ON THE EXISTENCE OF INFINITELY MANY SOLUTIONS OF THE DIRICHLET PROBLEM FOR SOME NONLINEAR ELLIPTIC EQUATIONS.

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ABSTRACT

This paper is concerned with an existence and multiplicity result for non-
linear elliptic equations of the type $Lu = g(x,u) + h(x,u)$ in $\Omega$, $u = 0$ on
$\partial \Omega$. $\Omega \subset \mathbb{R}^N$ is smooth and bounded. $L$ is a second order self-adjoint, uniformly
elliptic operator and the function $g; \Omega \times \mathbb{R} \to \mathbb{R}$ is odd with respect to $u$ and grows
like $f(x)|u|^{p-1}u$; the function $h(x,u) = 0(|u|^q)$ where $n, p$ and $q$ satisfy
certain inequalities.

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SIGNIFICANCE AND EXPLANATION

The existence of multiple solutions to nonlinear elliptic boundary value problems has been studied by many authors, especially when the nonlinear term is an odd function of the dependent variable. This paper shows, for a class of such equations, that when oddness is destroyed by adding a nonodd nonlinear perturbation to the equation, the resulting problem still possesses an infinite number of distinct solutions.
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In this paper we study the boundary value problem

\[
\begin{aligned}
Lu &= g(x,u) + h(x,u) \\
|u|_{\partial \Omega} &= 0.
\end{aligned}
\]

Let \( \Omega \subset \mathbb{R}^n (n > 1) \) be a smooth and bounded domain. \( L \) is a second order self-adjoint, uniformly elliptic operator

\[
L = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)
\]

with \( a_{ij} = a_{ji} \in C^1(\bar{\Omega}) \) and

\[
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \eta \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n
\]

\( \eta > 0 \) being the ellipticity constant.

g, h are given functions. We assume that

\( h : \bar{\Omega} \times \mathbb{R} + \mathbb{R} \) is a continuous function and \( h = h(x,s) \) is \( C^1 \)

with respect to \( s \in \mathbb{R} \) for all \( x \in \bar{\Omega} \). Moreover

\[
h(x,s) = 0(|s|^q), \ h_s(x,s) = 0(l)
\]

uniformly with respect to \( x \in \bar{\Omega} \) when \( |s| \) is large; \( 0 < q < 1 \).

We assume that

\( g : \bar{\Omega} \times \mathbb{R} + \mathbb{R} \) is a continuous function; \( g = g(x,s) \) is

\( C^1 \) and odd with respect to \( s \in \mathbb{R} \), for all \( x \in \bar{\Omega} \).

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The result of this paper is the following:

Theorem. Assume that \( g, h \) satisfying the above restrictions, then problem (1) possesses infinitely many distinct solutions.

In the case (1) is odd, that is \( h(x,u) \equiv 0 \), and \( 1 < p < \frac{n+2}{n-2} \) if \( n > 3 \) or \( 1 < p \leq \) if \( n = 2 \), it is well known that (1) has infinitely many distinct solutions, this follows easily from the well known Lusternik-Schnirelman theory, see Rabinowitz [1]. When \( h \) is independent of \( u \), i.e., \( h = h(x) \) and \( 1 < p < p_{n,0} \), (1) possesses infinitely many distinct solutions also, see Bahri and Berestycki [2], [3]. In this paper we use some of the same methods as in [2], [3], but we prove a more general result. Struwe [4], [5] has obtained a result similar to ours but with more restrictive conditions on \( g(x,u) \).

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Before proving the theorem we introduce some notations and lemmas.

---

Let \( f(p) = \frac{2(p+1)(p-q)}{(p-1)(2p+1-q)} - n \), when \( 0 \leq q < 1 \), we have \( f(1+0) = +\infty, f(\frac{n+2}{n-2}) < 0 \), hence there exists \( p_{n,q} \) satisfying \( 1 \leq p_{n,q} < \frac{n+2}{n-2} \), when \( 1 \leq q < p, f(p) = \frac{2(p+1)}{(p+1)q} \frac{p-1}{p-3} = n \), \( n = 2 - n < 0 \), i.e., in this case we can not find \( p \) satisfying (6). Hence we must restrict \( q \) to \( 0 \leq q < 1 \) such that there exists \( p \) satisfying (6).
Let $H$ be a real Hilbert space with norm $\| \cdot \|$. Let $S = \{ x \in H; \|x\| = 1 \}$ denote the unit sphere. We consider a functional $J \in C^0(S, \mathbb{R})$. For $a \in \mathbb{R}$ we denote

$$J_a = \{ x \in S; J(x) \leq a \}$$

$$J_a^+ = \{ x \in S; J(x) > a \} \ .$$

Assume that there exists a constant $M \in \mathbb{R}$ such that $J \in C^2(J_M, \mathbb{R})$. Assume also that $J$ satisfies the following Palais-Smale condition

$$(P.S)_{M} : \text{For any } M_1 > M \text{ and for any sequence } (x_n) \subset S \text{ such that } M \leq J(x_n) \leq M_1$$

and $\|J'(x_n)\| \to 0$, one can extract from $(x_n)$ a convergent subsequence.

Proposition 1. Suppose $H$ is infinite dimensional and $J \in C^0(S, \mathbb{R}) \cap C^2(J_M, \mathbb{R})$ satisfies condition $(P.S)_M$. If $a \in \mathbb{R}$, $a - M$ exists such that $J_a$ is not contractible in itself to a point, then $J$ has a critical value in $[a, +\infty)$. 

Proof: see [3].

Consider the following class of compact symmetric subsets of $S$

$$M_k = \{ A \subset S; A = g(S^k) \text{ where } g \text{ is odd and continuous} \} \quad (8)$$

where $k \in \mathbb{N}$ and $S^k = \{ x \in \mathbb{R}^{k+1}; \|x\| = 1 \}$ is the $k$-dimensional sphere. Let $J^* \in C^0(S, \mathbb{R}) \cap C^1(J_M^*, \mathbb{R})$ be an even functional. (The superscript $*$ will always be associated with evenness thereafter.) Define

$$C_k = \inf_{A \in M_k} \max_{x \in A} J^*(x) \quad (9)$$

then we have

Proposition 2. Let $J^* \in C^1(S, \mathbb{R})$ be even satisfying condition $(P.S)_M$ and bounded from below on $S$. Let $C_k$ be defined by (9) then:

(i) $C_k$ is a critical value of $J^* \forall C_k \geq M$

(ii) $C_k < C_{k+1} \forall C_k \geq M \quad (10)$

Proof. See [3].

Proposition 3. Let $J, J^* \in C^0(S, \mathbb{R}) \cap C^2(J_M, \mathbb{R})$, $J^* \in C^0(S, \mathbb{R}) \cap C^1(J_M^*, \mathbb{R})$ be two functionals both satisfying condition $(P.S)_M$. Assume furthermore that $J^*$ is even and
bounded from below on $S$. Let $C_k(k \in \mathbb{N})$ be defined as in (9). Suppose there exists $k \in \mathbb{N}$, $c > 0$ and $a \in \mathbb{R}$ such that $C_k > M$ and

$$
J^*_C(x) = \max_{a \in \mathbb{R}} J^*_C x (x) = J^*_C x (a) = C_{k+1} - c
$$

then $J$ has at least one critical value in $(a, +\infty)$.

Proof: See [3].

Henceforth we will be working in the space $H = H^1_0(\Omega)$ with the norm

$$
||u||^2 = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}
$$

Define for $u \in H^1_0(\Omega)$

$$
I(u) = \frac{1}{2} ||u||^2 - \int_{\Omega} G(x,u) - \int_{\Omega} H(x,u)
$$

$$
I^*(u) = \frac{1}{2} ||u||^2 - \int_{\Omega} G(x,u)
$$

($I^*$ is the even principal part of $I$) where

$$
G(x,s) = \int_{0}^{s} g(x,t) dt, \quad H(x,s) = \int_{0}^{s} h(x,t) dt
$$

From (3), (4) we have

$$
H(x,s) = O(|s|^{q+1})
$$

$$
g(x,s) \approx q_1(x)|s|^{p-1} s
$$

$$
G(x,s) \approx \frac{q_1(x)}{p+1} |s|^{p+1}
$$

Relations (16), (17), (18) are uniform with respect to $x \in \bar{\Omega}$.

Let

$$
J(v) = \max_{\lambda \geq 0} I(\lambda v), \quad J^*(v) = \max_{\lambda \geq 0} I^*(\lambda v) \quad \forall v \in S
$$

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At a point \( \lambda \) such that \( J(\lambda) = I(\lambda v) \), one has

\[
I(\lambda v) = \frac{\lambda^2}{2} - \int_\Omega G(x, \lambda v) - \int_\Omega H(x, \lambda v) \quad (20)
\]

\[
\frac{d}{d\lambda} I(\lambda v) = \lambda - \int_\Omega g(x, \lambda v)v - \int_\Omega h(x, \lambda v) = 0 \quad (21)
\]

\[
\frac{d^2}{d\lambda^2} I(\lambda v) = 1 - \int_\Omega g'(x, \lambda v)v^2 - \int_\Omega h'(x, \lambda v)v^2 \leq 0. \quad (22)
\]

Lemma 1. For any sequence \( v \in S, \lambda(v) \in \mathbb{R}^+ \) such that \( J(v) = I(\lambda(v)v) \), the following are equivalent

(i) \( J(v) \rightarrow +\infty \)

(ii) \( \lambda(v) \rightarrow +\infty \)

(iii) \( v \rightarrow 0 \) in \( L^{p+1}(\Omega) \).

Proof: Since

\[
J(v) = I(\lambda(v)v) \quad (23)
\]

it is clear that (i) \( \Leftrightarrow \) (ii).

Suppose that \( \lambda(v) \rightarrow +\infty \), from (3), (17), (21) we have

\[
\lambda(v)^{p+1} = \left[ \int_\Omega q_1(x)|v|^{p+1} \right]^{-1}. \quad (24)
\]

From (5) and (24) we have (iii), thus (ii) \( \Leftrightarrow \) (iii).

Conversely, suppose (iii) is true, we have

\[
\int_\Omega |v|^{p+1} \rightarrow 0 \Rightarrow \int_\Omega v^2 \rightarrow 0. \quad (25)
\]

If (ii) is false, i.e., \( \lambda(v) \) bounded, from (22) we get \( 1 \leq 0 \), a contradiction, thus \( \lambda(v) \rightarrow +\infty \).

Substituting (16), (18) into (20) and using (24) we have

\[
J(v) \approx \left( \frac{1}{2} - \frac{1}{p+1} \right) \lambda(v)^2 \approx \left( \frac{1}{2} - \frac{1}{p+1} \right) \left[ \int_\Omega q_1(x)|v|^{p+1} \right]^{-\frac{2}{p+1}}. \quad (26)
\]

Hence \( J(v) \rightarrow +\infty \) this proves the lemma.
Lemma 2. There exists a positive constant $M > 0$ such that $J(v) \in C^0(S, \mathbb{R}) \cap C^2(J_M, \mathbb{R})$

where $J_M = \{ v \in S; J(v) \geq M \}$.

Proof: For fixed $v \in S$, by (20), (16), (18) there exists $\lambda_0 > 0$ such that $I(\lambda_0 v) < 0$ as $\lambda > \lambda_0$.

Therefore the $\lambda(v)$ satisfying (23) must also satisfy $0 \leq \lambda(v) < \lambda_0$. In a small neighborhood of this fixed $v \in S$ we have

$$J(w) = \max_{\lambda \in [0, \lambda_0]} I(\lambda w), \ w \in S$$

But the maximum of a continuous functional on a compact set is continuous, so that $J(v) \in C^0(S, \mathbb{R})$.

By (3), (4), (22), (26) and lemma 1, we have

$$\left[ \frac{d}{d\lambda} I(\lambda v) \right]_{\lambda = \lambda(v)} = 1 - p + O(1) \quad \text{as} \quad J(v) \to +\infty$$

the estimation (27) is uniform for $x \in \tilde{\Omega}$. Hence a large positive constant $M$ exists such that

$$\left[ \frac{d^2}{d\lambda^2} I(\lambda v) \right]_{\lambda = \lambda(v)} < 0 \quad \text{as} \quad J(v) \geq M$$

Applying the implicit function theorem to $\frac{d}{d\lambda} I(\lambda v) = 0$ shows that $\lambda(v) \in C^1(J_M, \mathbb{R})$ and

$$J'(v, \psi) = \lambda(v)(I'(\lambda(v)v), \psi) + I'(\lambda(v)v)(\lambda'(v), \psi)$$

$$= I(v)(I'(\lambda(v)v), v)$$

where $(\cdot, \cdot)$ means the duality of $H^{-1}(\cdot)$ and $H_0^1(\cdot)$. (28) is valid in $H^{-1}(\cdot)$. Combining (28), $\lambda(v) \in C^1(J_M, \mathbb{R})$ and $I(u) \in C^2$ shows that $J(v) \in C^2(J_M, \mathbb{R})$.

Thus the proof of lemma 2 is completed.

Lemma 3. $J(v)$ satisfies (P.S) $\mathcal{M}$. 

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Proof: Assume that \((v_n) \in S\) satisfies

\[
M \leq J(v_n) \leq M_1
\]

(29)

where \(M_1 > M\) are constants and

\[
\|J'(v_n)\| \to 0 \quad (n \to -)
\]

(30)

where \(J'(v_n)\) only operates on \(S\).

We shall use (26), but in this case the meaning (26) has changed a little because \(M\) is fixed. The meaning of (26) is: having first fixed a positive \(\delta\), we can select a positive large \(M\) such that both (26) and

\[
(1-\delta)J(v) < \left(\frac{1}{2} - \frac{1}{p+1}\right) \lambda^2(v) < (1+\delta)J(v) \quad \forall v \in J_M
\]

(31)

\[
(1-\delta)J(v) < \left(\frac{1}{2} - \frac{1}{p+1}\right) \int \Omega g_1(x)|v|^{p+1} < (1+\delta)J(v) \quad \forall v \in J_M
\]

(32)

are true; the estimates (31), (32) are uniform for \(x \in \Omega\). Taking \(\delta = \frac{1}{2}\) is sufficient for our use. We prove the lemma with such a fixed \(M\).

Denote \(\lambda(v_n)v_n = u_n\). We denote hereafter by \(C\) various positive constants.

By (31) \(\|u_n\| = \lambda(v_n)\) is bounded by \(\frac{2(\delta p+1)}{p+1} (1-\delta)M \leq \|u_n\| \leq \frac{2(\delta p+1)}{p+1} (1+\delta)M\). By (6) the injection \(H^1_0(\Omega) \hookrightarrow L^p(\Omega)\) is compact. Therefore there exists a subsequence of \(u_n\), denoted again by \(u_n\), which converges strongly in \(L^p(\Omega)\).

For fixed \(\phi \in H^1_0(\Omega),\) we have the decomposition \(\phi = n_{n' n} + \psi_n\), where \(\psi_n || u_n\) and it is easy to see that \(\|\psi_n\| \leq \|\phi\|\). By (28) we have

\[
(I'(u_n), \phi) = n_{n'} (\lambda(v_n), u_n) + (I'(u_n), \psi_n) = (I'(u_n), \phi_n)
\]

\[
= \frac{1}{\lambda(v_n)} \langle J'(v_n), \psi_n \rangle.
\]

(33)

By (30), (33)

\[
I'(u_n) = -\frac{1}{\lambda(v_n)} \sum_{i,j=1}^n \frac{3}{3x_i} \left( a_{ij}(x) \frac{3u_n}{3x_j} - q(x, u_n) - h(x, u_n) \right)
\]

(34)
converges strongly to zero in $H^{-1}(\Omega)$, $g(x,u_n) + h(x,u_n)$ converges strongly in $L^1(\Omega)$ and also in $H^{-1}(\Omega)$. By (34) we have $u_n$ converges strongly in $H^1_0(\Omega)$, $u_n \to u$. Hence $v_n = \frac{u_n}{\|u_n\|} + \frac{u}{\|u\|}$ in $S$ if $\|u\| \neq 0$. But

$$\|u\| = \lim_{n \to \infty} \|u_n\| = \lim_{n \to \infty} \lambda(v_n) \geq \left[ \frac{2(p+1)}{p-1} \right]^{\frac{1}{2}} \frac{1}{(1-\delta)_M} > 0.$$ 

Therefore the proof of lemma 3 is complete.

Lemma 4. There exists constant $d$ such that

$$|J(v) - J^*(v)| \leq d \min\{J(v), J^*(v)\}^{\frac{1}{p+1}}$$ 

as $J(v) \geq M$. (35)

Proof: Let $u = \lambda(v)v$ we have

$$J(v) = I(\lambda(v)v) = I^*(\lambda(v)v) - \int_\Omega h(x,u)u.$$ 

(36)

From (36) by using (3), (26) we have

$$|J(v) - J^*(v)| \leq \int_\Omega h'(x,u)u|v| + C \int_\Omega |\lambda(v)v|^{q+1} + C \int_\Omega |\lambda(v)v|^p$$

$$\leq C \lambda(v)^{q+1} \int_\Omega |v|^{p+1} + C \int_\Omega |\lambda(v)v| \leq C \int_\Omega g_1(x)|v|^{p+1}$$

$$\leq C \lambda(v)^{q+1} \int_\Omega |v|^{p+1} + C \int_\Omega |\lambda(v)v| \leq C \int_\Omega g_1(x)|v|^{p+1}$$

$$\leq C \lambda(v)^{q+1} \int_\Omega |v|^{p+1} + C \int_\Omega |\lambda(v)v| \leq C \int_\Omega g_1(x)|v|^{p+1}$$

$$\leq C \lambda(v)^{q+1} \int_\Omega |v|^{p+1} + C \int_\Omega |\lambda(v)v| \leq C \int_\Omega g_1(x)|v|^{p+1}$$

a similar estimate can be obtained for $J^*(v) - J(v)$. Hence

$$|J(v) - J^*(v)| \leq C \int_\Omega g_1(x)|v|^{p+1}.$$ 

(37)

Combining (26) and a similar equivalent relation for $J^*(v)$ with (37) we obtain (35).

Proposition 4. For $J^*(v)$ defined by (19). Define $C_k$ by (9). Then we have

$$C_k \geq \left\{ \begin{array}{ll}
\frac{n+2-(n-2)p}{n(p-1)} & (n \geq 3) \\
\frac{2}{k^{p-1} - \epsilon_1} & (n = 2) \text{ (positive $\epsilon_1$ sufficiently small)}
\end{array} \right.$$ 

(38)

for $k$ large enough; $C$ is a positive constant.

Proof: See [3].
For the proof of (38), in [3] it was assumed that \( \exists u, s_0, 0 < u < 1, s_0 > 0 \) such that

\[
0 < \frac{g(x,s)}{s} \leq \mu q'(x,s) \quad \forall x \in \mathbb{R}, \forall s > s_0
\]

this condition is used in the proof of lemmas 7 and 9 in [3], which corresponds to our lemma 1 and (26). We do not need this condition.

We now come to the proof of our theorem. If the conclusion of the theorem were not true, i.e., if the number of solutions of problem (1) were finite, according to proposition 1, 2 and 3, (11) would not be true for any sufficiently large \( a \) and any \( k \).

Let

\[
\frac{g_{k+1}}{t} = \frac{\theta(c)}{t}
\]

where \( d \) is the constant in (35). If

\[
C_{k+1} - \epsilon \geq C_{k} + \epsilon + \theta(C_{k} + \epsilon) + \theta(C_{k} + \epsilon + \theta(C_{k} + \epsilon)) \quad (39)
\]

is true, we take \( a = C_{k} + \epsilon + \theta(C_{k} + \epsilon) \) by using (35) and (39) we obtain (11). Hence (39) cannot be true for \( k \) large enough. In other words, there exists a \( k_0 \) such that

\[
C_{k+1} - C_{k} \leq \theta(C_{k} + \epsilon) + \theta(C_{k} + \epsilon + \theta(C_{k} + \epsilon)) + 2\epsilon \quad \forall k \geq k_0
\]

so that

\[
C_{k} - C_{k_0} \leq \sum_{k=k_0}^{k-1} [\theta(C_{k} + \epsilon) + \theta(C_{k} + \epsilon + \theta(C_{k} + \epsilon)) + 2\epsilon]
\]

\[
\leq \sum_{k=k_0}^{k-1} \frac{g_{k+1}}{t} c \leq \frac{g_{k+1}}{t} k \leq C_{p+1}^{p+1}.
\]

Therefore

\[
C_{k} \leq C_{k} C_{p+1}^{p+1}
\]

or

\[
C_{k} \leq C_{p+1}^{p+1}.
\]

Combining with (38) we have

\[
-9-
\]
\[
\frac{n+2-(n-2)p}{n(p-1)} < p+1 - p-q + \epsilon_1
\]

or

\[p \geq P_{n,q} - \epsilon_1 (\epsilon_1 = 0 \text{ for } n \geq 3; \text{ for } n = 2, \epsilon_1 > 0 \text{ and sufficiently small})\]

where \(P_{n,q}\) is defined by (7). Hence (6) is false. Therefore when (6) is true, the number of solutions of problem (1) must be infinite. This proves the theorem.
REFERENCES


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