AD-A096 855 CALIFORNIA UNIV BERKELEY DEPT OF MECHANICAL ENGINEERING F/G 12/1
AN INVARIANT INFINITESIMAL THEORY OF MOTIONS SUPERPOSED ON A GI--ETC(U)
AUG 80 J CASEY, P M NAGHDI N00014-75-C-0148
UNCLASSIFIED UCB/AM-80-6 NL

1 OF 1
AD
A096855

END
DATE
FILMED
4-81
DTIC

UNCLASSIFIED

LEVEL II

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD A096855

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: UCB/AM-80-6
2. GOVT ACCESSION NO.: AD-A096 855
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): An Invariant Infinitesimal Theory of Motions Superposed on a Given Motion.
5. TYPE OF REPORT & PERIOD COVERED: Technical Report.
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): J. Casey, P. M. Naghdi
8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0148
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Mechanical Engineering, University of California, Berkeley, CA 94720
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR 064-436
11. CONTROLLING OFFICE NAME AND ADDRESS: Structural Mechanics Branch, Office of Naval Research, Arlington, VA 22217
12. REPORT DATE: Aug[redacted]80
13. NUMBER OF PAGES: 62
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

DTIC ELECTE 1981

18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Linear theory, small on large, properly invariant, elastic solids.
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

This paper is concerned with the construction of an invariant infinitesimal theory of motions superposed on a given motion. The development is applicable to any material but special attention is given to elastic solids. Included as a special case is an infinitesimal theory of elasticity with the following properties: (1) It is properly invariant under arbitrary (not necessarily infinitesimal) superposed rigid body motions, (2) it reduces by specialization to the theory (continued)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DTIC FILE COPY

81 3 25 004

20. Abstract (continued)

of rigid bodies undergoing finite motion, and (3) it can be brought into correspondence with the classical linear elasticity through a suitable reinterpretation of the symbols in the constitutive equation of the latter.

Accession For

X

A

An Invariant Infinitesimal Theory of Motions Superposed on a Given Motion

by

J. Casey and P. M. Naghdi

Abstract. This paper is concerned with the construction of an invariant infinitesimal theory of motions superposed on a given motion. The development is applicable to any material but special attention is given to elastic solids. Included as a special case is an infinitesimal theory of elasticity with the following properties: (1) It is properly invariant under arbitrary (not necessarily infinitesimal) superposed rigid body motions, (2) it reduces by specialization to the theory of rigid bodies undergoing finite motion, and (3) it can be brought into correspondence with the classical linear elasticity through a suitable reinterpretation of the symbols in the constitutive equation of the latter.

Contents

1. Introduction

The theories that describe the finite deformations of continuous media, most notably the theory of finite elasticity, all satisfy correct invariance requirements[†]. In contrast, as is well known, the classical theory of inifinitesimal elasticity does not satisfy correct invariance requirements; the theory of infinitesimal deformations superposed on a finite deformation is not an invariant theory either[‡]. A further example of a theory of practical importance that does not satisfy correct invariance requirements is that of "physically nonlinear" elasticity in which the deformation is assumed to be infinitesimal, while the constitutive equation is nonlinear in the infinitesimal strain.

Statements have occasionally been made in the literature[*] -- with evident justification -- regarding the physical meaninglessness of infinitesimal theories which fail to satisfy full invariance requirements, it being pointed out that such theories could not possibly apply to <u>any</u> material undergoing finite deformation. The purpose of this paper is to introduce an invariant infinitesimal theory of motions superposed on any given motion. This includes, of course, as a special case an invariant infinitesimal theory. While our method of approach and all of the kinematical and kinetical results hold for any material, we devote special attention to elastic solids.

[†] By this we mean <u>invariance requirements under superposed rigid body motions</u>, which embody the idea that all motions of a body which differ only by a rigid motion are mechanically equivalent. We do not employ here the <u>principle of material frame-indifference</u> (or <u>material objectivity</u>) which is used by some authors as an alternative to invariance requirements under superposed rigid body motions.

[‡] It is rather disconcerting that an infinitesimal theory of a deformable medium (such as the classical infinitesimal elasticity) does not include as a special case that for which the deformation is zero, i.e., the theory of rigid bodies.

[*] See Coleman and Noll (1961, p. 245) and Truesdell and Noll (1965, p. 117). Reference may also be made to a related remark by Truesdell and Toupin (1960, p. 724).

1.1 Examination of the usual infinitesimal theory

The main reason that infinitesimal theories do not satisfy invariance requirements, which are met by the finite theories, lies in the different behavior exhibited by the infinitesimal and the finite strain tensors when an arbitrary rigid motion is superposed on a given motion of a body $\mathcal{B}$. To elaborate, recall that the finite (relative) strain tensor defined by $\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})$, where $\underset{\sim}{C}$ is the Cauchy-Green measure of deformation and $\underset{\sim}{I}$ the identity tensor, is invariant under superposed rigid body motions. A consequence of such invariance is that $\underset{\sim}{E}$ must take on a constant value for all rigid motions of $\mathcal{B}$; by choice it is arranged in the definition of $\underset{\sim}{E}$ that this constant value be zero. Under certain conditions, the infinitesimal strain tensor $\underset{\sim}{e}$ furnishes a linear approximation to $\underset{\sim}{E}$ and is used in constitutive equations intended to describe the mechanical response of certain materials. This inevitably leads to the following difficulties: (1) the only rigid motions for which $\underset{\sim}{e}$ equals zero are the translations, and (2) the strain $\underset{\sim}{e}$ is not invariant under arbitrary superposed rigid motions of the body $\mathcal{B}$. Indeed, if contrary to (2) the tensor $\underset{\sim}{e}$ were invariant under arbitrary superposed rigid motions, then it would be zero for all rigid motions and this would contradict (1).

To illustrate points (1) and (2), consider the classical linear theory of elastic solids whose constitutive equation relative to a homogeneous unstressed reference configuration $_{o}\underset{\sim}{\kappa}$ can be written as

$$\underset{\sim}{T} = {}_{0}\rho\, \underset{\sim}{K}[\underset{\sim}{e}] \quad , \tag{1.1}$$

where $\underset{\sim}{K}[\underset{\sim}{e}]$ is linear in $\underset{\sim}{e}$, $\underset{\sim}{T}$ is the Cauchy stress tensor, $\underset{\sim}{K}$ is a constant fourth order tensor and ${}_{0}\rho$ is the mass density of the body $\mathcal{B}$ in configuration ${}_{0}\underset{\sim}{\kappa}$. When an arbitrary rigid motion is superposed on $\underset{\sim}{\chi}$, resulting in a motion $\underset{\sim}{\chi}^{+}$, it is generally regarded as a physically acceptable assumption that the stress vector $\underset{\sim}{t}$ (representing surface force per unit current area) be unaltered apart from orientation[†]. As a consequence of this, the Cauchy stress tensor $\underset{\sim}{T}^{+}$ in the motion $\underset{\sim}{\chi}^{+}$ is related to the stress tensor $\underset{\sim}{T}$ in the motion $\underset{\sim}{\chi}$ by

$$\underset{\sim}{T}^{+} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^{T} \quad . \tag{1.2}$$

In (1.2), $\underset{\sim}{Q}$ is a proper orthogonal tensor function of time which corresponds to the rigid rotation in the superposed motion, and $\underset{\sim}{Q}^{T}$ is the transpose of $\underset{\sim}{Q}$. For the special motion in which the body $\mathcal{B}$ remains always in its reference configuration, $\underset{\sim}{e} = \underset{\sim}{0}$ and hence $\underset{\sim}{T} = \underset{\sim}{0}$ by (1.1). Then, if (1.2) were satisfied, $\underset{\sim}{T}$ should equal zero for all rigid motions. However, if we use the definition of infinitesimal strain $\underset{\sim}{e}$ [see Eqs. $(2.8)_1$ and $(2.14)_5$], we see that the value of $\underset{\sim}{e}$ in a rigid motion is

$$\underset{\sim}{e} = \tfrac{1}{2}\{(\underset{\sim}{Q}-\underset{\sim}{I})^{T} + (\underset{\sim}{Q}-\underset{\sim}{I})\} = -\tfrac{1}{2}(\underset{\sim}{Q}-\underset{\sim}{I})^{T}(\underset{\sim}{Q}-\underset{\sim}{I}) \quad . \tag{1.3}$$

Hence, in view of (1.1), in a rigid motion we have

$$\underset{\sim}{T} = -\tfrac{1}{2}\, {}_{0}\rho\, \underset{\sim}{K}[(\underset{\sim}{Q}-\underset{\sim}{I})^{T}(\underset{\sim}{Q}-\underset{\sim}{I})] \quad , \tag{1.4}$$

which contradicts the result noted above. On this account in the classical linear theory of elasticity it is stipulated that only rigid motions which are themselves infinitesimal be allowed to enter the theory, because $\underset{\sim}{e}$ given by (1.3) is approximately zero for such motions; and, in turn, $\underset{\sim}{T}$ given by (1.4)

[†]See Green and Naghdi (1979) for the motivation and precise meaning of this terminology.

is approximately zero also. Thus, we see that the usual method of constructing the infinitesimal theory of elasticity -- which results in the choice of ~ as a strain measure -- forces us to exclude the class of finite rigid motions from the theory, despite the fact that general physical considerations [embodied in Eq. (1.2)] require that $\mathbf{T} = \mathbf{0}$ for <u>all</u> rigid motions.

The discussion in the preceding paragraph pertains to the effect of a purely rigid motion. We now turn to a related examination of the constitutive equation of a linearly elastic solid when a rigid motion is superposed on a given motion. To this end, we observe that even if $\mathbf{e}$ were zero for all rigid motions it would still seem undesirable that $\mathbf{e}$ be altered under arbitrary superposed rigid motions. For, if $\mathbf{e}$ is used as an ingredient of the constitutive equation characterizing the mechanical response of a material, the resulting theory will then predict that the response should in general change when the body is imparted an arbitrary (i.e., not necessarily infinitesimal) superposed rigid motion. It is easy to show that if $\boldsymbol{\chi}^+$ and $\boldsymbol{\chi}$ differ only by a rigid motion then the measure $\mathbf{e}^+$ calculated for the motion $\boldsymbol{\chi}^+$ is related to the measure $\mathbf{e}$ for the motion $\boldsymbol{\chi}$ by the equation

$$2\mathbf{e}^+ = 2\mathbf{e} - (\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I}) + (\mathbf{Q}-\mathbf{I})(\mathbf{F}-\mathbf{I}) + \{(\mathbf{Q}-\mathbf{I})(\mathbf{F}-\mathbf{I})\}^T \quad , \qquad (1.5)$$

where $\mathbf{F}$ is the deformation gradient in the motion $\boldsymbol{\chi}$. Utilizing (1.1) and (1.5) we find that the Cauchy stress in the motion $\boldsymbol{\chi}^+$ may be expressed as

$${}_0\rho\, \mathbf{K}[\mathbf{e}^+] = \mathbf{T} - \tfrac{1}{2}\, {}_0\rho\, \mathbf{K}[(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})] + \tfrac{1}{2}\, {}_0\rho\, \mathbf{K}[(\mathbf{Q}-\mathbf{I})(\mathbf{F}-\mathbf{I}) + \{(\mathbf{Q}-\mathbf{I})(\mathbf{F}-\mathbf{I})\}^T] \quad , \qquad (1.6)$$

with the constant tensor $\mathbf{K}$ remaining unaltered for a given material. However, if we use (1.2) and (1.1), we are led to the relation

$$\begin{aligned} \mathbf{T}^+ &= \mathbf{T} + (\mathbf{Q}-\mathbf{I})\mathbf{T} + \mathbf{T}(\mathbf{Q}-\mathbf{I})^T + (\mathbf{Q}-\mathbf{I})\mathbf{T}(\mathbf{Q}-\mathbf{I})^T \\ &= \mathbf{T} + {}_0\rho\{(\mathbf{Q}-\mathbf{I})\mathbf{K}[\mathbf{e}] + ((\mathbf{Q}-\mathbf{I})\mathbf{K}[\mathbf{e}])^T + (\mathbf{Q}-\mathbf{I})\mathbf{K}[\mathbf{e}](\mathbf{Q}-\mathbf{I})^T\} \quad . \qquad (1.7) \end{aligned}$$

The right-hand sides of (1.6) and (1.7) are in general unequal and consequently $\mathbf{T}^+$ is not given by $\mathbf{K}[\mathbf{e}^+]$ as it ought to be. Furthermore, this behavior will be reflected mathematically in the failure of the differential equations of motion to transform correctly into the equations describing motions which differ from a given motion only by a rigid motion. Of course, if both $(\underset{\sim}{F}-\underset{\sim}{I})$ and $(\underset{\sim}{Q}-\underset{\sim}{I})$ are taken to be negligible in (1.5), (1.6) and (1.7), then these equations yield the approximate relations

$$\underset{\sim}{e}^{+} \approx \underset{\sim}{e} \quad , \quad {}_{o}\rho\, \underset{\sim}{K}[\underset{\sim}{e}^{+}] \approx \underset{\sim}{T} \quad , \quad \underset{\sim}{T}^{+} \approx \underset{\sim}{T} \tag{1.8}$$

and hence the result

$$\underset{\sim}{T}^{+} \approx {}_{o}\rho\, \underset{\sim}{K}[\underset{\sim}{e}^{+}] \quad . \tag{1.9}$$

The latter procedure is the one adopted in the classical theory of (infinitesimal) elasticity. It is then said that the infinitesimal strain tensor is invariant under <u>infinitesimal</u> superposed rigid body motions and that the constitutive equation (1.1) is invariant under infinitesimal superposed rigid body motions. An analysis that parallels the foregoing can be carried out for the usual theory of infinitesimal deformations superposed on a given deformation[§].

1.2 <u>Nature of results for a properly invariant infinitesimal theory</u>

The construction of an invariant infinitesimal theory is effected here by the simple device of first removing from any given motion $\underset{\sim}{\chi}$ the translation and rotation at any one particle Y of $\mathcal{B}$, called a <u>pivot</u> in section 3, arriving thereby [see Eq. (3.2)] at a motion $\underset{\sim}{\chi}^{*}$. The invariant behavior of the finite strain tensor $\underset{\sim}{E}$ implies that the finite strain tensor $\underset{\sim}{E}^{*}$ in the motion $\underset{\sim}{\chi}^{*}$ takes on the same value as $\underset{\sim}{E}$ at corresponding values of their arguments. We next assume that the displacement gradient of $\underset{\sim}{\chi}^{*}$ is small. The linear approximation $\underset{\sim}{e}^{*}$ to $\underset{\sim}{E}^{*}$ is the measure upon which our infinitesimal theory is based. As shown in section 3, the strain tensor $\underset{\sim}{e}^{*}$ is invariant under arbitrary superposed rigid body motions of $\mathcal{B}$ and takes on the value zero for all rigid motions of $\mathcal{B}$.

[§]See Sec. 68 of Truesdell and Noll (1965), where additional references can be found.

Within the framework of the present paper, the constitutive equation of a linearly elastic solid takes the form[†]

$$\underset{\sim}{T} = {}_{o}\rho\, \underset{\sim}{R}(\underset{\sim}{Y},t)\underset{\sim}{K}[\underset{\sim}{e}^{*}]\underset{\sim}{R}^{T}(\underset{\sim}{Y},t) \quad , \tag{1.10}$$

where $\underset{\sim}{R}(\underset{\sim}{Y},t)$ is the rotation tensor at the particle Y at time t. When $\underset{\sim}{\chi}$ is transformed into $\underset{\sim}{\chi}^{+}$ by an arbitrary superposed rigid motion of $\mathcal{B}$, $\underset{\sim}{R}(\underset{\sim}{Y},t) \to \underset{\sim}{R}^{+}(\underset{\sim}{Y},t^{+}) = \underset{\sim}{Q}\ \underset{\sim}{R}(\underset{\sim}{Y},t)$, $\underset{\sim}{e}^{*} \to \underset{\sim}{e}^{+*} = \underset{\sim}{e}^{*}$ and $\underset{\sim}{T}^{+}$ is given by

$$\begin{aligned} \underset{\sim}{T}^{+} &= {}_{o}\rho\, \underset{\sim}{R}^{+}(\underset{\sim}{Y},t^{+})\underset{\sim}{K}^{+}[\underset{\sim}{e}^{+*}]\{\underset{\sim}{R}^{+}(\underset{\sim}{Y},t^{+})\}^{T} \\ &= {}_{o}\rho\, \underset{\sim}{Q}\underset{\sim}{R}(\underset{\sim}{Y},t)\underset{\sim}{K}[\underset{\sim}{e}^{*}]\{\underset{\sim}{Q}\underset{\sim}{R}(\underset{\sim}{Y},t)\}^{T} \\ &= \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^{T} \quad , \end{aligned} \tag{1.11}$$

so that (1.10) is indeed an invariant constitutive equation. Recalling (1.2) we denote the Cauchy stress tensor in the motion $\underset{\sim}{\chi}^{*}$ by

$$\underset{\sim}{T}^{*} = \underset{\sim}{R}^{T}(\underset{\sim}{Y},t)\,\underset{\sim}{T}\underset{\sim}{R}(\underset{\sim}{Y},t) \quad , \tag{1.12}$$

in which case (1.10) takes the appealing form

$$\underset{\sim}{T}^{*} = {}_{o}\rho\, \underset{\sim}{K}[\underset{\sim}{e}^{*}] \quad . \tag{1.13}$$

Happily the last equation is identical in form to (1.1), which allows the infinitesimal theory of section 4 to be brought into direct correspondence with the (classical) linear theory of elasticity by a simple reinterpretation of the symbols employed in the latter theory. We emphasize that, in contrast to (1.1), the constitutive equation (1.13) is invariant under arbitrary superposed rigid motions of $\mathcal{B}$.

The conceptual advantages of an approximate theory based on the strain measure $\underset{\sim}{e}^{*}$ rather than on $\underset{\sim}{e}$ lies in the fact that such a theory is properly

---

[†] We emphasize that for a particular choice of pivot, $\underset{\sim}{Y}$ is fixed in (1.10) and $\underset{\sim}{R}(\underset{\sim}{Y},t)$ varies with time only.

invariant under arbitrary superposed rigid motions, rendering the results physically meaningful, and that the theory includes as possible motions the entire class of rigid motions. In effect, by following the scheme outlined above, we are able to approximate the finite theory while keeping the invariance requirements intact. It is precisely the fact that the invariance requirements are themselves approximated in the classical infinitesimal theories that leads to the shortcomings of these theories mentioned earlier. While the usual method of linearizing a finite theory involves the systematic approximation of every equation of the theory, our results demonstrate that, if meaningful physical results are to be derived, it is essential to distinguish between invariance requirements which are to be kept intact, and the remainder of the finite theory which may be approximated. Indeed, by first removing (from all particles of the body) a rotation and translation from a motion $\underset{\sim}{\chi}$ and then performing approximation, the present method (sections 3 and 4) yields a properly invariant theory.

1.3 Outline of contents and additional remarks

Sections 2 and 3 are concerned, respectively, with some preliminary background material and the construction of an invariant infinitesimal strain measure. Some aspects of these make use of the notions of equivalence relations and equivalence classes, the relevant details of which are elaborated upon in Appendix A of the paper (following section 5). Using the results of section 3, an invariant infinitesimal theory of motions superposed on a given motion is developed in section 4 and begins with two independent motions ${}_1\underset{\sim}{\chi}$ and ${}_2\underset{\sim}{\chi}$ of the same body composed of the same material. Removing from both motions the translation and rotation at any one particle Y of $\mathcal{B}$ while maintaining the same stretch, we arrive at two motions ${}_1\underset{\sim}{\chi}^*$ and ${}_2\underset{\sim}{\chi}^*$. This is followed by introduction of the gradient $\underset{\sim}{H}$ of the relative displacement field ${}_2\underset{\sim}{\chi}^* - {}_1\underset{\sim}{\chi}^*$ taken, at each instant of time, with respect to the configuration occupied by $\mathcal{B}$ in the motion ${}_1\underset{\sim}{\chi}^*$ at that instant. The gradient $\underset{\sim}{H}$ is unaltered when rigid motions are superposed on either ${}_1\underset{\sim}{\chi}$ or ${}_2\underset{\sim}{\chi}$, or both. We next assume that $\underset{\sim}{H}$ is small and we derive a relationship [see Eq. (4.31)] between the Cauchy stress tensors for the motions ${}_1\mathbf{X}^*$ and ${}_2\mathbf{X}^*$ which is properly invariant under arbitrary superposed rigid body motions. This is our main result.

As noted earlier in this section, our method of construction involves the removal from all motions the translation and rotation at any particle Y. In section 5, we examine the nature of the results in section 4 when a pivot Y′ is chosen instead of Y. It is shown that the infinitesimal theory constructed with Y′ as pivot coincides, to within terms of $o(\epsilon^2)$, with that constructed with Y as pivot. The significance of this result is that it does not matter which particle is chosen as pivot.

Before closing this section, it is desirable to comment on a recent paper of Fosdick and Serrin (1979) regarding the impossibility of an exact linear constitutive theory for elastic solids. These authors consider a constitutive equation of the form $\mathbf{T} = \mathbf{\mathcal{A}}[\mathbf{H}]$, where $\mathbf{\mathcal{A}}$ is a constant fourth order tensor and[§] $\mathbf{H} = \mathbf{F} - \mathbf{I}$ is the displacement gradient. In the theories of finite deformation it is required that $\det(\mathbf{H}+\mathbf{I}) > 0$; and, hence, $\mathbf{H}$ is restricted to belong to some proper subset $\mathcal{D}$ of the set of all second order tensors. Under superposed rigid body motions $\mathbf{F}$ obeys (1.2), while $\mathbf{H}$ is transformed into $\mathbf{H}^+ = \mathbf{F}^+ - \mathbf{I} = \mathbf{Q}\mathbf{H} + \mathbf{Q} - \mathbf{I}$. We observe that $\mathbf{H}^+$ also belongs to $\mathcal{D}$ since $\det(\mathbf{H}^+ + \mathbf{I}) = (\det \mathbf{Q})\det(\mathbf{H} + \mathbf{I}) > 0$ for arbitrary proper orthogonal $\mathbf{Q}$. It is readily apparent that the constitutive equation $\mathbf{T} = \mathbf{\mathcal{A}}[\mathbf{H}]$, $\mathbf{H} \in \mathcal{D}$, could not generally be properly invariant under arbitrary superposed rigid body motions. Indeed, setting $\mathbf{H} = \mathbf{0}$, it follows that $\mathbf{\mathcal{A}}[\mathbf{Q}-\mathbf{I}] = \mathbf{0}$ for all proper orthogonal $\mathbf{Q}$. The only $\mathbf{\mathcal{A}}$ for which this is possible is $\mathbf{\mathcal{A}} = \mathbf{0}$.

In their development, Fosdick and Serrin place a stronger a priori restriction on $\mathbf{H}$ that requires $\mathbf{H}$ to belong to some subset $\mathcal{D}'$ of $\mathcal{D}$. If $\mathcal{D}'$ is a proper subset of $\mathcal{D}$, then only those $\mathbf{Q}$ are allowed to appear in the invariance requirements

---

§ Here the notation $\mathbf{H}$ is that of Fosdick and Serrin (1979). In the present paper, the corresponding quantity is denoted by $\mathbf{G}$.

that result in $\underset{\sim}{H}^{+} = \underset{\sim}{Q}\,\underset{\sim}{H} + \underset{\sim}{Q} - \underset{\sim}{I}$ also belonging to $\mathcal{D}'$. To accommodate this restriction on the choice of $\underset{\sim}{Q}$, Fosdick and Serrin (1979) replace the invariance requirements of the general finite theory with restricted invariance requirements which involve some proper subset of the set of proper orthogonal tensors. It may conceivably transpire that the constitutive equation $\underset{\sim}{T} = \underset{\sim}{\mathcal{Q}}[\underset{\sim}{H}]$, with $\underset{\sim}{H}, \underset{\sim}{H}^{+} \in \mathcal{D}'$ is then possible for non-zero $\underset{\sim}{\mathcal{Q}}$. Fosdick and Serrin show that this is in fact not the case and $\underset{\sim}{\mathcal{Q}}$ must still be zero.

In contrast to Fosdick and Serrin (1979), the point of view taken in the present paper is that there are compelling physical grounds for retaining the full set of proper orthogonal tensors in the invariance requirements of any theory of deformable media, including the approximate theories. Thus, the domain of a constitutive response function must be large enough to include all those tensors (e.g., $\underset{\sim}{F}^{+}$) derivable from any tensor in the domain (e.g., $\underset{\sim}{F}$) by a superposed rigid body motion with $\underset{\sim}{Q}$ an arbitrary proper orthogonal tensor. It is then a consequence of the development of the present paper that a linear constitutive equation of the form (1.13) is physically meaningful, in that it satisfies full invariance requirements.

2. General background. Preliminaries and notation

Consider a body $\mathcal{B}$ which, in a fixed reference configuration ${}_{o}\boldsymbol{\kappa}$, occupies a region[†] ${}_{o}\mathcal{R}$ embedded in a three-dimensional Euclidean space $\mathcal{E}$; we denote the boundary of ${}_{o}\mathcal{R}$ by $\partial_{o}\mathcal{R}$. Choosing a fixed origin $\mathcal{O}$ in $\mathcal{E}$, we identify each particle X of $\mathcal{B}$ by the position vector $\mathbf{X}$ of the place it occupies in ${}_{o}\mathcal{R}$. A motion of $\mathcal{B}$ is a mapping $\boldsymbol{\chi}$ which assigns a position vector $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$ to each particle X at each instant t of time $(-\infty < t < \infty)$. In what follows, we need to consider three separate motions of the body; and, for this purpose, we introduce the notation

$$ {}_{\alpha}\mathbf{x} = {}_{\alpha}\boldsymbol{\chi}(\mathbf{X},t) \quad , \quad (\alpha = o,1,2) \quad , \tag{2.1}$$

where ${}_{o}\mathbf{x} = \mathbf{X}$. Statements involving ${}_{\alpha}\boldsymbol{\chi}$ are understood to hold for values o,1,2, but we find it convenient to speak of ${}_{\alpha}\boldsymbol{\chi}$ in the singular. Thus, we say that the image of ${}_{o}\mathcal{R}$ in the motion ${}_{\alpha}\boldsymbol{\chi}$ will be denoted by ${}_{\alpha}\mathcal{R} = {}_{\alpha}\boldsymbol{\chi}({}_{o}\mathcal{R},t)$. The motion ${}_{o}\mathbf{x} = \mathbf{X} = {}_{o}\boldsymbol{\chi}(\mathbf{X},t)$ in which X remains at $\mathbf{X}$ for all t is called the <u>identity motion</u>. We note that ${}_{o}\mathbf{x} = {}_{o}\boldsymbol{\kappa}(X)$ and ${}_{o}\mathcal{R} = {}_{o}\boldsymbol{\kappa}(\mathcal{B})$. In our analysis of motions superposed on a given motion, ${}_{1}\boldsymbol{\chi}$ will represent the given motion and ${}_{2}\boldsymbol{\chi}$ a motion that is close to ${}_{1}\boldsymbol{\chi}$ in a sense to be made precise later. We assume that at each fixed t, the mapping of ${}_{o}\mathcal{R}$ into ${}_{\alpha}\mathcal{R}$ by (2.1) possesses a smooth inverse denoted by ${}_{\alpha}\boldsymbol{\chi}^{-1}$. Under these assumptions, ${}_{\alpha}\mathcal{R}$ is also a region with boundary $\partial_{\alpha}\mathcal{R} = {}_{\alpha}\boldsymbol{\chi}(\partial_{o}\mathcal{R},t)$. Clearly, the identity motion is its own inverse. The current configuration of $\mathcal{B}$ at each fixed t in the motion ${}_{\alpha}\boldsymbol{\chi}$ is the mapping ${}_{\alpha}\boldsymbol{\kappa}$ of $\mathcal{B}$ into $\mathcal{E}$ given by ${}_{\alpha}\boldsymbol{\kappa} = {}_{\alpha}\boldsymbol{\chi} \circ {}_{o}\boldsymbol{\kappa}$, where $\circ$ signifies the composition of mappings[*]. For any subset (or part) $\mathcal{S} \subseteq \mathcal{B}$ of the

---

[†] A region is regarded here as a nonempty connected and compact subset of $\mathcal{E}$ having a piecewise smooth boundary.

[*] It would be more correct if in our notation we distinguished the mapping $\boldsymbol{\chi}$ from the partial mapping $\boldsymbol{\chi}_t$ which at each fixed t takes $\mathbf{X}$ into $\mathbf{x}$. Strictly speaking, the mapping $\boldsymbol{\chi}_t$ has an inverse $\boldsymbol{\chi}_t^{-1}$ while $\boldsymbol{\chi}$ does not. Similarly, our notation ${}_{\alpha}\boldsymbol{\kappa} = {}_{\alpha}\boldsymbol{\chi} \circ {}_{o}\boldsymbol{\kappa}$ stands for the relation ${}_{\alpha}\boldsymbol{\kappa}_t = {}_{\alpha}\boldsymbol{\chi}_t \circ {}_{o}\boldsymbol{\kappa}$ involving the partial mappings ${}_{\alpha}\boldsymbol{\kappa}_t$ and ${}_{\alpha}\boldsymbol{\chi}_t$. However, such a notational distinction leads to undue clumsiness in later equations of the present paper.

body, we write ${}_o\mathcal{R} = {}_o\kappa(\mathcal{B})$, ${}_\alpha\mathcal{R} = {}_\alpha\chi({}_o\mathcal{R},t)$ and $\partial_\alpha\mathcal{R} = {}_\alpha\chi(\partial_o\mathcal{R},t)$, where $\partial_o\mathcal{R}$ is the boundary of the region ${}_o\mathcal{R}$ and $\partial_\alpha\mathcal{R}$ that of ${}_\alpha\mathcal{R}$.

2.1 Notation and terminology

Before continuing with the kinematics, we mention some mathematical terminology that will be needed in what follows. Any linear mapping from V, the three dimensional translation vector space associated with the point space $\mathcal{E}$, into V will be called a second order tensor. The trace and determinant functions are denoted respectively by tr and det. The transpose of a second order tensor $\underset{\sim}{A}$ will be denoted $\underset{\sim}{A}^T$, while the inverse of $\underset{\sim}{A}$ if it exists will be denoted by $\underset{\sim}{A}^{-1}$. The usual inner product on V is written $\underset{\sim}{a}\cdot\underset{\sim}{b}$ for any two vectors $\underset{\sim}{a},\underset{\sim}{b}\in V$ and the (induced) norm, or magnitude, of $\underset{\sim}{a}$ is given by $\|\underset{\sim}{a}\| = \sqrt{\underset{\sim}{a}\cdot\underset{\sim}{a}}$. The set of second order tensors can be provided with an inner product $\underset{\sim}{A}\cdot\underset{\sim}{B} = \mathrm{tr}(\underset{\sim}{A}^T\underset{\sim}{B})$ and a norm $\|\underset{\sim}{A}\| = \sqrt{\underset{\sim}{A}\cdot\underset{\sim}{A}}$ for any second order tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$. The tensor product $\underset{\sim}{a}\otimes\underset{\sim}{b}$ of any two vectors $\underset{\sim}{a},\underset{\sim}{b}\in V$ is the second order tensor defined by $(\underset{\sim}{a}\otimes\underset{\sim}{b})\underset{\sim}{u} = \underset{\sim}{b}\cdot\underset{\sim}{u}\,\underset{\sim}{a}$ for every vector $\underset{\sim}{u}$. We recall the formulae $\mathrm{tr}(\underset{\sim}{a}\otimes\underset{\sim}{b}) = \underset{\sim}{a}\cdot\underset{\sim}{b}$, $(\underset{\sim}{a}\otimes\underset{\sim}{b})^T = \underset{\sim}{b}\otimes\underset{\sim}{a}$ and $(\underset{\sim}{a}\otimes\underset{\sim}{b})(\underset{\sim}{c}\otimes\underset{\sim}{d}) = \underset{\sim}{b}\cdot\underset{\sim}{c}\,\underset{\sim}{a}\otimes\underset{\sim}{d} = (\underset{\sim}{a}\otimes\underset{\sim}{c})(\underset{\sim}{b}\otimes\underset{\sim}{d})$, $(\underset{\sim}{a}\otimes\underset{\sim}{b})\cdot(\underset{\sim}{c}\otimes\underset{\sim}{d}) = \underset{\sim}{a}\cdot\underset{\sim}{c}\;\underset{\sim}{b}\cdot\underset{\sim}{d}$, which hold for all vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{d}$ in V. The convention of summation over a repeated Latin index will be employed, but summation will not be performed over a repeated Greek index.

In order to express certain expressions in component form, it is convenient to employ two fixed right-handed orthonormal bases $\{\underset{\sim}{e}_A\}$ and $\{\underset{\sim}{e}_i\}$ in V, the former basis being used for vector fields defined on the region ${}_o\mathcal{R}$ and the latter for vector fields defined on other regions. Thus, for example, we write $\underset{\sim}{X} = X_A\underset{\sim}{e}_A$ and ${}_\alpha\underset{\sim}{x} = {}_\alpha x_i\,\underset{\sim}{e}_i$ $(\alpha = 1,2)$. Furthermore, a second order tensor $\underset{\sim}{A}$ may be represented by $A_{ij}\,\underset{\sim}{e}_i\otimes\underset{\sim}{e}_j$, $A_{iM}\,\underset{\sim}{e}_i\otimes\underset{\sim}{e}_M$ or $A_{MN}\,\underset{\sim}{e}_M\otimes\underset{\sim}{e}_N$ as appropriate, where $A_{ij} = \underset{\sim}{e}_i\cdot\underset{\sim}{A}\,\underset{\sim}{e}_j = \underset{\sim}{A}\cdot(\underset{\sim}{e}_i\otimes\underset{\sim}{e}_j)$, etc. Any linear mapping from the set of second order tensors into itself is a fourth order tensor. In particular, the tensor product $\underset{\sim}{a}\otimes\underset{\sim}{b}\otimes\underset{\sim}{c}\otimes\underset{\sim}{d}$

of any four vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{d} \in V$ is a fourth order tensor. It is useful to define an inner product of the fourth order tensor $\underset{\sim}{a}\otimes\underset{\sim}{b}\otimes\underset{\sim}{c}\otimes\underset{\sim}{d}$ and the second order tensor $\underset{\sim}{u}\otimes\underset{\sim}{v}$, $\underset{\sim}{u},\underset{\sim}{v} \in V$, by $(\underset{\sim}{a}\otimes\underset{\sim}{b}\otimes\underset{\sim}{c}\otimes\underset{\sim}{d})[\underset{\sim}{u}\otimes\underset{\sim}{v}] = \underset{\sim}{c}\cdot\underset{\sim}{u}\ \underset{\sim}{d}\cdot\underset{\sim}{v}\ \underset{\sim}{a}\otimes\underset{\sim}{b}$, which yields a second order tensor. Any fourth order tensor $\underset{\sim}{\mathcal{a}}$ may be represented as $\underset{\sim}{\mathcal{a}} = \mathcal{a}_{ijk\ell}\ \underset{\sim}{e}_i\otimes\underset{\sim}{e}_j\otimes\underset{\sim}{e}_k\otimes\underset{\sim}{e}_\ell = \mathcal{a}_{KLMN}\ \underset{\sim}{e}_K\otimes\underset{\sim}{e}_L\otimes\underset{\sim}{e}_M\otimes\underset{\sim}{e}_N = \mathcal{a}_{ijKL}\ \underset{\sim}{e}_i\otimes\underset{\sim}{e}_j\otimes\underset{\sim}{e}_K\otimes\underset{\sim}{e}_L$, etc., where, for example, $\mathcal{a}_{ijk\ell} = \underset{\sim}{e}_i\cdot\underset{\sim}{\mathcal{a}}[\underset{\sim}{e}_k\otimes\underset{\sim}{e}_\ell]\underset{\sim}{e}_j = (\underset{\sim}{e}_i\otimes\underset{\sim}{e}_j)\cdot\underset{\sim}{\mathcal{a}}[\underset{\sim}{e}_k\otimes\underset{\sim}{e}_\ell]$. The transpose $\underset{\sim}{\mathcal{a}}^T$ of a fourth order tensor $\underset{\sim}{\mathcal{a}}$ is that unique fourth order tensor with the property that $\underset{\sim}{B}\cdot\underset{\sim}{\mathcal{a}}[\underset{\sim}{A}] = \underset{\sim}{A}\cdot\underset{\sim}{\mathcal{a}}^T[\underset{\sim}{B}]$ for all second order tensors $\underset{\sim}{A},\underset{\sim}{B}$. Clearly, $\underset{\sim}{\mathcal{a}}^T = (\mathcal{a}_{ijk\ell}\ \underset{\sim}{e}_i\otimes\underset{\sim}{e}_j\otimes\underset{\sim}{e}_k\otimes\underset{\sim}{e}_\ell)^T = \mathcal{a}_{k\ell ij}\ \underset{\sim}{e}_i\otimes\underset{\sim}{e}_j\otimes\underset{\sim}{e}_k\otimes\underset{\sim}{e}_\ell$.

2.2 Kinematical and kinetical results associated with the motions (2.1)

Having disposed of the foregoing notational preliminaries, we return to further consideration of kinematics. The deformation gradient ${}_\alpha\underset{\sim}{F}$, associated with the motion ${}_\alpha\underset{\sim}{X}$, relative to $\underset{\sim}{X}$ is defined by

$$ {}_\alpha\underset{\sim}{F} = \frac{\partial\,{}_\alpha\underset{\sim}{X}}{\partial\underset{\sim}{X}}(\underset{\sim}{X},t) \quad , \quad {}_\alpha J = \det\,({}_\alpha\underset{\sim}{F}) > 0 \quad . \tag{2.2} $$

For the identity motion ${}_o\underset{\sim}{X}$, ${}_o\underset{\sim}{F} = \underset{\sim}{I}$ and ${}_oJ = 1$. Being invertible, ${}_\alpha F$ possesses a unique polar decomposition in the form

$$ {}_\alpha\underset{\sim}{F} = {}_\alpha\underset{\sim}{R}\ {}_\alpha\underset{\sim}{U} \quad , \tag{2.3} $$

where the (local) rotation ${}_\alpha\underset{\sim}{R}$ is a proper orthogonal second order tensor and the right stretch ${}_\alpha\mathbf{U}$ is a symmetric positive-definite second order tensor. Also, the right Cauchy-Green measure of deformation ${}_\alpha\mathbf{C}$ and the Lagrangian finite strain tensor ${}_\alpha\mathbf{E}$ are given by

$$ {}_\alpha\mathbf{C} = {}_\alpha\mathbf{U}^2 = {}_\alpha\mathbf{F}^T\,{}_\alpha\mathbf{F} \quad , \quad {}_\alpha\mathbf{E} = \tfrac{1}{2}({}_\alpha\mathbf{C} - \mathbf{I}) \quad . \tag{2.4} $$

We note that ${}_o\mathbf{R} = {}_o\mathbf{U} = {}_o\mathbf{C} = \mathbf{I}$ and ${}_o\mathbf{E} = \mathbf{0}$. The relative displacement field associated with the motion ${}_\alpha\boldsymbol{\chi}$ is the mapping ${}_\alpha\boldsymbol{\chi} - {}_o\boldsymbol{\chi}$ with the values

$$ {}_\alpha\mathbf{u} = ({}_\alpha\boldsymbol{\chi} - {}_o\boldsymbol{\chi})(\mathbf{X},t) = {}_\alpha\mathbf{x} - \mathbf{X} \tag{2.5} $$

and its gradient, namely

$$ {}_\alpha\mathbf{G} = \frac{\partial({}_\alpha\boldsymbol{\chi} - {}_o\boldsymbol{\chi})}{\partial\mathbf{X}}(\mathbf{X},t) = {}_\alpha\mathbf{F} - \mathbf{I} \quad , \tag{2.6} $$

is called the displacement gradient. In the case of the identity motion, we have ${}_o\mathbf{u} = \mathbf{o}$ and ${}_o\mathbf{G} = \mathbf{0}$.

A motion ${}_\alpha\boldsymbol{\chi}^+$ *of* $\mathcal{B}$ *is said to differ from* ${}_\alpha\boldsymbol{\chi}$ by a superposed rigid body motion (or simply by a rigid motion) if and only if

$$ {}_\alpha\boldsymbol{\chi}^+(\mathbf{X},{}_\alpha t^+) = {}_\alpha\mathbf{Q}(t)\,{}_\alpha\boldsymbol{\chi}(\mathbf{X},t) + {}_\alpha\mathbf{a}(t) \quad , \quad {}_\alpha t^+ = t + {}_\alpha a \tag{2.7} $$

for some proper orthogonal second order tensor-valued function ${}_\alpha\mathbf{Q}(t)$ of time, some vector-valued function ${}_\alpha\mathbf{a}(t)$ of time, and some real constant ${}_\alpha a$. The configuration of $\mathcal{B}$, at time ${}_\alpha t^+$, in the motion ${}_\alpha\boldsymbol{\chi}^+$ is ${}_\alpha\boldsymbol{\kappa}^+ = {}_\alpha\boldsymbol{\chi}^+ \circ {}_o\boldsymbol{\kappa}$. The class of rigid motions of $\mathcal{B}$ consists of those motions ${}_o\boldsymbol{\chi}^+$ which differ from the identity motion ${}_o\boldsymbol{\chi}$ by a rigid motion, being given by (2.7) with $\alpha = o$. A translation is a rigid motion whose rotation is $\mathbf{I}$.

It is demonstrated in Appendix A that the statement "differs by a rigid motion" is an equivalence relation on the set $\mathfrak{M}$ of all motions of $\mathcal{B}$. This allows $\mathfrak{M}$ to be partitioned into disjoint subsets (equivalence classes) each of which comprises all motions of $\mathcal{B}$ and only those, which differ from one another by a rigid motion. Thus, each equivalence class comprises those motions of $\mathcal{B}$ which are regarded as mechanically indistinguishable. The equivalence class which contains all motions that differ from a given motion $\underset{\sim}{\chi}$ by a rigid motion is denoted by $K(\underset{\sim}{\chi})$. For example, the equivalence class $K({}_{o}\underset{\sim}{\chi})$ contains the entire set of rigid motions of $\mathcal{B}$.

We also recall from Appendix A that an equivalence class is determined by any one of its members. If, instead of a motion $\underset{\sim}{\chi}$, we begin with a motion $\underset{\sim}{\theta}$ and place all the members of $\mathfrak{m}$ that are equivalent to $\underset{\sim}{\theta}$ in a class $K(\underset{\sim}{\theta})$, we find that $K(\underset{\sim}{\theta}) = K(\underset{\sim}{\chi})$.

We may regard (2.7) as defining a function $\underset{\sim}{\omega}$ taking $\mathfrak{m}$ into $\mathfrak{m}$ such that for fixed values of ${}_{\alpha}\underset{\sim}{Q}$, ${}_{\alpha}\underset{\sim}{a}$ and ${}_{\alpha}a$ in (2.7) ${}_{\alpha}\underset{\sim}{\chi}^{+} = \underset{\sim}{\omega}({}_{\alpha}\underset{\sim}{\chi})$.

The symmetric and skew-symmetric parts of ${}_{\alpha}\underset{\sim}{G}$ are defined by

$$ {}_{\alpha}\underset{\sim}{e} = \tfrac{1}{2}({}_{\alpha}\underset{\sim}{G} + {}_{\alpha}\underset{\sim}{G}^{T}) \quad , \quad {}_{\alpha}\underset{\sim}{w} = \tfrac{1}{2}({}_{\alpha}\underset{\sim}{G} - {}_{\alpha}\underset{\sim}{G}^{T}) \quad , \tag{2.8} $$

respectively, and ${}_{o}\underset{\sim}{e} = {}_{o}\underset{\sim}{w} = \underset{\sim}{0}$ since ${}_{o}\underset{\sim}{G} = \underset{\sim}{0}$. It follows from (2.4), (2.6) and (2.8) that

$$ {}_{\alpha}\underset{\sim}{E} = {}_{\alpha}\underset{\sim}{e} + \tfrac{1}{2}\, {}_{\alpha}\underset{\sim}{G}^{T}\, {}_{\alpha}\underset{\sim}{G} \quad . \tag{2.9} $$

Recalling that $\mathrm{tr}({}_{\alpha}\underset{\sim}{G}^{T}\, {}_{\alpha}\underset{\sim}{G}) = \|{}_{\alpha}\underset{\sim}{G}\|^{2}$ which equals zero if and only if ${}_{\alpha}\underset{\sim}{G} = 0$, it is clear from (2.9) and (2.6) that ${}_{\alpha}\underset{\sim}{E} = {}_{\alpha}\underset{\sim}{e}$ if and only if ${}_{\alpha}\underset{\sim}{F} = \underset{\sim}{I}$ and hence if and only if ${}_{\alpha}\underset{\sim}{\chi}$ is a translation.

From the deformation gradient ${}_{\alpha}\underset{\sim}{F}^{+} = \dfrac{\partial\, {}_{\alpha}\underset{\sim}{\chi}^{+}}{\partial \underset{\sim}{X}}(\underset{\sim}{X}, {}_{\alpha}t^{+})$ of ${}_{\alpha}\underset{\sim}{\chi}^{+}$ in $(2.7)_1$ and (2.2), we readily obtain

$$ {}_{\alpha}\underset{\sim}{F}^{+} = {}_{\alpha}\underset{\sim}{Q}(t)\, {}_{\alpha}\underset{\sim}{F} \quad , \quad {}_{\alpha}J^{+} = \det({}_{\alpha}\underset{\sim}{F}^{+}) = {}_{\alpha}J > 0 \quad . \tag{2.10} $$

Then, using $\underset{\alpha\sim}{F}^{+}$ to define, as in (2.3) and (2.4), the tensors $\underset{\alpha\sim}{R}^{+}, \underset{\alpha\sim}{U}^{+}, \underset{\alpha\sim}{C}^{+}$ and $\underset{\alpha\sim}{E}^{+}$, it follows at once that

$$\underset{\alpha\sim}{U}^{+} = \underset{\alpha\sim}{U} \quad , \quad \underset{\alpha\sim}{C}^{+} = \underset{\alpha\sim}{C} \quad , \quad \underset{\alpha\sim}{E}^{+} = \underset{\alpha\sim}{E} \quad , \quad \underset{\alpha\sim}{R}^{+} = \underset{\alpha\sim}{Q}(t)\ \underset{\alpha\sim}{R} \quad . \tag{2.11}$$

The relative displacement field associated with $\underset{\alpha\sim}{\chi}^{+}$ is $\underset{\alpha\sim}{u}^{+} = (\underset{\alpha\sim}{\chi}^{+} - \underset{o\sim}{\chi})(\underset{\sim}{X}, {}_{\alpha}t^{+})$ and its gradient is $\underset{\alpha\sim}{G}^{+} = \underset{\alpha\sim}{F}^{+} - \underset{\sim}{I}$. Hence, in view of $(2.10)_1$ and (2.6),

$$\underset{\alpha\sim}{G}^{+} = \underset{\alpha\sim}{Q}(t)\ \underset{\alpha\sim}{G} + \underset{\alpha\sim}{Q}(t) - \underset{\sim}{I} \quad , \tag{2.12}$$

so that $\underset{\alpha\sim}{G}^{+}$ is neither unaltered, nor unaltered apart from orientation under all superposed rigid body motions of $\mathcal{B}$. Similarly, the symmetric and skew-symmetric parts of $\underset{\alpha\sim}{G}^{+}$, i.e., $\underset{\alpha\sim}{e}^{+}$ and $\underset{\alpha\sim}{w}^{+}$ are related to $\underset{\alpha\sim}{e}$ and $\underset{\alpha\sim}{w}$ in (2.8) by

$$\begin{aligned} 2\ \underset{\alpha\sim}{e}^{+} &= 2\ \underset{\alpha\sim}{e} - \{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\}^T\{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\} + \{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\}\underset{\alpha\sim}{G} + (\{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\}\underset{\alpha\sim}{G})^T \quad , \\ 2\ \underset{\alpha\sim}{w}^{+} &= 2\ \underset{\alpha\sim}{w} + \underset{\alpha\sim}{Q}(t) - \underset{\alpha\sim}{Q}^T(t) + \{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\}\underset{\alpha\sim}{G} - (\{\underset{\alpha\sim}{Q}(t) - \underset{\sim}{I}\}\underset{\alpha\sim}{G})^T \quad , \end{aligned} \tag{2.13}$$

where use has been made of the identity $(\underset{\sim}{Q} - \underset{\sim}{I})^T + (\underset{\sim}{Q} - \underset{\sim}{I}) = -(\underset{\sim}{Q} - \underset{\sim}{I})^T(\underset{\sim}{Q} - \underset{\sim}{I})$, for any orthogonal tensor $\underset{\sim}{Q}$.

Since in the identity motion $\underset{o\sim}{\chi}$, $\underset{o\sim}{F} = \underset{o\sim}{R} = \underset{o\sim}{U} = \underset{o\sim}{C} = \underset{\sim}{I}$ while $\underset{o\sim}{E} = \underset{o\sim}{G} = \underset{o\sim}{e} = \underset{o\sim}{w} = \underset{\sim}{0}$, it follows from (2.7), (2.10), (2.11), (2.12) and (2.13) that in any rigid motion, denoted for convenience by $\underset{o\sim}{\chi}^{+}$, we have $\underset{o\sim}{\chi}^{+}(\underset{\sim}{X}, {}_{o}t^{+}) = \underset{o\sim}{Q}(t)\,\underset{o\sim}{\chi}(\underset{\sim}{X}, t) + \underset{o\sim}{a}(t)$ and

$$\begin{aligned} \underset{o\sim}{F}^{+} &= \underset{o\sim}{R}^{+} = \underset{o\sim}{Q}(t) \quad , \\ \underset{o\sim}{C}^{+} = \underset{o\sim}{U}^{+} = \underset{\sim}{I} \quad , \quad & \underset{o\sim}{E}^{+} = \underset{\sim}{0} \quad , \quad \underset{o\sim}{G}^{+} = \underset{o\sim}{Q}(t) - \underset{\sim}{I} \quad , \\ \underset{o\sim}{e}^{+} &= -\tfrac{1}{2}\{\underset{o\sim}{Q}(t) - \underset{\sim}{I}\}^T\{\underset{o\sim}{Q}(t) - \underset{\sim}{I}\} \quad , \\ \underset{o\sim}{w}^{+} &= \tfrac{1}{2}\{\underset{o\sim}{Q}(t) - \underset{o\sim}{Q}^T(t)\} \quad . \end{aligned} \tag{2.14}$$

Clearly, by $(2.14)_5$, ${}_{\alpha}\underset{\sim}{e}^{+} = \underset{\sim}{0}$ if and only if ${}_{0}\underset{\sim}{Q}(t) = \underset{\sim}{I}$. Hence, the only rigid motions for which the tensor ${}_{0}\underset{\sim}{e}^{+}$ vanishes are the translations. It follows from this that ${}_{\alpha}\underset{\sim}{e}$ is not invariant under arbitrary superposed rigid body motions of $\mathcal{B}$. For, if it were invariant, then it would equal zero for <u>all</u> rigid motions of $\mathcal{B}$.

In the language of equivalence classes, if ${}_{\alpha}\underset{\sim}{\chi}$ and ${}_{\alpha}\underset{\sim}{\chi}^{+}$ belong to the same equivalence class, then $(2.11)_3$ holds. As is well known, the converse is also true. The finite strain tensor ${}_{\alpha}\underset{\sim}{E}$ can therefore be used to characterize the strain of all motions in the equivalence class $K({}_{\alpha}\underset{\sim}{\chi})$. In particular ${}_{\alpha}\underset{\sim}{E} = \underset{\sim}{0}$ for all rigid motions, i.e., for the equivalence class $K({}_{0}\underset{\sim}{\chi})$. In contrast, we have just seen that ${}_{\alpha}\underset{\sim}{e}$ does not give the same value for all motions in $K({}_{\alpha}\underset{\sim}{\chi})$; in particular, ${}_{\alpha}\underset{\sim}{e}$ is not zero for all motions in $K({}_{0}\underset{\sim}{\chi})$ but only for the translations.

Let ${}_{\alpha}\rho$ be the mass density in the configuration ${}_{\alpha}\underset{\sim}{\kappa}$, ${}_{\alpha}\underset{\sim}{b}$ the body force field per unit mass in the configuration ${}_{\alpha}\underset{\sim}{\kappa}$, ${}_{\alpha}\underset{\sim}{n}$ the outward unit normal to the surface $\partial\, {}_{\alpha}\mathcal{P}$, ${}_{\alpha}\underset{\sim}{t}$ the stress vector acting on this surface and ${}_{\alpha}\underset{\sim}{T}$ the associated Cauchy stress tensor. Then, in any motion ${}_{\alpha}\underset{\sim}{\chi}$, from conservation laws for mass, linear and angular momentum follow the results

$$
\begin{gathered}
{}_{0}\rho = {}_{\alpha}\rho \; {}_{\alpha}J \;, \\
{}_{\alpha}\underset{\sim}{t} = {}_{\alpha}\underset{\sim}{T} \; {}_{\alpha}\underset{\sim}{n} \;, \quad {}_{\alpha}\underset{\sim}{T}^{T} = {}_{\alpha}\underset{\sim}{T} \;, \\
{}_{\alpha}\mathrm{div}\; {}_{\alpha}\underset{\sim}{T} + {}_{\alpha}\rho \; {}_{\alpha}\underset{\sim}{b} = {}_{\alpha}\rho \; {}_{\alpha}\dot{\underset{\sim}{v}} \;.
\end{gathered}
\tag{2.15}
$$

In (2.15), ${}_{\alpha}\mathrm{div}$ is the (right) divergence operator with respect to ${}_{\alpha}\underset{\sim}{x}$, having a component representation

$$
{}_{\alpha}\mathrm{div}\; {}_{\alpha}\underset{\sim}{T} = \frac{\partial}{\partial\, {}_{\alpha}x_k} \left({}_{\alpha}T_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \right) \underset{\sim}{e}_k = \frac{\partial\, {}_{\alpha}T_{ij}}{\partial\, {}_{\alpha}x_j} \underset{\sim}{e}_i \tag{2.16}
$$

and ${}_{\alpha}\mathbf{v}$ is the particle velocity in the motion ${}_{\alpha}\boldsymbol{\chi}$ and is given by

$$ {}_{\alpha}\mathbf{v} = {}_{\alpha}\dot{\mathbf{x}} = \frac{\partial\, {}_{\alpha}\boldsymbol{\chi}}{\partial t}(\mathbf{X}, t) \quad , \tag{2.17} $$

with a superposed dot signifying material time differentiation. For $\alpha = 0$, ${}_{0}\dot{\mathbf{x}} = {}_{0}\dot{\mathbf{v}} = \mathbf{0}$. The unit normal ${}_{0}\mathbf{n}$ is carried by the motion ${}_{\alpha}\boldsymbol{\chi}$ into ${}_{\alpha}\mathbf{n}$ in accordance with the formula

$$ {}_{\alpha}\mathbf{n} = \frac{({}_{\alpha}\mathbf{F}^{-1})^{T}\, {}_{0}\mathbf{n}}{\|({}_{\alpha}\mathbf{F}^{-1})^{T}\, {}_{0}\mathbf{n}\|} \quad , \tag{2.18} $$

the inverse of which is

$$ {}_{0}\mathbf{n} = \frac{{}_{\alpha}\mathbf{F}^{T}\, {}_{\alpha}\mathbf{n}}{\|{}_{\alpha}\mathbf{F}^{T}\, {}_{\alpha}\mathbf{n}\|} \quad . \tag{2.19} $$

Denoting the mass density in the configuration ${}_{\alpha}\boldsymbol{\kappa}^{+}$ by ${}_{\alpha}\rho^{+}$ and applying $(2.15)_1$ to the motion ${}_{\alpha}\boldsymbol{\chi}^{+}$, we obtain

$$ {}_{0}\rho = {}_{\alpha}\rho^{+}\, {}_{\alpha}J^{+} \tag{2.20} $$

and hence, in view of $(2.10)_2$ and $(2.15)_1$, we have

$$ {}_{\alpha}\rho^{+} = {}_{\alpha}\rho \quad . \tag{2.21} $$

It follows from (2.18), (2.19) and $(2.10)_1$ that under the transformations (2.7), ${}_{\alpha}\mathbf{n}$ is carried into ${}_{\alpha}\mathbf{n}^{+}$, the outward unit normal to $\partial\, {}_{\alpha}\mathcal{P}^{+}$, with

$$ {}_{\alpha}\mathbf{n}^{+} = {}_{\alpha}\mathbf{Q}(t)\, {}_{\alpha}\mathbf{n} \quad . \tag{2.22} $$

We adopt the usual assumption that the stress vector ${}_{\alpha}\mathbf{t}^{+}$ for the motion ${}_{\alpha}\boldsymbol{\chi}^{+}$ is related to ${}_{\alpha}\mathbf{t}$ by

$$ {}_{\alpha}\mathbf{t}^{+} = {}_{\alpha}\mathbf{Q}(t)\, {}_{\alpha}\mathbf{t} \tag{2.23} $$

and it then follows with the aid of $(2.15)_2$ and (2.22) that the Cauchy stress tensor ${}_{\alpha}\underset{\sim}{T}^{+}$ in the motion ${}_{\alpha}\underset{\sim}{\chi}^{+}$ is related to ${}_{\alpha}\underset{\sim}{T}$ by

$$ {}_{\alpha}\underset{\sim}{T}^{+} = {}_{\alpha}\underset{\sim}{Q}(t)\ {}_{\alpha}\underset{\sim}{T}\ {}_{\alpha}\underset{\sim}{Q}^{T}(t) \quad . \tag{2.24} $$

The balance of linear momentum in the motion ${}_{\alpha}\underset{\sim}{\chi}^{+}$ is written as

$$ {}_{\alpha}\mathrm{div}^{+}\ {}_{\alpha}\underset{\sim}{T}^{+} + {}_{\alpha}\rho^{+}\ {}_{\alpha}\underset{\sim}{b}^{+} = {}_{\alpha}\rho^{+}\ {}_{\alpha}\underset{\sim}{\dot{v}}^{+} \quad , \tag{2.25} $$

where

$$ \begin{aligned} {}_{\alpha}\mathrm{div}^{+}\ {}_{\alpha}\underset{\sim}{T}^{+} &= \frac{\partial\ {}_{\alpha}T^{+}_{ij}}{\partial\ {}_{\alpha}x^{+}_{j}}\ \underset{\sim}{e}_{i} \quad , \\ {}_{\alpha}\underset{\sim}{v}^{+} &= \frac{\partial\ {}_{\alpha}\underset{\sim}{\chi}^{+}}{\partial\ {}_{\alpha}t^{+}}\ (\underset{\sim}{X}, {}_{\alpha}t^{+}) \quad . \end{aligned} \tag{2.26} $$

For later reference, we note that by $(2.7)_1$, (2.16), $(2.26)_1$, (2.21), $(2.15)_4$ and (2.25),

$$ \begin{aligned} {}_{\alpha}\mathrm{div}^{+}\ {}_{\alpha}\underset{\sim}{T}^{+} &= {}_{\alpha}\underset{\sim}{Q}(t)\ {}_{\alpha}\mathrm{div}\ {}_{\alpha}\underset{\sim}{T} \quad , \\ {}_{\alpha}\underset{\sim}{\dot{v}}^{+} - {}_{\alpha}\underset{\sim}{b}^{+} &= {}_{\alpha}\underset{\sim}{Q}(t)({}_{\alpha}\underset{\sim}{\dot{v}} - {}_{\alpha}\underset{\sim}{b}) \quad . \end{aligned} \tag{2.27} $$

2.3 Classical infinitesimal deformation

Having disposed of the above preliminaries, in the remainder of this section we discuss the main ingredients of the usual method of constructing infinitesimal theories. The theory of infinitesimal elasticity is derived from the finite theory by setting* ${}_{1}\underset{\sim}{\chi} = {}_{2}\underset{\sim}{\chi} = \underset{\sim}{\chi}$ and introducing as a measure of smallness the nonnegative real function†

$$ \epsilon = \epsilon(t) = \sup_{\underset{\sim}{X} \in {}_{0}\mathcal{R}} \|\underset{\sim}{G}(\underset{\sim}{X},t)\| \quad , \tag{2.28} $$

*In the infinitesimal theory it suffices to consider two separate motions $\underset{\sim}{\chi}$ and ${}_{0}\underset{\sim}{\chi}$. Accordingly, in this case, we drop the subscripts 1,2 from quantities associated with $\underset{\sim}{\chi}$.

†The smoothness of $\underset{\sim}{\chi}$ and the compactness of ${}_{0}\mathcal{R}$ ensure the existence of $\epsilon(t)$.

where sup stands for the supremum (or least upper bound) of a nonempty bounded set of real numbers. If $\underset{\sim}{Z}(\underset{\sim}{G})$ is any vector- or tensor-valued function of $\underset{\sim}{G}$ defined in a neighborhood of $\underset{\sim}{G} = \underset{\sim}{O}$ and satisfying the condition that there exists a nonnegative real constant C such that $\|\underset{\sim}{Z}(\underset{\sim}{G})\| < C\epsilon^n$ as $\epsilon \to 0$, then we write $\underset{\sim}{Z} = \underset{\sim}{O}(\epsilon^n)$ as $\epsilon \to 0$

Before proceeding further, we recall the following well-known results:

$$
\begin{aligned}
&(a)\quad \underset{\sim}{F} - \underset{\sim}{I} = \underset{\sim}{G} = \underset{\sim}{O}(\epsilon) \quad , \quad (b)\quad \underset{\sim}{F}^{-1} - \underset{\sim}{I} = -\underset{\sim}{G} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \\
&(c)\quad \underset{\sim}{U} - \underset{\sim}{I} = \underset{\sim}{e} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \quad (d)\quad \underset{\sim}{C} - \underset{\sim}{I} = 2\underset{\sim}{e} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \\
&(e)\quad \underset{\sim}{E} = \underset{\sim}{e} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \quad (f)\quad \underset{\sim}{U}^{-1} - \underset{\sim}{I} = -\underset{\sim}{e} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \\
&(g)\quad \underset{\sim}{R} - \underset{\sim}{I} = \underset{\sim}{w} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad , \quad (h)\quad \underset{\sim}{R}^T - \underset{\sim}{I} = -\underset{\sim}{w} + \underset{\sim}{O}(\epsilon^2) = \underset{\sim}{O}(\epsilon) \quad ,
\end{aligned}
\tag{2.29}
$$

as $\epsilon \to 0$. In view of (2.29e) and (2.29g), $\underset{\sim}{e}$ and $\underset{\sim}{w}$ are referred to as the infinitesimal strain tensor and infinitesimal rotation tensor, respectively. For sufficiently small values of ϵ, $\underset{\sim}{e}$ approximates the finite strain $\underset{\sim}{E}$ and $\underset{\sim}{I} + \underset{\sim}{w}$ approximates the finite rotation $\underset{\sim}{R}$.

Again in the notation of Appendix A, a motion $\underset{\sim}{\theta} \in \mathfrak{m}$ is said to differ from a motion $\underset{\sim}{\chi}$ by an infinitesimal rigid motion if and only if

$$\underset{\sim}{\theta}(\underset{\sim}{X},\tau) = \{\underset{\sim}{I} + \underset{\sim}{W}(t)\}\underset{\sim}{\chi}(\underset{\sim}{X},t) + \underset{\sim}{d}(t) \quad , \quad \tau = t + d \tag{2.30}$$

for some skew-symmetric tensor-valued function $\underset{\sim}{W}(t)$ of time, some vector-valued function $\underset{\sim}{d}(t)$ of time and some real constant d. It is shown in Appendix A that the statement "differs by an infinitesimal rigid motion" is <u>not</u> an equivalence relation on $\mathfrak{m}$. It is further shown that if a motion $\underset{\sim}{\theta}$ differs from a motion $\underset{\sim}{\chi}$ by a rigid motion, as well as by an infinitesimal rigid motion, then $\underset{\sim}{\theta}$ must differ from $\underset{\sim}{\chi}$ by a translation in which case $\underset{\sim}{W} = \underset{\sim}{O}$

in (2.30) and $\underset{\sim}{Q} = \underset{\sim}{I}$ in (2.7). An infinitesimal rigid motion is a motion that differs from the identity motion ${}_{o}\underset{\sim}{X}$ by an infinitesimal rigid motion; and, hence, is of the form

$$\underset{\sim}{\theta}(\underset{\sim}{X}, \tau) = \{\underset{\sim}{I} + \underset{\sim}{W}(t)\}\underset{\sim}{X} + \underset{\sim}{d}(t) \quad . \tag{2.31}$$

We note that the determinant of the deformation gradient of the motion $\underset{\sim}{\theta}$ in (2.30) is equal to $\det(\underset{\sim}{I} + \underset{\sim}{W}(t))\det \underset{\sim}{F} = (1 + \frac{1}{2}\|\underset{\sim}{W}(t)\|^2)J > 0$, so that the condition of the form $(2.2)_2$ is satisfied. It follows from the result mentioned earlier in this paragraph that the only motions in $\mathfrak{M}$ that are both rigid and infinitesimal rigid are the translations. We observe that the deformation gradient, displacement gradient, finite strain tensor, infinitesimal strain and infinitesimal rotation tensor associated with the infinitesimal rigid motion (2.31) are:

$$\begin{gathered} \underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{W}(t) \quad , \quad \underset{\sim}{G} = \underset{\sim}{W}(t) \quad , \quad \underset{\sim}{E} = \tfrac{1}{2}\,\underset{\sim}{W}^T(t)\underset{\sim}{W}(t) \quad , \\ \underset{\sim}{e} = \underset{\sim}{0} \quad , \quad \underset{\sim}{w} = \underset{\sim}{W}(t) \quad . \end{gathered} \tag{2.32}$$

It follows from $(2.14)_{3,4}$,(2.28) and (2.29e) that in any rigid motion the infinitesimal strain tensor satisfies

$$\underset{\sim}{e} = \underset{\sim}{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0 \quad , \tag{2.33}$$

with

$$\epsilon = \|{}_{o}\underset{\sim}{Q}(t) - \underset{\sim}{I}\| \quad . \tag{2.34}$$

Furthermore, to within terms of $\underset{\sim}{O}(\epsilon^2)$ as $\epsilon \to 0$, $\underset{\sim}{G}$ is skew-symmetric and coincides with $\underset{\sim}{w}$, in view of $(2.14)_1$, (2.29a,g) and (2.34). The relationship between rigid motions and infinitesimal rigid motions is now apparent: the limit of a rigid motion as ϵ of (2.34) tends to zero is an infinitesimal rigid motion.

Under superposed rigid body motions (2.7), from $(2.11)_3$ and (2.9) follows that

$$\mathbf{e}^+ + \tfrac{1}{2}(\mathbf{G}^+)^T\mathbf{G}^+ = \mathbf{e} + \tfrac{1}{2}\,\mathbf{G}^T\mathbf{G} \tag{2.35}$$

and by applying (2.28) to both $\mathbf{G}$ and $\mathbf{G}^+$ we obtain

$$\mathbf{e}^+ = \mathbf{e} + \mathbf{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0 \quad , \tag{2.36}$$

i.e., if in every motion (including the motions $\mathbf{\chi}^+$ in (2.7)) the norm $\|\mathbf{G}\|$ is kept small, then $\mathbf{e}$ is approximately equal to $\mathbf{e}^+$. However, if a finite rigid body motion is superposed on a small deformation, then ϵ given by (2.28) is small while $\|\mathbf{G}^+\|$ computed from (2.12) need not be small. It is then clear from $(2.13)_1$ that

$$\mathbf{e}^+ = \mathbf{e} - \tfrac{1}{2}\{\mathbf{Q}(t) - \mathbf{I}\}^T\{\mathbf{Q}(t) - \mathbf{I}\} + \mathbf{O}(\epsilon) \quad \text{as} \quad \epsilon \to 0 \quad . \tag{2.37}$$

3. A properly invariant infinitesimal strain tensor

3.1 Construction of the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$

From among the particles of $\mathcal{B}$, let one denoted by Y and called a pivot be chosen. Then, by (2.1), (2.2) and (2.3), we have

$$ {}_{\alpha}\underset{\sim}{y} = {}_{\alpha}\underset{\sim}{\chi}(\underset{\sim}{Y},t) \quad , \quad \frac{\partial\, {}_{\alpha}\underset{\sim}{\chi}}{\partial \underset{\sim}{X}}(\underset{\sim}{Y},t) = {}_{\alpha}\underset{\sim}{R}(\underset{\sim}{Y},t)\, {}_{\alpha}\underset{\sim}{U}(\underset{\sim}{Y},t) \quad , \tag{3.1} $$

where $\underset{\sim}{Y} = {}_{o}\underset{\sim}{y} = {}_{o}\underset{\sim}{\kappa}(Y)$. For any motion ${}_{\alpha}\underset{\sim}{\chi}$ we can construct a motion ${}_{\alpha}\underset{\sim}{\chi}^{*} = \underset{\sim}{\pi}({}_{\alpha}\underset{\sim}{\chi})$ by removing from ${}_{\alpha}\underset{\sim}{\chi}$ the translation and rotation at the pivot Y, while maintaining at all particles of $\mathcal{B}$ the stretch (and hence finite strain) experienced in the motion ${}_{\alpha}\underset{\sim}{\chi}$. In order to achieve this construction, it is necessary and sufficient that[†]

$$ \begin{aligned} {}_{\alpha}\underset{\sim}{x}^{*} &= {}_{\alpha}\underset{\sim}{\chi}^{*}(\underset{\sim}{X}, {}_{\alpha}t^{*}) = {}_{\alpha}\underset{\sim}{R}^{T}(\underset{\sim}{Y},t)\{ {}_{\alpha}\underset{\sim}{\chi}(\underset{\sim}{X},t) - {}_{\alpha}\underset{\sim}{\chi}(\underset{\sim}{Y},t)\} + \underset{\sim}{Y} \quad , \\ {}_{\alpha}t^{*} &= t - {}_{\alpha}c \quad , \end{aligned} \tag{3.2} $$

where ${}_{\alpha}c$ is a real constant. The configuration of $\mathcal{B}$ at time ${}_{\alpha}t^{*}$ in the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$ is denoted by ${}_{\alpha}\underset{\sim}{\kappa}^{*} = {}_{\alpha}\underset{\sim}{\chi}^{*} \circ {}_{o}\underset{\sim}{\kappa}$. In line with the notation of section 2, we write ${}_{\alpha}\mathcal{P}^{*} = {}_{\alpha}\underset{\sim}{\chi}^{*}({}_{o}\mathcal{P}, {}_{\alpha}t^{*})$ and $\partial\, {}_{\alpha}\mathcal{P}^{*} = {}_{\alpha}\underset{\sim}{\chi}^{*}(\partial\, {}_{o}\mathcal{P}, {}_{\alpha}t^{*})$ for the region and its boundary occupied by any part $\mathcal{S} \subseteq \mathcal{B}$ in the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$. It may be noted that application of (3.2) to the identity motion reproduces the identity motion, i.e.,

$$ {}_{o}\underset{\sim}{\chi}^{*} = \underset{\sim}{\pi}({}_{o}\underset{\sim}{\chi}) = {}_{o}\underset{\sim}{\chi} \quad . $$

Different choices of the constant ${}_{\alpha}c$ in $(3.2)_2$ merely result in a reparametrization of ${}_{\alpha}\underset{\sim}{\chi}^{*}$ with ${}_{\alpha}t^{*}$ replaced by ${}_{\alpha}t$ plus a constant; and the same position ${}_{\alpha}\underset{\sim}{x}^{*}$ is reached, except possibly earlier or later depending on the value of the constant ${}_{\alpha}c$. We regard all such parametrizations -- corresponding to different choices of ${}_{\alpha}c$ -- as representing the same motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$.

We observe that (3.2) is of the form (2.7) with ${}_{\alpha}\underset{\sim}{Q}(t) = {}_{\alpha}\underset{\sim}{R}^{T}(\underset{\sim}{Y},t)$,

---

[†] This will be made clear by (3.15) and the remarks following it.

${}_\alpha\mathbf{a}(t) = -{}_\alpha\mathbf{R}^T(\mathbf{Y},t)\,{}_\alpha\mathbf{x}(\mathbf{Y},t) + \mathbf{Y}$ and ${}_\alpha a = -c_\alpha$. Consequently, ${}_\alpha\mathbf{x}^*$ is a member of the equivalence class $K({}_\alpha\mathbf{x})$, i.e., the class of motions which differ from ${}_\alpha\mathbf{x}$ by a rigid motion. We now state and prove an easy

Theorem 3.1. Two motions ${}_1\mathbf{x}$ and ${}_2\mathbf{x}$ of $\mathcal{B}$ differ by a rigid motion if and only if $\boldsymbol{\pi}({}_1\mathbf{x}) = \boldsymbol{\pi}({}_2\mathbf{x})$, or equivalently ${}_1\mathbf{x}^* = {}_2\mathbf{x}^*$.

Proof: We first prove the necessity. If ${}_1\mathbf{x}$ and ${}_2\mathbf{x}$ differ by a rigid motion, then by (2.7) we have

$$ {}_1\mathbf{x}(\mathbf{X},t^+) = \mathbf{Q}(t)\,{}_2\mathbf{x}(\mathbf{X},t) + \mathbf{a}(t) \quad , \quad t^+ = t + a \tag{3.3} $$

for some proper orthogonal tensor-valued function $\mathbf{Q}(t)$ of time, some vector-valued function $\mathbf{a}(t)$ of time and some constant a. As in $(2.11)_4$, it follows from $(3.3)_1$ that ${}_1\mathbf{R}(\mathbf{X},t^+) = \mathbf{Q}(t)\;{}_2\mathbf{R}(\mathbf{X},t)$; and, in particular, at the pivot

$$ {}_1\mathbf{R}(\mathbf{Y},t^+) = \mathbf{Q}(t)\;{}_2\mathbf{R}(\mathbf{Y},t) \quad . \tag{3.4} $$

Applying (3.2) to the motion ${}_2\mathbf{x}$ we obtain ${}_2\mathbf{x}^* = \boldsymbol{\pi}({}_2\mathbf{x})$ with

$$ {}_2\mathbf{x}^*(\mathbf{X},t) = {}_2\mathbf{R}^T(\mathbf{Y},t)\{{}_2\mathbf{x}(\mathbf{X},t) - {}_2\mathbf{x}(\mathbf{Y},t)\} + \mathbf{Y} \quad , \tag{3.5} $$

where we have chosen the parameter ${}_2c$ in $(3.2)_2$ to be zero. Again, ${}_1\mathbf{x}^* = \boldsymbol{\pi}({}_1\mathbf{x})$ is obtained from (3.2), with a choice of ${}_1c = -a$, so that

$$ \begin{aligned} {}_1\mathbf{x}^*(\mathbf{X},t) &= {}_1\mathbf{R}^T(\mathbf{Y},t^+)\{{}_1\mathbf{x}(\mathbf{X},t^+) - {}_1\mathbf{x}(\mathbf{Y},t^+)\} + \mathbf{Y} \\ &= {}_1\mathbf{R}^T(\mathbf{Y},t^+)\mathbf{Q}(t)\{{}_2\mathbf{x}(\mathbf{X},t) - {}_2\mathbf{x}(\mathbf{Y},t)\} + \mathbf{Y} \\ &= {}_2\mathbf{x}^*(\mathbf{X},t) \quad , \end{aligned} \tag{3.6} $$

where (3.3), (3.4) and (3.5) have been used.

We now turn to the sufficiency. Suppose

$$ {}_1\mathbf{x}^*(\mathbf{X},t) = {}_2\mathbf{x}^*(\mathbf{X},t) \tag{3.7} $$

and denote the rotation tensors in the motions ${}_1\mathbf{X}$ and ${}_2\mathbf{X}$ by ${}_1\mathbf{R}$ and ${}_2\mathbf{R}$, respectively. Then, it follows from (3.2) that

$$\begin{aligned} {}_1\mathbf{X}^*(\mathbf{X},t) &= {}_1\mathbf{R}^T(\mathbf{Y},t+{}_1c)\{{}_1\mathbf{X}(\mathbf{X},t+{}_1c) - {}_1\mathbf{X}(\mathbf{Y},t+{}_1c)\} + \mathbf{Y} \quad , \\ {}_2\mathbf{X}^*(\mathbf{X},t) &= {}_2\mathbf{R}^T(\mathbf{Y},t+{}_2c)\{{}_2\mathbf{X}(\mathbf{X},t+{}_2c) - {}_2\mathbf{X}(\mathbf{Y},t+{}_2c)\} + \mathbf{Y} \quad , \end{aligned} \tag{3.8}$$

for some constants ${}_1c$ and ${}_2c$. Also, from (3.7) and (3.8) we have

$$\begin{aligned} {}_1\mathbf{X}(\mathbf{X},t+{}_1c) = {}& {}_1\mathbf{R}(\mathbf{Y},t+{}_1c)\,{}_2\mathbf{R}^T(\mathbf{Y},t+{}_2c)\,{}_2\mathbf{X}(\mathbf{X},t+{}_2c) \\ & + {}_1\mathbf{X}(\mathbf{Y},t+{}_1c) \\ & - {}_1\mathbf{R}(\mathbf{Y},t+{}_1c)\,{}_2\mathbf{R}^T(\mathbf{Y},t+{}_2c)\,{}_2\mathbf{X}(\mathbf{Y},t+{}_2c) \quad , \end{aligned}$$

which is of the form (3.3) with

$$\begin{aligned} a &= {}_1c - {}_2c \quad , \\ \mathbf{Q}(t) &= {}_1\mathbf{R}(\mathbf{Y},t+{}_1c-{}_2c)\,{}_2\mathbf{R}^T(\mathbf{Y},t) \quad , \\ \mathbf{a}(t) &= {}_1\mathbf{X}(\mathbf{Y},t+{}_1c-{}_2c) - {}_1\mathbf{R}(\mathbf{Y},t+{}_1c-{}_2c)\,{}_2\mathbf{R}^T(\mathbf{Y},t)\,{}_2\mathbf{X}(\mathbf{Y},t) \quad . \end{aligned} \tag{3.10}$$

This completes the proof.

In the language of equivalence classes (see Appendix A), Theorem 3.1 may be stated as $K({}_1\mathbf{X}) = K({}_2\mathbf{X})$ if and only if $\boldsymbol{\pi}({}_1\mathbf{X}) = \boldsymbol{\pi}({}_2\mathbf{X})$. It is clear from Theorem 3.1 that the function $\boldsymbol{\pi}$, defined on $\mathfrak{M}$ through (3.2), maps every motion in an equivalence class $K(\boldsymbol{\theta})$ -- consisting of motions that differ from $\boldsymbol{\theta}$ by a rigid motion -- into the motion $\boldsymbol{\theta}^* = \boldsymbol{\pi}(\boldsymbol{\theta})$. Since $\boldsymbol{\theta}^*$ is itself a member of the class $K(\boldsymbol{\theta})$, we may write $K(\boldsymbol{\theta}^*) = K(\boldsymbol{\theta})$ and note that $\boldsymbol{\theta}^*$ can be used to determine the equivalence class. Recalling the function $\boldsymbol{\omega}$ defined following (2.7) we may write

$$\boldsymbol{\pi} \circ \boldsymbol{\omega} = \boldsymbol{\pi} \tag{3.11}$$

for every choice of ${}_{\alpha}\mathbf{Q}$, ${}_{\alpha}\mathbf{a}$ and ${}_{\alpha}a$ in (2.7). The mapping $\boldsymbol{\pi}$ extracts from $\mathfrak{m}$ a proper subset

$$\mathfrak{n} = \{\boldsymbol{\pi}(\boldsymbol{\theta}) \,|\, \boldsymbol{\theta} \in \mathfrak{m}\} = \boldsymbol{\pi}(\mathfrak{m}) \quad .$$

The notion of invariance under superposed rigid body motions implies that the mechanical response of a body $\mathcal{B}$ in the entire set of motions $\mathfrak{m}$ is completely determined by its response in the subset $\mathfrak{n} \subset \mathfrak{m}$.

Recalling the definitions $(2.2)_1$, (2.5) and (2.6), the deformation gradient, the relative displacement and the displacement gradient in the motion ${}_{\alpha}\mathbf{X}^*$ are, respectively, given by

$$\begin{aligned}
{}_{\alpha}\mathbf{F}^* &= \frac{\partial\, {}_{\alpha}\mathbf{X}^*}{\partial \mathbf{X}}(\mathbf{X}, {}_{\alpha}t^*) \quad , \\
{}_{\alpha}\mathbf{u}^* &= ({}_{\alpha}\mathbf{X}^* - {}_{o}\mathbf{X})(\mathbf{X}, {}_{\alpha}t^*) = {}_{\alpha}\mathbf{x}^* - \mathbf{X} \quad , \\
{}_{\alpha}\mathbf{G}^* &= {}_{\alpha}\mathbf{F}^* - \mathbf{I} \quad .
\end{aligned} \tag{3.12}$$

Also, similar to (2.3), (2.4), (2.8) and (2.9), associated with the motion ${}_{\alpha}\mathbf{X}^*$ we have

$$\begin{aligned}
{}_{\alpha}\mathbf{F}^* = {}_{\alpha}\mathbf{R}^*\, {}_{\alpha}\mathbf{U}^* \quad &, \quad {}_{\alpha}\mathbf{C}^* = ({}_{\alpha}\mathbf{F}^*)^T\, {}_{\alpha}\mathbf{F}^* = ({}_{\alpha}\mathbf{U}^*)^2 \quad , \\
{}_{\alpha}\mathbf{E}^* = \tfrac{1}{2}({}_{\alpha}\mathbf{C}^* - \mathbf{I}) &= {}_{\alpha}\mathbf{e}^* + \tfrac{1}{2}({}_{\alpha}\mathbf{G}^*)^T\, {}_{\alpha}\mathbf{G}^* \quad , \\
{}_{\alpha}\mathbf{e}^* = \tfrac{1}{2}({}_{\alpha}\mathbf{G}^* + ({}_{\alpha}\mathbf{G}^*)^T) \quad &, \quad {}_{\alpha}\mathbf{w}^* = \tfrac{1}{2}({}_{\alpha}\mathbf{G}^* - ({}_{\alpha}\mathbf{G}^*)^T) \quad .
\end{aligned} \tag{3.13}$$

Since (3.2) is of the form (2.7) with ${}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)$ playing the role of ${}_{\alpha}\mathbf{Q}(t)$, it follows from (2.10), (2.11) and (3.13) that

$$
\begin{aligned}
&{}_\alpha\mathbf{F}^* = {}_\alpha\mathbf{R}^T(\mathbf{Y},t)\,{}_\alpha\mathbf{F} \quad , \quad {}_\alpha J^* = \det({}_\alpha\mathbf{F}^*) = {}_\alpha J > 0 \quad , \\
&{}_\alpha\mathbf{U}^* = {}_\alpha\mathbf{U} \quad , \quad {}_\alpha\mathbf{C}^* = {}_\alpha\mathbf{C} \quad , \quad {}_\alpha\mathbf{E}^* = {}_\alpha\mathbf{E} \quad , \quad {}_\alpha\mathbf{R}^* = {}_\alpha\mathbf{R}^T(\mathbf{Y},t)\,{}_\alpha\mathbf{R} \quad .
\end{aligned}
\tag{3.14}
$$

We note that for the identity motion ${}_0\mathbf{X}$, ${}_0\mathbf{F}^* = {}_0\mathbf{R}^* = {}_0\mathbf{C}^* = {}_0\mathbf{U}^* = \mathbf{I}$ and ${}_0\mathbf{E}^* = {}_0\mathbf{G}^* = {}_0\mathbf{e}^* = {}_0\mathbf{w}^* = \mathbf{0}$.

The position vector and rotation at the pivot Y, in the motion ${}_\alpha\mathbf{X}^*$ by $(3.2)_1$ and $(3.14)_6$ are:

$$
{}_\alpha\mathbf{y}^* = {}_\alpha\mathbf{X}^*(\mathbf{Y},{}_\alpha t^*) = \mathbf{Y} \quad , \quad {}_\alpha\mathbf{R}^*(\mathbf{Y},{}_\alpha t^*) = \mathbf{I} \quad ,
\tag{3.15}
$$

while $(3.14)_{3,5}$ show that for every particle of $\mathcal{B}$ the stretch and finite strain in the motion ${}_\alpha\mathbf{X}^*$ retain the values they had in the motion ${}_\alpha\mathbf{X}$. It is clear from the foregoing that (3.15) and $(3.14)_3$ are necessary conditions for the validity of (3.2). It is easy to prove that they are also sufficient. In this connection, we recall the well-known fact that a condition of the form $(3.14)_3$ for the stretch implies that the configuration ${}_\alpha\boldsymbol{\kappa}^*$ is related to ${}_\alpha\boldsymbol{\kappa}$ by a rigid displacement, so that

$$
{}_\alpha\mathbf{X}^*(\mathbf{X},{}_\alpha t^*) = {}_\alpha\bar{\mathbf{Q}}(t)\,{}_\alpha\mathbf{X}(\mathbf{X},t) + {}_\alpha\bar{\mathbf{a}}(t) \quad , \quad {}_\alpha t^* = {}_\alpha t + {}_\alpha\bar{a}
\tag{3.16}
$$

for some proper orthogonal tensor-valued function ${}_\alpha\bar{\mathbf{Q}}(t)$, some vector-valued function ${}_\alpha\bar{\mathbf{a}}(t)$ and some real constant ${}_\alpha\bar{a}$. Consequently, applying the gradient operator to $(3.16)_1$ and evaluating the result at the pivot Y, we obtain

$$
{}_\alpha\mathbf{F}^*(\mathbf{Y},{}_\alpha t^*) = {}_\alpha\bar{\mathbf{Q}}(t)\,{}_\alpha\mathbf{F}(\mathbf{Y},t) \quad .
\tag{3.17}
$$

Then, application of the polar decomposition theorem to (3.17) and use of $(3.14)_3$ results in

$$
{}_\alpha\mathbf{R}^*(\mathbf{Y},{}_\alpha t^*) = {}_\alpha\bar{\mathbf{Q}}(t)\,{}_\alpha\mathbf{R}(\mathbf{Y},t)
\tag{3.18}
$$

and hence by $(3.15)_2$ we deduce

$$\underset{\alpha\sim}{\bar{Q}}(t) = \underset{\alpha\sim}{R}^T(\underset{\sim}{Y},t) \quad . \tag{3.19}$$

Next, substitution of (3.19) into $(3.16)_1$ gives

$$\underset{\alpha\sim}{x}^*(\underset{\sim}{X},{}_{\alpha}t^*) = \underset{\alpha\sim}{R}^T(\underset{\sim}{Y},t)\,\underset{\alpha\sim}{x}(\underset{\sim}{X},t) + \underset{\alpha\sim}{\bar{a}}(t) \tag{3.20}$$

and after imposing $(3.15)_1$ on (3.20), we have

$$\underset{\alpha\sim}{\bar{a}}(t) = \underset{\sim}{Y} - \underset{\alpha\sim}{R}^T(\underset{\sim}{Y},t)\,\underset{\alpha\sim}{x}(\underset{\sim}{Y},t) \quad , \tag{3.21}$$

so that $\underset{\alpha\sim}{x}^*$ is of the form (3.2). This concludes the proof that (3.15) and $(3.14)_3$ are sufficient as well as necessary conditions for the validity of (3.2).

Thus, the motion $\underset{\alpha\sim}{x}^*$ can be obtained directly from $\underset{\alpha\sim}{x}$ in accordance with (3.2) or indirectly by imposing the condition (3.15) and $(3.14)_3$, which are equivalent to (3.2). The conditions (3.15) and $(3.14)_3$ involve the idea that the translation and rotation may be removed at a particle of the body while maintaining the stretch (and finite strain) at all particles‡. For later reference, we record here certain results at the pivot Y, namely

$$\underset{\alpha\sim}{u}^*(\underset{\sim}{Y},{}_{\alpha}t^*) = \underset{\sim}{o} \quad , \quad \underset{\alpha\sim}{e}^*(Y,{}_{\alpha}t^*) = \underset{\sim}{U}(Y,t) - \underset{\sim}{I} \quad , \quad \underset{\alpha\sim}{w}^* = \underset{\sim}{O} \quad , \tag{3.22}$$

obtained with the use of (3.15), $(3.14)_{1,3}$, $(3.12)_3$ and $(3.13)_{4,5}$.

It is important to note a property of the formula (3.2): Having obtained $\underset{\alpha\sim}{x}^*$ from $\underset{\alpha\sim}{x}$, if (3.2) is now applied to $\underset{\alpha\sim}{x}^*$ itself no motion different from $\underset{\alpha\sim}{x}^*$ will emerge, since $\underset{\sim}{\pi}(\underset{\alpha\sim}{x}^*) = \underset{\alpha\sim}{x}^*$ by virtue of (3.15). Indeed for any positive integer m

‡A similar idea has been often stated in the context of classical infinitesimal elasticity; see, e.g., the last few lines in section 18 of Love (1927).

$$\pi^m = \pi \quad , \tag{3.23}$$

where π^m stands for m applications of the function π. This conclusion also follows from Theorem 3.1 since ${}_\alpha\chi^*$, being a member of the equivalence class $K({}_\alpha\chi^*)$, must be mapped into ${}_\alpha\chi^*$.

3.2 Invariance properties of the motion ${}_\alpha\chi^*$

Consider next the motions ${}_\alpha\chi$ and ${}_\alpha\chi^+$ in (2.7). It follows from (3.11) that

$$({}_\alpha\chi^+)^* = (\pi \circ \omega)({}_\alpha\chi) = \pi({}_\alpha\chi) = {}_\alpha\chi^* \quad , \tag{3.24}$$

where‡ by (3.2)

$$({}_\alpha\chi^+)^*(X,t) = \{{}_\alpha R^+(Y,{}_\alpha t^+)\}^T\{{}_\alpha\chi^+(X,{}_\alpha t^+) - {}_\alpha\chi^+(Y,{}_\alpha t^+)\} + Y \tag{3.25}$$

with ${}_\alpha c$ chosen equal to $-{}_\alpha a$. Thus, according to (3.24), when a rigid body motion is superposed on a given motion ${}_\alpha\chi$ resulting in a motion ${}_\alpha\chi^+$, by applying (3.2) to ${}_\alpha\chi^+$ we again arrive at ${}_\alpha\chi^*$.

By substituting ${}_\alpha\chi^*$ in $(2.7)_1$, we can generate the equivalence class $K({}_\alpha\chi^*)$ which coincides with $K({}_\alpha\chi)$ and $K({}_\alpha\chi^+)$. We note that $\pi(({}_\alpha\chi^*)^+) = {}_\alpha\chi^*$ or $(({}_\alpha\chi^*)^+)^* = {}_\alpha\chi^*$, where $({}_\alpha\chi^*)^+$ is the motion given by (2.7) when ${}_\alpha\chi$ is replaced by ${}_\alpha\chi^*$, i.e., $({}_\alpha\chi^*)^+ = \omega({}_\alpha\chi^*)$.

In subsequent developments we need to have available explicit relationships between various kinematical quantities calculated from the motions ${}_\alpha\chi^*$ and $({}_\alpha\chi^+)^*$. With the notations

$${}_\alpha F^{+*} = \frac{\partial({}_\alpha\chi^+)^*}{\partial X}(X,t) \quad , \quad {}_\alpha u^{+*} = (({}_\alpha\chi^+)^* - {}_0\chi)(X,t) \quad , \tag{3.26}$$

along with definitions paralleling (3.13) for ${}_\alpha R^{+*}$, ${}_\alpha U^{+*}$, ${}_\alpha C^{+*}$, ${}_\alpha E^{+*}$, ${}_\alpha e^{+*}$ and ${}_\alpha w^{+*}$, it follows at once from (3.24) that

---

‡ Clearly, (3.24) also follows directly from (3.25), $(2.11)_4$, $(2.7)_1$ and (3.2). Indeed these were the very equations which led (through Theorem 3.1) to (3.15).

$$
\begin{aligned}
&{}_{\alpha}\underset{\sim}{F}^{+*} = {}_{\alpha}\underset{\sim}{F}^{*} \;, \quad {}_{\alpha}\underset{\sim}{u}^{+*} = {}_{\alpha}\underset{\sim}{u}^{*} \;, \quad {}_{\alpha}\underset{\sim}{G}^{+*} = {}_{\alpha}\underset{\sim}{G}^{*} \;, \\
&{}_{\alpha}\underset{\sim}{U}^{+*} = {}_{\alpha}\underset{\sim}{U}^{*} \;, \quad {}_{\alpha}\underset{\sim}{C}^{+*} = {}_{\alpha}\underset{\sim}{C}^{*} \;, \quad {}_{\alpha}\underset{\sim}{E}^{+*} = {}_{\alpha}\underset{\sim}{E}^{*} \;, \\
&{}_{\alpha}\underset{\sim}{R}^{+*} = {}_{\alpha}\underset{\sim}{R}^{*} \;, \quad {}_{\alpha}\underset{\sim}{e}^{+*} = {}_{\alpha}\underset{\sim}{e}^{*} \;, \quad {}_{\alpha}\underset{\sim}{w}^{+*} = {}_{\alpha}\underset{\sim}{w}^{*} \;.
\end{aligned}
\tag{3.27}
$$

We note that in the notation of $(3.26)_1$, ${}_{\alpha}\underset{\sim}{F}^{+*}$ denotes the gradient of the motion $({}_{\alpha}\underset{\sim}{\chi}^{+})^{*} = \underset{\sim}{\pi}({}_{\alpha}\underset{\sim}{\chi}^{+})$, while ${}_{\alpha}\underset{\sim}{F}^{*+} = ({}_{\alpha}\underset{\sim}{F}^{*})^{+}$ in keeping with $(2.10)_1$ stands for the gradient of a motion $({}_{\alpha}\underset{\sim}{\chi}^{*})^{+}$ which differs from ${}_{\alpha}\underset{\sim}{\chi}^{*}$ by a superposed rigid body motion. The significance of the results (3.27) lies in the fact that while motions which differ from each other by a rigid motion (and thereby belong to the same equivalence class), in general have different values for $\underset{\sim}{G}$, $\underset{\sim}{e}$ and $\underset{\sim}{w}$ [see (2.12) and (2.13)], but have the same values for $\underset{\sim}{G}^{*}$, $\underset{\sim}{e}^{*}$ and $\underset{\sim}{w}^{*}$.

It is worth making an observation here for the special case of rigid body motions. Since for a rigid motion ${}_{o}\underset{\sim}{\chi}^{+} = \underset{\sim}{\omega}({}_{o}\underset{\sim}{\chi})$, it follows from (3.11) and (3.2) that

$$
\underset{\sim}{\pi}({}_{o}\underset{\sim}{\chi}^{+}) = (\underset{\sim}{\pi} \circ \underset{\sim}{\omega})({}_{o}\underset{\sim}{\chi}) = \underset{\sim}{\pi}({}_{o}\underset{\sim}{\chi}) = {}_{o}\underset{\sim}{\chi}
\tag{3.28}
$$

so that the entire equivalence class $K({}_{o}\underset{\sim}{\chi})$ of rigid body motions is mapped into the identity motion ${}_{o}\underset{\sim}{\chi}$. Consequently, the values of $\underset{\sim}{F}^{*}$, $\underset{\sim}{R}^{*}$, $\underset{\sim}{C}^{*}$, $\underset{\sim}{U}^{*}$, $\underset{\sim}{E}^{*}$, $\underset{\sim}{G}^{*}$, $\underset{\sim}{e}^{*}$ and $\underset{\sim}{w}^{*}$ in any rigid motion coincide with the values of these fields in the identity motion ${}_{o}\underset{\sim}{\chi}$. Thus, using the notations of (3.26) and (3.27), for any rigid motion:

$$
\begin{aligned}
&{}_{o}\underset{\sim}{F}^{+*} = {}_{o}\underset{\sim}{F}^{*} = \underset{\sim}{I} \;, \quad {}_{o}\underset{\sim}{R}^{+*} = {}_{o}\underset{\sim}{R}^{*} = \underset{\sim}{I} \;, \\
&{}_{o}\underset{\sim}{C}^{+*} = {}_{o}\underset{\sim}{C}^{*} = \underset{\sim}{I} \;, \quad {}_{o}\underset{\sim}{U}^{+*} = {}_{o}\underset{\sim}{U}^{*} = \underset{\sim}{I} \;, \\
&{}_{o}\underset{\sim}{E}^{+*} = {}_{o}\underset{\sim}{E}^{*} = \underset{\sim}{0} \;, \quad {}_{o}\underset{\sim}{G}^{+*} = {}_{o}\underset{\sim}{G}^{*} = \underset{\sim}{0} \;, \\
&{}_{o}\underset{\sim}{e}^{+*} = {}_{o}\underset{\sim}{e}^{*} = \underset{\sim}{0} \;, \quad {}_{o}\underset{\sim}{w}^{+*} = {}_{o}\underset{\sim}{w}^{*} = \underset{\sim}{0} \;.
\end{aligned}
\tag{3.29}
$$

The results in (3.29), especially the last two of (3.29), should be contrasted with those in (2.14).

We now proceed to establish two theorems which are, respectively, the converses of $(3.27)_6$ and $(3.27)_8$.

<u>Theorem 3.2</u>. Let ${}_1\underset{\sim}{\chi}$ and ${}_2\underset{\sim}{\chi} \in \mathfrak{M}$ and ${}_\alpha\underset{\sim}{\chi}^* = \underset{\sim}{\pi}({}_\alpha\underset{\sim}{\chi})$, $(\alpha = 1,2)$. If ${}_1\underset{\sim}{E}^*(\underset{\sim}{X},t+a) = {}_2\underset{\sim}{E}^*(\underset{\sim}{X},t)$, where a is a real constant, then (i) ${}_1\underset{\sim}{\chi}^* = {}_2\underset{\sim}{\chi}^*$ and (ii) ${}_1\underset{\sim}{\chi}$ and ${}_2\underset{\sim}{\chi}$ differ by a rigid motion.

<u>Proof</u>. If ${}_1\underset{\sim}{E}^*(\underset{\sim}{X},t+a) = {}_2\underset{\sim}{E}^*(\underset{\sim}{X},t)$, then ${}_1\underset{\sim}{E}(\underset{\sim}{X},t+a) = {}_2\underset{\sim}{E}(\underset{\sim}{X},t)$ by (3.14) and (ii) is obtained as a well-known result and then (i) follows by Theorem 3.1.

<u>Theorem 3.3</u>. Let ${}_1\underset{\sim}{\chi}, {}_2\underset{\sim}{\chi} \in \mathfrak{M}$ and ${}_\alpha\underset{\sim}{\chi}^* = \underset{\sim}{\pi}({}_\alpha\underset{\sim}{\chi})$ $(\alpha = 1,2)$. If ${}_1\underset{\sim}{e}^*(\underset{\sim}{X},t+a) = {}_2\underset{\sim}{e}^*(\underset{\sim}{X},t)$, where a is a real constant, then (i) ${}_1\underset{\sim}{\chi}^* = {}_2\underset{\sim}{\chi}^*$ and (ii) ${}_1\underset{\sim}{\chi}$ and ${}_2\underset{\sim}{\chi}$ differ by a rigid motion.

<u>Proof</u>. Taking a component representation of $(3.12)_1$ relative to the basis $\{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_A\}$,

$$ {}_\alpha\underset{\sim}{F}^* = {}_\alpha F^*_{iA}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_A = \frac{\partial\, {}_\alpha\chi^*_i}{\partial X_A}(\underset{\sim}{X}, {}_\alpha t^*)\underset{\sim}{e}_i \otimes \underset{\sim}{e}_A \quad , \tag{3.30} $$

where ${}_1t^* = t + a$, ${}_2t^* = t$, it follows from the smoothness properties of the motion ${}_\alpha\underset{\sim}{\chi}$ that

$$ {}_\alpha F^*_{iA,B} = {}_\alpha F^*_{iB,A} \quad , \tag{3.31} $$

where $(\ )_{,A}$ stands for $\partial(\ )/\partial X_A$. These are the compatibility conditions for existence of the deformation gradient. Now put

$$ \underset{\sim}{V}(\underset{\sim}{X},t) = V_{iA}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_A = {}_1\underset{\sim}{F}^*(\underset{\sim}{X},t+a) - {}_2\underset{\sim}{F}^*(\underset{\sim}{X},t) \quad . \tag{3.32} $$

Then,

$$ V_{iA,B} = V_{iB,A} \quad . \tag{3.33} $$

Furthermore, decomposition of $\underset{\sim}{V}$ into its symmetric part $\underset{\sim}{A}$ and skew-symmetric part $\underset{\sim}{B}$, namely $\underset{\sim}{A} = \frac{1}{2}(\underset{\sim}{V}+\underset{\sim}{V}^T)$ and $\underset{\sim}{B} = \frac{1}{2}(\underset{\sim}{V}-\underset{\sim}{V}^T)$, when referred to the basis $\{\underset{\sim}{e}_M \otimes \underset{\sim}{e}_N\}$

gives

$$A_{MN} = \mathbf{e}_M \cdot \mathbf{A}\, \mathbf{e}_N = \tfrac{1}{2}(g_{iM}V_{iN} + g_{iN}V_{iM}) \quad ,$$
$$B_{MN} = \tfrac{1}{2}(g_{iM}V_{iN} - g_{iN}V_{iM}) \quad , \tag{3.34}$$

where $g_{iM} = \mathbf{e}_i \cdot \mathbf{e}_M$. Hence, by (3.33) and (3.34),

$$A_{KM,N} - A_{NK,M} = B_{MN,K} \quad . \tag{3.35}$$

By hypothesis, ${}_1\mathbf{e}^*(\mathbf{X},t+a) = {}_2\mathbf{e}^*(\mathbf{X},t)$ and it follows from $(3.13)_5$, (3.12) and (3.32) that

$$\mathbf{V}^T = -\mathbf{V} \tag{3.36}$$

and hence

$$\mathbf{A} = \mathbf{0} \quad , \quad \mathbf{B} = \mathbf{V} \quad . \tag{3.37}$$

Moreover, in view of (3.35) and $(3.37)_1$,

$$B_{MN,K} = 0 \quad , \tag{3.38}$$

so that $\mathbf{B}$ is independent of $\mathbf{X}$. Consequently, we may write (3.32) in the form

$$\mathbf{B}(t) = {}_1\mathbf{F}^*(\mathbf{X},t+a) - {}_2\mathbf{F}^*(\mathbf{X},t) \quad , \tag{3.39}$$

where $(3.37)_2$ has been used. Evaluating (3.39) at the pivot Y, it follows from $(3.13)_1$ and $(3.15)_2$ that

$$\mathbf{B}(t) = {}_1\mathbf{U}^*(\mathbf{Y},t+a) - {}_2\mathbf{U}^*(\mathbf{Y},t) \quad , \tag{3.40}$$

so that $\mathbf{B}^T(t) = \mathbf{B}(t)$. But by definition $\mathbf{B}^T(t) = -\mathbf{B}(t)$ and we conclude that

$$\mathbf{B}(t) = \mathbf{0} \quad . \tag{3.41}$$

With the help of (3.41) and (3.39), integration of (3.39) yields

$$ {}_{1}\underset{\sim}{\chi}^{*}(\underset{\sim}{X},t+a) = {}_{2}\underset{\sim}{\chi}^{*}(\underset{\sim}{X},t) + \underset{\sim}{a}(t) \quad , \tag{3.42} $$

where $\underset{\sim}{a}(t)$ is some vector-valued function of time. Evaluating (3.42) at the pivot Y and invoking $(3.15)_1$, we conclude that

$$ \underset{\sim}{a}(t) = \underset{\sim}{o} \quad . \tag{3.43} $$

Recalling the remark on parametrization following (3.2), part (i) of the theorem is proved and part (ii) follows immediately from part (i) and Theorem 3.1.

Corollary 3.1. If $\underset{\sim}{e}^{*} = \underset{\sim}{0}$ in a motion $\underset{\sim}{\chi}$, then $\underset{\sim}{\chi}^{*} = {}_{0}\underset{\sim}{\chi}$ and $\underset{\sim}{\chi}$ is a rigid motion. The proof follows at once by setting ${}_{2}\underset{\sim}{\chi} = {}_{0}\underset{\sim}{\chi}$ and ${}_{1}\underset{\sim}{\chi} = \underset{\sim}{\chi}$ in Theorem 3.3.

In anticipation of our later use of $\underset{\alpha\sim}{e}^{*}$ as an infinitesimal strain approximating the finite strain tensor $\underset{\alpha\sim}{E}^{*}$, we need to make some observations here. First we recall that two essential properties of any satisfactory finite strain measure are[†]: (i) It should give the same value for two motions if and only if these motions differ by a rigid motion (in this sense it can be used to characterize all motions that belong to the same equivalence class); and (ii) with its use one should be able to calculate exactly the change in length of material lines in a given motion.

In view of Theorems 3.2 and 3.3, along with $(3.27)_{6,8}$, both $\underset{\alpha\sim}{E}^{*}$ and $\underset{\alpha\sim}{e}^{*}$ satisfy property (i). Also, the tensor $\underset{\alpha\sim}{E}^{*}$ satisfies property (ii), but, in general, $\underset{\alpha\sim}{e}^{*}$ does not[‡]. Indeed, we may deduce from $(3.13)_3$ that[§] $\underset{\alpha\sim}{e}^{*} = \underset{\alpha\sim}{E}^{*}$ if

---

[†] Property (ii) implies property (i) but not conversely.

[‡] The classical infinitesimal strain tensor $\underset{\alpha\sim}{e}$ in $(2.8)_1$ [or $\underset{\sim}{e}$ in (2.29e)] satisfies neither (i) nor (ii).

[§] The argument here parallels that used immediately following (2.9).

and only if ${}_{\alpha}\boldsymbol{\chi}$ is a rigid motion in which case both ${}_{\alpha}\mathbf{e}^*$ and ${}_{\alpha}\mathbf{E}^*$ are zero so that ${}_{\alpha}\mathbf{e}^*$ satisfies property (ii) only for a motion which is rigid.

Turning now to the kinetics, we denote by ${}_{\alpha}\rho^*$ the mass density in the configuration ${}_{\alpha}\boldsymbol{\kappa}^*$, ${}_{\alpha}\mathbf{b}^*$ the body force per unit mass in ${}_{\alpha}\boldsymbol{\kappa}^*$, ${}_{\alpha}\mathbf{n}^*$ the outward unit normal to the surface $\partial\,{}_{\alpha}\mathcal{P}^*$, ${}_{\alpha}\mathbf{t}^*$ the stress vector on $\partial\,{}_{\alpha}\mathcal{P}^*$ and ${}_{\alpha}\mathbf{T}^*$ the associated Cauchy stress tensor. Field equations of the form (2.15) with an asterisk added then hold for the motion ${}_{\alpha}\boldsymbol{\chi}^*$. Furthermore, recalling that ${}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)$ in $(3.2)_1$ plays the role of ${}_{\alpha}\mathbf{Q}(t)$ in (2.7), it follows from (2.21), (2.22), (2.23), (2.24) and (2.27) that

$$
\begin{aligned}
{}_{\alpha}\rho^* = {}_{\alpha}\rho \quad , \quad {}_{\alpha}\mathbf{n}^* &= {}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)\,{}_{\alpha}\mathbf{n} \quad , \quad {}_{\alpha}\mathbf{t}^* = {}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)\,{}_{\alpha}\mathbf{t} \quad , \\
{}_{\alpha}\mathbf{T}^* &= {}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)\;{}_{\alpha}\mathbf{T}\;{}_{\alpha}\mathbf{R}(\mathbf{Y},t) \quad , \\
{}_{\alpha}\mathrm{div}^*\;{}_{\alpha}\mathbf{T}^* &= {}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)\;{}_{\alpha}\mathrm{div}\;{}_{\alpha}\mathbf{T} \quad , \\
\dot{\mathbf{v}}^* - {}_{\alpha}\mathbf{b}^* &= {}_{\alpha}\mathbf{R}^T(\mathbf{Y},t)({}_{\alpha}\dot{\mathbf{v}} - {}_{\alpha}\mathbf{b}) \quad ,
\end{aligned}
\tag{3.44}
$$

where ${}_{\alpha}\mathrm{div}^*$ is the divergence operator with respect to position in the configuration ${}_{\alpha}\boldsymbol{\kappa}^*$ and is defined in a manner paralleling (2.16).

Similarly, associated with the motion $({}_{\alpha}\boldsymbol{\chi}^+)^*$ we have the quantities ${}_{\alpha}J^{+*}$, ${}_{\alpha}\rho^{+*}$, ${}_{\alpha}\mathbf{b}^{+*}$, ${}_{\alpha}\dot{\mathbf{v}}^{+*}$, ${}_{\alpha}\mathbf{n}^{+*}$, ${}_{\alpha}\mathbf{t}^{+*}$, ${}_{\alpha}\mathbf{T}^{+*}$ and the operator ${}_{\alpha}\mathrm{div}^{+*}$. Then, remembering $(3.27)_1$ and the conservation of mass, it follows that

$$
{}_{\alpha}J^{+*} = {}_{\alpha}J^* \quad , \quad {}_{\alpha}\rho^{+*} = {}_{\alpha}\rho^* \quad . \tag{3.45}
$$

Also, with the help of $(3.44)_{2,3,4}$, (2.24) and $(2.11)_4$, we obtain

$$\underset{\alpha\sim}{T}^{+\cdot} = \{\underset{\alpha\sim}{Q}(t)\ \underset{\alpha\sim}{R}(\underset{\sim}{Y},t)\}^{T}\ \underset{\alpha\sim}{Q}(t)\ \underset{\alpha\sim}{T}\ \underset{\alpha\sim}{Q}^{T}(t)\ \underset{\alpha\sim}{Q}(t)\ \underset{\alpha\sim}{R}(\underset{\sim}{Y},t) = \underset{\alpha\sim}{T}^{*} \ ,$$

$$\underset{\alpha\sim}{t}^{+*} = \underset{\alpha\sim}{t}^{*} \ , \quad \underset{\alpha\sim}{n}^{+*} = \underset{\alpha\sim}{n}^{*} \ . \tag{3.46}$$

With the use of (3.24), $(3.46)_1$ and the balance of linear momentum, we also have

$$\underset{\alpha}{\mathrm{div}}^{+*}\ \underset{\alpha\sim}{T}^{+*} = \underset{\alpha}{\mathrm{div}}^{*}\ \underset{\alpha\sim}{T}^{*} \ ,$$

$$\underset{\alpha\sim}{\dot{v}}^{+*} = \underset{\alpha\sim}{\dot{v}}^{*} \ , \quad \underset{\alpha\sim}{b}^{+*} = \underset{\alpha\sim}{b}^{*} \ . \tag{3.47}$$

3.3 Properly invariant infinitesimal theory

In the next section, we construct an infinitesimal theory of motions superposed on any given motion. This includes, as a special case, an infinitesimal theory of motions, i.e., motions superposed on the identity motion $\underset{0\sim}{\chi}$. However, it is instructive at this stage to elaborate briefly on the infinitesimal theory.

The infinitesimal theory developed here is derived from the finite theory by considering as a measure of smallness (associated with the motion $\underset{\sim}{\chi}$) the nonnegative real function

$$\epsilon^{*} = \epsilon^{*}(t^{*}) = \sup_{\underset{\sim}{X} \in {}_{0}\mathcal{R}} \|\underset{\sim}{G}^{*}(\underset{\sim}{X},t^{*})\| \ . \tag{3.48}$$

It is important to note that, in defining a measure of smallness, we do not as in (2.27) use the displacement gradient $\underset{\sim}{G}$ in the motion $\underset{\sim}{\chi}$. Instead, we first map $\underset{\sim}{\chi}$ into $\underset{\sim}{\chi}^{*} = \underset{\sim}{\pi}(\underset{\sim}{\chi})$ and use the displacement gradient $\underset{\sim}{G}^{*}$ in our definition (3.48). In this way, the same ϵ^{*} is associated with every motion in the equivalence class $K(\underset{\sim}{\chi}^{*})$ and the motion $\underset{\sim}{\chi}^{*}$ is made to represent this entire class in the infinitesimal theory as well. As in (2.29), we can readily obtain the expressions

$$
\begin{aligned}
&(a) && \underset{\sim}{F}^{*} - \underset{\sim}{I} = \underset{\sim}{G}^{*} = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(b) && (\underset{\sim}{F}^{*})^{-1} - \underset{\sim}{I} = -\underset{\sim}{G}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(c) && \underset{\sim}{U}^{*} - \underset{\sim}{I} = \underset{\sim}{e}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(d) && \underset{\sim}{C}^{*} - \underset{\sim}{I} = 2\underset{\sim}{e}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(e) && \underset{\sim}{E}^{*} = \underset{\sim}{e}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(f) && (\underset{\sim}{U}^{*})^{-1} - \underset{\sim}{I} = -\underset{\sim}{e}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(g) && \underset{\sim}{R}^{*} - \underset{\sim}{I} = \underset{\sim}{w}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad , \\
&(h) && (\underset{\sim}{R}^{*})^{T} - \underset{\sim}{I} = -\underset{\sim}{w}^{*} + \underset{\sim}{O}((\epsilon^{*})^{2}) = \underset{\sim}{O}(\epsilon^{*}) \quad ,
\end{aligned}
\tag{3.49}
$$

as $\epsilon^{*} \to 0$. In particular, we note that $\underset{\sim}{e}^{*}$ is a linear approximation‡ to $\underset{\sim}{E}^{*} = \underset{\sim}{E}$. In view of this and the invariance condition $(3.27)_8$ of $\underset{\sim}{e}^{*}$, we refer to $\underset{\sim}{e}^{*}$ as an <u>invariant infinitesimal strain tensor</u>. Likewise $\underset{\sim}{w}^{*}$ is an <u>invariant infinitesimal rotation tensor</u>.

Consider any smooth curve $\bar{C}$ in ${}_{0}\mathcal{R}$ that joins $\underset{\sim}{Y}$ to a given $\underset{\sim}{X}$. Let $\bar{C}$ be parametrized by its arclength $S \geqq 0$ so chosen that $S = 0$ at $\underset{\sim}{Y}$. The unit tangent vector $\underset{\sim}{\tau}(S)$ at a point $\underset{\sim}{\bar{R}}(S)$ on $\bar{C}$ is given by

$$
\underset{\sim}{\tau}(S) = \frac{d\underset{\sim}{\bar{R}}}{dS}(S) \quad . \tag{3.50}
$$

Denoting the value of S at $\underset{\sim}{X}$ by L, we may, in view of (3.12), calculate the displacement $\underset{\sim}{u}^{*}(\underset{\sim}{X},t^{*})$ of the particle X in the configuration $\underset{\sim}{\kappa}^{*}$ by means of the expression

$$
\begin{aligned}
\underset{\sim}{u}^{*}(\underset{\sim}{X},t^{*}) &= \underset{\sim}{u}^{*}(\underset{\sim}{Y},t^{*}) + \int_{0}^{L} \underset{\sim}{G}^{*}(\underset{\sim}{\bar{R}}(S),t^{*})\underset{\sim}{\tau}(S)dS \\
&= \int_{0}^{L} \underset{\sim}{G}^{*}(\underset{\sim}{\bar{R}}(S),t^{*})\underset{\sim}{\tau}(S)dS \quad ,
\end{aligned}
\tag{3.51}
$$

‡See also the remark following the Corollary 3.1.

where [illegible]$_1$ has been used. Then by the usual inequalities for integrals together with $\|\mathbf{A}\mathbf{a}\| \leq \|\mathbf{A}\|\,\|\mathbf{a}\|$ for all second order tensors $\mathbf{A}$ and all vectors $\mathbf{a}$, and the fact that $\|\boldsymbol{\tau}(S)\| = 1$, it follows from (3.51) that

$$\|\mathbf{u}^*(\mathbf{X},t^*)\| \leq \int_0^L \|\mathbf{G}^*(\overline{\mathbf{R}}(S),t^*)\boldsymbol{\tau}(S)\|dS \leq \epsilon^* L \quad . \tag{3.52}$$

Since, at each t^*, for every $\mathbf{X}$ in ${}_0\mathfrak{R}$, a smooth curve joining $\mathbf{Y}$ to $\mathbf{X}$ can be found whose length is finite, it follows from (3.52) that

$$\mathbf{u}^*(\mathbf{X},t^*) = \mathbf{O}(\epsilon^*) \text{ as } \epsilon^* \to 0 \quad . \tag{3.53}$$

Hence, in view of $(3.12)_2$, $\mathbf{X}^*$ approximates ${}_0\mathbf{X}$ to within terms of $\mathbf{O}(\epsilon^*)$ as $\epsilon^* \to 0$.

Finally, we note that it follows from (2.12) and (2.13) that

$$\mathbf{G}^* = \mathbf{R}^T(\mathbf{Y},t)\mathbf{G} + \mathbf{R}^T(\mathbf{Y},t) - \mathbf{I} \quad , \tag{3.54}$$

while

$$\begin{aligned} 2\mathbf{e}^* &= 2\mathbf{e} - \{\mathbf{R}(\mathbf{Y},t) - \mathbf{I}\}\{\mathbf{R}^T(\mathbf{Y},t) - \mathbf{I}\} \\ &\quad + \{\mathbf{R}^T(\mathbf{Y},t) - \mathbf{I}\}\mathbf{G} + (\{\mathbf{R}^T(\mathbf{Y},t) - \mathbf{I}\}\mathbf{G})^T \quad , \\ 2\mathbf{w}^* &= 2\mathbf{w} + \mathbf{R}^T(\mathbf{Y},t) - \mathbf{R}(\mathbf{Y},t) \\ &\quad + \{\mathbf{R}^T(\mathbf{Y},t) - \mathbf{I}\}\mathbf{G} - (\{\mathbf{R}^T(\mathbf{Y},t) - \mathbf{I}\}\mathbf{G})^T \quad . \end{aligned} \tag{3.55}$$

We now make a comparison between the invariant infinitesimal strain measure $\mathbf{e}^*$ and the usual infinitesimal measure $\mathbf{e}$. Suppose that ϵ in (2.28) is small for all displacement gradients. Then, $\mathbf{G}^*$ satisfies

$$\mathbf{G}^* = \mathbf{O}(\epsilon) \text{ as } \epsilon \to 0 \quad , \tag{3.56}$$

in view of (2.29a) and

$$\underset{\sim}{G}^{*} = \underset{\sim}{G} - \underset{\sim}{w}(Y,t^{*}) + \underset{\sim}{O}(\epsilon^{2}) = \underset{\sim}{O}(\epsilon) \quad \text{as } \epsilon \to 0 \quad , \tag{3.57}$$

derived from (3.54) with the help of (2.29a,h). It follows from (3.55), together with (2.29a,h), that*

$$\begin{aligned} \underset{\sim}{e}^{*} - \underset{\sim}{e} + \underset{\sim}{O}(\epsilon^{2}) &= \underset{\sim}{O}(\epsilon) \text{ as } \epsilon \to 0 \quad , \\ \underset{\sim}{w}^{*} = \underset{\sim}{w} - \underset{\sim}{w}(\underset{\sim}{Y},t^{*}) + \underset{\sim}{O}(\epsilon^{2}) &= \underset{\sim}{O}(\epsilon) \text{ as } \epsilon \to 0 \quad . \end{aligned} \tag{3.58}$$

---

* Since the linearized $\underset{\sim}{G}^{*}$, $\underset{\sim}{e}^{*}$ and $\underset{\sim}{w}^{*}$ were obtained by separate linearization procedures, we observe that (3.57) is consistent with the sum of $(3.58)_1$ and $(3.58)_2$.

4. An invariant theory of small on large

We develop here in the sense of section 3, a properly invariant infinitesimal theory of motions superposed on a given motion and then consider the case of an elastic material.

4.1 General results for motions superposed on a given motion

It is convenient to define the "difference motions" $\boldsymbol{\chi}'$ and $\boldsymbol{\chi}^{*\prime}$ by

$$\boldsymbol{\chi}' = {}_2\boldsymbol{\chi} \circ {}_1\boldsymbol{\chi}^{-1} \quad , \quad \boldsymbol{\chi}^{*\prime} = {}_2\boldsymbol{\chi}^{*} \circ ({}_1\boldsymbol{\chi}^{*})^{-1} \quad . \tag{4.1}$$

Then, by (2.1) and (3.2),

$${}_2\mathbf{x} = \boldsymbol{\chi}'({}_1\mathbf{x},t) \quad , \quad {}_2\mathbf{x}^{*} = \boldsymbol{\chi}^{*\prime}({}_1\mathbf{x}^{*},t^{*}) \quad , \tag{4.2}$$

where we have taken ${}_2c = {}_1c$ in $(3.2)_2$ so that ${}_2t^{*} = {}_1t^{*} = t^{*}$. In (4.1), $\boldsymbol{\chi}'$ represents a deformation whose superposition on ${}_1\boldsymbol{\chi}$ results in ${}_2\boldsymbol{\chi}$. Similarly, the superposition of $\boldsymbol{\chi}^{*\prime}$ on ${}_1\boldsymbol{\chi}^{*}$ yields ${}_2\boldsymbol{\chi}^{*}$.

An application of the chain rule of differentiation to (4.1), together with the use of (2.2), $(3.12)_1$, $(3.14)_{1,2}$, (2.18), (2.19) and $(3.44)_2$ leads to

$$\begin{gathered}
\mathbf{F}' = \frac{\partial \boldsymbol{\chi}'}{\partial_1 \mathbf{x}}({}_1\mathbf{x},t) = {}_2\mathbf{F}\,{}_1\mathbf{F}^{-1} \quad , \quad J' = \det(\mathbf{F}') = {}_2J/{}_1J > 0 \quad , \\
\mathbf{F}^{*\prime} = \frac{\partial \boldsymbol{\chi}^{*\prime}}{\partial_1 \mathbf{x}^{*}}({}_1\mathbf{x}^{*},t^{*}) = {}_2\mathbf{F}^{*}({}_1\mathbf{F}^{*})^{-1} \quad , \quad J^{*\prime} = \det(\mathbf{F}^{*\prime}) = {}_2J^{*}/{}_1J^{*} > 0 \quad , \\
\mathbf{F}^{*\prime} = {}_2\mathbf{R}^T(\mathbf{Y},t)\mathbf{F}'\,{}_1\mathbf{R}(\mathbf{Y},t) \quad , \\
{}_2\mathbf{n} = \frac{((\mathbf{F}')^{-1})^T\,{}_1\mathbf{n}}{\|((\mathbf{F}')^{-1})^T\,{}_1\mathbf{n}\|} \quad , \quad {}_2\mathbf{n}^{*} = \frac{((\mathbf{F}^{*\prime})^{-1})^T\,{}_1\mathbf{n}^{*}}{\|((\mathbf{F}^{*\prime})^{-1})^T\,{}_1\mathbf{n}^{*}\|}
\end{gathered} \tag{4.3}$$

Next, we introduce the difference motion

$$\boldsymbol{\chi}^{+\prime} = {}_2\boldsymbol{\chi}^{+} \circ ({}_1\boldsymbol{\chi}^{+})^{-1} \tag{4.4}$$

associated with the motions on the right-hand side of $(2.7)_1$. The gradient of $\boldsymbol{\chi}^{+\prime}$ with respect to ${}_1\mathbf{x}^{+}$ will be denoted by $\mathbf{F}^{+\prime}$. In view of (4.4), $(2.10)_1$ and [illegible]

$$\underset{\sim}{F}^{+\prime} = {}_2\underset{\sim}{F}^{+}({}_1\underset{\sim}{F}^{+})^{-1} = {}_2\underset{\sim}{Q}(t)\underset{\sim}{F}'{}_1\underset{\sim}{Q}^{T}(t) \quad . \tag{4.5}$$

Further, considering the difference motion

$$\underset{\sim}{X}^{+*\prime} = ({}_2\underset{\sim}{X}^{+})^{*} \circ (({}_1\underset{\sim}{X}^{+})^{*})^{-1} \tag{4.6}$$

whose gradient with respect to $({}_1\underset{\sim}{x}^{+})^{*}$ is denoted by $\underset{\sim}{F}^{+*\prime}$, by (3.24), $(4.1)_2$ and $(4.3)_3$ we have

$$\underset{\sim}{X}^{+*\prime} = \underset{\sim}{X}^{*\prime} \quad , \quad \underset{\sim}{F}^{+*\prime} = \underset{\sim}{F}^{*\prime} \tag{4.7}$$

Now the displacement field $\underset{\sim}{h}^{*}$ of the configuration ${}_2\underset{\sim}{\kappa}^{*}$ relative to ${}_1\underset{\sim}{\kappa}^{*}$, is given by

$$\underset{\sim}{h}^{*} = {}_2\underset{\sim}{X}^{*} - {}_1\underset{\sim}{X}^{*} \quad , \quad \underset{\sim}{h}^{*}(\underset{\sim}{X},t^{*}) = {}_2\underset{\sim}{x}^{*} - {}_1\underset{\sim}{x}^{*} \quad . \tag{4.8}$$

For a fixed value of t^{*}, $\underset{\sim}{h}^{*}(\underset{\sim}{X},t^{*})$ can be expressed as a function of $({}_1\underset{\sim}{x}^{*},t^{*})$ in the form

$$\begin{aligned}\underset{\sim}{h}^{*}(\underset{\sim}{X},t^{*}) &= (\underset{\sim}{h}^{*} \circ ({}_1\underset{\sim}{X}^{*})^{-1})({}_1\underset{\sim}{x}^{*},t^{*}) \\ &= \{{}_2\underset{\sim}{X}^{*} \circ ({}_1\underset{\sim}{X}^{*})^{-1} - {}_1\underset{\sim}{X}^{*} \circ ({}_1\underset{\sim}{X}^{*})^{-1}\}({}_1\underset{\sim}{x}^{*},t^{*}) \quad .\end{aligned} \tag{4.9}$$

Also, let a function $\underset{\sim}{h}$ at each time t^{*} be defined by

$$\underset{\sim}{h} = \underset{\sim}{h}^{*} \circ ({}_1\underset{\sim}{X}^{*})^{-1} \quad . \tag{4.10}$$

Then, recalling $(4.1)_2$, from $(4.8)_2$ and (4.9), we obtain

$${}_2\underset{\sim}{x}^{*} - {}_1\underset{\sim}{x}^{*} = \underset{\sim}{h}({}_1\underset{\sim}{x}^{*},t^{*}) = (\underset{\sim}{X}^{*\prime} - {}_0\underset{\sim}{\bar{X}})({}_1\underset{\sim}{x}^{*},t^{*}) \quad , \tag{4.11}$$

where ${}_0\underset{\sim}{\bar{X}}$ is the identity mapping[+] on the region ${}_1\mathcal{R}^{*}$. We note that at the current position of the pivot

---

[+] An overbar is used to distinguish between this mapping and the identity motion ${}_0\underset{\sim}{X}$.

$$\underset{\sim}{h}({}_1\underset{\sim}{y}^*, t^*) = \underset{\sim}{0} \tag{4.12}$$

by $(3.15)_1$. By (4.11), $(4.3)_3$ and (4.10), the relative displacement gradient of $\underset{\sim}{h}$ with respect to ${}_1\underset{\sim}{x}^*$, namely

$$\underset{\sim}{H} = \frac{\partial \underset{\sim}{h}}{\partial\, {}_1\underset{\sim}{x}^*}({}_1\underset{\sim}{x}^*, t^*) \quad , \tag{4.13}$$

and the gradient of $\underset{\sim}{h}^*$ with respect to $\underset{\sim}{X}$ are given by

$$\underset{\sim}{H} = {}_2\underset{\sim}{F}^*({}_1\underset{\sim}{F}^*)^{-1} - \underset{\sim}{I} = \underset{\sim}{F}^{*\prime} - \underset{\sim}{I} \quad , \quad \frac{\partial \underset{\sim}{h}^*}{\partial \underset{\sim}{X}} = \underset{\sim}{H}\, {}_1\underset{\sim}{F}^* = {}_2\underset{\sim}{F}^* - {}_1\underset{\sim}{F}^* \quad . \tag{4.14}$$

Then, the difference ${}_2\underset{\sim}{E}^* - {}_1\underset{\sim}{E}^*$ of strains, with the help of $(3.13)_{2,3}$ and $(4.14)_2$, is of the form

$${}_2\underset{\sim}{E}^* - {}_1\underset{\sim}{E}^* = \tfrac{1}{2}({}_1\underset{\sim}{F}^*)^T\{\underset{\sim}{H} + \underset{\sim}{H}^T + \underset{\sim}{H}^T\underset{\sim}{H}\}\,{}_1\underset{\sim}{F}^* \quad . \tag{4.15}$$

The displacement field $\underset{\sim}{h}^{+*}$ associated with the motions $({}_\alpha\underset{\sim}{\chi}^+)^*$ $(\alpha = 1,2)$ is defined in a manner paralleling (4.8). Thus,

$$\underset{\sim}{h}^{+*} = ({}_2\underset{\sim}{\chi}^+)^* - ({}_1\underset{\sim}{\chi}^+)^* \quad , \quad \underset{\sim}{h}^{+*}(\underset{\sim}{X}, t) = ({}_2\underset{\sim}{x}^+)^* - ({}_1\underset{\sim}{x}^+)^* \quad . \tag{4.16}$$

Then, in view of (3.24) and $(4.8)_1$,

$$\underset{\sim}{h}^{+*} = \underset{\sim}{h}^* \quad . \tag{4.17}$$

It is convenient for our present purpose to express $\underset{\sim}{h}^{+*}(\underset{\sim}{X}, t)$ as a function $\underset{\sim}{h}^+$ of $({}_1\underset{\sim}{x}^+)^*, t$:

$$\begin{gathered} \underset{\sim}{h}^+ = \underset{\sim}{h}^{+*} \circ (({}_1\underset{\sim}{\chi}^+)^*)^{-1} \quad , \\ ({}_2\underset{\sim}{x}^+)^* - ({}_1\underset{\sim}{x}^+)^* = \underset{\sim}{h}^+(({}_1\underset{\sim}{x}^+)^*, t) = (\underset{\sim}{\chi}^{+*\prime} - {}_0\overline{\underset{\sim}{X}}^+)(({}_1\underset{\sim}{x}^+)^*, t) \quad , \end{gathered} \tag{4.18}$$

where (4.6) has been used and where ${}_0\overline{\underset{\sim}{X}}^+$ denotes the identity mapping on the region $\mathcal{R}^{+*} = ({}_1\underset{\sim}{\chi}^+)^*(\mathcal{R}, t)$. Inspection of (3.24), (4.17), $(4.18)_1$ and (4.10) reveals that

$$\underset{\sim}{h}^{+} = \underset{\sim}{h} \ . \tag{4.19}$$

Also in view of (3.24), ${}_{0}\bar{X}^{+} = {}_{0}\bar{X}$. It now follows from (4.13) and (4.19) that the relative displacement gradient $\underset{\sim}{H}^{+}$, namely

$$\underset{\sim}{H}^{+} = \frac{\partial \underset{\sim}{h}^{+}}{\partial ({}_{1}\underset{\sim}{x}^{+})^{*}} \left(({}_{1}\underset{\sim}{x}^{+})^{*}, t \right) \ , \tag{4.20}$$

satisfies

$$\underset{\sim}{H}^{+} = \underset{\sim}{H} \ . \tag{4.21}$$

4.2 Results for an elastic material

Now consider the body $\mathcal{B}$ to be composed of an elastic material[‡]. Thus, let ${}_{\alpha}\epsilon$ be the elastic strain energy per unit mass in the configuration ${}_{\alpha}\underset{\sim}{\kappa}$. Furthermore, let ${}_{\alpha}\epsilon^{+}$ and ${}_{\alpha}\epsilon^{*}$ denote the strain energy per unit mass in the configurations ${}_{\alpha}\underset{\sim}{\kappa}^{+}$ and ${}_{\alpha}\underset{\sim}{\kappa}^{*}$, respectively. We assume that ${}_{\alpha}\epsilon^{+} = {}_{\alpha}\epsilon$ and it then follows that ${}_{\alpha}\epsilon^{*} = {}_{\alpha}\epsilon$. A nonlinearly elastic solid may be characterized by the constitutive equation

$${}_{\alpha}\underset{\sim}{T} = \tfrac{1}{2}\,{}_{\alpha}\rho\,{}_{\alpha}\underset{\sim}{F}\{D\hat{\epsilon}({}_{\alpha}\underset{\sim}{E}) + D^{T}\hat{\epsilon}({}_{\alpha}\underset{\sim}{E})\}\,{}_{\alpha}\underset{\sim}{F}^{T} \ , \tag{4.22}$$

where ${}_{\alpha}\epsilon = \hat{\epsilon}({}_{\alpha}\underset{\sim}{E})$ and the notation $D\hat{\epsilon}({}_{\alpha}\underset{\sim}{E})$ stands for the derivative of the function $\hat{\epsilon}$ at the point ${}_{\alpha}\underset{\sim}{E}$, while $D^{T}\hat{\epsilon}({}_{\alpha}\underset{\sim}{E}) = \{D\hat{\epsilon}({}_{\alpha}\underset{\sim}{E})\}^{T}$. We observe that in view of $(2.10)_{1}$, $(2.11)_{3}$ and (2.21) the value ${}_{\alpha}\underset{\sim}{T}^{+}$ of the stress tensor given by (4.22) for the motion ${}_{\alpha}\underset{\sim}{\chi}^{+}$ satisfies (2.24), so that (4.22) is a properly invariant constitutive equation. The Cauchy stress ${}_{\alpha}\underset{\sim}{T}^{*}$ in the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$ has the form

$${}_{\alpha}\underset{\sim}{T}^{*} = \tfrac{1}{2}\,{}_{\alpha}\rho^{*}\,{}_{\alpha}\underset{\sim}{F}^{*}\{D\hat{\epsilon}({}_{\alpha}\underset{\sim}{E}^{*}) + D^{T}\hat{\epsilon}({}_{\alpha}\underset{\sim}{E}^{*})\}({}_{\alpha}\underset{\sim}{F}^{*})^{T} \tag{4.23}$$

and we observe that $(3.46)_{1}$ is satisfied. To continue the discussion, let ${}_{1}\underset{\sim}{\chi}$ be an arbitrary known motion of $\mathcal{B}$ and ${}_{2}\underset{\sim}{\chi}$ some general motion. Having constructed the motions ${}_{1}\underset{\sim}{\chi}^{*}$ and ${}_{2}\underset{\sim}{\chi}^{*}$, we employ as our measure of smallness associated with

---

[‡] By an elastic material we mean a Green elastic material for which a potential function $\hat{\epsilon}$ is assumed to exist. In this subsection, the symbol ϵ is employed to represent the elastic strain energy ${}_{\alpha}\epsilon$, as well as the quantities ${}_{\alpha}\epsilon^{+}$ and ${}_{\alpha}\epsilon^{*}$; but this need not be confused with the use of ϵ for a different purpose in earlier parts of the paper, e.g., in Eqs. (2.28) and (2.29), or with the use of $\bar{\epsilon}$ in (4.24) and elsewhere in subsection 4.2.

${}_1\underset{\sim}{x}$ and ${}_2\underset{\sim}{x}$ the nonnegative real function

$$\bar{\epsilon} = \bar{\epsilon}(t^*) = \sup_{{}_1\underset{\sim}{x}^* \in {}_1\mathcal{R}^*} \|\underset{\sim}{H}({}_1\underset{\sim}{x}^*, t^*)\| \quad . \tag{4.24}$$

Following the same line of reasoning that led to (3.53), it may be deduced from (4.12) and (4.24) that

$$\underset{\sim}{h}({}_1\underset{\sim}{x}^*, t^*) = \underset{\sim}{O}(\bar{\epsilon}) \quad \text{as} \quad \bar{\epsilon} \to 0 \quad . \tag{4.25}$$

Next, with the use of the polar decomposition of $F^{*\prime}$, i.e.,

$$\underset{\sim}{F}^{*\prime} = \underset{\sim}{R}^{*\prime}\, \underset{\sim}{U}^{*\prime} \quad , \tag{4.26}$$

where $\underset{\sim}{R}^{*\prime}$ and $\underset{\sim}{U}^{*\prime}$ are proper orthogonal and symmetric positive definite tensors, respectively, we obtain estimates for various kinematical quantities. These are:

$$\text{(a)}\quad \underset{\sim}{F}^{*\prime} - \underset{\sim}{I} = \underset{\sim}{H} = \underset{\sim}{O}(\bar{\epsilon}) \quad , \quad \text{(b)}\quad (\underset{\sim}{F}^{*\prime})^{-1} - \underset{\sim}{I} = -\underset{\sim}{H} + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad ,$$

$$\text{(c)}\quad \underset{\sim}{U}^{*\prime} - \underset{\sim}{I} = \tfrac{1}{2}(\underset{\sim}{H}+\underset{\sim}{H}^T) + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad , \quad \text{(d)}\quad (\underset{\sim}{U}^{*\prime})^{-1} - \underset{\sim}{I} = -\tfrac{1}{2}(\underset{\sim}{H}+\underset{\sim}{H}^T) + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad , \tag{4.27}$$

$$\text{(e)}\quad \underset{\sim}{R}^{*\prime} - \underset{\sim}{I} = \tfrac{1}{2}(\underset{\sim}{H}-\underset{\sim}{H}^T) + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad , \quad \text{(f)}\quad (\underset{\sim}{R}^{*\prime})^T - \underset{\sim}{I} = -\tfrac{1}{2}(\underset{\sim}{H}-\underset{\sim}{H}^T) + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad ,$$

$$\text{(g)}\quad {}_2\underset{\sim}{E}^* - {}_1\underset{\sim}{E}^* = \tfrac{1}{2}({}_1\underset{\sim}{F}^*)^T\{\underset{\sim}{H}+\underset{\sim}{H}^T\}{}_1\underset{\sim}{F}^* + \underset{\sim}{O}(\bar{\epsilon}^2) = \underset{\sim}{O}(\bar{\epsilon}) \quad ,$$

as $\bar{\epsilon} \to 0$, where (4.15) has been used in deducing (4.27g).

Assuming sufficient smoothness, we expand $D\hat{\epsilon}({}_2\underset{\sim}{E}^*)$ in a Taylor series about the point ${}_1\underset{\sim}{E}^*$ and invoke (4.24) and (4.27g) to obtain

$$D\hat{\epsilon}({}_2\underset{\sim}{E}^*) = D\hat{\epsilon}({}_1\underset{\sim}{E}^*) + \tfrac{1}{2}D^2\hat{\epsilon}({}_1\underset{\sim}{E}^*)[({}_1\underset{\sim}{F}^*)^T(\underset{\sim}{H}+\underset{\sim}{H}^T){}_1\underset{\sim}{F}^*] + \underset{\sim}{O}(\bar{\epsilon}^2) \text{ as } \bar{\epsilon} \to 0 \quad , \tag{4.28}$$

where $D^2\hat{\epsilon}({}_1\underset{\sim}{E}^*)$ is the second derivative of $\hat{\epsilon}$ at ${}_1\underset{\sim}{E}^*$. Recalling the definition of the transpose of a fourth order tensor, we note the symmetry condition

$$(D^2\hat{\epsilon}({}_1\underset{\sim}{E}^*))^T = D^2\hat{\epsilon}({}_1\underset{\sim}{E}^*) \quad , \quad D^2_{ABCD}\hat{\epsilon}({}_1\underset{\sim}{E}^*) = D^2_{CDAB}\hat{\epsilon}({}_1\underset{\sim}{E}^*) \quad , \tag{4.29}$$

which follows from the assumed smoothness of $\hat{\epsilon}$. With regard to the relationship between the mass densities ${}_1\rho^*$ and ${}_2\rho^*$, from $(2.15)_1$, $(3.14)_2$, $(3.44)_1$, $(4.3)_{3,4}$ and (4.14) it may be deduced that

$${}_2\rho^* = {}_1\rho^*\{\det(\underset{\sim}{I}+\underset{\sim}{H})\}^{-1} = {}_1\rho^*\{1 - \mathrm{tr}\ \underset{\sim}{H} + O(\bar{\epsilon}^2)\} \text{ as } \bar{\epsilon}\to 0 \quad . \tag{4.30}$$

Substitution of (4.28) and (4.30) into (4.23), with $\alpha=2$, and use of $(4.14)_1$ together with (4.23), with $\alpha=1$, leads to the approximation

$${}_2\underset{\sim}{T}^* = (1 - \mathrm{tr}\ \underset{\sim}{H})\,{}_1\underset{\sim}{T}^* + {}_1\underset{\sim}{T}^*\underset{\sim}{H}^T + \underset{\sim}{H}\,{}_1\underset{\sim}{T}^* + \tfrac{1}{2}\,{}_1\rho^*\,{}_1\underset{\sim}{F}^*\bar{\underset{\sim}{K}}[({}_1\underset{\sim}{F}^*)^T(\underset{\sim}{H}+\underset{\sim}{H}^T)\,{}_1\underset{\sim}{F}^*]({}_1\underset{\sim}{F}^*)^T \quad , \tag{4.31}$$

where terms of $\underset{\sim}{O}(\bar{\epsilon}^2)$ as $\bar{\epsilon}\to 0$ have been omitted. The fourth order tensor $\bar{\underset{\sim}{K}} = \bar{K}_{ABCD}\underset{\sim}{e}_A\otimes\underset{\sim}{e}_B\otimes\underset{\sim}{e}_C\otimes\underset{\sim}{e}_D$ is defined by‡

$$\bar{K}_{ABCD} = \tfrac{1}{4}\{D^2_{ABCD}\hat{\epsilon}({}_1\underset{\sim}{E}^*) + D^2_{ABDC}\hat{\epsilon}({}_1\underset{\sim}{E}^*) + D^2_{BACD}\hat{\epsilon}({}_1\underset{\sim}{E}^*) + D^2_{BADC}\hat{\epsilon}({}_1\underset{\sim}{E}^*)\} \quad . \tag{4.32}$$

In view of (4.29) and (4.32), $\bar{\underset{\sim}{K}}$ possesses the symmetries

$$\bar{K}_{ABCD} = \bar{K}_{BACD} = \bar{K}_{ABDC} = \bar{K}_{CDAB} \quad . \tag{4.33}$$

We now proceed to show that the expression (4.31) is unaltered when, in accordance with the transformations (2.7), rigid body motions are superposed on ${}_\alpha\underset{\sim}{X}$ resulting in the motions ${}_\alpha\underset{\sim}{X}^+$. First, defining $\bar{\underset{\sim}{K}}^+$ in a manner analogous to that in which $\bar{\underset{\sim}{K}}$ was defined by (4.32), we observe that $\bar{\underset{\sim}{K}}^+ = \bar{\underset{\sim}{K}}$. Using (4.31), the stress tensor ${}_2\underset{\sim}{T}^{+*}$ in the motion $({}_2\underset{\sim}{X}^+)^*$ is given by:

‡Due to the symmetry of the term in square brackets in (4.28), it is only the part $\frac{1}{2}(D^2_{ABCD}\hat{\epsilon}({}_1\underset{\sim}{E}^*) + D^2_{ABDC}\hat{\epsilon}({}_1\underset{\sim}{E}^*))$ that contributes to the expression.

$$
\begin{aligned}
{}_2\mathbf{T}^{+*} &= (1-\mathrm{tr}\,\mathbf{H}^{+})\,{}_1\mathbf{T}^{+*} + {}_1\mathbf{T}^{+*}(\mathbf{H}^{+})^{T} + \mathbf{H}^{+}\,{}_1\mathbf{T}^{+*} \\
&\quad + \tfrac{1}{2}\,{}_1\rho^{+*}\,{}_1\mathbf{F}^{+*}\overline{\mathbf{K}}^{+}[({}_1\mathbf{F}^{+*})^{T}(\mathbf{H}^{+}+(\mathbf{H}^{+})^{T})\,{}_1\mathbf{F}^{+*}]({}_1\mathbf{F}^{+*})^{T} \\
&= (1-\mathrm{tr}\,\mathbf{H})\,{}_1\mathbf{T}^{*} + {}_1\mathbf{T}^{*}\,\mathbf{H}^{T} + \mathbf{H}\,{}_1\mathbf{T}^{*} \\
&\quad + \tfrac{1}{2}\,{}_1\rho^{*}\,{}_1\mathbf{F}^{*}\overline{\mathbf{K}}[({}_1\mathbf{F}^{*})^{T}(\mathbf{H}+\mathbf{H}^{T})\,{}_1\mathbf{F}^{*}]({}_1\mathbf{F}^{*})^{T} \\
&= {}_2\mathbf{T}^{*}\ ,
\end{aligned}
\tag{4.34}
$$

where (4.20) has been used along with equations (3.46), $(3.44)_1$ and $(3.27)_1$ with $\alpha = 1$. Therefore, by (4.34), the transformation property $(3.46)_1$ with $\alpha = 2$ is satisfied when ${}_2\mathbf{T}^*$ is given by (4.31), so that (4.31) is a properly invariant expression. The significance of (4.34) is that the relation (4.31) transforms correctly when arbitrary finite rigid motions are superposed independently (and possibly simultaneously) on both ${}_1\mathbf{X}$ and ${}_2\mathbf{X}$.

In order to derive an expression for the traction vector ${}_2\mathbf{t}^*$, we note by (4.27b) that

$$
\|((\mathbf{F}^{*\prime})^{-1})^{T}\,{}_1\mathbf{n}^{*}\|^{-1} = 1 + \tfrac{1}{2}\,{}_1\mathbf{n}^{*}\cdot(\mathbf{H}+\mathbf{H}^{T})\,{}_1\mathbf{n}^{*} + O(\bar{\epsilon}^{2}) \text{ as } \bar{\epsilon}\to 0 \tag{4.35}
$$

and hence by $(4.3)_7$

$$
{}_2\mathbf{n}^{*} = {}_1\mathbf{n}^{*} + \tfrac{1}{2}\{{}_1\mathbf{n}^{*}\cdot(\mathbf{H}+\mathbf{H}^{T})\,{}_1\mathbf{n}^{*}\}\,{}_1\mathbf{n}^{*} - \mathbf{H}^{T}\,{}_1\mathbf{n}^{*} + \mathbf{O}(\bar{\epsilon}^{2}) \text{ as } \bar{\epsilon}\to 0\ . \tag{4.36}
$$

It then follows from $(3.44)_{1,2,3}$, (2.15), (4.31) and (4.36) that when terms of $O(\bar{\epsilon}^2)$ are omitted

$$
\begin{aligned}
{}_2\mathbf{t}^{*} &= (1-\mathrm{tr}\,\mathbf{H} + \tfrac{1}{2}\,{}_1\mathbf{n}^{*}\cdot(\mathbf{H}+\mathbf{H}^{T})\,{}_1\mathbf{n}^{*})\,{}_1\mathbf{t}^{*} \\
&\quad + \mathbf{H}\,{}_1\mathbf{t}^{*} + \tfrac{1}{2}\,{}_1\rho^{*}\,{}_1\mathbf{F}^{*}\overline{\mathbf{K}}[({}_1\mathbf{F}^{*})^{T}(\mathbf{H}+\mathbf{H}^{T})\,{}_1\mathbf{F}^{*}]({}_1\mathbf{F}^{*})^{T}\,{}_1\mathbf{n}^{*}\ .
\end{aligned}
\tag{4.37}
$$

4.3 Invariant infinitesimal elasticity

The results of the subsection 4.2 can be easily specialized to yield a properly invariant infinitesimal theory of elasticity as follows.

$$ {}_1\mathbf{X} = {}_0\mathbf{X} \quad , \quad {}_2\mathbf{X} = \mathbf{X} \quad . \tag{4.38} $$

Then, clearly

$$ {}_1\mathbf{F} = {}_1\mathbf{R} = \mathbf{I} \quad , \quad {}_1\mathbf{X}^* = {}_0\mathbf{X} \quad , \quad {}_1\mathbf{F}^* = \mathbf{I} \quad , \quad {}_1\rho^* = {}_0\rho \quad . \tag{4.39} $$

The stress tensor in the motion ${}_1\mathbf{X} = {}_0\mathbf{X}$ is obtained from (4.22) with ${}_0\mathbf{F} = \mathbf{I}$, ${}_0\mathbf{R} = \mathbf{I}$ and is given by

$$ {}_1\mathbf{T} = {}_0\mathbf{T} + {}_0\rho\{D\hat{\epsilon}(\mathbf{0}) + D^T\hat{\epsilon}(\mathbf{0})\} \quad . \tag{4.40} $$

Assuming a stress-free reference configuration ${}_0\mathbf{\kappa}$, we take

$$ {}_1\mathbf{T} = {}_0\mathbf{T} = \mathbf{0} \quad , \tag{4.41} $$

and hence by $(3.44)_4$

$$ {}_1\mathbf{T}^* = \mathbf{0} \quad . \tag{4.42} $$

In keeping with $(4.38)_2$ we suppress the subscript 2 in all quantities associated with the motion ${}_2\mathbf{X} = \mathbf{X}$. Thus ${}_2\mathbf{X}^*$ becomes $\mathbf{X}^*$, ${}_2\mathbf{F}^*$ becomes $\mathbf{F}^*$, etc. Then, by (4.1), $(4.38)_1$ and $(4.39)_2$, we have

$$ \mathbf{X}' = \mathbf{X} \quad , \quad \mathbf{X}^{*\prime} = \mathbf{X}^* \quad , \tag{4.43} $$

since ${}_0\mathbf{X}^{-1} = {}_0\mathbf{X}$. Furthermore, by $(4.3)_3$ and $(4.39)_3$,

$$ \mathbf{F}^{*\prime} = \mathbf{F}^* \quad . \tag{4.44} $$

From $(4.8)_1$, $(4.39)_2$ and $(3.12)_2$ we obtain

$$ \mathbf{h}^*(\mathbf{X},t^*) = (\mathbf{X}^* - {}_0\mathbf{X})(\mathbf{X},t^*) = \mathbf{u}^* \tag{4.45} $$

and we see from (4.44), $(4.14)_1$ and $(3.12)_3$ that

$$\underset{\sim}{H} = \underset{\sim}{F}^{*} - \underset{\sim}{I} - \underset{\sim}{j}^{*} \quad . \tag{4.46}$$

Consequently $\bar{\boldsymbol{\epsilon}}$ in (4.24) becomes equal to $\boldsymbol{\epsilon}^{*}$ of (3.48). In view of (4.39) and $(3.2[illegible])_{2,3}$ the tensor $\bar{\underset{\sim}{K}}$ defined in (4.32) becomes $\underset{\sim}{K}$, where

$$K_{ABCD} = \tfrac{1}{4}\{D^{2}_{ABCD}\hat{\boldsymbol{\epsilon}}(\underset{\sim}{0}) + D^{2}_{ABCD}\hat{\boldsymbol{\epsilon}}(\underset{\sim}{0}) + D^{2}_{BACD}\hat{\boldsymbol{\epsilon}}(\underset{\sim}{0}) + D^{2}_{BADC}\hat{\boldsymbol{\epsilon}}(\underset{\sim}{0})\} \quad . \tag{4.47}$$

This is the same quantity that appears in equation (1.1). With the help of [illegible], $(4.39)_{3,4}$, (4.47), (4.46) and $(3.13)_4$, specializing (4.31) we deduce the desired constitutive equation for linearly elastic solid, which was recorded earlier (see Eq. (1.13)). Having been obtained as a special case of (4.31), clearly (1.13) is properly invariant under the transformation (2.7) with $\alpha = 2$ suppressed. Alternatively, the invariance of (1.13) can be established at once from $(3.27)_8$. It is then seen that $(3.46)_1$, with $\alpha = 2$ suppressed, is satisfied when $\underset{\sim}{T}^{*}$ is given by (1.13).

Next we obtain an expression for the traction vector $\underset{\sim}{t}^{*} = {}_2\underset{\sim}{t}^{*}$. First, by $(4.3[illegible])_1$, $(4.39)_{2,3}$ and (2.18) we note that

$${}_1\underset{\sim}{n}^{*} = {}_0\underset{\sim}{n} \quad . \tag{4.48}$$

It then follows from (4.36), (4.46) and $(3.13)_4$ that

$$\underset{\sim}{n}^{*} = {}_2\underset{\sim}{n}^{*} = (1 + {}_0\underset{\sim}{n} \cdot \underset{\sim}{e}^{*}\,{}_0\underset{\sim}{n})\,{}_0\underset{\sim}{n} - (\underset{\sim}{G}^{*})^{T}\,{}_0\underset{\sim}{n} \tag{4.49}$$

where terms of $\underset{\sim}{O}(\bar{\boldsymbol{\epsilon}}^{2})$ or equivalently of $\underset{\sim}{O}((\boldsymbol{\epsilon}^{*})^{2})$ have been omitted. Now it follows from (4.42), together with (2.15) in the form ${}_1\underset{\sim}{t}^{*} = {}_1\underset{\sim}{T}^{*}\,{}_1\underset{\sim}{n}^{*}$, that

$${}_1\underset{\sim}{t}^{*} = \underset{\sim}{0} \quad . \tag{4.50}$$

With the help of $(4.39)_{3,4}$, (4.46), $(3.13)_4$, (4.48), (4.50) and recalling that $\bar{\underset{\sim}{K}}$ reduces to $\underset{\sim}{K}$ it follows from (4.37) that

$$\underset{\sim}{t}^{*} = {}_0\rho\, \underset{\sim}{K}[\underset{\sim}{e}^{*}]\,{}_0\underset{\sim}{n} \quad . \tag{4.51}$$

This expression agrees with that derived from (1.13) and (4.49) when terms of $\underset{\sim}{O}(\bar{\epsilon}^2)$ are omitted.

To complete the infinitesimal theory of motions superposed on a given motion, it is necessary to insert (4.31) in the equations of motion written in terms of the quantities appearing on the right-hand side of $(3.44)_{5,6}$, and to express all quantities in terms of the variable ${}_1\underset{\sim}{x}^*$.

5. Consequence of a change of pivot

It is evident that the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$ depends on the choice of pivot. In this section, we examine how our results behave when one particle Y' is chosen for pivot rather than another Y. We temporarily attach a subscript Y to quantities associated with the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}$ introduced in (3.2). In a manner paralleling (3.2), we define a motion ${}_{\alpha}\underset{\sim}{\chi}^{*}_{Y'} = \underset{\sim}{\pi}_{Y'}({}_{\alpha}\underset{\sim}{\chi}^{*})$ associated with the pivot Y' as follows†:

$$ {}_{\alpha}\underset{\sim}{x}^{*}_{Y'} = {}_{\alpha}\underset{\sim}{\chi}^{*}_{Y'}(\underset{\sim}{X},t^{*}) = {}_{\alpha}\underset{\sim}{R}^{T}(\underset{\sim}{Y}',t)\{{}_{\alpha}\underset{\sim}{\chi}(\underset{\sim}{X},t) - {}_{\alpha}\underset{\sim}{\chi}(\underset{\sim}{Y}',t)\} + \underset{\sim}{Y}' \quad , \tag{5.1} $$

$\underset{\sim}{Y}' = {}_{0}\underset{\sim}{\kappa}(Y')$ being the position vector of the particle Y' in the reference configuration ${}_{0}\underset{\sim}{\kappa}$. The deformation gradient ${}_{\alpha}\underset{\sim}{F}^{*}_{Y'}$ in the motion (5.1) satisfies the relations

$$ {}_{\alpha}\underset{\sim}{F}^{*}_{Y'} = {}_{\alpha}\underset{\sim}{R}^{T}(\underset{\sim}{Y}',t)\,{}_{\alpha}\underset{\sim}{F} = {}_{\alpha}\underset{\sim}{\bar{R}}^{T}\,{}_{\alpha}\underset{\sim}{F}^{*}_{Y} \quad , \tag{5.2} $$

where

$$ {}_{\alpha}\underset{\sim}{\bar{R}} = {}_{\alpha}\underset{\sim}{R}^{T}(\underset{\sim}{Y},t)\,{}_{\alpha}\underset{\sim}{R}(\underset{\sim}{Y}',t) = {}_{\alpha}\underset{\sim}{R}^{*}_{Y}(\underset{\sim}{Y}',t^{*}) \tag{5.3} $$

is the rotation at the particle Y' in the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}_{Y}$, and where use has been made of $(3.14)_{1,6}$. We may, as in (3.13) and $(3.12)_{3}$, define tensors ${}_{\alpha}\underset{\sim}{R}^{*}_{Y'}$, ${}_{\alpha}\underset{\sim}{U}^{*}_{Y'}$, ${}_{\alpha}\underset{\sim}{C}^{*}_{Y'}$, ${}_{\alpha}\underset{\sim}{E}^{*}_{Y'}$, ${}_{\alpha}\underset{\sim}{G}^{*}_{Y'}$, ${}_{\alpha}\underset{\sim}{e}^{*}_{Y'}$ and ${}_{\alpha}\underset{\sim}{w}^{*}_{Y'}$ associated with the motion ${}_{\alpha}\underset{\sim}{\chi}^{*}_{Y'}$. It is then readily seen that

†We could replace ${}_{\alpha}t^{*} = {}_{\alpha}t^{*}_{Y}$ of $(3.2)_{2}$ by ${}_{\alpha}t^{*}_{Y'}$, with ${}_{\alpha}t^{*}_{Y}$ and ${}_{\alpha}t^{*}_{Y'}$ differing from one another by a constant, but in view of our remark (following (3.2)) on parameterization, we take ${}_{\alpha}t^{*}_{Y'} = {}_{\alpha}t^{*}_{Y} = t^{*}$.

$$
\begin{gathered}
{}_{\alpha}\mathbf{C}^{*}_{Y'} = {}_{\alpha}\mathbf{C}^{*}_{Y} \ , \quad {}_{\alpha}\mathbf{U}^{*}_{Y'} = {}_{\alpha}\mathbf{U}^{*}_{Y} \ , \quad {}_{\alpha}\mathbf{E}^{*}_{Y'} = {}_{\alpha}\mathbf{E}^{*}_{Y} \ , \\
{}_{\alpha}\mathbf{R}^{*}_{Y'} = {}_{\alpha}\bar{\mathbf{R}}^{T} \, {}_{\alpha}\mathbf{R}^{*}_{Y} \ , \\
{}_{\alpha}\mathbf{G}^{*}_{Y'} = {}_{\alpha}\bar{\mathbf{R}}^{T}\{{}_{\alpha}\mathbf{G}^{*}_{Y}+\mathbf{I}\} - \mathbf{I} \ , \qquad (5.4) \\
2\,{}_{\alpha}\mathbf{e}^{*}_{Y'} = 2\,{}_{\alpha}\mathbf{e}^{*}_{Y} - \{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\}^{T}\{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\} + \{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\}\,{}_{\alpha}\mathbf{G}^{*}_{Y} + (\{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\}\,{}_{\alpha}\mathbf{G}^{*}_{Y})^{T} \ , \\
2\,{}_{\alpha}\mathbf{w}^{*}_{Y'} = 2\,{}_{\alpha}\mathbf{w}^{*}_{Y} + {}_{\alpha}\bar{\mathbf{R}}^{T} - {}_{\alpha}\bar{\mathbf{R}} + \{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\}\,{}_{\alpha}\mathbf{G}^{*}_{Y} - (\{{}_{\alpha}\bar{\mathbf{R}}^{T}-\mathbf{I}\}\,{}_{\alpha}\mathbf{G}^{*}_{Y})^{T} \ .
\end{gathered}
$$

Also, let ${}_{\alpha}\rho^{*}_{Y'}$ be the mass density in the configuration ${}_{\alpha}\kappa^{*}_{Y'} = {}_{\alpha}\chi^{*}_{Y'} \circ {}_{0}\kappa$ and ${}_{\alpha}\mathbf{T}^{*}_{Y'}$ the Cauchy stress tensor associated with the motion ${}_{\alpha}\chi^{*}_{Y'}$. Then,

$$
\begin{gathered}
{}_{\alpha}\rho^{*}_{Y'} = {}_{\alpha}\rho^{*}_{Y} \ , \qquad (5.5) \\
{}_{\alpha}\mathbf{T}^{*}_{Y'} = {}_{\alpha}\mathbf{R}^{T}(\mathbf{Y}',t)\,{}_{\alpha}\mathbf{T}\,{}_{\alpha}\mathbf{R}(\mathbf{Y}',t) = {}_{\alpha}\bar{\mathbf{R}}^{T}\,{}_{\alpha}\mathbf{T}^{*}_{Y}\,{}_{\alpha}\bar{\mathbf{R}} \ ,
\end{gathered}
$$

where $(3.44)_{1,4}$ and (5.3) have been used.

It follows from (3.49h) and (5.3) that

$$
{}_{\alpha}\bar{\mathbf{R}}^{T} - \mathbf{I} = -\,{}_{\alpha}\bar{\mathbf{w}} + \mathbf{O}((\epsilon^{*}_{Y})^{2}) = \mathbf{O}(\epsilon^{*}_{Y}) \text{ as } \epsilon^{*}_{Y} \to 0 \ , \qquad (5.6)
$$

where

$$
{}_{\alpha}\bar{\mathbf{w}} = -\,{}_{\alpha}\mathbf{w}^{*}_{Y}(\mathbf{Y}',{}_{\alpha}t^{*}) \qquad (5.7)
$$

Hence, in view of $(5.4)_{5,6,7}$,

$$
\begin{gathered}
{}_{\alpha}\mathbf{G}^{*}_{Y'} = {}_{\alpha}\mathbf{G}^{*}_{Y} - {}_{\alpha}\bar{\mathbf{w}} + \mathbf{O}((\epsilon^{*}_{Y})^{2}) = \mathbf{O}(\epsilon^{*}_{Y}) \ , \\
{}_{\alpha}\mathbf{e}^{*}_{Y'} = {}_{\alpha}\mathbf{e}^{*}_{Y} + \mathbf{O}((\epsilon^{*}_{Y})^{2}) = \mathbf{O}(\epsilon^{*}_{Y}) \ , \qquad (5.8) \\
{}_{\alpha}\mathbf{w}^{*}_{Y'} = {}_{\alpha}\mathbf{w}^{*}_{Y} - {}_{\alpha}\bar{\mathbf{w}} + \mathbf{O}((\epsilon^{*}_{Y})^{2}) = \mathbf{O}(\epsilon^{*}_{Y})
\end{gathered}
$$

as $\epsilon^{*}_{Y} \to 0$. In accordance with $(5.8)_2$, when Y' is chosen as pivot, the infinitesimal strain tensor ${}_{\alpha}\mathbf{e}^{*}_{Y'}$ coincides with ${}_{\alpha}\mathbf{e}^{*}_{Y}$ (corresponding to the pivot Y) to within terms of $O((\overset{*}{\epsilon}_Y)^2)$. Furthermore, if ${}_\alpha\overset{*}{\mathbf{T}}_Y = O(\overset{*}{\epsilon}_Y)$ as it is in (1.15), it follows from (5.[illegible]) and (5.6) that

$$ {}_\alpha\overset{*}{\mathbf{T}}_{Y'} = {}_\alpha\overset{*}{\mathbf{T}}_Y + O((\overset{*}{\epsilon}_Y)^2) \text{ as } \overset{*}{\epsilon}_Y \to 0 \quad . \tag{5.9} $$

Therefore, the choice of pivot is immaterial in the infinitesimal theory.

Returning to the more general case of small on large, associated with the motions ${}_\alpha\overset{*}{\chi}_{Y'}$ $(\alpha = 1,2)$ we may introduce a difference motion $\overset{*}{\chi}{}'_Y$ in the same manner as the difference motion $\overset{*}{\chi}{}'_Y$ was defined in $(4.1)_1$. It is then readily seen that $\overset{*}{\mathbf{F}}{}'_Y$, the deformation gradient of $\overset{*}{\chi}{}'_Y$, satisfies the relations

$$ \overset{*}{\mathbf{F}}_{Y'} = {}_2\mathbf{R}^T(Y',t)\mathbf{F}'\,{}_1\mathbf{R}(Y',t) = {}_2\bar{\mathbf{R}}^T\,\overset{*}{\mathbf{F}}{}'_Y\,{}_1\bar{\mathbf{R}} = \overset{*}{\mathbf{R}}{}'_Y\,\overset{*}{\mathbf{U}}{}'_Y \quad , \tag{5.10} $$

where $(4.5)_1$ and (5.5) have been used and where $\overset{*}{\mathbf{R}}{}'$ and $\overset{*}{\mathbf{U}}{}'$ are proper orthogonal and symmetric positive definite tensors, respectively. Likewise, considering displacement fields $\mathbf{h}_{Y'}$ and $\mathbf{h}_Y$ in the form (4.11) and taking their gradients $\mathbf{H}_Y$ and $\mathbf{H}_{Y'}$ as in (4.13), it follows at once that

$$ \mathbf{H}_{Y'} = \overset{*}{\mathbf{F}}{}'_Y - \mathbf{I} = {}_2\bar{\mathbf{R}}^T(\mathbf{H}_Y + \mathbf{I})\,{}_1\bar{\mathbf{R}} - \mathbf{I} \quad , \tag{5.11} $$

where $(4.14)_1$ and (5.1[illegible]) have been used. The measure of smallness in (4.24) is now denoted by $\bar{\epsilon}_Y$ and, in addition, with $\mathbf{H}_{Y'}$ we associate a measure of smallness $\bar{\epsilon}_{Y'}$ by means of the formula

$$ \bar{\epsilon}_{Y'} = \sup_{{}_1\overset{*}{\chi}_{Y'} \in {}_1\overset{*}{\mathcal{R}}_{Y'}} |\mathbf{H}_{Y'}({}_1\overset{*}{\chi}_{Y'}, \overset{*}{t})| \quad , \tag{5.12} $$

where ${}_\alpha\overset{*}{\mathcal{R}}_{Y'} = \overset{*}{\chi}_{Y'}(\mathcal{R}, \overset{*}{t})$. It follows from (4.2[illegible]), (5.10) and (4.2[illegible]) that

$$ \begin{aligned} (\overset{*}{\mathbf{U}}_{Y'})^2 &= {}_1\bar{\mathbf{R}}^T(\overset{*}{\mathbf{U}}{}'_Y)^2\,{}_1\bar{\mathbf{R}} = \mathbf{I} + {}_1\bar{\mathbf{R}}^T(\mathbf{H}_Y + \mathbf{H}_Y^T)\,{}_1\bar{\mathbf{R}} + O(\bar{\epsilon}_Y^2) \\ (\overset{*}{\mathbf{U}}_{Y'})^{-1} &= \mathbf{I} - \tfrac{1}{2}\,{}_1\bar{\mathbf{R}}^T(\mathbf{H}_Y + \mathbf{H}_Y^T)\,{}_1\bar{\mathbf{R}} + O(\bar{\epsilon}_Y^2) \end{aligned} \tag{5.13} $$

as $\bar{\epsilon}_Y \to 0$ and, hence by (5.13) and (4.27a,c),

$$\mathbf{R}^{*\prime}_Y = {}_2\bar{\mathbf{R}}^{-1}\{\mathbf{I} + \tfrac{1}{2}(\mathbf{H}_Y - \mathbf{H}_Y^T)\}\,{}_1\bar{\mathbf{R}} + \mathbf{O}(\bar{\epsilon}_Y^2)$$

$$= {}_2\bar{\mathbf{R}}^{-1}\,\mathbf{R}^{*\prime}_Y\,{}_1\bar{\mathbf{R}} + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \tag{5.14}$$

Next we observe that $(3.13)_1$, $(4.14)_1$ and (4.24) lead to the relations

$$({}_1\mathbf{U}^*_Y)^{-1}({}_2\mathbf{U}^*_Y)^2({}_1\mathbf{U}^*_Y)^{-1} = ({}_1\mathbf{R}^*_Y)^T(\mathbf{I} + \mathbf{H}_Y + \mathbf{H}_Y^T + \mathbf{H}_Y^T\mathbf{H}_Y)\,{}_1\mathbf{R}^*_Y$$

$$= \mathbf{I} + ({}_1\mathbf{R}^*_Y)^T(\mathbf{H}_Y + \mathbf{H}_Y^T)\,{}_1\mathbf{R}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \tag{5.15}$$

Therefore $({}_1\mathbf{U}^*_Y)^{-1}\,{}_2\mathbf{U}^*_Y$ may be written in the form

$$({}_1\mathbf{U}^*_Y)^{-1}\,{}_2\mathbf{U}^*_Y = ({}_1\mathbf{R}^*_Y)^T\{\mathbf{I} + \tfrac{1}{2}(\mathbf{H}_Y + \mathbf{H}_Y^T) + \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad , \tag{5.16}$$

where[+] $\mathbf{\Psi}$ is a skew-symmetric function of $\mathbf{O}(\bar{\epsilon}_Y)$. The symmetry of ${}_2\mathbf{U}^*_Y$ places a further restriction on $\mathbf{\Psi}$, namely[‡]

$${}_1\mathbf{U}^*_Y({}_1\mathbf{R}^*_Y)^T\{\tfrac{1}{2}(\mathbf{H}_Y + \mathbf{H}_Y^T) + \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y = ({}_1\mathbf{R}^*_Y)^T\{\tfrac{1}{2}(\mathbf{H}_Y + \mathbf{H}_Y^T) - \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y\,{}_1\mathbf{U}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \tag{5.17}$$

Using (5.16) and (5.17), we may express ${}_2\mathbf{U}^*_Y$ in the form

$${}_2\mathbf{U}^*_Y = ({}_1\mathbf{R}^*_Y)^T\{\mathbf{I} + \tfrac{1}{2}(\mathbf{H}_Y + \mathbf{H}_Y^T) - \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y\,{}_1\mathbf{U}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \tag{5.18}$$

Furthermore, in view of $(3.13)_1$, $(4.14)_1$, (5.16) and (4.27e)

$${}_2\mathbf{R}^*_Y = \{\mathbf{I} + \tfrac{1}{2}(\mathbf{H}_Y - \mathbf{H}_Y^T) + \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) = \{\mathbf{R}^{*\prime}_Y + \mathbf{\Psi}\}\,{}_1\mathbf{R}^*_Y + \mathbf{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \tag{5.19}$$

Adopting the notation

[+] The function $\mathbf{\Psi}$ may be written as a linear function of $\mathbf{H}_Y$. It is not difficult to show that $\mathbf{\Psi}$ always exists and is unique.

[‡] In particular, if ${}_1\mathbf{U}^*_Y = \mathbf{I}$, (5.17) implies that $\mathbf{\Psi} = \mathbf{0}$.

$$\bar{\underset{\sim}{y}} = {}_1\underset{\sim}{X}^*(\underset{\sim}{Y}',t^*) \quad ,$$

$$\underset{\sim}{W}(YY') = \tfrac{1}{2}\{\underset{\sim}{H}_Y(\bar{\underset{\sim}{y}},t^*) - \underset{\sim}{H}_Y^T(\bar{\underset{\sim}{y}},t^*)\} + \underset{\sim}{\Psi}(\bar{\underset{\sim}{y}},t^*) = -\underset{\sim}{W}^T(YY') = \underset{\sim}{O}(\bar{\epsilon}_Y) \text{ as } \bar{\epsilon}_Y \to 0 \qquad (5.20)$$

it follows from (5.19) and (5.3) that

$${}_2\bar{\underset{\sim}{R}} = \{\underset{\sim}{I} + \underset{\sim}{W}(YY')\}{}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \qquad (5.21)$$

Then, by (5.14) and (4.27e),

$$\underset{\sim}{R}_{Y'}^{*} = \underset{\sim}{I} + {}_1\bar{\underset{\sim}{R}}^T\{\tfrac{1}{2}(\underset{\sim}{H}_Y - \underset{\sim}{H}_Y^T) - \underset{\sim}{W}(YY')\}{}_1\bar{\underset{\sim}{R}}_Y + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \qquad (5.22)$$

In particular, it is clear from (5.20) and (5.22) that

$$\underset{\sim}{R}_{Y'}^{*}(\bar{\underset{\sim}{y}},t^*) = \underset{\sim}{I} - {}_1\bar{\underset{\sim}{R}}^T\,\underset{\sim}{\Psi}(\bar{\underset{\sim}{y}},t^*){}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_y^2) = \underset{\sim}{I} + \underset{\sim}{O}(\bar{\epsilon}_Y) \text{ as } \bar{\epsilon}_Y \to 0 \quad . \qquad (5.23)$$

Next, we return to (5.11) and make use of (5.21) to obtain the following estimates for $\underset{\sim}{H}_{Y'}$, its symmetric and skew-symmetric parts and for $\underset{\sim}{F}_{Y'}^{*}$:

$$\begin{aligned}
\underset{\sim}{H}_{Y'} &= {}_1\bar{\underset{\sim}{R}}^T\{\underset{\sim}{H}_Y - \underset{\sim}{W}(YY')\}{}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_Y^2) = \underset{\sim}{O}(\bar{\epsilon}_Y) \quad , \\
\tfrac{1}{2}(\underset{\sim}{H}_{Y'} + \underset{\sim}{H}_{Y'}^T) &= \tfrac{1}{2}{}_1\bar{\underset{\sim}{R}}^T\{\underset{\sim}{H}_Y + \underset{\sim}{H}_Y^T\}{}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \quad , \\
\tfrac{1}{2}(\underset{\sim}{H}_{Y'} - \underset{\sim}{H}_{Y'}^T) &= \tfrac{1}{2}{}_1\bar{\underset{\sim}{R}}^T\{\underset{\sim}{H}_Y - \underset{\sim}{H}_Y^T - \underset{\sim}{W}(YY')\}{}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \quad , \\
\underset{\sim}{F}_{Y'}^{*} &= {}_1\bar{\underset{\sim}{R}}^T\{\underset{\sim}{F}_Y^{*} - \underset{\sim}{W}(YY')\}{}_1\bar{\underset{\sim}{R}} + \underset{\sim}{O}(\bar{\epsilon}_Y^2)
\end{aligned} \qquad (5.24)$$

as $\bar{\epsilon}_Y \to 0$.

If Y is used as a pivot, then the stress tensor ${}_2\underset{\sim}{T}_Y^*$ is given by (4.31) (with a subscript Y attached to all quantities) while if Y′ is used, ${}_2\underset{\sim}{T}_{Y'}^*$ is given by

$$
\begin{aligned}
{}_2\mathbf{T}^*_{Y'} = {} & (1 - \mathrm{tr}\,\mathbf{H}_{Y'})\,{}_1\mathbf{T}^*_{Y'} + {}_1\mathbf{T}^*_{Y'}\,\mathbf{H}^T_{Y'} + \mathbf{H}_{Y'}\,{}_1\mathbf{T}^*_{Y'} \\
& + \tfrac{1}{2}\,{}_1\rho^*_{Y'}\,{}_1\mathbf{F}^*_{Y'}\,\bar{\mathbf{K}}_{Y'}[({}_1\mathbf{F}^*_{Y'})^T(\mathbf{H}_{Y'}+\mathbf{H}^T_{Y'})\,{}_1\mathbf{F}^*_{Y'}]({}_1\mathbf{F}^*_{Y'})^T \\
& + \mathbf{O}(\bar{\epsilon}^2_{Y'}) \quad \text{as} \quad \bar{\epsilon}_{Y'} \to 0 \quad ,
\end{aligned}
\tag{5.25}
$$

where $\bar{\mathbf{K}}_{Y'}$ is defined similar to that in (4.32). We now proceed to show that (5.25) implies (4.31). To this end, we first note from (4.32) and $(5.4)_3$ that $\bar{\mathbf{K}}_{Y'} = \bar{\mathbf{K}}_Y$ and also recall from (5.24) that $\mathbf{H}_{Y'}$ is of $\mathbf{O}(\bar{\epsilon}_Y)$ so that the error term in (5.25) is of $\mathbf{O}(\bar{\epsilon}^2_Y)$. Then, with the use of (5.5) we deduce from (5.25) that

$$
\begin{aligned}
{}_2\mathbf{T}^*_{Y} = {} & (1 - \mathrm{tr}\,\mathbf{H}_{Y'})\,{}_2\bar{\mathbf{R}}\,{}_1\bar{\mathbf{R}}^T\,{}_1\mathbf{T}^*_{Y}\,{}_1\bar{\mathbf{R}}\,{}_2\bar{\mathbf{R}}^T \\
& + {}_2\bar{\mathbf{R}}\,{}_1\bar{\mathbf{R}}^T\,{}_1\mathbf{T}^*_{Y}\,{}_1\bar{\mathbf{R}}\,\mathbf{H}^T_{Y'}\,{}_2\bar{\mathbf{R}}^T \\
& + {}_2\bar{\mathbf{R}}\,\mathbf{H}_{Y'}\,{}_1\bar{\mathbf{R}}^T\,{}_1\mathbf{T}^*_{Y}\,{}_1\bar{\mathbf{R}}\,{}_2\bar{\mathbf{R}}^T \\
& + \tfrac{1}{2}\,{}_1\rho^*_{Y}\,{}_2\bar{\mathbf{R}}\,{}_1\mathbf{F}^*_{Y'}\,\bar{\mathbf{K}}_{Y}[({}_1\mathbf{F}^*_{Y'})^T(\mathbf{H}_{Y'}+\mathbf{H}^T_{Y'})\,{}_1\mathbf{F}^*_{Y'}]({}_1\mathbf{F}^*_{Y'})^T\,{}_2\bar{\mathbf{R}}^T \\
& + \mathbf{O}(\bar{\epsilon}^2_{Y}) \quad \text{as} \quad \bar{\epsilon}_{Y} \to 0 \quad ,
\end{aligned}
\tag{5.26}
$$

where (5.21) has been noted in writing the error term. With the help of (5.24), (5.21) and (5.2) it is readily seen that

$$
\begin{aligned}
\mathrm{tr}\,\mathbf{H}_{Y'} &= \mathrm{tr}\,\mathbf{H}_{Y} + \mathbf{O}(\bar{\epsilon}^2_{Y}) \quad , \\
{}_2\bar{\mathbf{R}}\,{}_1\bar{\mathbf{R}}^T &= \mathbf{I} + \mathbf{W}(YY') + \mathbf{O}(\bar{\epsilon}^2_{Y}) \quad , \\
{}_2\bar{\mathbf{R}}\,\mathbf{H}_{Y'}\,{}_1\bar{\mathbf{R}}^T &= \mathbf{H}_{Y} - \mathbf{W}(YY') + \mathbf{O}(\bar{\epsilon}^2_{Y}) \quad , \\
{}_2\bar{\mathbf{R}}\,{}_1\mathbf{F}^*_{Y'} &= \{\mathbf{I} + \mathbf{W}(YY')\}\,{}_1\mathbf{F}^*_{Y} + \mathbf{O}(\bar{\epsilon}^2_{Y}) \quad , \\
({}_1\mathbf{F}^*_{Y'})^T(\mathbf{H}_{Y'}+\mathbf{H}^T_{Y'})\,{}_1\mathbf{F}^*_{Y'} &= ({}_1\mathbf{F}^*_{Y})^T(\mathbf{H}_{Y}+\mathbf{H}^T_{Y})\,{}_1\mathbf{F}^*_{Y} + \mathbf{O}(\bar{\epsilon}^2_{Y})
\end{aligned}
\tag{5.27}
$$

as $\bar{\epsilon}_Y \to 0$. Substituting the latter results in the appropriate terms of (5.26) we find that

$$(1 - \mathrm{tr}\ \underset{\sim}{H}_{Y'})\,{}_2\bar{\underset{\sim}{R}}\ {}_1\bar{\underset{\sim}{R}}^T\ {}_1\underset{\sim}{T}_Y^*\ {}_1\bar{\underset{\sim}{R}}\ {}_2\bar{\underset{\sim}{R}}^T = (1 - \mathrm{tr}\ \underset{\sim}{H}_Y)\,{}_1\underset{\sim}{T}_Y^* + \underset{\sim}{W}(YY')\,{}_1\underset{\sim}{T}_Y^*$$

$$- {}_1\underset{\sim}{T}_Y^*\ \underset{\sim}{W}(YY') + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \quad ,$$

$${}_2\bar{\underset{\sim}{R}}\ {}_1\bar{\underset{\sim}{R}}^T\ {}_1\underset{\sim}{T}_Y^*\ {}_1\bar{\underset{\sim}{R}}\ \underset{\sim}{H}_{Y'}^T\ {}_2\bar{\underset{\sim}{R}}^T = {}_1\underset{\sim}{T}_Y^*\ \underset{\sim}{H}_Y^T + {}_1\underset{\sim}{T}_Y^*\ \underset{\sim}{W}(YY') + \underset{\sim}{O}(\bar{\epsilon}_Y^2) \quad , \qquad (5.28)$$

$${}_2\bar{\underset{\sim}{R}}\ {}_1\underset{\sim}{F}_{Y'}^*\ \bar{\underset{\sim}{K}}_Y[({}_1\underset{\sim}{F}_{Y'}^*)^T(\underset{\sim}{H}_{Y'}+\underset{\sim}{H}_{Y'}^T)\,{}_1\underset{\sim}{F}_{Y'}^*]({}_1\underset{\sim}{F}_{Y'}^*)^T\ {}_2\bar{\underset{\sim}{R}}^T$$

$$= {}_1\underset{\sim}{F}_Y^*\ \bar{\underset{\sim}{K}}_Y[({}_1\underset{\sim}{F}_Y^*)^T(\underset{\sim}{H}_Y+\underset{\sim}{H}_Y^T)\,{}_1\underset{\sim}{F}_Y^*]({}_1\underset{\sim}{F}_Y^*)^T + \underset{\sim}{O}(\bar{\epsilon}_Y^2)$$

as $\bar{\epsilon}_Y \to 0$. Inserting the results (5.28) in (5.26) we conclude that

$${}_2\underset{\sim}{T}_Y^* = (1 - \mathrm{tr}\ \underset{\sim}{H}_Y)\,{}_1\underset{\sim}{T}_Y^* + \underset{\sim}{H}_Y\ {}_1\underset{\sim}{T}_Y^*$$

$$+ {}_1\underset{\sim}{T}_Y^*\ \underset{\sim}{H}_Y^T$$

$$+ \tfrac{1}{2}\,{}_1\rho_Y^*\ {}_1\underset{\sim}{F}_Y^*\ \bar{\underset{\sim}{K}}_Y[({}_1\underset{\sim}{F}_Y^*)^T(\underset{\sim}{H}_Y+\underset{\sim}{H}_Y^T)\,{}_1\underset{\sim}{F}_Y^*]({}_1\underset{\sim}{F}_Y^*)^T$$

$$+ \underset{\sim}{O}(\bar{\epsilon}_Y^2) \text{ as } \bar{\epsilon}_Y \to 0 \quad , \qquad (5.29)$$

which was to be shown.

Appendix A

This appendix provides certain mathematical developments concerning equivalence relations and equivalence classes (used in sections 2 and 3), which pertain to the procedure employed in the construction of invariant infinitesimal theories. In particular, we discuss the two relations "differs by a rigid motion" and "differs by an infinitesimal rigid motion."

Theorem A.1. The relation $\sim$ = "differs by a rigid motion" defined in[†] (2.7) is an equivalence relation on $\mathfrak{M}$, i.e.,

(a) $\chi \sim \chi$ for every $\chi \in \mathfrak{M}$ (Reflexivity).

(b) If $\chi, \theta \in \mathfrak{M}$ and $\chi \sim \theta$, then $\theta \sim \chi$ (Symmetry).

(c) If $\chi, \phi, \theta \in \mathfrak{M}$ and $\chi \sim \phi$, $\phi \sim \theta$, then $\chi \sim \theta$ (Transitivity).

Proof:

(a) $\chi(X,t+0) = I\,\chi(X,t) + 0$, so that (2.7) is satisfied with the choices $Q(t) = I$, $a(t) = 0$, $a = 0$.

(b) If $\chi \sim \theta$, then $\chi(X,t+a) = Q(t)\theta(X,t) + a(t)$ by (2.7). Hence, $\theta(X,\tau+b) = P(\tau)\chi(X,\tau) + b(\tau)$ with $\tau = t - b$, $b = -a$, $P(\tau) = Q^T(t)$, $b(\tau) = -Q^T(t)a(t)$ so that $\theta \sim \chi$.

(c) If $\chi \sim \phi$ and $\phi \sim \theta$, then $\chi(X,t+a) = Q(t)\phi(X,t) + a(t)$, $\phi(X,t) = P(t-b)\theta(X,t-b) + b(t-b)$ with $Q, P \in \mathfrak{G}^+$ the set of proper orthogonal tensors and a,b constants. Hence, $\chi(X,\tau+c) = S(\tau)\theta(X,\tau) + c(\tau)$ with $\tau = t-b$, $c = b+a$, $S(\tau) = Q(t)P(\tau) \in \mathfrak{G}^+$, $c(\tau) = Q(t)b(\tau) + a(t)$, so that $\chi \sim \theta$.

The set $K(\chi) = \{\phi \in \mathfrak{M} \mid \phi \sim \chi\}$ is called the equivalence class of χ in the equivalence relation $\sim$ and any member of it is called a representative of $K(\chi)$. We recall the standard results[‡]

[†] We suppress the index α in this appendix.

[‡] See, for example, van der Waerden (1970, p. 10).

(i) $K(\underset{\sim}{\chi}) = K(\underset{\sim}{\theta})$ if and only if $\underset{\sim}{\chi} \sim \underset{\sim}{\theta}$

(ii) $\bigcup_{\underset{\sim}{\chi} \in \mathfrak{m}} K(\underset{\sim}{\chi}) = \mathfrak{m}$

(iii) $K(\underset{\sim}{\chi}) \neq K(\underset{\sim}{\theta})$ implies $K(\underset{\sim}{\chi}) \cap K(\underset{\sim}{\theta}) = \emptyset$.

Thus all the motions in $\mathfrak{m}$ which are equivalent to one another (i.e., differ from one another by a rigid motion) and are regarded as being mechanically indistinguishable, belong to the same equivalence class. Clearly, an equivalence class is determined by any one of its members: if instead of $\underset{\sim}{\chi}$, we begin with the motion $\underset{\sim}{\theta}$ and place all the members of $\mathfrak{m}$ that are equivalent to $\underset{\sim}{\theta}$ in the same class we arrive at a class $K(\underset{\sim}{\theta})$ which is identical to $K(\underset{\sim}{\chi})$. Furthermore, the equivalence classes cover $\mathfrak{m}$, and distinct equivalence classes are disjoint. We may therefore partition $\mathfrak{m}$ into disjoint subsets, each of which contains all those motions, and those only, which differ from one another by a rigid motion.

As was pointed out in section 2, since the Lagrangian finite strain tensor $\underset{\sim}{E}$ remains unaltered under superposed rigid body motions, it may be used to characterize the equivalence classes of $\mathfrak{m}$. Adopting the convenient notation $\underset{\sim}{E}(\underset{\sim}{X},t\,;\underset{\sim}{\chi})$ for the Lagrangian strain at $\underset{\sim}{X}$ and t in the motion $\underset{\sim}{\chi}$, we record the following

Theorem A2. For any $\underset{\sim}{\theta},\underset{\sim}{\chi} \in \mathfrak{m}$, $\underset{\sim}{\theta} \sim \underset{\sim}{\chi}$ if and only if $\underset{\sim}{E}(\underset{\sim}{X},t+a\,;\underset{\sim}{\theta}) = \underset{\sim}{E}(\underset{\sim}{X},t\,;\underset{\sim}{\chi})$ for some constant a. The necessity part of the proof follows immediately from $(2.4)_2$ and (2.7), while (as remarked in the proof of Theorem 3.2) the sufficiency part is well known.

In view of Theorem A2, and the result (i) noted above, we may state

Theorem A3. For any $\underset{\sim}{\chi},\underset{\sim}{\theta} \in \mathfrak{m}$, $K(\underset{\sim}{\chi}) = K(\underset{\sim}{\theta})$ if and only if $\underset{\sim}{E}(\underset{\sim}{X},t\,;\underset{\sim}{\chi}) = \underset{\sim}{E}(\underset{\sim}{X},t+a\,;\underset{\sim}{\theta})$ for some constant a. In fact, we may now say that the relation "$\underset{\sim}{\theta}$ has the same Lagrangian finite strain as $\underset{\sim}{\chi}$" is an equivalence relation on $\mathfrak{m}$ which generates the same partition as the equivalence relation $\sim$.

Next, recalling the definition of $\underset{\sim}{E}$ in $(2.4)_2$ we observe at once

Theorem A4. The Lagrangian finite strain tensor has a value zero for the identity motion, i.e., $\mathbf{E}(\mathbf{X},t\,;\,{}_0\boldsymbol{\chi}) = \mathbf{0}$.

We may use Theorems A2 and A4 to show that the value $\mathbf{E} = \mathbf{0}$ characterizes the equivalence class of rigid motions:

Theorem A5. $\boldsymbol{\chi} \in \mathfrak{M}$ is a rigid motion if and only if $\mathbf{E}(\mathbf{X},t\,;\,\boldsymbol{\chi}) = \mathbf{0}$ (for all $(\mathbf{X},t)$).

Proof: If $\boldsymbol{\chi}$ is rigid then $\boldsymbol{\chi} \sim {}_0\boldsymbol{\chi}$ and hence by Theorems A2 and A4, $\mathbf{E}(\mathbf{X},t\,;\,\boldsymbol{\chi}) = \mathbf{0}$. Conversely, if $\mathbf{E}(\mathbf{X},t\,;\,\boldsymbol{\chi}) = \mathbf{0}$, then $\mathbf{E}(\mathbf{X},t\,;\,\boldsymbol{\chi}) = \mathbf{E}(\mathbf{X},t\,;\,{}_0\boldsymbol{\chi})$ by Theorem A4 and hence by Theorem A2, $\boldsymbol{\chi} \sim {}_0\boldsymbol{\chi}$ so that $\boldsymbol{\chi}$ is rigid.

We have employed the formula $(2.4)_2$ in the proofs of Theorems A2 and A4. Alternatively, we could characterize the notion of strain in a rather general way by assuming that our strain measure satisfies Theorem A2[†]. The strain associated with the class of rigid motions would then be some constant (tensor), not necessarily zero. The tensors $\mathbf{C}$ and $\mathbf{U}$ in (2.4) satisfy theorems paralleling Theorems A2 and A4 with both these tensors having a value $\mathbf{I}$ for the class of rigid motions. From the foregoing theorems and remarks, it is evident that an essential feature of the notion of strain is that it characterizes an entire class of motions rather than simply a motion. In particular, the Lagrangian strain tensor $\mathbf{E}$ defined in $(2.4)_2$ characterizes equivalence classes consisting of motions that differ from one another by a rigid motion and which are regarded as being mechanically equivalent.

Turning next to the infinitesimal strain tensor $\mathbf{e} = \mathbf{e}(\mathbf{X},t\,;\,\boldsymbol{\chi})$ in the motion $\boldsymbol{\chi}$, which is defined by $(2.8)_1$, and the relation "differs by an infinitesimal rigid motion" we establish the following three results of interest:

[†] Of course, it would not be possible without other assumptions to relate such a concept of strain to the change in length of material line elements. While this may appear strange, we remark that in the theory of elastic-plastic materials a tensor $\mathbf{E}_p$, called plastic strain appears, which only by introducing an addition assumption can be related to the (permanent) change in length of line elements.

Theorem A6. The relation "differs by an infinitesimal rigid motion" is not an equivalence relation on $\mathfrak{M}$.

Proof. While the relation satisfies the reflexivity property, it fails to satisfy both the symmetry and transitivity properties of an equivalence relation. To elaborate, let $\underset{\sim}{\theta} \in \mathfrak{M}$ differ from $\underset{\sim}{\chi} \in \mathfrak{M}$ by an infinitesimal rigid motion. Then, (2.30) hold and $\underset{\sim}{\chi}(\underset{\sim}{X},t) = \{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}(\theta(\underset{\sim}{X},\tau)-d(t))$. If $\underset{\sim}{\chi}$ were to differ from $\underset{\sim}{\theta}$ by an infinitesimal rigid motion, then it would be possible to express $\{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}$ as the sum of $\underset{\sim}{I}$ and a skew-symmetric tensor and it would then follow that $\mathrm{tr}(\{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}-\underset{\sim}{I}) = 0$. However, $\mathrm{tr}(\{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}-\underset{\sim}{I}) = \mathrm{tr}(\{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}) - 3 = \frac{-2\omega^2}{1+\omega^2} = \frac{-\|\underset{\sim}{W}(t)\|^2}{1+\frac{1}{2}\|\underset{\sim}{W}(t)\|^2}$ where $\underset{\sim}{W}(t) = \omega_{12}(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 - \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1) + \omega_{13}(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3 - \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_1) + \omega_{23}(\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3 - \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_1)$ and $\omega^2 = \omega_{12}^2 + \omega_{13}^2 + \omega_{23}^2 = \frac{1}{2}\|\underset{\sim}{W}(t)\|^2$. Therefore $\mathrm{tr}(\{\underset{\sim}{I}+\underset{\sim}{W}(t)\}^{-1}-\underset{\sim}{I}) = 0$ implies $\underset{\sim}{W}(t) = \underset{\sim}{0}$. Since $\underset{\sim}{\theta}$ can be chosen with $\underset{\sim}{W}(t) \neq \underset{\sim}{0}$, it follows that the symmetry property does not hold. That the transitivity property does not hold may be shown by observing that for any two skew-symmetric tensors $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$, $\mathrm{tr}\{(\underset{\sim}{I}+\underset{\sim}{W}_1)(\underset{\sim}{I}+\underset{\sim}{W}_2)-\underset{\sim}{I}\} = \mathrm{tr}\{\underset{\sim}{W}_1\underset{\sim}{W}_2\} = -2\underset{\sim}{\omega}_1 \cdot \underset{\sim}{\omega}_2$, where $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_2$ are the axial vectors of $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$, respectively. Since $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$ can be chosen so that $\underset{\sim}{\omega}_1 \cdot \underset{\sim}{\omega}_2$ is non-zero [e.g., $\underset{\sim}{W}_1 = \underset{\sim}{W}_2 \neq \underset{\sim}{0}$] it follows that the product $(\underset{\sim}{I}+\underset{\sim}{W}_1)(\underset{\sim}{I}+\underset{\sim}{W}_2)$ cannot always be expressed as the sum of $\underset{\sim}{I}$ and a skew-symmetric tensor.

It is clear from the definition (2.30) that if $\underset{\sim}{\theta}$ differs from $\underset{\sim}{\chi}$ by an infinitesimal rigid motion, then the displacement gradient $\underset{\sim}{H} = \frac{\partial\underset{\sim}{\theta}}{\partial\underset{\sim}{x}} - \underset{\sim}{I}$ of $\underset{\sim}{\theta}$ with respect to position $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X},t)$ in the motion $\underset{\sim}{\chi}$ is skew-symmetric and the associated infinitesimal strain $\frac{1}{2}(\underset{\sim}{H}+\underset{\sim}{H}^T)$ is zero.

Before proceeding further, we recall that in the finite theory, one is concerned with a set of motions which differs from a given motion $\underset{\sim}{\chi}$ by rigid motions. In contrast, in the infinitesimal theory one is concerned with a set of motions which differ from $\underset{\sim}{\chi}$ by infinitesimal rigid motions. It is natural to ask to what extent these two sets overlap, the answer to which is contained in

Theorem A7. If a motion $\underset{\sim}{\theta} \in \mathfrak{M}$ differs from $\underset{\sim}{\chi} \in \mathfrak{M}$ by a rigid motion and if it also differs from $\underset{\sim}{\chi}$ by an infinitesimal rigid motion, then $\underset{\sim}{\theta}$ must differ from $\underset{\sim}{\chi}$ only by translation.

Proof. Let $\underset{\sim}{\theta}$ differ from $\underset{\sim}{\chi}$ by a rigid motion and separately consider $\underset{\sim}{\theta}$ differing from $\underset{\sim}{\chi}$ by an infinitesimal rigid motion. Then, from (2.7) and (2.30), we have $\underset{\sim}{Q}(t-a) = \underset{\sim}{I} + \underset{\sim}{W}(t-b)$ for some proper orthogonal $\underset{\sim}{Q}$, skew-symmetric $\underset{\sim}{W}$ and real constants a,b. Taking the determinant of both sides of the latter equation, and recalling that $\det\{\underset{\sim}{Q}(t-a)\} = 1$, $\det\{\underset{\sim}{I} + \underset{\sim}{W}(t-b)\} = 1 + \frac{1}{2}\|\underset{\sim}{W}(t-b)\|^2$, yields $\|\underset{\sim}{W}(t-b)\| = 0$ and hence $\underset{\sim}{W}(t-b) = \underset{\sim}{O}$, $\underset{\sim}{Q}(t-a) = \underset{\sim}{I}$. Consequently, $\underset{\sim}{\theta}(\underset{\sim}{X},t+a) = \underset{\sim}{\chi}(\underset{\sim}{X},t) + \underset{\sim}{a}(t)$, i.e., $\underset{\sim}{\theta}$ differs from $\underset{\sim}{\chi}$ only by translation.

By setting $\underset{\sim}{\chi} = {}_{o}\underset{\sim}{\chi}$ in Theorem A7, it follows at once that the only motions which are both rigid and infinitesimal rigid are the translations, i.e., $\theta(\underset{\sim}{X},t+a) = {}_{o}\underset{\sim}{\chi}(\underset{\sim}{X},t) + \underset{\sim}{a}(t)$. In view of Theorem A7, the equivalence class $K(\underset{\sim}{\chi})$ and the set of motions that differ from $\underset{\sim}{\chi}$ by an infinitesimal rigid motion have a non-empty intersection comprising those motions which differ from motions in $K(\underset{\sim}{\chi})$ by a translation, but neither of the two is a subset of the other. In particular, the set $K({}_{o}\underset{\sim}{\chi})$ of rigid motions and the set of infinitesimal rigid motions intersect in the set of translations, but neither of the two sets contains the other.

As noted in (2.32), the infinitesimal strain tensor $\underset{\sim}{e}$ vanishes in an infinitesimal rigid motion; the converse is well known and may be proved by a simpler version of the argument used in the proof of Theorem 3.3. It was shown following (2.9) that $\underset{\sim}{E}(\underset{\sim}{X},t\,;\underset{\sim}{\chi}) = \underset{\sim}{e}(\underset{\sim}{X},t\,;\underset{\sim}{\chi})$ if and only if $\underset{\sim}{\chi}$ is a translation. More generally, we can prove the following

Theorem A8. Suppose $\chi, \theta \in \mathfrak{M}$ and $\chi \sim \theta$. Then, the following three statements are equivalent: (1) $E(X,t+a : \chi) = e(X,t ; \theta)$; (2) $\theta(X,t) = {}_{o}\chi(X,t) + a(t)$, $\chi(X,t+a) = Q(t)\,{}_{o}\chi(X,t) + b(t)$; (3) $E(X,t+a ; \chi) = e(X,t ; \theta) = 0$, where Q is proper orthogonal, a, b are vectors and a is a real constant.

Proof. In what follows, it will be shown that (1) implies (2), (2) implies (3) and (3) implies (1). By Theorem A2, $\chi \sim \theta$ implies $E(X,t+a ; \chi) = E(X,t : \theta)$ $= e(X,t ; \theta) + \frac{1}{2}\{\frac{\partial \theta}{\partial X}(X,t) - I\}^{T}\{\frac{\partial \theta}{\partial X}(X,t) - I\}$, where a formula of the type (2.) has been used. If $E(X,t+a : \chi) = e(X,t ; \theta)$, then $\|\frac{\partial \theta}{\partial X}(X,t) - I\| = 0$ and hence $\frac{\partial \theta}{\partial X}(X,t) = I$. Consequently $\theta(X,t) = {}_{o}\chi(X,t) + a(t)$ and $\chi(X,t+a) = Q(t)\,{}_{o}\chi(X,t) + b(t)$, establishing (2). Statement (3) then follows at once. Likewise, Statement (1) follows trivially from (3).

An immediate corollary of Theorem A8 is that the only rigid motions for which $e = 0$ are the translations. (This was shown by a different method following (2.14).)

The significance of Theorem A8 is that the infinitesimal strain tensor e, in contrast to the finite strain tensor E, cannot be used to characterize the equivalence classes $K(\chi)$ of $\mathfrak{M}$. The usual method of constructing infinitesimal theories, which involves the use of e as a strain measure, destroys the special structure consisting of the partition of $\mathfrak{M}$ into disjoint sets of motions that differ from one another by a rigid motion. If the infinitesimal theories are to be invariant under arbitrary superposed rigid body motions, this special structure must be preserved. The method introduced in section 3 does preserve this structure.

In the context of this appendix, our construction of an invariant infinitesimal theory may be viewed as follows: By means of the mapping π of $\mathfrak{M}$ into $\mathfrak{M}$, a particular member χ^{*} is singled out to represent the entire class $K(\chi)$ in the infinitesimal theory. The invariant infinitesimal strain $e^{*}(X,t : \chi)$ in the motion χ is defined to be the usual infinitesimal strain tensor evaluated for the motion $\underset{\sim}{\chi}^*$, i.e., $\underset{\sim}{e}^*(\underset{\sim}{X},t\,;\underset{\sim}{\chi}) = \underset{\sim}{e}(\underset{\sim}{X},t\,;\underset{\sim}{\pi}(\underset{\sim}{\chi}))$. By Theorem 3.2, this construction preserves the structure induced by $\mathfrak{m}$ by the equivalence relation "differs by a rigid motion" and hence the infinitesimal strain measure $\underset{\sim}{e}^*$ may be used to characterize the equivalence classes $K(\underset{\sim}{\chi})$ in $\mathfrak{m}$. In particular, the entire class of rigid motions is characterized by $\underset{\sim}{e}^* = \underset{\sim}{0}$.

Acknowledgment. The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under Contract N00014-75-C-0148, Project NR 064-436, with the University of California, Berkeley.

References

Coleman, B. D., and W. Noll, "Foundations of linear viscoelasticity," Revs. Modern Phys. **33**, 239-249 (1961).

Fosdick, R. L., and J. Serrin, "On the impossibility of linear Cauchy and Piola-Kirchhoff constitutive theories for stress in solids," J. Elasticity **9**, 83-89 (1979).

Green, A. E., and P. M. Naghdi, "A note on invariance under superposed rigid body motions," J. Elasticity **9**, 1-8 (1979).

Love, A. E. H., *A treatise on the mathematical theory of elasticity*, 4th ed. Cambridge University Press, 1927.

Truesdell, C., and R. A. Toupin, "The classical field theories" in S. Flügge's Handbuch der Physik, Vol. III/1, Springer-Verlag, 1960.

Truesdell, C., and W. Noll, "The non-linear field theories of mechanics" in S. Flügge's Handbuch der Physik, Vol. III/3, Springer-Verlag, 1965.

van der Waerden, B. L., *Algebra* (trans. from the 7th German ed.), Vol. 1, F. Ungar Publ. Co., New York, N. Y., 1970.