AFOSR-TR- 81-0128

LEVEL II

AD A095911

Approved for public release;
distribution unlimited.

81 2 27 129

LEVEL

(4)

A CHARACTERIZATION OF BIB DESIGNS BASED ON v TREATMENTS IN BLOCKS OF SIZE k WHOSE NUMBER OF BLOCKS IS AT LEAST ${}_vC_k$ (*)

by

A. Hedayat(**)
Department of Statistics
University of California, Berkeley

and

H. L. Hwang
Department of Mathematics
University of Illinois, Chicago

January, 1981

(*) Research supported by Grant AFOSR-76-3050.

(**) On leave from Department of Mathematics, University of Illinois, Chicago.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) UNCLASSIFIED

18 19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR TR-81-0128

2. GOVT ACCESSION NO.: AD-A095911

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): 6 A CHARACTERIZATION OF BIB DESIGNS BASED ON v TREATMENTS IN BLOCKS OF SIZE k WHOSE NUMBER OF BLOCKS IS AT LEAST vCk.

5. TYPE OF REPORT & PERIOD COVERED: 9 Interim rept.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): 10 A. Hedayat H. L. Hwang

8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-76-3050

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
University of Illinois at Chicago Circle
Department of Mathematics
Chicago, IL 60680

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 16 61102F 2304 A5

11. CONTROLLING OFFICE NAME AND ADDRESS:
Air Force Office of Scientific Research/NM
Bolling AFB, Washington, DC 20332

11

12. REPORT DATE: Jan[redacted]81

13. NUMBER OF PAGES: 12

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 17 A5 12 14

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distrbution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

BIB designs, number of blocks, support sizes.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For certain number of treatments, v, it is impossible to form a reduced BIB design in blocks of size k. Based on the necessary conditions $vr = bk$ and $\lambda(v-1) = r(k-1)$ which must hold in any BIB(v,b,r,k,λ) design, it is shown that $v = 8$ and $k = 3$ is the only case for which no reduced BIB design can be constructed of k 2. A table of BOB designs for $v = 8$ and $k = 3$ with support sizes from 22 to 56 is provided.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 409950

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

-0-

A CHARACTERIZATION OF BIB DESIGNS BASED ON v TREATMENTS IN BLOCKS OF SIZE k WHOSE NUMBER OF BLOCKS IS AT LEAST $_vC_k$ (*)

by

A. Hedayat (**)
Department of Statistics
University of California, Berkeley

and

H. L. Hwang
Department of Mathematics
University of Illinois, Chicago

Lambda Lambda

ABSTRACT

For certain number of treatments, v, it is impossible to form a reduced BIB design in blocks of size k. Based on the necessary conditions $vr = bk$ and $\lambda(v-1) = r(k-1)$ which must hold in any BIB(v,b,r,k,λ) design, it is shown that $v = 8$ and $k = 3$ is the only case for which no reduced BIB design can be constructed if $k > 2$. A table of BIB designs for $v = 8$ and $k = 3$ with support sizes from 22 to 56 is provided.

AMS 1970 Subject Classification: Primary 62K10; Secondary 05B05.

Key Phrases: BIB designs, number of blocks, support sizes.

(*) Research supported by Grant AFOSR 76-3050C.

(**) On leave from Department of Mathematics, University of Illinois, Chicago.

1. Introduction. For comparing v treatments in b blocks of size $k < v$ it is well known that a BIB design, when it exists, is the best possible design for any reasonable statistical criterion given that the model of response is the usual homoscedastic linear additive model. To minimize the cost of experimentation it is desirable (i) to conduct the experiment in a BIB design with as few blocks as possible and (ii) to assign the treatments to the blocks very judiciously since to the experimenter the implementation of different treatment compositions in a block may cost differently.

Unfortunately, the inherent strigent symmetry of a BIB design does not allow us to choose b with as much freedom as we wish. Indeed, it is necessary that the number of blocks, b, satisfies the following conditions.

$$(1.1) \quad \begin{array}{l} b \geq v, \text{ the Fisher inequaility} \\ b = vr/k \text{ for some integer } r \text{ such that } r(k-1) \equiv 0 (\mathrm{mod}\ v-1). \end{array}$$

An obvious, yet very important, point overlooked by some research workers is this: Conditions (1.1) impose restrictions on the number of blocks not on the treatment compositions of the blocks of a design (see the last paragraph in this section).

With no other considerations, one way to form a BIB design for v treatments in blocks of size k is to take all ${}_vC_k$ possible subsets of size k out of v treatments as the blocks of the design. For an obvious reason such a BIB design (or its multiple copies) is called the <u>unreduced</u> or the trivial BIB design. Throughout the paper the symbol ${}_mC_n$ stands for the binomial coefficient $\binom{m}{n}$. Clearly, an unreduced BIB design has a large number of blocks and it is very costly to implement it in many practical situations. Thus it is highly desirable to have <u>reduced</u> BIB designs, i.e., designs whose number of blocks are less than ${}_vC_k$. Fortunately, the literature of BIB design abounds with such designs. But strangely the literature lacks any systematic study of those cases where no reduced BIB designs are possible. Knowledge of such cases is indeed useful in planning the experiment if the experimenter insists on running a BIB design. Therefore, in this context, a problem of interest is to characterize all (v,k) for which the only b which satisfies conditions (1.1) is $b \equiv 0 \pmod{{}_vC_k}$. Thus for such v's and k's no reduced BIB designs are possible. Fortunately, as we shall prove there is only a single case to confront with, namely $v = 8$ and $k = 3$, besides the obvious and the unavoidable cases when the block size is 2. To establish this result we first show that for any v and k with $v > k = 2$, a BIB design exists only if the triple (b,r,λ) is an integer multiple of

$$({}_vC_k/d,\ \ {}_{v-1}C_{k-1}/d,\ \ {}_{v-2}C_{k-2}/d)$$

where $\lambda = r(k-1)/(v-1)$ and d is the greatest common divisor of ${}_{v-i}C_{k-i}$, $i = 0,1,2$. Thus we notice that a BIB(v,b,r,k,λ) has the smallest number of blocks among all BIB designs based on v

and k if its b, r and λ are relatively prime. van Lint (1973) has obtained many interesting results for such BIB designs, but mainly for $b < {}_vC_k$. We then show that the case $v = 8$ and $k = 3$ is the only case when $k \geq 3$ for which ${}_vC_k$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$ are relatively prime. Therefore $(v,k) = (v,2)$ or $(8,3)$ are the only solutions to our problem.

Before closing this section we would like to emphasize that though for $v = 8$ and $k = 3$ we need $b \equiv 0 \pmod{56}$ blocks in order to form a BIB design but fortunately we do not have to limit ourselves to the unreduced BIB design. In response to point (ii) above, Foody and Hedayat (1977) have shown that in this case we can build a BIB design with repeated blocks with the number of distinct blocks as low as 22 and indeed for any other number between 22 and 56. Since $v = 8$ and $k = 3$ is the only answer to our characterization when $k \geq 3$ we have included a Table of BIB designs for $v = 8$ and $k = 3$ based on the results of Foody and Hedayat (1977, 1979).

2. Definitions and notation. Let $V = \{1,2,...,v\}$ and let $v\Sigma k$ be the set of all distinct subsets of size k based on V. Elements of $v\Sigma k$ will be called blocks. A block of size 2 will be referred to as a pair.

A <u>balanced incomplete block design</u>, D, with parameters v, b, r, k and λ, written BIB(v,b,r,k,λ), is a collection of b elements of $v\Sigma k$ with the properties that

(i) each element of V occurs in exactly r blocks;

(ii) each element of $v\Sigma 2$ occurs in exactly λ blocks.

Note that the above definition does not require that the blocks of a BIB design be distinct. The collection of distinct blocks in a BIB design, D, is called the <u>support</u> of the design and the number of distinct blocks in D is denoted by b^* and called the <u>support size</u> of D.

3. <u>The necessary conditions for existence of a BIB design based on v and k.</u> For $v > k \geq 2$ it is easy to verify that the necessary conditions for the existence of a BIB(v,b,r,k,λ) design are:

$$\text{(3.1)} \quad \begin{array}{ll} \text{(i)} & bk = vr \\ \text{(ii)} & \lambda(v-1) = r(k-1). \end{array}$$

A very useful version of (3.1) is given below.

<u>Theorem 3.1</u>. For given v and k with $v > k \geq 2$, the necessary condition for the existence of a BIB(v,b,r,k,λ) design is that the triple (b,r,λ) must be an integer multiple of

$({}_{v}C_{k}/d,\ {}_{v-1}C_{k-1}/d,\ {}_{v-2}C_{k-2}/d)$, where d is the greatest common divisor of ${}_{v}C_{k}$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$.

Proof: From (3.1), $b = (v/1)\cdot r = [{}_{v}C_{k}/{}_{v-1}C_{k-1}]\cdot r$ and $\lambda = ((k-1)/(v-1))\cdot r = ({}_{v-2}C_{k-2}/{}_{v-1}C_{k-1})\cdot r$.

Hence $b/{}_{v}C_{k} = r/{}_{v-1}C_{k-1} = \lambda/{}_{v-2}C_{k-2} = t$.

Note that the common ratio t must be a rational number so that the numbers $b = {}_{v}C_{k}\cdot t$, $r = {}_{v-1}C_{k-1}\cdot t$ and $\lambda = {}_{v-2}C_{k-2}\cdot t$ are integers.

Let d be the greatest common divisor of ${}_{v}C_{k}$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$, then ${}_{v}C_{k}/d$, ${}_{v-1}C_{k-1}/d$ and ${}_{v-2}C_{k-2}/d$ are relatively prime. In order that b, r, λ remain integer values, t must be equal to t'/d for some integer t'. Hence the result.

Corollary 3.1. For given v and k with $v > k \geq 2$, let b_{min} be the minimum solution for b satisfying (3.1). Then $b_{min} = {}_{v}C_{k}$ if and only if ${}_{v}C_{k}$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$ are relatively prime.

4. Characterization of v and k with $v \geq 2k$ such that ${}_vC_k$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$ are relatively prime.

Lemma 4.1. Let v and k be two positive integers such that $v \geq 2k$ and $k \geq 3$, then $\prod_{i=2}^{k-1} (v-i) > k!$ unless $k = 3$ and $v \leq 8$.

Proof: Case (i). If $k = 3$, $\prod_{i=2}^{k-1} (v-i) = v-2$ and $k! = 3! = 6$. Thus $\prod_{i=2}^{k-1} (v-i) > k!$ unless $v \leq 8$.

Case (ii). If $k > 3$, then $v \geq 2k \geq 8$. For all i such that $2 \leq i \leq k-1$ we have $v-i \geq v-k+1 \geq 2k-k+1 = k+1$. Moreover $v-2 \geq 2k-2 = 2(k-1)$ and $v-4 \geq 2k-4 = 2(k-2)$.

Hence

$$\begin{aligned}
\prod_{i=2}^{k-1} (v-i) &= (v-2)(v-3)(v-4)\cdots(v-k+1) \\
&\geq 2(k-1)\cdot k\cdot 2(k-2)\cdot \underbrace{k\cdots k}_{(k-5)\text{-factors}} \\
&\geq 2k(k-1)(k-2)\cdots 3\cdot 2\cdot 1 \\
&> k!
\end{aligned}$$

Theorem 4.1. For $v \geq 2k$ and $k \geq 2$, the only solution for (v,k) such that the binomial coefficients ${}_vC_k$, ${}_{v-1}C_{k-1}$ and ${}_{v-2}C_{k-2}$ are relatively prime are $(v,k) = (v,2)$ or $(8,3)$.

Proof: Let $k \geq 3$, since ${}_vC_k = \{[v(v-1)]/[k(k-1)]\} \cdot {}_{v-2}C_{k-2}$ the numbers ${}_{v-i}C_{k-i}$, $i = 0,1,2$ are relatively prime only if ${}_{v-2}C_{k-2}$ divides $k(k-1)$. This could possibly occur only if ${}_{v-2}C_{k-2} \leq k(k-1)$ which is equivalent to $\prod_{i=2}^{k-1} (v-i) \leq k!$.

Lemma 4.1 implies that the possible v and k which satisfy the above inequality are $(v,k) = (6,3)$, $(7,3)$ and $(8,3)$. But it is easy to check that among these, $v = 8$ and $k = 3$ is the only pair with the desired property.

For $k = 2$, it is obvious that for any v the number ${}_{v-2}C_{k-2}$ is equal to one, hence the numbers ${}_{v-i}C_{k-i}$, $i = 0,1,2$ are relatively prime.

For characterizing all v and k for which b_{min} satisfying (3.1) is at least ${}_vC_k$ it is enough to search among those v and k with $v \geq 2k$ due to the fact that the complementary design of a BIB design is also a BIB design with the same number of blocks. By Corollary 3.1 and Theorem 4.1, $(v,k) = (v,2)$ and $(8,3)$ are the only solutions.

5. Final Remarks. Since $v = 8$ and $k = 3$ is the only nontrivial case for which $b_{min} = {}_vC_k$ according to (3.1) it would be interesting to study BIB designs with these parameters. Any BIB design with $v = 8$ and $k = 3$ has multiple of ${}_8C_3 = 56$ blocks. We do not have to take all 56 distinct blocks to form the designs.

Indeed it is possible to construct BIB designs with $v = 8$ and $k = 3$ with support sizes $22 \leq b^* \leq 56$. Table 1 (from Foody and Hedayat (1977, 1979), including a new correction) provides one example for each stated support size. Utilizing the method of trade off of Hedayat and Li (1979, 1980), one can construct many more such designs. Whether or not $b^* = 22$ is the minimum support size in case of $v = 8$, $k = 3$ is under investigation.

Before closing the paper we should mention that van Lint and Ryser (1972) and van Lint (1973) have studied some related problems in BIB designs. Their discoveries are very useful for further studies in this area.

Table 1

BIB Designs With v = 8 and k = 3
Support Sizes 22 to 56

b*	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56
b	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	56	112	56	112	112	112	56
123	-	-	-	3	1	-	-	2	-	-	3	-	-	-	2	-	3	1	3	-	1	1	2	1	1	1	1	1	1	2	1	2	2	2	1
124	-	-	-	-	-	2	2	-	-	3	-	-	-	-	-	-	-	1	3	1	1	1	-	1	1	1	1	1	1	2	1	2	2	1	1
125	1	1	1	1	-	-	-	-	4	-	1	3	-	1	-	1	1	2	-	3	2	1	1	-	2	-	1	1	1	2	1	-	1	2	1
126	-	-	-	-	4	2	2	3	-	2	-	2	-	-	2	-	-	-	-	1	-	1	1	3	-	2	1	1	1	2	1	3	3	2	1
127	2	2	2	-	1	2	2	1	-	1	-	-	3	1	1	1	1	1	-	-	1	1	1	1	1	2	2	1	1	2	1	3	2	2	1
128	3	3	3	2	-	-	-	-	2	-	2	1	3	4	1	4	1	1	-	1	1	1	1	-	1	-	-	1	1	2	1	2	2	3	1
134	-	-	-	2	-	-	-	1	1	-	2	1	1	1	1	1	1	1	-	1	1	1	1	1	1	1	1	1	1	2	1	1	1	3	1
135	-	-	-	1	3	4	4	2	-	3	1	-	1	1	2	1	1	1	2	1	1	1	1	1	1	1	2	1	1	3	1	3	3	3	1
136	5	3	5	-	2	1	1	1	2	1	-	2	1	3	1	2	1	1	1	1	1	-	1	1	1	1	-	1	1	1	1	2	2	2	1
137	1	3	1	-	-	1	1	-	2	2	-	2	1	1	-	1	-	1	-	2	1	2	1	1	1	1	1	1	1	2	1	2	2	2	1
138	-	-	-	-	-	-	-	-	1	-	-	1	2	-	-	1	-	1	-	1	1	1	-	1	1	1	1	1	1	2	1	2	2	-	1
145	5	5	5	-	3	-	-	2	2	-	-	3	3	3	2	3	-	2	-	1	2	1	1	1	1	1	1	1	1	4	2	2	2	1	1
146	1	1	1	-	-	3	3	-	2	2	-	1	2	2	-	2	-	-	2	1	-	2	-	1	1	1	2	1	1	3	1	2	2	2	1
147	-	-	-	3	1	-	-	1	-	-	3	-	-	-	2	-	2	1	1	1	1	-	1	1	1	1	-	1	-	-	-	2	2	2	1
148	-	-	-	1	2	1	1	2	1	1	1	1	-	-	1	-	3	1	-	1	1	1	3	1	1	1	1	1	2	1	1	3	3	3	1
156	-	-	-	4	-	-	-	2	-	1	4	-	1	-	2	-	3	1	1	-	1	-	2	1	1	1	1	1	1	2	1	3	2	2	1
157	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	1	1	-	2	-	-	1	1	1	1	1	-	1	1	1	1	2	2	2	1
158	-	-	-	-	-	2	2	-	-	2	-	-	1	-	-	-	-	-	1	1	-	2	-	2	-	2	1	1	1	-	-	2	2	2	1
167	-	-	-	1	-	-	-	-	2	-	1	1	2	1	-	2	1	2	-	2	2	2	1	-	1	-	1	1	2	2	1	1	2	2	1
168	-	2	-	1	-	-	-	-	-	-	1	-	-	-	1	-	1	2	2	1	2	1	1	-	2	1	1	1	-	2	1	1	1	2	1
178	3	1	3	2	4	3	3	4	2	3	2	3	-	2	3	1	1	1	3	1	1	-	1	2	1	1	2	1	1	5	2	2	2	2	1
234	4	4	4	3	2	2	2	2	3	1	2	3	1	3	1	3	2	-	-	2	-	2	2	1	-	1	1	2	1	2	1	3	3	2	1
235	-	-	-	-	-	2	1	-	-	2	-	-	3	-	-	-	-	2	-	-	1	1	-	2	1	2	1	-	1	1	1	1	1	2	1
236	-	-	-	-	-	-	-	-	3	-	1	2	1	1	-	1	1	-	1	3	1	1	1	-	1	-	1	2	1	2	1	3	3	2	1
237	2	2	2	-	-	-	-	-	-	-	-	1	-	2	1	2	-	1	1	1	1	-	-	-	1	-	1	1	-	3	1	2	2	2	1
238	-	-	-	-	3	2	3	2	-	3	-	-	-	-	2	-	-	2	1	-	2	1	1	2	2	2	1	-	2	2	1	1	1	2	1
245	1	1	1	1	2	-	-	2	-	-	1	-	2	1	2	1	1	-	2	-	1	-	1	1	1	1	1	1	1	2	1	2	1	3	1
246	1	1	1	2	1	-	-	1	-	-	1	-	1	1	1	1	1	4	-	-	3	1	1	-	3	1	1	-	1	2	1	2	2	2	1
247	-	-	-	-	-	2	2	-	2	2	1	1	-	1	1	1	1	1	-	1	1	1	1	2	1	1	1	1	1	2	1	1	2	3	1
248	-	-	-	-	1	-	-	1	1	-	1	2	2	-	1	-	1	-	1	2	-	1	1	1	-	1	1	1	1	2	1	2	2	1	1
256	2	2	2	2	1	2	3	1	-	2	1	-	1	3	2	3	1	1	1	-	1	1	1	1	1	1	1	1	1	3	1	2	2	2	1
257	2	2	2	1	3	1	1	3	1	1	2	2	-	1	2	1	1	-	1	2	-	-	2	1	-	1	1	1	2	2	1	4	4	1	1
258	-	-	-	1	-	1	1	-	1	1	1	1	-	-	-	-	2	1	2	1	1	2	1	1	1	1	1	2	-	2	1	3	3	2	1
267	-	-	-	2	-	-	-	-	2	-	2	1	2	-	-	-	2	1	3	1	1	2	1	1	1	1	-	1	1	1	1	-	-	2	1
268	3	3	3	-	-	2	1	1	1	1	1	1	1	1	1	1	1	-	1	1	-	-	1	1	-	1	2	1	1	2	1	2	2	2	1
278	-	-	-	3	2	1	1	2	1	1	1	1	-	1	1	1	1	2	1	1	2	1	1	1	2	1	1	1	1	2	1	2	2	2	1
345	-	-	-	-	-	-	1	-	-	1	-	-	-	-	1	-	1	-	1	1	-	1	1	1	1	1	-	1	1	-	-	2	2	1	1
346	-	2	-	1	1	-	-	1	-	1	2	-	-	-	1	1	1	2	1	-	2	1	1	1	1	1	1	-	1	3	1	2	2	2	1
347	2	-	-	-	3	-	-	2	-	-	-	-	3	-	1	-	1	2	2	-	1	1	1	1	1	1	1	-	2	3	2	2	2	1	1
348	-	-	2	-	-	4	3	-	2	3	-	2	1	2	1	1	-	1	2	2	2	-	-	1	2	1	2	2	-	2	1	2	2	3	1
356	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	1	1	1	1	1	1	-	1	-	1	1	1	2	1	1	1	2	1
357	-	-	2	5	-	-	-	-	4	-	4	3	1	2	-	2	3	2	1	2	3	2	2	1	2	1	1	2	1	2	1	2	2	2	1
358	6	6	4	-	3	-	-	4	2	-	1	2	-	3	3	3	1	-	1	1	-	-	1	1	-	1	1	1	1	4	2	3	3	2	1
367	1	1	1	-	3	5	5	4	-	4	-	-	-	1	4	1	-	-	1	-	-	-	-	3	1	3	2	1	1	2	1	2	2	2	1
368	-	-	-	5	-	-	-	-	1	-	3	1	3	1	-	1	3	2	1	1	1	3	2	1	1	1	1	1	1	2	1	2	2	2	1
378	-	-	-	1	-	-	-	-	-	-	2	-	-	-	-	-	2	-	1	1	-	1	2	-	-	-	-	1	1	-	-	2	2	3	1
456	-	-	-	-	-	2	1	-	3	1	1	2	1	-	-	-	2	-	1	3	-	2	1	2	-	2	1	2	1	-	1	2	3	2	1
457	-	-	-	-	1	4	4	2	1	4	-	1	-	1	1	1	-	1	1	1	1	1	-	1	1	1	2	1	1	3	1	2	2	3	1
458	-	-	-	5	-	-	-	-	-	-	4	-	-	1	-	1	2	3	1	-	2	1	2	-	2	-	1	-	1	3	1	2	2	2	1
467	1	1	3	3	1	-	-	1	1	-	2	3	1	2	1	1	2	-	1	2	1	-	3	-	1	-	1	2	1	2	1	3	2	2	1
468	3	1	1	-	3	1	2	3	-	2	-	-	1	1	3	1	-	-	1	-	-	-	-	2	-	1	-	1	1	2	1	1	1	2	1
478	3	5	3	-	-	-	-	-	2	-	-	1	2	2	-	3	-	1	1	1	1	3	-	1	1	2	1	1	1	2	1	2	2	1	1
567	4	4	2	-	2	-	-	1	-	-	-	-	1	1	1	1	-	2	1	-	1	1	-	1	1	1	1	-	-	3	1	2	2	2	1
568	-	-	2	-	3	2	2	2	3	2	-	3	1	2	1	2	-	1	1	2	2	1	1	1	2	1	1	1	2	2	1	2	2	2	1
578	-	-	-	-	-	1	1	-	-	1	-	-	4	-	2	-	1	1	-	1	1	-	1	1	1	1	1	1	1	1	1	-	-	2	1
678	-	-	-	-	-	1	1	-	1	1	1	1	-	1	-	1	1	1	-	1	1	1	1	1	1	1	1	1	1	2	1	4	4	2	1

Note: For each support size the number of blocks, b, is a minimum.

REFERENCES

Hedayat, A. and S.-Y. R. Li, (1979). The trade off method in the construction of BIB designs with variable support size. Ann. Statist. 7, 1277-1287.

Hedayat, A. and S.-Y. R. Li, (1980). Combinatorial topology and the trade off method in BIB designs. Ann. Discrete Math. 6, 189-200.

Foody, W. and A. Hedayat, (1977). On theory and applications of BIB designs with repeated blocks. Ann. Statist. 5, 932-945.
Corrections: Ann. Statist. 7 (1979), 925.

van Lint, J.H. and H.J. Ryser (1972). Block designs with repeated blocks. Discrete Math., 3, 381-396.

van Lint, J.H., (1973). Block designs with repeated blocks and $(b,r,\lambda) = 1$. J. Combinatorial Theory (A). 15, 288-309.