A NORMAL APPROXIMATION FOR THE MULTIVARIATE LIKELIHOOD RATIO STATISTICS

by

Govind S. Mudholkar* Madhusudan C. Trivedi
Department of Statistics Pennwalt Corporation
and
University of Rochester Pharmaceutical Division
Rochester, N.Y. 14627 P.O. Box 1710
Rochester, N.Y. 14603

ABSTRACT

For many multivariate hypotheses, under the normality assumptions, the likelihood ratio tests are optimal in the sense of having maximal exact slopes. The exact distributions needed for implementing these tests are complex and their tabulation is limited in scope and accessibility. In this paper, a method of constructing normal approximations to these distributions is described, and illustrated using the problems of testing sphericity and independence between two sets of variates. The normal approximations are compared with well known competing approximations and are seen to fare well.

Key Words: Sphericity, independence between two sets of variates.

*Research supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF under Grant No. AFOSR-77-3360. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.
1. INTRODUCTION

For most testing of hypothesis problems in multivariate analysis, under the normality assumption, several reasonable solutions of comparable merit exist. These include the tests resulting from the union-intersection principle, the class of likelihood ratio criteria and adhoc statistics such as Bartlett-Pillai trace for MANOVA. The Neyman-Pearson theory provides some information on the operating characteristics of these procedures, but does not indicate any of the contenders as superior. However, as demonstrated by Hsieh (1979), the likelihood ratio tests for many of the multivariate hypotheses have maximal exact slopes, i.e., they are asymptotically optimal according to Bahadur's (1967) method of comparing tests. From a practical standpoint the null distributions of the likelihood ratio statistics or of their competitors, are of crucial importance. These distributions, where available, are complex, their tables are generally limited in scope and not often accessible. Moreover, the tabulations concern only selected percentiles and are inadequate for computing the p-values needed in practice. The pragmatic approach to such distribution problems from early days (e.g., Neyman and Pearson, 1931) is to seek reasonably accurate and convenient approximations to the distributions.

The principal methods of approximating a likelihood ratio use the fact that, in large samples, its distribution is approximately of Pearson type I form and that of its negative logarithm is of type III, i.e., chi-square, form. Nayer (1936) following a suggestion by Neyman and Pearson (1931) used the moments to approximate
mate the percentiles for testing the homogeneity of variances in this manner. Bishop (1939), on the other hand, obtained empirical expressions for the parameters for a type I approximation by passing the intermediate stage of computing the moments. Bartlett (1937) pursuing the asymptotic chi-square character of a negative multiple of loglikelihood ratio, pointed out by Neyman and Pearson (1931), used moments to approximate it by a scaled chi-square variable for samples of moderate size. This approximation deteriorates as the size of the problem, as measured by the dimension of the multivariate normal distribution or by the number of populations in the problem increases, or when the effective sample size is small.

A comprehensive investigation of various approximations was conducted by Box (1949), in which he introduced new widely known and used asymptotic chi-square series approximations for the distributions of likelihood ratios. Box studied his series approximations, in the context of two multivariate problems, comparing them with the exact distributions and with several other approximations including one based on the F-distribution.

The purpose of this essay is to describe a method for constructing a Gaussian approximation to the null distribution of the likelihood ratio, and to demonstrate its efficacy and relevance in testing multivariate hypotheses. The normal approximation is outlined in Section 2. It is illustrated using two common multivariate problems, namely testing independence of two sets of variates and testing the sphericity hypotheses. Section 3 contains the likelihood ratio statistics for the two problems together.
with current approximations for their null distributions. These approximations are then numerically compared with the new normal approximation in Section 4.

2. A NORMAL APPROXIMATION FOR THE LIKELIHOOD RATIO $\Lambda$

Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of asymptotically normally distributed nonnegative random variables. The convergence of the distribution of $Y_n$ to normality can be accelerated by approximately symmetrizing it with a transformation as follows:

Let $\kappa_r = \kappa_r(n), r = 1, 2, \ldots$, denote the cumulants of $Y = Y_n$ and suppose that $\kappa_1 \to \infty$ and $\kappa_r/\kappa_1 = \phi_r, r \geq 2$, are bounded as $n \to \infty$. Then using the Taylor series it is easy to obtain the following asymptotic expansion for the expectation $E(Y/\kappa_1)^h$ of a power of $Y$ as

$$E(Y/\kappa_1)^h = 1 + \frac{h(h-1)\phi_2}{2\kappa_1} + \frac{h(h-1)(h-2)}{24\kappa_1^2} (4\phi_3$$

$$+ 3(h-3)\phi_2^2) + O(\kappa_1^{-3}).$$ (1)

From this the $r^{th}$ moment of $(Y/\kappa_1)^h$ can be obtained by substituting $(rh)$ for $h$ in (1). The following central moments of $(Y/\kappa_1)^h$ are then obtained in a routine manner:

$$\mu_2(h) = \frac{h^2\phi_2}{\kappa_1} + \frac{h^2(h-1)(2\phi_3 + (3h-5)\phi_2^2)}{2\kappa_1^2} + O(\kappa_1^{-3}),$$ (2)

$$\mu_3(h) = \frac{h^3\phi_3}{\kappa_1} + 3(h-1)\phi_2^2) + O(\kappa_1^{-3}),$$ (3)

$$\mu_4(h) = \frac{3h^4\phi_2^2}{\kappa_1^2} + O(\kappa_1^{-3}).$$
Since $Y$ is asymptotically normally distributed as $n \to \infty$, by the Mann-Wald (1943) theorem so is an appropriately normalized $\left(\frac{Y}{\kappa_1}\right)^h$.

This convergence to normality is accelerated if $h$ is chosen so that the leading term in the expansion (3) for $\mu_3(h)$ vanishes. This value $h_0$ of $h$ which approximately symmetrizes $\left(\frac{Y}{\kappa_1}\right)^h$ is obtained from (3) as

$$h_0 = 1 - \kappa_1 \kappa_3/(3 \kappa_2^2).$$

The distribution of $\left(\frac{Y}{\kappa_1}\right)^{h_0}$ may be approximated by the normal distribution with mean $\mu_1(h_0)$ and variance $\mu_2(h_0)$ given in (1) and (2), respectively. That is,

$$\Pr(\bar{Y} \leq y) = \Phi\left[\left(\frac{Y}{\kappa_1}\right)^{h_0} - \mu_1(h_0)\right]/\sigma(h_0),$$

where $\sigma^2(h_0) = \mu_2(h_0)$ is given by (2).

It is well known, e.g., see Anderson (1958) or Srivastava and Khatri (1979), that for many likelihood ratio statistics $\Lambda$ appearing in multivariate analysis under the normality assumption, $U = \Lambda^2/N$ is distributed as a product $\Pi X_i$ of independent beta variates $X_i$, $i = 1, 2, \ldots, k$, distributed according to $B(X_i; a_i, b_i)$, where $N$ is the number of observations. Equivalently, we have

$$-\log U = \sum_{i=1}^{k} (-\log X_i)$$

in distribution. Now, it can be shown that, as $a_i$ and $b_i \to \infty$, $-\log X_i$ converges in law to normality. Hence, it is possible to construct a normal approximation for $U$ as described above. Towards this end we need the cumulants of $-\log X_i$. The moment generating function of $-\log X_i$ is easily seen to be $M(t) = B(a_i - t, b_i)/B(a_i, b_i)$. Hence, the cumulant generating function is
-5-

\[ K(t) = \log \{ \Gamma(a_i + b_i) / \Gamma(a_i) \} - \log \{ \Gamma(a_i + b_i - t) / \Gamma(a_i - t) \}. \]

Differentiating and using \( \Psi(Z) = \frac{d}{dZ} \log \Gamma(Z) \), the \( r \) th cumulant of \(-\log X_i\) is

\[ C_{ri} = (-1)^r \{ \Psi(r-1)(a_i) - \Psi(r-1)(a_i + b_i) \}. \]

But \( \Psi'(Z) = - \sum_{j=0}^{\infty} (Z+j)^{-1} \), giving

\[ C_{ri} = (r-1)! \left\{ \sum_{j=0}^{\infty} (a_i + j)^{-r} \right\} \]

where \( m \) denotes the largest integer in \( b_i \) and \( v = b_i - m \). The cumulants of \(-\log U'\) obtained using (5) are,

\[ k \]

\[ \kappa_r(U') = (r-1)! \left\{ \sum_{i=1}^{k} \sum_{j=0}^{m-1} (a_i + j)^{-r} \right\} \]

\[ + \sum_{i=1}^{k} \sum_{j=m}^{\infty} ((a_i + j)^{-r} - (a_i + j + v)^{-r}) \]. \hspace{1cm} (6)

If \( b_i \) is an integer then the second sum in (6) vanishes and

\[ k \]

\[ \kappa_r(U') = (r-1)! \sum_{i=1}^{k} \sum_{j=0}^{m-1} (a_i + j)^{-r}. \hspace{1cm} (7) \]

From (7) we observe that as either \( k \) or \( b_i \) or both \( \rightarrow \infty \), \( \kappa_1 \) diverges, but \( \kappa_r, r > 1 \), are bounded. That is, \( \phi_r = \kappa_r / \kappa_1 \rightarrow 0 \). Hence, it is possible to construct the normal approximation to the distribution of \( A \) as described above. Thus, from (4) we get

\[ \Pr(\Lambda > \lambda) = \Phi \left[ \frac{\lambda'}{\kappa_1^{\cdot} h_0} - \mu_1'(h_0) / \sigma(h_0) \right], \hspace{1cm} (8) \]

where \( \lambda' = 2(\log \lambda) / N \). The 100(1-\( \alpha \))\th percentile \( A_{1-\alpha} \) can be approximated as

\[ A_{1-\alpha} \approx \kappa_1 \left( Z_\alpha \sigma(h_0) + \mu_1'(h_0) \right)^{1/h_0}, \hspace{1cm} (9) \]
where $Z_\alpha$ denotes the $100\alpha$th percentile of the standard normal variate.

3. TWO APPLICATIONS IN MULTIVARIATE ANALYSIS

The normal approximation derived in the previous section is now illustrated and later examined in the context of the multivariate problems of testing independence between two sets of normal variates and testing the sphericity hypothesis.

3.1 Independence Between Two Sets. Let $X' = (X_1, X_2, \ldots, \ldots, X_p)$ and $Y' = (Y_1, Y_2, \ldots, \ldots, Y_p)$, $p_1 \times p_2$, $p_1 + p_2 = p$, be jointly normally distributed with $\text{Var}(X) = \Sigma_{11}$, $\text{Var}(Y) = \Sigma_{22}$ and $\text{Cov}(X,Y) = \Sigma_{12}$. The hypothesis of independence between $X$ and $Y$ is $H_0: \Sigma_{12} = 0$. If $S$ is the usual estimate of $\Sigma$ based on a sample of size $N$, then the likelihood ratio statistic for testing $H_0$ is

$$\Lambda = \left[ \frac{|S|/\det(S_{11}) \det(S_{22})} \right]^{N/2},$$

where $S_{11}$ and $S_{22}$ are the submatrices of $S$ corresponding to $\Sigma_{11}$ and $\Sigma_{22}$, respectively. The exact distribution of the statistic $\Lambda$ is given, and tabulated for some values of $p_1$ and $p_2$, by several authors (e.g., see Krishnaiah 1979 and Consul 1967a). Among various approximations proposed for the null distribution of $\Lambda$ two are well known and widely used in statistical packages such as BMDP (see Engelman, et al., 1977). These are (i) the chi-square series approximation due to Box (1949) and (ii) the $F$ approximation due to Rao (1948).

Box-Approximation. Let $w = p_1 p_2$, $m = N - (p_1 + p_2 + 3)/2$, $\gamma_2 = w(p_1^2 + p_2^2 - 5)/48$, $\gamma_4 = \gamma_2^2/2 + w(3(p_1^4 + p_2^4) + 10w^2 - 50(p_1^2 p_2^2))$
+ 159}/1920. Then,
\[
\Pr(-m \log U \leq z) = \Pr(X^2 \leq z) + \frac{\gamma_2[\Pr(X^2_{+4} \leq z) - \Pr(X^2 \leq z)]}{m^2}
\]
\[
+ \left[\gamma_4[\Pr(X^2_{+8} \leq z) - \Pr(X^2 \leq z)] - \frac{\gamma_2^2[\Pr(X^2_{+4} \leq z) - \Pr(X^2 \leq z)]}{m^4 + O(N^{-6})}\right],
\]
where $X^2_k$ denotes a chi-square variable with $k$ degrees of freedom and $U = \Lambda^2/N$.

**Rao-Approximation.** Let $m' = N - (p_1+p_2+3)/2$, $L = (p_1p_2 - 2)/4$, $s = \sqrt{(p_1^2p_2^2 - 4)/(p_1^2+p_2^2 - 5)}$. Then,
\[
Q = (m's - 2L)(1 - U^{1/S})/(p_1p_2U^{1/S}),
\]
has an F-distribution with $p_1p_2$ and $m's - 2L$ degrees of freedom.

Now, it is well known that (e.g., see Anderson, 1958, p. 236) under $H_0$ the likelihood ratio statistic $\Lambda$ satisfies the equivalence $\Lambda^2/N = U = \prod X_i$ in law, where $X_i(i = 1, 2, \ldots, p_2)$ are independently distributed according to beta distributions $B(X_i; (N - p_1 - i)/2, p_1/2)$. The normal approximation developed in the previous section can be specialized in this case by taking $k = p_2$, $a_1 = (N - p_1 - i)/2$, $b_1 = p_1/2$ in the expressions (6) for the cumulants, (8) for the probabilities, and (9) for the percentiles of $\Lambda$.

### 3.2 Testing the Sphericity Hypothesis.
Let $X_1, X_2, \ldots, X_N$ be a random sample from a $p$-variate normal population with mean $\mu$ and covariance matrix $\Sigma$. The hypothesis that the $p$ components of the random vector $X$ are independent with the same variance i.e., $H_0: \Sigma = \sigma^2 I_p$, $\sigma^2 > 0$ unknown, is known as the sphericity hypothesis. The hypothesis also arises in the analysis of data from experiments
consisting of repeated measurements. In these experiments, the measurements on a subject are assumed to have compound symmetry, i.e., have the same variances and same correlations. The problem of testing the hypothesis of compound symmetry \( H_0 : \Sigma = \sigma^2 (\rho I + (1-\rho)I) \) for the covariance structure of \((p+1)\) repeated measurements \( Y \) can be reduced to the sphericity hypothesis by an orthogonal transformation \( Y' = Y'(1/\sqrt{(p+1)}:T_1) \) where \( 1 \) is the vector of 1's. \( Y \) satisfies compound symmetry if and only if \( X = T_1 Y \) satisfies the sphericity hypothesis. The likelihood ratio criterion for the sphericity hypothesis was proposed by Mauchly (1940) as

\[
U = \frac{\Lambda^2}{N} = |S|((\text{tr}S)/p)^{-p},
\]

where \( S \) is the covariance matrix of the sample of size \( N \). He also derived its null distribution for \( p = 2 \). The exact null distribution of \( U \) for \( p = 3, 4 \) and 6 was obtained by Consul (1967b). The 5% and 1% points for \( p = 4(1)10 \) were given by Nagarsanker and Pillai (1973). The series approximation due to Box can be expressed in this case as follows.

**Box-Approximation.** Let \( e = p(p+1)/2 - 1, f = n - (2p^2 + p+2)/(6p) \) and \( g = (p+2)(p-1)(p-2)(2p^3+6p^2+3p+2)/(288p^2) \) for \( n = N-1 \). Then,

\[
\Pr(- f \log U \leq z) = \Pr(\chi^2_e \leq z) + g(\Pr(\chi^2_{e+4} \leq z) - \Pr(\chi^2_e \leq z))/f^2 + O(f^{-3}). \tag{12}
\]

It is well known (e.g. see Srivastava and Khatri, 1979) that the distribution of \( U \) under \( H_0 \) is the same as that of the product \( \Pi X_i^2 \), where \( X_i \) \((i = 1, 2, \ldots, p-1)\) are independent beta random
variables distributed according to $\mathcal{B}(x_i; (n-i)/2, i(p+2)/(2p))$, $n = N - 1$. Again, we can obtain the cumulants of $U' = -2(\log U)/N$ using (6) with $a_i = (n-i)/2, b_i = i(p+2)/(2p)$ and $k = p - 1$. Hence, the probabilities and the percentiles of the likelihood ratio $\Lambda$ may be obtained from (8) and (9), respectively.

4. NUMERICAL COMPARISONS

The quality of the normal approximations for the two multivariate likelihood ratio statistics discussed in the previous section and the other two approximations, was examined by computing the probabilities corresponding to the tabulated percentiles of the statistics. Thus, in the case of the null distribution of $\Lambda$ for testing independence, the approximations due to Box (10), due to Rao (11), and the normal approximation given in Section (3.1) were used to compute the probabilities corresponding to all 5% and 1% points of $\Lambda$ given in Pearson and Hartley (1972, p. 99 and 333). Similarly, in case of the sphericity problem, all percentiles given by Nagarsanker and Pillai (1973) were used to examine the approximation due to Box given by (12) and the relevant normal approximation. In both cases, the series approximation due to Box was used in two steps: 1) only the first term; and 2) all terms given in (10) and (12). Also the percentiles approximated using the normal approximations were compared with the competing approximations using the first term of the Box series and the F-approximation. A selection of errors, i.e., $(\text{Approximation} - \text{Exact value}) \times 10^5$, in various cases is presented in Tables 1 and 2.
Conclusions. Let New, Rao, Box 1 and Box 3 denote the normal approximation, the F-approximation due to Rao, the first term approximation due to Box and the three term approximation due to Box, respectively. From Tables 1 and 2 it may be observed that (i) Rao, Box 1 and Box 3 have errors in second through fifth decimal place, they are especially large for small $N$ and decreasing rapidly as $N$ increases. The normal approximation has errors in fourth or fifth decimal place. (ii) As $p_1$, $p_2$ or $p$ increases, errors due to Rao, Box 1 and Box 3 increase while those due to the normal approximation either decrease or maintain the same level. Overall, the normal approximation is superior for small $N$ and is comparable with the others when the $N$ is large.
TABLE 1. Errors of the Approximations for the Likelihood Ratio Statistic for Testing Independence Between Two Sets

\[
\alpha = .05
\]

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>N</th>
<th>λ</th>
<th>PERCENTILES</th>
<th>ERRORS*</th>
<th>PROBABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>NEW</td>
<td>RAO</td>
<td>BOX1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>12</td>
<td>0.00001</td>
<td>0</td>
<td>0</td>
<td>67</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>0.04107</td>
<td>4</td>
<td>7</td>
<td>698</td>
<td>-17</td>
</tr>
<tr>
<td>22</td>
<td>30</td>
<td>0.00107</td>
<td>0</td>
<td>1</td>
<td>356</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>37</td>
<td>0.01620</td>
<td>0</td>
<td>0</td>
<td>760</td>
<td>-4</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>18</td>
<td>0.00217</td>
<td>0</td>
<td>5</td>
<td>161</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
<td>0.04111</td>
<td>1</td>
<td>3</td>
<td>352</td>
<td>-8</td>
</tr>
<tr>
<td>16</td>
<td>26</td>
<td>0.00019</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>12</td>
</tr>
<tr>
<td>16</td>
<td>33</td>
<td>0.00568</td>
<td>0</td>
<td>1</td>
<td>213</td>
<td>-6</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>19</td>
<td>0.00021</td>
<td>0</td>
<td>1</td>
<td>37</td>
</tr>
<tr>
<td>8</td>
<td>23</td>
<td>0.00356</td>
<td>0</td>
<td>3</td>
<td>133</td>
<td>-3</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>0.00007</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>25</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>0.00157</td>
<td>0</td>
<td>2</td>
<td>81</td>
<td>9</td>
</tr>
</tbody>
</table>

\[
\alpha = .01
\]

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>N</th>
<th>λ</th>
<th>PERCENTILES</th>
<th>ERRORS*</th>
<th>PROBABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>NEW</td>
<td>RAO</td>
<td>BOX1</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>12</td>
<td>0.00000</td>
<td>0</td>
<td>0</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>0.02261</td>
<td>0</td>
<td>7</td>
<td>514</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>30</td>
<td>0.00043</td>
<td>0</td>
<td>0</td>
<td>209</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>37</td>
<td>0.00990</td>
<td>-2</td>
<td>-1</td>
<td>572</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>18</td>
<td>0.00085</td>
<td>0</td>
<td>3</td>
<td>85</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
<td>0.02086</td>
<td>0</td>
<td>6</td>
<td>270</td>
<td>-2</td>
</tr>
<tr>
<td>16</td>
<td>26</td>
<td>0.00007</td>
<td>0</td>
<td>0</td>
<td>25</td>
<td>-10</td>
</tr>
<tr>
<td>16</td>
<td>33</td>
<td>0.00327</td>
<td>0</td>
<td>0</td>
<td>147</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>19</td>
<td>0.00007</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>23</td>
<td>0.00174</td>
<td>0</td>
<td>2</td>
<td>81</td>
<td>-1</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>0.00002</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>0.00075</td>
<td>0</td>
<td>1</td>
<td>48</td>
<td>5</td>
</tr>
</tbody>
</table>

TABLE 2. Errors of Approximations for the Likelihood Ratio Statistic for Testing Sphericity

\[
\alpha = .05
\]

<table>
<thead>
<tr>
<th>P</th>
<th>N</th>
<th>λ</th>
<th>PERCENTILES</th>
<th>ERRORS*</th>
<th>λ</th>
<th>PERCENTILES</th>
<th>ERRORS*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>NEW</td>
<td>RAO</td>
<td>BOX1</td>
<td>NEW</td>
<td>RAO</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>0.09739</td>
<td>16</td>
<td>467</td>
<td>-21</td>
<td>-539</td>
<td>-57</td>
</tr>
<tr>
<td>15</td>
<td>0.25350</td>
<td>35</td>
<td>236</td>
<td>-29</td>
<td>-190</td>
<td>-10</td>
<td>0.17210</td>
</tr>
<tr>
<td>20</td>
<td>0.37720</td>
<td>43</td>
<td>120</td>
<td>-33</td>
<td>-97</td>
<td>-5</td>
<td>0.28670</td>
</tr>
<tr>
<td>30</td>
<td>0.53900</td>
<td>46</td>
<td>37</td>
<td>-38</td>
<td>-40</td>
<td>-2</td>
<td>0.45310</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.03110</td>
<td>1</td>
<td>444</td>
<td>-7</td>
<td>-1171</td>
<td>-236</td>
</tr>
<tr>
<td>15</td>
<td>0.13780</td>
<td>7</td>
<td>334</td>
<td>-10</td>
<td>-393</td>
<td>-29</td>
<td>0.08685</td>
</tr>
<tr>
<td>20</td>
<td>0.24820</td>
<td>17</td>
<td>201</td>
<td>-17</td>
<td>-194</td>
<td>-7</td>
<td>0.17970</td>
</tr>
<tr>
<td>30</td>
<td>0.41640</td>
<td>28</td>
<td>79</td>
<td>-27</td>
<td>-78</td>
<td>-2</td>
<td>0.34020</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>0.00994</td>
<td>0</td>
<td>104</td>
<td>20</td>
<td>-3352</td>
<td>-1789</td>
</tr>
<tr>
<td>15</td>
<td>0.02712</td>
<td>0</td>
<td>297</td>
<td>-4</td>
<td>-1158</td>
<td>-228</td>
<td>0.01444</td>
</tr>
<tr>
<td>20</td>
<td>0.08446</td>
<td>5</td>
<td>282</td>
<td>-12</td>
<td>-562</td>
<td>-63</td>
<td>0.05514</td>
</tr>
<tr>
<td>30</td>
<td>0.21780</td>
<td>15</td>
<td>163</td>
<td>-21</td>
<td>-216</td>
<td>-13</td>
<td>0.16770</td>
</tr>
</tbody>
</table>

*Error in prob. = (Approx. value - α)x10^5 and in perc. (Approx. value - λ)x10^5.
REFERENCES


A normal approximation for the multivariate likelihood ratio statistics

For many multivariate hypotheses, under the normality assumptions, the likelihood ratio tests are optimal in the sense of having maximal exact slopes. The exact distributions needed for implementing these tests are complex and their tabulation is limited in scope and accessibility. In this paper, a method of constructing normal approximations to these distributions is described, and illustrated using the problems of testing sphericity and independence between two sets of variates. The normal approximations are compared with well known competing approximations and seen to fare well.