, et al [1976]. It was found that when the wave frequency is in the right range, the electromagnetic wave is negative energy in the rotating frame of the beam electron and undergoes a convective instability. The basic requirements for the instability were found to be:

(1) Minimum beam density:

\[
\left( \frac{n_b}{n_o} \right) > \frac{k_z (\Delta v)^2}{2 \omega_c v_b} \]

(1)

where \( n_b \) and \( n_o \) are the beam and plasma density, respectively, \( v_b \) and \( \Delta v \) the beam velocity and thermal spread in velocity space, \( k_z \) the wave vector component along the magnetic field, and \( \omega_c \) the electron cyclotron frequency.

(2) Accessibility to free space \((\omega > \omega_R\) where \( \omega_R \) is the right hand cutoff):

\[
\omega^2 < \frac{k_z v_b \omega_c}{\omega_{pe}}
\]

(2)

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(3) Frequency range:

$$\omega < \omega_{ce} + k z v_b$$

(3)

Condition (1) typically requires $n_b \geq 10^{-3} n_o$ and condition (2) typically requires local depletion of the plasma density such that $\omega_{pe} < 0.2 \omega_{ce}$. The latter is in very good agreement with the observed density depletion [Benson and Calvert, 1979].

Combining conditions (1) - (3) gives the following limits on the frequency and propagation directions of the wave for amplification

$$\omega_{ce} + \frac{\omega_{pe}}{\omega_{ce}} < \omega < \omega_{ce} + k z v_b$$

(4)

$$\frac{\omega_{pe}^2}{k v_b \omega_{ce}} < \cos \theta < \frac{2\omega_{ce} v_b n_b}{k (\Delta \nu)^n_o}$$

(5)

Eq. (4) gives radiation in a narrow frequency just above the right band cutoff, while Eq. (5) normally limits the propagation to be almost (but not quite) perpendicular to the magnetic field [Grabbe, 1980]. Furthermore, the 0-mode has no such unstable frequency range, hence the X-mode is the predicted polarization. All of these predictions are in excellent agreement with observation.

The above conclusions were based on a steady state model, in which amplitude of the density fluctuations was assumed to be approximately constant. This is valid if the energy in the density fluctuations is replenished by the beam or other sources at approximately the same rate as it is being used up. However, the Feynman diagram
for the three-wave process (Fig. 1) reveals that a more dynamical process is taking place. Not only is energy being resonantly transferred from the density fluctuation to the electromagnetic wave in the appropriate frequency band because of the beam, but the energy the beam injects into the beat wave is being transferred back to the density fluctuations and the electromagnetic wave because of a finite three wave coupling coefficient. This coupling coefficient was ignored in the steady state theory, but must be included to understand the full dynamical process.

The purpose of this paper is to study the dynamics of the full three wave process, including temporal variations in the density fluctuations and a finite three-wave coupling coefficient, in order to confirm the predictions of the steady state theory. In Sec. II we introduce a set of coupled nonlinear rate equations for the evolution of each of the three waves in the Feynman diagram, and discuss steady state solutions. Several numerical solutions of these equations are presented and discussed in Sec. III for various typical parameters. The principle conclusions are summarized in Sec. IV.
Fig. 1 — Feynman diagram for the three-wave interaction involved in the amplification of radiation to produce auroral kilometric radiation. The processes numbered are (1) induced absorption, (2) beam amplification, (3) induced (and spontaneous) emission, and (4) resonant amplification. The rate coefficient for each process is shown.
II. Dynamical Model

The dynamical processes involved in the amplification process which produces the AKR are summarized in the three-wave Feynman diagram in Fig. 1. These processes are (1) Induced absorption of the X mode by the density fluctuations to create the quasimode or beat wave (2) Beam amplification of the beat wave (3) Spontaneous and induced emission of the X-mode by the beat mode (4) Resonant interaction between the density fluctuations and the X mode in which energy is transferred to the X mode. The conditions for the last process were given in the introduction, and the resulting spatial growth rate was given by Eq. (28) in Grabbe, et al [1980]:

\[ \kappa = \left( \frac{\omega}{c} \right) \left[ \left( \frac{\alpha_2}{\alpha_1} \right)^2 - \left( \frac{\sqrt{\alpha_1}}{k_1} - 1 \right)^2 \right]^{1/2} \]  

(6)

We want to formulate the rate equations for the aforementioned processes in terms of the (quantum) occupation number density of the waves.

\[ N_j = \left| \hat{\mathbf{E}}_j \right|^2 / 8\pi \omega_1 \]  

(7)

where \( \hat{\mathbf{E}} \) is the wave electric field. We designate \( N_o \) as the number density of the X-mode, \( N_1 \) for the density fluctuations, and \( N_b \) for the beat wave. We then have the following contributions to the rate equation for each process numbered in Fig. 1:
(1) Induced absorption.

\[ \frac{dN_b}{dt} = -WN_0N_1 \] (8a)

\[ \frac{dN_1}{dt} = -WN_0N_1 \] (8b)

\[ \frac{dN_b}{dt} = 2WN_0N_1 \] (8c)

where \( W \) is the three wave coupling coefficient [Tsytovich, 1974]:

\[ W = \frac{e^2}{8\pi m_e^2} \frac{\omega e^2}{\omega_{pe}^2} \frac{m_i}{n_i} \] (9)

(2) Beam amplification.

\[ \frac{dN_b}{dt} = \Gamma_b N_b \] (10)

where we take the beam driven growth rate of the quasimode \( \Gamma_b \) to be the usual form for a beam plasma instability [Briggs, 1964]:

\[ \Gamma_b = 2 \text{Im} \omega \sim 1.4 \left( \frac{n_b}{n_o} \right)^{1/3} \omega_{pe} \] (11)

(3) Spontaneous and induced emission.

\[ \frac{dN_o}{dt} = WN_b(1+N_o+N_1) \] (12a)
\[
\frac{dN_o}{dt} = W N_o (1+N_o+N_1)
\]
(12b)

\[
\frac{dN_1}{dt} = -2W N_o (1+N_o+N_1)
\]
(12c)

The spontaneous emission term (first term on the right hand) is normally negligible, since normally \( N_o >> 1 \) or \( N_1 >> 1 \) (\( N_o \) and \( N_1 \), taken as dimensionless occupation numbers).

4. Resonant interaction.

\[
\frac{dN_o}{dt} = \gamma N_o N_1^{1/2}
\]
(13a)

\[
\frac{dN_1}{dt} = -\gamma N_o N_1^{1/2}
\]
(13b)

Here the dependence of the resonant growth rate \( \Gamma \) of the X-mode on the density fluctuation \( N_1 \) has been explicitly factored out:

\[
\Gamma = \gamma N_1^{1/2}
\]
(14)

Here the temporal growth rate can be expressed in terms of the spatial growth rate by multiplying by the group velocity

\[
\Gamma = 2 \kappa c
\]
(15)

where the factor 2 represents the conversion between electric field growth rate and the quantum density growth rate. Thus from Eq. (6)

\[
\gamma \sim \frac{(2\pi)^{3/2}\omega_0^{3/2} e}{\alpha_{\perp}^{1/2} T_e}
\]
(16)

at its maximum value.

Combining, (1), (2), (3), (4) we find the complete set of equations to be
\[
\frac{dN_0}{dt} = W(N_bN_o + N_bN_1 - N_0N_1) + \gamma N_o^{1/2} \tag{17a}
\]

\[
\frac{dN_1}{dt} = W(N_bN_o + N_bN_1 - N_0N_1) - \gamma N_o^{1/2} \tag{17b}
\]

\[
\frac{dN_b}{dt} = 2W(N_bN_o - N_bN_1 - N_0N_1) + \Gamma N_b \tag{17c}
\]

These are the central equations we want to solve for AKR evolution.

Before obtaining numerical solutions of the equations we want to first examine them for steady state solutions. If we consider the limit of constant density fluctuations \(dN_1/dt = 0\), the case analyzed in our steady state theory, we find

\[
\frac{dN_0}{dt} = 2\gamma N_o^{1/2} \tag{18}
\]

This result shows that the X-mode grows at twice the rate determined in our steady state \(\text{[Grabbe, et al., 1980]}\). This result can be understood by noting the two processes which transfer energy to the X mode:

(3) transfers energy from the beat wave to the X mode and density fluctuations at equal rates; (4) transfers energy from the density fluctuation to the X-mode. Both processes (3) and (4) must occur at equal rates for the density fluctuation to achieve a steady state.

Since the steady state model only considers the contribution of process (4), it only gives one-half of the growth rate.
III. Numerical Results

To obtain typical values for the growth rates and coupling coefficient we use Eqs. (9), (11) and (16). If we take typical source region values $n_0 > 10^{-3} n_o$, $\nu_p < 0.2 \omega_{ce}$, $n_1 > 0.3 - 0.5 n_o$, we find the following normalized values when the $N_o$'s are normalized to dimensionless values:

\[
\Gamma_b > 6 \times 10^4 \text{ sec}^{-1}
\]
\[
\Gamma = \gamma N_1 > 6 \times 10^4 \text{ sec}^{-1}
\]
\[
\hat{W} \sim 0.05 \text{ sec}^{-1}
\]

Here $\gamma = (e^2/m_e c^2 \omega) N_a$, $\gamma = (e/m_e c \nu_p)$ and $\hat{W} = (m^2 c^2/e^2) W$. These values will be used as a guide for input to the numerical calculations.

The kinetic equations in Eq. (17) were solved for several values of the growth rate and initial conditions, although the initial value of $N_1$ was always set to the normalized value on 1. A classical Runge-Kutta integration routine was used initially but proved to be too inefficient, so it was replaced by a stiff integration scheme called CHEMEQ [Young, 1980]. A sampling of the results is shown in Figs. 2-4. Included in the graphs is the ratio of the time dependent growth rate $\Gamma^*(t)$ to the resonant growth rate $\Gamma$, where

\[
\frac{dN_0}{dt} = \Gamma^*(t) N_0.
\]  

The graphs show that there are two principle stages of growth of the X-mode. In the initial stage the growth rate is just the resonant growth rate $dN_0/dt = \gamma N_0 N_1$. The reason for this stage is that not much energy has been transferred into the beat wave, so all of the X-mode energy is coming from the density fluctuations. This stage
Fig. 2 — Growth and saturation of the X mode radiation for the initial conditions and rate coefficients shown. The values shown are for the normalized form given by Eq. (19). Note that the radiation grows in two principal stages: in the first, the wave grows primarily on energy from the EIC density fluctuations, and temporarily saturates; in the second the wave grows to very large amplitudes because of energy coupled in from the beam via a beat wave then finally saturates.
Fig. 3 — Same as Fig. 2 for a slightly different set of rate coefficients.
Fig. 4 — Graph of the ratio of the growth rate of the wave as function of time to that found in the steady state theory, for the conditions in Fig. 3. Note that it is normally $2^n$, for various non-negative integer values of $n$. 
normally saturates when most of the energy in the density fluctuations is absorbed. The second stage starts when a significant amount of energy has been pumped into the beat wave by the beam, and this energy begins being transferred to the density fluctuations and the X-mode. This second stage grows at \( \frac{dN_o}{dt} = 2\gamma N_0 \), or twice the resonant growth rate. The graphs of \( \Gamma(t)/\Gamma = (t)/\gamma \) show that there may also exist subsequent stages in which the growth rate occurs at \( \frac{dN_o}{dt} = 4\gamma N_0 \), \( \frac{dN_o}{dt} = 8\gamma N_0 \), etc. Then the process reaches a final level of saturation when almost all of the energy of the density fluctuations and the beat wave has been depleted. The stages of growth confirm that the rate of growth is governed by the resonant growth rate \( \Gamma = \gamma N_1 \), as predicted by the steady state theory.

A comparison of the saturation amplitude of the X-mode shows that it always is several orders of magnitude above the initial value of the density fluctuations \( N_1(t=0) \). This shows that virtually all of the energy comes from the beam, rather than from the initial level of the density fluctuations. Furthermore, when \( \Gamma_b \) is varied on sample runs with all other parameters being held constant, then the saturation amplitude increases as \( \Gamma_b \) is increased; thus, \( \Gamma_b \) determines the total energy the electromagnetic wave can absorb. A statistical analysis of the saturation level \( N_o^{\text{sat}} \) from many sample runs reveals the scaling law

\[
N_o^{\text{sat}} \propto \Gamma_b^{4/3}
\]  

From Eq. (11), we have the dependence on the relative beam density.
A derivation of this scaling law is given in the Appendix.

A comparison has been made on sample runs for different initial values of $N_1$, all other parameters held constant. It is found that although the growth rate of the X mode increases linearly with $N_1$, the saturation amplitude is relatively independent of the initial value of $N_1$. The significance of this for the growth process in the finite density depleted cavity introduced in the steady state model [Grabbe, et al., 1980] is the following. If the X-mode saturates before propagating out of the growth region in the cavity, then an increase in the initial levels of the density fluctuations does not have a very important effect on level of AKR produced. However, if the X-mode propagates out of the growth region before reaching its saturation level, increasing the initial level of the EIC density fluctuations would bring the AKR closer to its saturation value. Calculation done in the steady state model would suggest the former case occurs more often than the latter.
IV. Summary and Conclusions

We have formulated the dynamics of the three-wave process involved in amplification of the X-mode to produce AKR in terms of a set of coupled rate equations. An analysis of these coupled rate equations has confirmed the conclusions drawn from the steady state theory. It was found that the growth rate was determined by the resonant growth rate found in the steady state. The growth was seen to occur in stages: an initial stage in which the growth rate was just the resonant growth rate, and subsequent stage of growth at $2^n$ times the resonant growth rate, where $n=1,2,3,...$. This is followed by a saturation of the wave.

It was found that the AKR could saturate at $10^8-10^{10}$ times the initial (noise) levels. This is adequate to produce the observed levels of AKR. Almost all of this energy comes from the beam, and the saturation amplitude was seen to scale as $r_b^{4/3}$, so that the growth rate of the beat wave determines the total energy the electromagnetic wave can absorb. Finally, it was found that for sufficiently large density cavities with growth regions, the saturation level of AKR is relatively independent of the initial level of the EIC density fluctuations.

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