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**AN APPROXIMATE METHOD FOR CALCULATING CONCENTRATION-PATHLENGTH -- ETC(U)**

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AN APPROXIMATE METHOD FOR CALCULATING CONCENTRATION-PATHLENGTH THROUGH A GAUSSIAN PLUME

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**Title:** An Approximate Method for Calculating Concentration-Pathlength through a Gaussian Plume

**Author:** Bruce W. Fowler

**Abstract:**

The calculation of concentration-pathlength along a general line-of-sight is one of the most time and memory consuming calculations in most obscuration models. An approximate method for calculating the concentration-pathlength for a general, finite length line-of-sight through a Gaussian plume cloud in closed form is developed. Comparisons of the approximation to detailed numerical calculations are presented for selected white phosphorus munitions.
I. INTRODUCTION

Most models for simulating the concentration of obscurant in a smoke or dust cloud make use of gaussian plumes. This concentration must be integrated along some line-of-sight that connects an observer to a target. The complexity of this integration usually dictates that it be performed numerically at great cost in terms of both computer memory and execution time. In this report, an approximate method for calculating this concentration-pathlength in closed form is described.

This investigation is part of an ongoing examination of the effect of battlefield obscuration on the performance of missile (and selected nonmissile) weapon systems as part of the Concepts Analysis and Validation work area of the A214 Missile Technology Program. The results of this examination will be used in the formulation, analysis, and evaluation of present and conceptual missile weapon systems.

II. CONCENTRATION-PATHLENGTH DEVELOPMENT

The transmission through an obscurant from an observer to a target (point in space to point in space) is commonly expressed as

\[ T = \exp \left[ -\alpha C_z \right] \] (1)

where: \( T \) = transmission,
\( \alpha \) = extinction coefficient of the obscurant (cm\(^2\)/g), and
\( C_z \) = concentration-pathlength through the obscurant (g/cm\(^2\)).

The concentration-pathlength is usually given by

\[ C_z = \int_{\text{target}}^{\text{observer}} C(x,y,z) \, d\xi(x,y,z) \] (2)

where: \( C(x,y,z) \) = concentration of obscurant at point with coordinates \( x,y,z \) (g/cm\(^3\)), and
\( \xi(x,y,z) \) = line-of-sight (cm).

The most common coordinate system used is a cartesian coordinate system with origin at the formation point of the obscurant cloud (the cloud coordinate system). The x axis of this system lies in the ground plane along the wind direction, positive downwind. The y axis is vertical and the z axis is oriented so that the system is right-handed. An alternate coordinate system, called the cloud centroid coordinate system, has the same orientation, but the origin is at the cloud centroid. This coordinate system is time dependent.
The observer and target have coordinates \( x_o, y_o, z_o \) and \( x_t, y_t, z_t \) in the cloud coordinate system. The centroid of the cloud has coordinates \( u(t-t_0), 0, z(t-t_0) \), where \( u \) is the ground plane wind speed, \( t_0 \) is the initial formation time of the obscurant, and \( z(t) \) is the centroid rise function.

The concentration of obscurant is most commonly expressed as some form of trivariate gaussian. One of the simplest of these which contains the complexity needed for this report is that of the Smoke Effectiveness Manual Model (SEMM).\(^{10}\) This model is

\[
C = \frac{2Q\lambda\Omega}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp \left[ -1/2 \left( \frac{(x-ut)^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2} \right) \right]
\]

where:

- \( Q \) = munition load of obscurant producing material (g),
- \( \lambda \) = efficiency factor,
- \( \Omega \) = yield factor,
- \( \sigma_x, \sigma_y, \sigma_z \) = cloud standard deviations (time dependent) (m),
- \( x, y, z \) = cartesian coordinates of points in cloud coordinate system (m),
- \( u \) = wind speed (m/s), and
- \( t \) = time since cloud formation (s).

The cloud rise function is zero. Equation 3 is appropriate for an instantaneous cloud.

Two methods are commonly used to develop the explicit form of the concentration-pathlength: line-of-sight (LOS) parametrization and coordinate system rotation. Each shall be reviewed separately. In LOS parametrization, the coordinates along the LOS are written as

\[
x = x_o + \Delta x_n \tag{4}
\]
\[
y = y_o + \Delta y_n \tag{5}
\]
\[
z = z_o + \Delta z_n \tag{6}
\]

where:

\[
\Delta x_n = \frac{\xi_t - x_o}{d} \tag{7}
\]
\[
\Delta y_n = \frac{\xi_t - y_o}{c} \tag{8}
\]
\[
\Delta z_n = \frac{\zeta_t - z_o}{d} \tag{9}
\]

The distance between the observer and the target is
\[ d = \left[ (x_t - x_o)^2 + (y_t - y_o)^2 + (z_t - z_o)^2 \right]^{1/2} \]  

(10)

The parameter \( \eta \) takes on all values between 0 and \( d \). Equations 4 through 6 are substituted into Equation 3 to yield:

\[
C(\eta) = \frac{2\eta \lambda \Omega}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_z} \exp \left[ -\frac{1}{2} \left\{ \frac{(x - ut)^2}{\sigma_x^2} + \frac{y_o^2}{\sigma_y^2} + \frac{z_o^2}{\sigma_z^2} \right\} \right] 
\exp \left[ -\eta \left\{ \frac{(x - ut) \Delta x}{\sigma_x^2} + \frac{y_o \Delta y}{\sigma_y^2} + \frac{z_o \Delta z}{\sigma_z^2} \right\} \right] 
\exp \left[ -\frac{\eta}{2} \left( \frac{\Delta x^2}{\sigma_x^2} + \frac{\sigma_x^2}{\sigma_y^2} + \frac{\Delta z^2}{\sigma_z^2} \right) \right] 
\]

(11)

after some minor rearrangement. The explicit form of the concentration-pathlength is then

\[
C(z) = \int_0^d C(\eta) d\eta 
\]

(12)

Although only the last two terms of the right hand side of Equation 11 are \( \eta \) dependent, the integral of Equation 12 must either be performed numerically, the square of the exponent completed and the result extracted from a table, or if \( d \) is sufficiently large, an approximation that \( d = \infty \) may be introduced and the integral performed analytically.11

The technique of coordinate system rotation first translates into cloud centroid coordinates, and then defines the scaled coordinate system given by

\[
x' = \frac{x}{\sigma_x} \\
y' = \frac{y}{\sigma_y} \\
z' = \frac{z}{\sigma_z} 
\]

(13)

where \( x, y, z \) are in cloud centroid coordinates. Thus, the observer coordinates in this system are
and the concentration, Equation 3, is

\[ C(x', y', z') = \frac{20\lambda_{\mu}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left[ -\frac{1}{2} \left( \frac{x'^2}{\sigma_x^2} + \frac{y'^2}{\sigma_y^2} + \frac{z'^2}{\sigma_z^2} \right) \right] \]  

(15)

Next, the ground plane distance (scaled) between observer and target is defined as

\[ d' = \left[ (x'_t - x'_o)^2 + (y'_t - y'_o)^2 \right]^{1/2} \]  

(16)

and two angles

\[ \phi = \arctan \left[ \frac{y'_t - y'_o}{x'_t - x'_o} \right] \]  

(17)

and

\[ \theta = \arctan \left[ \frac{z'_t - z'_o}{d'} \right] \]  

(18)

These two angles define a rotation matrix

\[ R = \begin{bmatrix}
\cos(\theta) \cos(\phi) & \cos(\theta) \sin(\phi) & \sin(\theta) \\
-sin(\phi) & \cos(\phi) & 0 \\
-sin(\theta) \cos(\phi) & -sin(\theta) \sin(\phi) & \cos(\theta)
\end{bmatrix} \]  

(19)

and a new coordinate system by

\[ \eta = R \cdot \tau' \]  

(20)

where:
The new coordinate system has the advantage that \( \eta_y = \eta_{y0} \) and \( \eta_z = \eta_{z0} \). Only \( \eta_x \) varies. The concentration is given by

\[
C = \frac{2Q_0\Omega}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp \left[ -\frac{\eta_x^2}{2} \right] \tag{22}
\]

The concentration-pathlength is then given by

\[
C_k = \frac{2Q_0\Omega}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \left( \frac{\cos^2(\theta) \cos^2(\phi)}{\sigma_x} + \frac{\cos^2(\theta) \sin^2(\phi)}{\sigma_y} + \frac{\sin^2(\phi)}{\sigma_z} \right)^{-1} \exp \left[ -\frac{1}{2} \left( \eta_y^2 + \eta_z^2 \right) \right] \int_{\eta_{x0}}^{\eta_x} \exp \left[ -\frac{\eta_x^2}{2} \right] d\eta_x \tag{23}
\]

which is equivalent to Equation 12 if the square of the argument is completed and the variable of integration shifted. The value of Equation 23 is that it may be written in terms of error functions (erf's).

The error function is given by

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp \left[ -t^2 \right] dt \tag{24}
\]

Equation 23 may be rewritten in this form by defining a new variable of integration

\[
\zeta = \frac{\eta_x}{\sqrt{2}} \tag{25}
\]

and simple algebra. This gives
\[
C_x = \frac{\Omega}{2\sigma_x \sigma_y \sigma_z} \left\{ \frac{\cos^2(\theta) \cos^2(\phi)}{\sigma_x} + \frac{\cos^2(\theta) \sin^2(\phi)}{\sigma_y} + \frac{\sin^2(\theta)}{\sigma_z} \right\}^{-1} \\
\exp\left[-\frac{1}{2} \left( \eta_y^2 + \eta_z^2 \right) \right] \left[ \text{sgn}(\eta_{x/t}) \text{erf}\left(\frac{\eta_{x/t}}{\sqrt{2}}\right) \right] - \text{sgn}(\eta_{x/o}) \text{erf}\left(\frac{\eta_{x/o}}{\sqrt{2}}\right) 
\]

where:

\[
\text{sgn}(x) = 1, \ x > 0, \\
= 1, \ x = 0, \ \text{and} \\
= 0, \ x < 0. 
\]

The use of the \text{sgn} function arises from the fact that the integrand of the error function is even (non-negative) so that the sign of the integral depends only on the sign of the end point.

Equation 26 is usually the most efficient form to use in computing concentration-pathlength since the error function may be compactly computed using a single \text{Pade} approximant.\textsuperscript{12} Alternately, if \( |\eta_{x/t} - \eta_{x/o}| \gg |\eta_{x/t}| \), and \( |\eta_{x/o}| \gg |\eta_{x/t}| \), then the integral of Equation 23 may be analytically solved for infinite limits.\textsuperscript{11}

\section*{III. APPROXIMATION TO THE ERROR FUNCTION}

A function that is well-known to solid state physicists and statistical mechanicians is that used to describe the distribution of states of particles obeying Fermi-Dirac statistics.\textsuperscript{13} This function has the form

\[
f(x) = [1 + \exp(x)]^{-1} 
\]

which has the property:

\[
f(x) = 1, \ x \ll 0, \\
= 0, \ x \gg 0, \ \text{and} \\
= 1/2, \ x = 0. 
\]

For simplicity, \( f(x) \) is referred to as a Fermi function.

Less well-known are the properties of the derivative of the Fermi function. The derivative may be calculated in straightforward manner as

\[
\frac{df(x)}{dx} = -\exp(x) f(x)^2 \\
= -f(x) f(-x). 
\]
The derivative of the Fermi function has a maximum value of 1/4 at $x = 0$. It is also approximately gaussian in shape. This is shown in Figure 1.

This shape leads to the suspicion that the Fermi function may be used as an approximation to the error function

$$\exp \left[ -x^2 \right] = -4 \cdot \frac{df(ax)}{d(ax)}$$

(31)

where $a$ is a parameter chosen to give agreement with the error function.

IV. PERFORMANCE OF THE APPROXIMATION

As an investigation of the relative accuracy of the approximation for the error function developed in the last section, Equation 35, the SEMM model was exercised for several WP munitions. HC (hexachloroethana) munitions were not exercised because of the additional computational burden involved. The approximation is, of course, still as valid for HC, and reduces the computational burden, but the time dependence of the calculation for HC only detracts from the examination of the approximation.

Several geometries of target and observer were considered, both symmetric and unsymmetric with respect to the cloud. The quantity of comparison used was the visual transmission through the cloud, Equation 1, between observer and target. The worse cases found were when both target and observer were on the same side of the cloud relative to the cloud centroid. An example of these results are shown in Figure 2 through 8.

These figures are plots of transmission for a 4.2-in. WP round, neutral meteorology, and head wind. The observer and target are located on the $x$ axis. Thus, the LOS either passes through the centroid or may be extended through it. The observer is always located $3 \sigma_x$ from the centroid. The distance between observer and target is varied from 0 to $6 \sigma_x$.

It may be seen that the error is greatest for early times when the observer and target are within $\sigma_x$ of each other. At time $t = 0$, this distance apart ($\sigma_x$) is about 4 meters, increasing approximately linearly with time. The error decreases with time. For reasonable observer target distances of $10^3$ m, approximately $5 \times 10^3$ seconds are required for $\sigma_x$ to grow to $10^3$ m in size. This is interpreted to indicate that the error in this approximation is not stressing for most simulation uses.

By substituting Equation 31 into Equation 24, we may get
erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \left\{ -f \frac{df(x)}{dx} \right\} dx \tag{32}

= \frac{8}{\sqrt{\pi}} \int_{0}^{a} \frac{df(y)}{dy} dy \tag{33}

= \frac{8}{\sqrt{\pi}} [f(0) - f(ax)] .

By using erf(\infty) = 1, Equation 33 may be written as

1 = \frac{8}{\sqrt{\pi}} f(0)

= \frac{4}{\sqrt{\pi}}

which yields

a = \frac{4}{\sqrt{\pi}} \tag{34}

and

erf(x) = 1 - 2f \left( \frac{4x}{\sqrt{\pi}} \right) \tag{35}

Equations 26 and 35 may be combined to yield an approximate concentration-pathlength of

\[ C_{\perp} = O \frac{Q \Omega}{2\pi \sigma_x \sigma_y \sigma_z} \left[ \cos^2(\theta) \cos^2(\phi) + \cos^2(\theta) \sin^2(\phi) + \sin^2(\phi) \right]^{-1} \tag{36} \]

\[ \exp \left[ -1/2 \left( \frac{\eta_y^2 + \eta_z^2}{\sigma_y} \right) \right] \left[ \text{sgn}(\eta_{xt}) \left[ 1 - f \left( \frac{4|\eta_{xt}|}{\sqrt{2\pi}} \right) \right] - \text{sgn}(\eta_{xo}) \left[ 1 - f \left( \frac{4|\eta_{xo}|}{\sqrt{2\pi}} \right) \right] \right] \]

The advantage of Equation 36 lies in its simple evaluation.

V. SUMMARY AND CONCLUSIONS

A closed form integrable approximation for the error function has been developed. This approximation has the advantage that it can significantly reduce the computational burden in existing obscuration models for the calculation of concentration-pathlength. This computational burden currently
represents a large fraction of the computer memory and execution time required to exercise these models.

The error implicit in this approximation has been shown to be such that it is not excessive for most geometries that are of interest to obscuration modelers.

Figure 1. Comparison of gaussian and derivative of Fermi function.
Figure 2. Comparison of transmission calculated using error function and Fermi function through smoke cloud at $t = 0$.

Figure 3. Comparison of transmission calculated using error function and Fermi function through smoke cloud at $t = 30$. 

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Figure 4. Comparison of transmission calculated using error function and Fermi function through smoke cloud at \( t = 60 \).

Figure 5. Comparison of transmission calculated using error function and Fermi function through smoke at \( t = 120 \).
Figure 6. Comparison of transmission calculated using error function and Fermi function through smoke at $t = 240$.

Figure 7. Comparison of transmission calculated using error function and Fermi function through smoke at $t = 480$. 
Figure 8. Comparison of transmission calculated using error function and Fermi function smoke at $t = 360$. 
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