



AD A093610

MRC Technical Summary Report # 2127

ON PERIODIC WATER-WAVES AND THEIR  
CONVERGENCE TO SOLITARY WAVES  
IN THE LONG-WAVE LIMIT

1  
B.S.

C. J. Amick and J. P. Toland

LEVEL II

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

DTIC  
SELECTED  
JAN 9 1981  
C

October 1980

(Received September 12, 1980)

Approved for public release  
Distribution unlimited

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

FILE COPY

80 12 22 038

14) MRZ-TSR-2227

11) Oct 80

UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

6) ON PERIODIC WATER WAVES AND THEIR CONVERGENCE  
TO SOLITARY WAVES IN THE LONG-WAVE LIMIT.

10) C. J. Amick<sup>†</sup> and J. P. Toland<sup>††</sup>

12) 77

9) Technical Summary Report #2127  
October 1980

ABSTRACT 15) DAAG-29-80-C-0042

A detailed discussion of Nekrasov's approach to the steady water-wave problems leads to a new integral equation formulation of the periodic problem. This development allows the adaptation of the methods of [1] to show the global convergence of periodic waves to solitary waves in the long-wave limit.

In addition, it is shown how the classical integral equation formulation due to Nekrasov leads, via the Maximum Principle, to new results about qualitative features of periodic waves for which there has long been a global existence theory ([9], [12]).

AMS (MOS) Subject Classifications: 76.45, 45G05, 45C05, 47H15

Key Words: Nekrasov's Integral Equation; asymptotic convergence; long-wave limit; solitary wave; periodic wave; global bifurcation; a priori bounds.

Work Unit Number 1 - Applied Analysis

† Department of Mathematics, University of Chicago.

†† Research supported in part by the United Kingdom Science Research Council. Department of Mathematics, University College London.

222200 *elt*

## SIGNIFICANCE AND EXPLANATION

Previous work [1] showed the existence of large-amplitude solitary waves by invoking the modern theory of global bifurcation to a sequence of approximate problems (none of which had any physical significance in their own right), and then passing to the limit.

The results in [14], [26], [27] proved the existence of small-amplitude solitary waves by showing the convergence of small-amplitude periodic waves to solitary waves as their wavelength increases indefinitely. In this paper, we show that large-amplitude solitary waves, up to and including a wave of 'greatest height', arise in the long wave limit of periodic waves.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	<input type="checkbox"/>
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

---

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON PERIODIC WATER-WAVES AND THEIR CONVERGENCE  
TO SOLITARY WAVES IN THE LONG-WAVE LIMIT

C. J. Amick<sup>†</sup> and J. F. Toland<sup>††</sup>

1. INTRODUCTION

1.1. Introductory remarks.

Under consideration are the steady two-dimensional waves which can arise as the free surface of a heavy, ideal liquid acted on by gravity, and contained in a channel of infinite extent with a horizontal bottom, in the absence of surface tension effects. It is well-known that both periodic waves [9], [12] and solitary waves [1] of large amplitude may occur in these circumstances. A precise account of the free boundary-value problem presented by this situation is given in the next section, and various physical parameters describing the flow are introduced. After some basic results about conformal mappings and Jacobi elliptic functions have been recorded in section 1.3, the method of Nekrasov [21] is used to reduce the existence question for these free boundary-value problems to a similar question for nonlinear integral equations. Throughout this section, we emphasise the role which various physical parameters play in these integral equation formulations. For example, in the periodic case, the wavelength and the mean depth are specified a priori and appear as constants in the equations, whereas the

---

<sup>†</sup>Department of Mathematics, University of Chicago.

<sup>††</sup>Research supported in part by the United Kingdom Science Research Council. Department of Mathematics, University College London.

mean velocity, the flux and the flow velocity at the crest depend on the solution of the equation being considered. An account of this is given in Theorems 1.5 and 1.6.

Of the two integral equation formulations (1.31) and (1.32) of the periodic problems given in section 1.3, equation (1.32) is perhaps the more familiar. It was used in [9] to prove a global existence theorem for periodic water-waves (though the physical interpretation of its solutions there is different from ours). Equation (1.31), which is equivalent to the usual integral equations for periodic waves ([9], [12], [20], [28]), is introduced because it has distinct advantages for our purposes in section 3. The most important of these is its striking resemblance to the approximation used in [1; section 3.2] to prove the existence of large-amplitude solitary waves.

After a few remarks in section 2.1 about recent developments in the theory of large-amplitude periodic water-waves, section 2.2 is devoted to a summary and sketch of the proofs of a global bifurcation theorem for periodic waves of wavelength  $\lambda$  on a flow of mean depth  $h$ , where  $\lambda$  and  $h$  are any given positive real numbers. Among these results is the existence of a connected set of such waves containing waves of all amplitudes up to that of a 'wave of greatest height'. This connected set contains a wave whose maximum angle of inclination to the horizontal is  $\beta$ , for any  $\beta \in [0, \frac{\pi}{6} + \epsilon]$  where  $\epsilon > 0$  is sufficiently small, and the mean velocity of all such waves is bounded away from zero and infinity. Some of these results are already known in a different context, while for others the proof given here is new. For the sake of clarity, we have collected them here and expressed them in terms of equation (1.31), which is the form in which we shall need them again in section 3.

In section 2.3 we show that solutions of equation (1.32) lie in a cone which is smaller than the cone of non-negative functions in  $C_0[0, \lambda/2]$ , namely the cone  $\hat{K}$  of non-negative functions  $u$  which are decreasing on  $[\lambda/4, \lambda/2]$  and such that  $u(\chi) \geq u(\lambda/2 - \chi)$ ,  $\chi \in [0, \lambda/4]$ . This leads to a considerable improvement in the global bifurcation theory for (1.32). We show that the maximal connected subset of non-trivial solutions which bifurcates from the curve of trivial solutions  $\{(\mu, 0) : \mu \in \mathbb{R}\}$  at the first characteristic value,  $6\pi\lambda^{-1} \coth(2\pi h/\lambda)$ , of the linearised problem is unbounded, and lies in  $(6\pi\lambda^{-1} \coth(2\pi h/\lambda), \infty) \times \hat{K}$ . Then using the strong maximum principle, we argue that if  $(\mu, \theta)$  lies in it, then  $\theta'(\chi) < 0$  on  $[\lambda/4, \lambda/2]$ . The significance of this observation, which lies in the fact that  $\theta$  represents the angle of inclination of the free surface (suitably parameterized) with the horizontal, is discussed, and the possibility of extending the method to get information about the shape of the highest wave is mentioned, but no firm conclusion is reached. Using an idea of Benjamin, we show that the maximum angle of inclination of any periodic or solitary water-wave under consideration (those in the sets  $C_\lambda$  or  $C'$  in Theorems 2.2 and 3.5, respectively) is less than  $\pi/3$ .

Finally, the main result of this paper is proved in section 3, and is summarised as follows: if  $h$  is fixed, then as  $\lambda \rightarrow \infty$  the connected sets of periodic waves of wavelength  $\lambda$  on a flow of mean depth  $h$  converge, in a certain sense, to a connected set of solitary waves whose asymptotic height is  $h$ . This connected set enjoys all the properties of the connected set  $C$  mentioned in [1; Theorem 3.9], and the behaviour of the corresponding waves is described in [1; section 4]. The global existence of solitary waves is already known [1]; what is new here is that periodic waves converge to solitary waves in the long-wave limit. An easy corollary of our general result in this direction is the following:

COROLLARY 3.4. For each  $\beta$ ,  $0 < \beta < \pi/2$ , and  $h, \epsilon > 0$  there exists on a flow of mean depth  $h$ , a periodic, symmetric, irrotational, wave of wavelength  $\lambda$  such that the free surface of which subtends an angle  $\beta$  with the horizontal at its steepest point. If  $h$  is fixed and  $\lambda \rightarrow \infty$  as  $\epsilon \rightarrow 0$ , then a subsequence of the periodic wave profiles converge uniformly on compact subsets of  $\mathbb{R}^2$  to the profile of a steady solitary water-wave whose free surface subtends a maximum angle of  $\beta$  with the horizontal, and whose asymptotic depth is  $h$ .

Such results as these may be regarded as global versions of the theorem of Ter-Krikerov [26], [27] and Laurentiev [14], who proved the existence of small-amplitude solitary waves by showing the convergence of small-amplitude periodic waves to solitary waves as their wavelength increases indefinitely. (See [3] for a similar global treatment of a different problem.) It is worth noting that because the mathematical theory of large-amplitude water-waves lacks any form of global uniqueness result, we cannot claim that all solitary waves may be described as the long-wave limit of a sequence of periodic waves. The results of section 3 follow immediately by the methods of [1], section 3, once the similarity between (1.31) and equation (2.14) of [1] has been noted. The analysis presented here has the advantage that the linearization about the zero solution of (1.31) is well understood, and the nonlinear problem is found. It may be regarded as a small step towards a rigorous theoretical confirmation of the very striking numerical results of [2].

## 1.2. The water-wave problem.

The problem being considered is that of the existence of two-dimensional irrotational waves on the surface of an inviscid, incompressible fluid of mean depth  $h$ . In this section two-dimensional irrotational waves are considered.

(a) A symmetric, periodic flow of wavelength  $\lambda$  whose mean depth is  $h$ .

If such a flow exists and if the free surface has a unique maximum per wavelength, then a cross-section of the flow perpendicular to the wave crests may be identified with a region in the complex  $z$ -plane between the line  $y = 0$  and a curve  $\{x + iH_\lambda(x) : x \in \mathbb{R}\}$ . Here  $H_\lambda : \mathbb{R} \rightarrow (0, \infty)$  is a function of period  $\lambda$  which is even and is decreasing on the interval  $(0, \lambda/2)$  (see Figure 1). One wavelength of this flow then occupies the region  $S_\lambda$  bounded by the lines  $x = \pm \lambda/2, y = 0$  and the free surface  $\Gamma_\lambda = \{x + iH_\lambda(x) : x \in (-\lambda/2, \lambda/2)\}$ . Since the fluid is supposed to be incompressible and the flow irrotational, there exists an analytic function, the complex potential,  $\omega = \phi + i\psi$ , which is related to the velocity  $(u(z), v(z))$  of the flow at a point  $z \in S_\lambda$  by the expression

$$u(z) - iv(z) = -\frac{d\omega}{dz} = -\phi_x + i\phi_y = -\psi_y - i\psi_x \quad (1.1)$$

Since the flow is symmetric about  $x = 0$ ,  $\omega$  must satisfy the relationship

$$\overline{\frac{d\omega}{dz}}(z) = \frac{d\omega}{dz}(-\bar{z}) \quad (1.2)$$

whence

$$\psi_x(z) = -\psi_x(-\bar{z}) \quad (1.3)$$

and

$$\psi(z) = \psi(-\bar{z}) \quad (1.4)$$

In particular,  $\psi_x$  is zero on the imaginary axis and, by periodicity,

$$\psi_x(z) = -\phi_y(z) = 0 \quad \text{if } \text{Real } z = \pm \lambda/2 \quad (1.5)$$

Let  $C = \{z(t) : t \in [0, 1]\}$  be any simple curve in  $S_\lambda$  directed from  $-\lambda/2 + iy$  to  $\lambda/2 + iy$ . Then

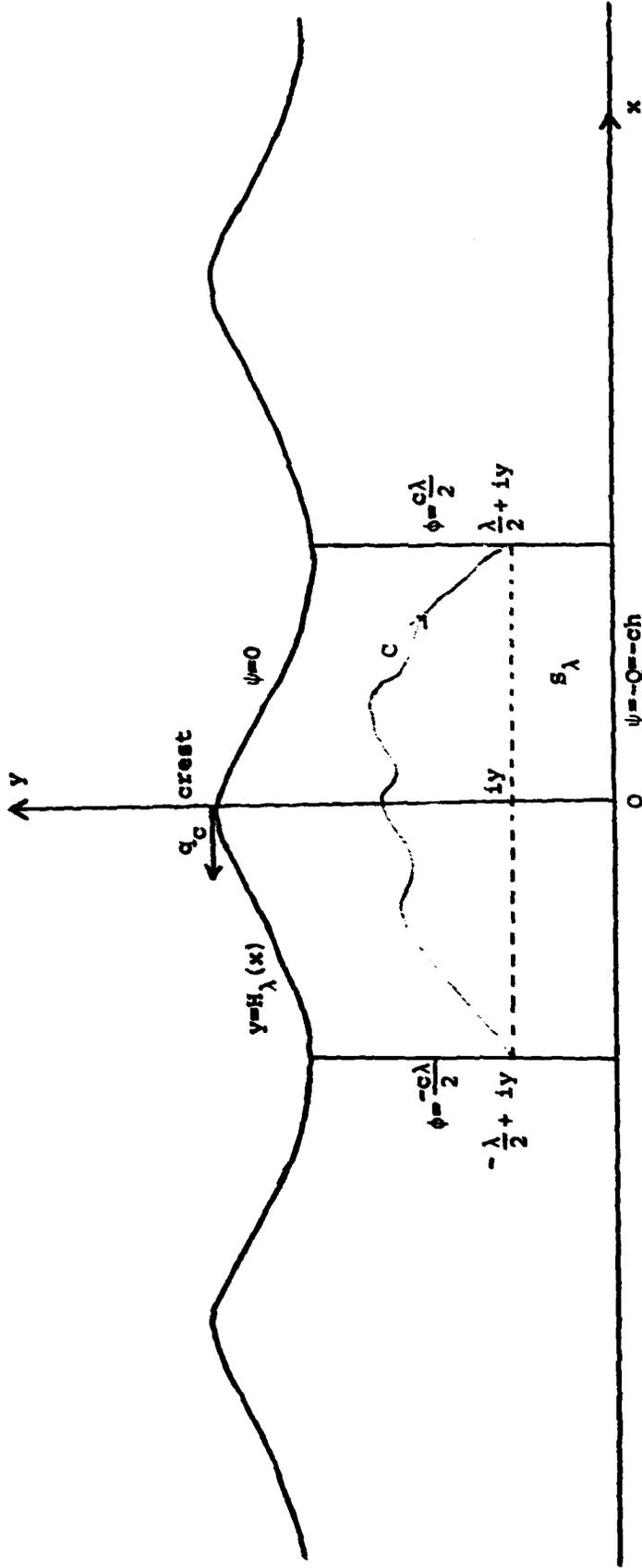


Figure 1. A steady periodic wave of wavelength  $\lambda$  on a flow of mean depth  $h$ . The region occupied by one wavelength is  $S_\lambda$ . The mean velocity of the flow is  $-c$ , and its speed at the wave crest is  $q_c$ .

$$\int_C \{u(z) - iv(z)\} dz = \omega(-\lambda/2 + iy) - \omega(\lambda/2 + iy)$$

$$= \phi(-\lambda/2 + iy) - \phi(\lambda/2 + iy) ,$$

by (1.4),

$$= -(\phi(\lambda/2) - \phi(-\lambda/2))$$

by (1.5). In particular, if  $C$  is chosen to be a horizontal line (for example, the bottom of the domain  $S_\lambda$ ), we find that

$$\frac{1}{\lambda} \int_C u(z) dz = \frac{1}{\lambda} \int_C \{u(z) + iv(z)\} dz = -\{\phi(\lambda/2) - \phi(-\lambda/2)\}/\lambda , \quad (1.6)$$

which is called the mean velocity and is denoted by  $-c$ . (If the flow is considered in a frame of reference relative to which the mean velocity is zero, then  $c$  is the phase speed of the wave.) Since the bottom ( $y = 0$ ) and the free surface  $\Gamma_\lambda$  are streamlines, the stream-function  $\psi$  must be constant on both, and without loss of generality, we may suppose that

$$\psi(z) = 0 \quad \text{if } z \in \Gamma_\lambda . \quad (1.7)$$

Since  $h$  is the mean depth of the flow,

$$\psi(z) = -Q \quad \text{if } \text{Imag } z = 0 , \quad (1.8)$$

where

$$Q = ch . \quad (1.9)$$

(Note that for a given flow the mean depth is not to be confused with an integral average of the height of the free surface. It is defined by (1.9) once the flux  $Q$  of the flow is known. By definition,  $Q$  is the value of  $\psi$  on the bottom when  $\psi$  has been normalized so that  $\psi = 0$  on the free surface.) Finally, since  $\Gamma_\lambda$  is a free streamline, the pressure is a constant there, and Bernoulli's theorem then implies that

$$\frac{1}{2} |\nabla\phi(z)|^2 + g \operatorname{Im} z = \text{constant} \quad (1.10)$$

for all  $z \in \Gamma_\lambda$ , where  $g$  is the acceleration due to gravity.

The existence question for this type of periodic flow is first one of finding the region  $S_\lambda$  occupied by one wavelength of the flow, and then one of finding  $\phi$  and  $\psi$  such that a periodic flow of wavelength  $\lambda$  and mean depth  $h$  occupies  $S_\lambda$ . It must be shown that  $\phi$  and  $\psi$  satisfies all the conditions (1.1) - (1.5), (1.7), (1.8) and (1.10) in  $S_\lambda$  where  $Q$  is given by (1.9) and  $c$  is given by (1.6).

(b) Solitary waves on a flow of asymptotic depth  $h$ . By a steady solitary wave is meant a symmetric two-dimensional flow whose free surface is in the form of a single symmetric wave of elevation, whose extent is infinite, and which is asymptotic to a finite height at  $\pm\infty$  (see Figure 2). The flow at  $\pm\infty$  is supposed to be approximately uniform horizontal flow from right to left in the channel. The boundary-value problem posed by this situation is first to find the flow domain  $S$  bounded by the line  $y = 0$  and a curve  $\Gamma = \{x + iH(x) : x \in \mathbb{R}\}$ , where the even function  $H$  is decreasing on  $(0, \infty)$  and

$$\lim_{|x| \rightarrow \infty} H(x) = h, \quad (1.11)$$

and then to find a complex potential  $\omega$  satisfying all the boundary conditions, which, in this case, take the following form. The relationship between the complex potential and the velocity field is given by (1.1), and since the flow is symmetrical (1.2) must also be satisfied. Since the flow is supposed approximately uniform and horizontal at points of  $S$  far from the crest, there results that



and by (1.12),  $\lim_{|z| \rightarrow \infty} \psi(z) = c$ . Finally, since  $\Gamma$  is a free streamline, Bernoulli's theorem requires that (1.10) must hold on  $\Gamma$ .

If, by analogy with the periodic case, the mean velocity of a solitary wave is calculated using the formula  $\lim_{h \rightarrow \infty} \frac{1}{h} \int_{-h/2}^{h/2} u(x+iy) dx$  for any  $y \in (0, h)$ , then it follows from (1.12) that this value coincides with the asymptotic velocity. Since the flux of the solitary wave is  $ch$ , it follows that the mean depth and the asymptotic depth coincide.

It is appropriate at this stage to mention one further parameter associated with a flow, solitary or periodic. In either case,  $u_c$  is used to denote the speed of the steady flow at the crest of a wave.

As is well-known, both the free boundary-value problems described above can be reformulated as nonlinear integral equations [20], [21]. In section 1.4, this formulation is discussed in detail, but before we can do that we need to introduce some conformal mappings. The approach of the next two sections is suggested by the remarks in the appendix of [9].

### 1.3. Preliminary mapping theorems.

A treatment of Jacobi's elliptic functions which is adequate for our purposes is given in [6]. Throughout the section and in all of the sequel,  $k$  is an arbitrary, but fixed, positive real number.

For  $k \in (0, 1)$ ,  $\operatorname{sn}(\cdot, k)$  denotes the odd Jacobi elliptic function of modulus  $k$  with primitive periods  $4K$  and  $2iK'$  and simple poles with residues  $1/k$  or  $-1/k$  at points congruent to  $iK'$  or to  $2K + iK'$  (mod  $4K, 2iK'$ ), respectively. The primitive periods are given in terms of  $k$  by the formulas

$$K = \int_0^1 (1-x^2)^{-\frac{1}{2}} (1-k^2x^2)^{-\frac{1}{2}} dx, \quad (1.15)$$

and

$$K' = \int_0^1 (1-x^2)^{-\frac{1}{2}} (1-(1-k^2)x^2)^{-\frac{1}{2}} dx. \quad (1.16)$$

Clearly  $K$  and  $K'$  are monotone functions of  $k \in (0, 1)$ , with  $K \downarrow \frac{\pi}{2}$  and  $K' \uparrow \infty$  as  $k \downarrow 0$ , while  $K \uparrow \infty$  and  $K' \downarrow \frac{\pi}{2}$  as  $k \uparrow 1$ .

For any  $\lambda > 0$ , let  $k_\lambda$  be the modulus of the unique function  $\text{sn}(\cdot, k_\lambda)$  such that

$$\frac{\lambda}{h} = 4 \frac{K_\lambda}{K'_\lambda}, \quad (1.17)$$

where  $K_\lambda$  and  $K'_\lambda$  are defined in terms of  $k_\lambda$  by (1.15) and (1.16). Since  $h$  is fixed,  $k_\lambda$ ,  $K_\lambda$  and  $K'_\lambda$  are monotone functions of  $\lambda$  and

$$\frac{2K_\lambda}{\lambda} \rightarrow \frac{\pi}{4h} \quad (1.18)$$

as  $\lambda \rightarrow \infty$ . For convenience with notation, we shall use  $s_\lambda$  to denote the elliptic function  $\text{sn}(\cdot, k_\lambda)$  for all  $\lambda > 0$ , and  $s_\infty$  to denote the analytic function  $\tanh$  which is the pointwise limit of  $s_\lambda$  as  $\lambda \rightarrow \infty$  [6; page 414, ex. 1].

It is well known [6; page 414, ex. 4] that the mapping  $\tilde{p}_\lambda$  from the complex  $\zeta$ -plane into the complex  $\xi$ -plane defined by

$$\tilde{p}_\lambda(\zeta) = -k_\lambda s_\lambda^2(2K_\lambda(\zeta + ih)/\lambda)$$

is a conformal mapping of the region  $R_\lambda = \{\zeta = \chi + i\eta : -\lambda/2 < \chi < \lambda/2, -h < \eta < 0\}$  onto the region  $D' = \{\xi = re^{is} : 0 < r < 1, -\pi < s < \pi\}$ . The function  $\tilde{p}_\lambda$  is analytic on  $\bar{R}_\lambda$  and maps the boundary portion  $A_\lambda = \{\zeta \in \bar{R}_\lambda : \zeta = \chi + i0, \chi \in [-\lambda/2, \lambda/2]\}$  onto the set

$\{\xi = e^{is} : -\pi < s \leq \pi\}$ , and maps  $\partial R_\lambda \setminus A_\lambda$  onto the non-positive real axis in the unit disc. Let  $p_\lambda : [-\lambda/2, \lambda/2) \rightarrow (-\pi, -]$  be defined as follows:

$$p_\lambda(\chi) = s \quad \text{for } \chi \in [-\lambda/2, \lambda/2) ,$$

if and only if  $s \in (-\pi, \pi]$  and

$$\tilde{p}_\lambda(\chi + i0) = e^{is} .$$

Another, more convenient way of saying this is that for  $\chi \in [-\lambda/2, \lambda/2)$ ,

$p_\lambda(\chi) = s$  if and only if  $s \in (-\pi, -]$  and

$$\begin{aligned} \cos \frac{s}{2} + i \sin \frac{s}{2} &= -ik_\lambda^{1/2} s_\lambda(2K_\lambda(\chi + ih)/\lambda) \\ &= -i \left\{ \frac{(1 + k_\lambda) s_\lambda(2K_\lambda \chi/\lambda) + ic_\lambda(2K_\lambda \chi/\lambda) d_\lambda(2K_\lambda \chi/\lambda)}{1 + k_\lambda s_\lambda^2(2K_\lambda \chi/\lambda)} \right\} , \end{aligned} \quad (1.19)$$

where  $c_\lambda$  and  $d_\lambda$  denote the even Jacobi elliptic functions  $\text{cn}(\cdot, k_\lambda)$  and  $\text{dn}(\cdot, k_\lambda)$ , respectively. (Algebraic identities and rules for differentiating the functions  $c_\lambda$ ,  $d_\lambda$  and  $s_\lambda$  and given in [6; page 384]. The expression (1.19) follows from the relation (1.17) and [6; page 396, eq. 3].) Let  $\tilde{q}_\lambda : D' \rightarrow R_\lambda$  denote the inverse of  $\tilde{p}_\lambda$ , and let  $q_\lambda : (-\pi, -] \cup [-\lambda/2, \lambda/2)$  denote the inverse of  $p_\lambda$ . From equation (1.19) it follows that

$$\sin \frac{s}{2} = - \frac{(1 + k_\lambda) s_\lambda(2K_\lambda q_\lambda(s)/\lambda)}{1 + k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)} ,$$

which, upon differentiating with respect to  $s$  and using (1.19) along with the identities in [5; page 384], yields

$$\frac{1}{2} \cos \frac{s}{2} = - \frac{2K_\lambda (1 + k_\lambda)}{\lambda} \left\{ \frac{1 - k_\lambda s_\lambda^2 (2K_\lambda q_\lambda(s)/\lambda)}{1 + k_\lambda s_\lambda^2 (2K_\lambda q_\lambda(s)/\lambda)} \right\} \cos \frac{s}{2} q'_\lambda \quad (1.20)$$

But, by the algebraic identities relating  $s_\lambda$ ,  $c_\lambda$  and  $d_\lambda$ , there results that

$$(1 - k_\lambda s_\lambda^2)^2 = c_\lambda^2 d_\lambda^2 + (1 - k_\lambda)^2 s_\lambda^2 ,$$

and so (1.19) and (1.20) together yield the following expression for  $q'_\lambda$  :

$$q'_\lambda(s) = -\{\lambda/(4K_\lambda (1 + k_\lambda))\} \left[ \cos^2 \frac{s}{2} + \left( \frac{1 - k_\lambda}{1 + k_\lambda} \right)^2 \sin^2 \frac{s}{2} \right]^{-1/2} . \quad (1.21)$$

For convenience with notation, we define the following expressions:

$$\left. \begin{aligned} f_\lambda(s) &= \frac{1}{2} \left[ \cos^2 \frac{s}{2} + \left( \frac{1 - k_\lambda}{1 + k_\lambda} \right)^2 \sin^2 \frac{s}{2} \right]^{-1/2} , \\ f(s) &= \frac{1}{2} \sec \frac{s}{2} , \end{aligned} \right\} \quad (1.22)$$

for all  $s \in (-\pi, \pi)$ , and

$$\Lambda = \lambda/(2K_\lambda (1 + k_\lambda)) . \quad (1.23)$$

Recall from (1.18) that

$$\Lambda \rightarrow 2h/\pi \text{ as } \lambda \rightarrow \infty . \quad (1.24)$$

Since the only zeros of  $dp_\lambda/d\zeta$  occur at  $\zeta = -ih$  and at  $\zeta = \pm \lambda/2 - ih$ , the real and imaginary parts of  $\tilde{q}_\lambda$  satisfy the Cauchy-Riemann conditions on the boundary portion  $\{e^{it} : -\pi < t < \pi\}$  of  $\partial D'$ .

Hence

$$\begin{aligned} \frac{\partial}{\partial s} (\text{Imag } \tilde{q}_\lambda) \Big|_{e^{is}} &= - \frac{\partial}{\partial s} (\text{Real } \tilde{q}_\lambda) \Big|_{e^{is}} = -q'_\lambda(s) \\ &= \Lambda f_\lambda(s) \end{aligned} \quad (1.25)$$

for all  $s \in (-\pi, \pi)$ .

Before finishing this discussion of conformal mappings, we note that in the limiting case when  $\lambda \rightarrow \infty$  a mapping which takes the region  $R_\infty = \{\chi + i\eta : \chi \in (-\infty, \infty), -h < \eta < 0\}$  conformally onto  $\mathcal{D}'$  and the boundary portion  $A_\infty = \{\chi + i0 : \chi \in (-\infty, \infty)\}$  onto  $\{e^{is} : -\pi < s < \pi\}$ , is given by

$$\begin{aligned}\tilde{p}(\zeta) &= -\tanh^2(\pi(\zeta + ih)/4h) \\ &= -s_\infty^2(\pi(\zeta + ih)/4h) .\end{aligned}$$

If the inverse of  $\tilde{p}$  is denoted by  $\tilde{q}$ , then it follows just as before that

$$\begin{aligned}\frac{\partial}{\partial r} (\text{Imag } \tilde{q}) \Big|_{e^{is}} &= -\frac{\partial}{\partial s} (\text{Real } \tilde{q}) \Big|_{e^{is}} \\ &= \frac{h}{\pi} \sec \frac{s}{2} = \frac{2h}{\pi} f(s) .\end{aligned}\tag{1.26}$$

If  $v : [-\pi, \pi] \rightarrow \mathbb{R}$  is a continuous, odd function with  $v(\pi) = 0$ , then there exists a unique harmonic function  $\tilde{u}$  on the unit disc  $\mathcal{D} = \{\xi : |\xi| < 1\}$  which satisfies the Neumann boundary condition  $\frac{\partial \tilde{u}}{\partial r} \Big|_{e^{is}} = v(s)$ ,  $s \in (-\pi, \pi]$ , and the normalization condition  $\int_{\partial \mathcal{D}} u = 0$ . It is easy to see that for all  $s \in (-\pi, \pi]$ ,

$$\tilde{u}(e^{is}) = \int_{-\pi}^{\pi} G(s,t)v(t)dt ,$$

where

$$\begin{aligned}G(s,t) &= \frac{1}{\pi} \sum_{\lambda=1}^{\infty} \frac{\sin \lambda s \sin \lambda t}{\lambda} \\ &= \frac{1}{2\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right|\end{aligned}\tag{1.27}$$

for all  $(s, t) \in (-\pi, \pi] \times (-\pi, \pi]$ ,  $s \neq t$ , and  $\tilde{u}$  is zero on the real axis in  $\mathcal{D}$ . (The identity (1.27) and further properties of  $G$  are from [1, Theorem 2.5].) Note that (1.27) ensures that  $G(s,t) \geq 0$  for all  $(s,t) \in [0,\pi] \times [0,\pi]$ ,  $s \neq t$ .

The next theorem concerns the change of variables which enables the convergence result of section 3 to be deduced from the work of [1].

**THEOREM 1.1.** Let  $V : [-\lambda/2, \lambda/2] \rightarrow \mathbb{R}$  be a continuous, odd function which is positive on  $(0, \lambda/2)$  with  $V(-\lambda/2) = 0$ . Then putting  $v(s) = -V(q_\lambda(s))$ , for all  $s \in (-\pi, \pi]$ , defines a continuous, odd function which is positive on  $(0, \pi)$ , and  $v(\pi) = 0$ .

Moreover, if  $\Lambda$  is given by (1.23), then

$$u(s) = \Lambda \int_{-\pi}^{\pi} G(s,t)v(t)f_\lambda(t)dt \quad (1.28)$$

for all  $s \in (-\pi, \pi]$ , if and only if

$$u(s) = -U(q_\lambda(s)) \quad ,$$

where

$$U(x) = \int_{-\lambda/2}^{\lambda/2} \frac{1}{2\pi} \ln \left| \frac{s_\lambda(2K_\lambda(x+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(x-\epsilon)/\lambda)} \right| V(\epsilon) d\epsilon \quad . \quad (1.29)$$

Furthermore, there exists a harmonic function  $\tilde{U}$  on  $R_\lambda$  such that

$$\begin{aligned} \tilde{U}(x+i0) &= U(x) \quad , \quad \text{for } x \in [-\lambda/2, \lambda/2) \quad , \\ \left. \frac{\partial \tilde{U}}{\partial \bar{n}} \right|_{x+i0} &= V(x) \quad , \quad \text{for } x \in (-\lambda/2, \lambda/2) \quad , \end{aligned}$$

and  $\tilde{U} = 0$  on  $\partial R_\lambda \setminus A_\lambda$ .

Proof. It follows from (1.19) and from the formula for the elliptic function of a sum that, under the change of variables

$$x = q_\lambda(s) \quad \text{and} \quad \epsilon = q_\lambda(t) \quad , \quad s, t \in (-\pi, \pi] \quad ,$$

the kernel

$$\frac{1}{2\pi} \ln \left| \frac{s_\lambda (2K_\lambda (\lambda + \epsilon)/\lambda)}{s_\lambda (2K_\lambda (\lambda - \epsilon)/\lambda)} \right|$$

becomes

$$\frac{1}{2\pi} \ln \left| \frac{\sin((s + t)/2)}{\sin((s - t)/2)} \right| .$$

Since  $q_\lambda'(s) = -\Lambda f_\lambda(s)$  on  $(-\pi, \pi]$ , the result for the first part of the theorem is immediate.

Because  $v$  is continuous and odd on  $(-\pi, \pi]$  and  $v(\pi) = 0$ , it follows that there exists a unique function  $\tilde{u}$ , harmonic on  $\mathcal{D}$  and continuous on  $\bar{\mathcal{D}}$ , such that

$$\frac{\partial \tilde{u}}{\partial r} \Big|_{e^{it}} = \Lambda f_\lambda(t) v(t)$$

and

$$\tilde{u}(e^{it}) = u(t)$$

for all  $t \in (-\pi, \pi]$ . Since  $v$  is odd,  $\tilde{u}$  is zero on the real axis in  $\mathcal{D}$ . Therefore  $\tilde{U}$  defined by

$$\tilde{U}(z) = -\tilde{u}(\tilde{p}_\lambda(z))$$

is harmonic on  $R_\lambda$  and continuous on  $\bar{R}_\lambda$ . Since  $\tilde{p}_\lambda$  maps  $\partial R_\lambda \setminus A_\lambda$  onto the non-positive real axis in  $\mathcal{D}$ , where  $\tilde{u}$  vanishes, it follows that  $\tilde{U}$  vanishes on  $\partial R_\lambda \setminus A_\lambda$ . The results for  $\tilde{U}$  on  $A_\lambda$  follow by (1.25).

q.e.d.

LEMMA 1.2. For all  $x, \epsilon \in [-\lambda/2, \lambda/2] \times [-\lambda/2, \lambda/2]$ ,  $\lambda \neq 0$ ,

$$\frac{1}{2\pi} \ln \left| \frac{s_\lambda (2K_\lambda (\lambda + \epsilon)/\lambda)}{s_\lambda (2K_\lambda (\lambda - \epsilon)/\lambda)} \right| = \frac{1}{\pi} \sum_{h=1}^{\infty} \epsilon^{-1} \tanh\left(\frac{2\pi h}{\lambda}\right) \sin\left(\frac{2\pi h x}{\lambda}\right) \sin\left(\frac{2\pi h \epsilon}{\lambda}\right) .$$

Proof. This follows by a simple calculation from the expansion for  $\operatorname{sn}(u, k)$  [8; page 912(20)]:

$$2n \operatorname{sn}(u, k) = 2n \frac{2K}{\pi} + 2n \sin \frac{\pi u}{2K} - 4 \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{q^{\ell}}{1+q^{\ell}} \left\{ \sin \left( \frac{\pi \ell u}{2K} \right) \right\}^2,$$

where  $q = e^{-\pi K'/K}$ .

q.e.d.

**THEOREM 1.3.** The solutions of the linear characteristic value problem

$$u(s) = \frac{\mu}{3} \int_{-\pi}^{\pi} G(s,t) f_{\lambda}(t) u(t) dt$$

consist precisely of the set of characteristic values  $\left\{ \frac{6\lambda\pi\ell}{\lambda} \coth \left( \frac{2\pi\ell h}{\lambda} \right) \right\}_{\ell=1}^{\infty}$

with corresponding eigenfunctions  $\left\{ \sin \frac{2\pi\ell q_{\lambda}(s)}{\lambda} \right\}_{\ell=1}^{\infty}$ . In particular, the

smallest characteristic value,  $\mu_{\lambda} = \frac{6\lambda\pi}{\lambda} \coth \left( \frac{2\pi h}{\lambda} \right) + \frac{6}{\pi}$  as  $\lambda \rightarrow \infty$ .

Proof. From Lemma 1.2 it follows that the set of characteristic values of the operator defined by the right-hand side of (1.29) comprise the set

$$\left\{ \frac{2\pi\ell}{\lambda} \coth \left( \frac{2\pi\ell h}{\lambda} \right) \right\}_{\ell=1}^{\infty}, \text{ and the corresponding eigenvectors are } \left\{ \sin(2\pi\ell x/\lambda) \right\}_{\ell=1}^{\infty}.$$

The result is then an immediate consequence of Theorem 1.1 and the fact that  $\Lambda \rightarrow 2h/\pi$  as  $\lambda \rightarrow \infty$  by (1.24). Since  $f_{\lambda_1}(x) \leq f_{\lambda_2}(x)$  if  $\lambda_1 \leq \lambda_2$ , it follows that  $\mu_{\lambda_1} \geq \mu_{\lambda_2}$  by [1; Theorems A.1 and A.2]. (See also Lemma 3.3.)

q.e.d.

#### 1.4. On integral equations for water-waves.

The purpose of this section is to show the equivalence of two non-linear integral equations, each of which is a formulation of the periodic water-wave problem when the mean depth and the wavelength are given. Theorem 1.4 is a statement of this equivalence, while in Theorem 1.5 a precise description of the wave which corresponds to a solution of equation (1.31) is given.

Theorem 1.6, which is taken without proof from [1], is a statement of the corresponding result for solitary waves.

Let  $h$  be fixed, as in the previous section, and let  $\lambda$  be any positive real number.

THEOREM 1.4. (i) If  $\theta : [-\lambda/2, \lambda/2] \rightarrow \mathbb{R}$  is continuous, odd, and  $0 < \theta(x) < \pi/2$  on  $(0, \lambda/2)$ , and if for all  $s \in (-\pi, \pi)$ ,

$$\theta(s) = -\theta(q_\lambda(s)) \quad , \quad (1.30)$$

then  $\theta : (-\pi, \pi) \rightarrow \mathbb{R}$  is continuous, odd, and  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ .

Moreover, for some  $\mu > 0$ ,  $\theta$  satisfies the equation

$$\theta(s) = \frac{1}{6} \int_{-\pi}^{\pi} \frac{1}{\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \frac{f_\lambda(t) \sin \theta(t)}{\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw} dt \quad (1.31)$$

for all  $s \in [-\pi, \pi]$ , if and only if  $\theta$  satisfies the equation

$$\theta(x) = \frac{1}{6} \int_{-\lambda/2}^{\lambda/2} \frac{1}{\pi} \ln \left| \frac{s_\lambda(2K_\lambda(x+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(x-\epsilon)/\lambda)} \right| \frac{\sin \theta(\epsilon)}{\Lambda/\mu + \int_0^\epsilon \sin \theta(w) dw} d\epsilon \quad (1.32)$$

for all  $x \in [-\lambda/2, \lambda/2]$ . Here  $\Lambda$  is given by (1.23) and  $f_\lambda$  by (1.22).

(ii) If  $\theta$  is as in (i) and satisfies (1.32), then there exists a harmonic function on  $R_\lambda$  which coincides with  $\theta$  on the boundary portion  $A_\lambda$ , and which is zero on  $\partial R_\lambda \setminus A_\lambda$ . If  $\tilde{\theta}$  is used to denote this harmonic function on  $R_\lambda$ , then

$$\frac{\partial \tilde{\theta}}{\partial n} \Big|_{x+i0} = \frac{1}{3} \frac{\sin \theta(x)}{\frac{\Lambda}{\mu} + \int_0^x \sin \theta(w) dw} \quad (1.33)$$

for all  $x + i0 \in A_\lambda$ .

Proof. This result is immediate from Theorem 1.1 and equation (1.25).  
q.e.d.

In order to use the methods of [1] to prove that connected sets of periodic waves converge to solitary waves in the long-wave limit, it is necessary to be explicit about the waves to which solutions of (1.31) correspond.

THEOREM 1.5. Suppose that  $\theta$  is an odd, continuous function on  $[-\pi, \pi]$  with  $0 < \theta(s) \leq \pi$  on  $(0, -)$  and  $\theta(\pi) = 0$ , which satisfies the integral equation (1.31) on  $[-\pi, \pi]$  for some  $\mu > 0$ . Then  $\theta$  is real-analytic on  $[-\pi, \pi]$  and satisfies  $0 < \theta(s) < \pi/2$  on  $(0, -)$ . Moreover, there exists a solution of the periodic water-wave problem of period  $\lambda$  on a flow of mean depth  $h$ . The mean velocity of the flow is given by

$$c = \frac{\sqrt{3g}}{L} \left( \frac{2}{\lambda} \int_0^{\pi} \frac{f_{\lambda}(t) \cos \theta(t)}{\left( \frac{1}{\mu} + \int_0^t f_{\lambda}(w) \sin \theta(w) dw \right)^{1/3}} dt \right)^{-3/2}, \quad (1.34)$$

from which the flux  $Q$  and the speed at the crest  $q_c$  may be calculated as follows:

$$Q = ch \quad (1.35)$$

and

$$q_c = \left( \frac{3g/c}{\mu} \right)^{1/3}. \quad (1.36)$$

The free surface  $\Gamma_{\lambda}$  is then given by  $\{(x, H_{\lambda}(x)) : x \in (-\lambda/2, \lambda/2)\}$ , where for  $x \in [0, \lambda/2]$ ,

$$H_{\lambda}(x) - H_{\lambda}(0) = \left( \frac{2}{3g} \right)^{1/3} \int_{s^{-1}(x)}^0 \frac{f_{\lambda}(t) \sin \theta(t)}{\left( \frac{1}{\mu} + \int_0^t f_{\lambda}(w) \sin \theta(w) dw \right)^{1/3}} dt, \quad (1.37)$$

and for  $s \in [-\pi, 0]$ ,

$$\alpha(s) = \left( \frac{c^2 \lambda^2}{3g} \right)^{1/3} \int_s^0 \frac{f_\lambda(t) \cos \theta(t)}{\left( \frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw \right)^{1/3}} dt \quad (1.38)$$

Remark. In this expression for the free surface profile, the value of  $H_\lambda(0)$  is given by  $\int_{-h}^0 \exp(\tilde{T}(0 + i\eta)) d\eta$ , where  $\tilde{T}$  is the function in (1.39) below. Since  $\tilde{T}$  is uniquely determined in  $R_\lambda$  by  $\tilde{\theta}$  and (1.39), one may determine  $H_\lambda(0)$  explicitly in terms of  $\mu$ ,  $\theta$ ,  $h$ , and  $\lambda$ . Unfortunately, it does not appear possible to put this result in as neat a form as (1.34), (1.37), or (1.38).

In Theorem 2.4, we show that the upper bound of  $\pi/2$  for  $\theta$  may be replaced by  $\pi/3$ .

Proof. The method of proof of Theorem 2.2 (ii), (iii) applies to any solution of (1.31), and not just to those in  $C_\lambda$ . Hence, the real-analyticity of  $\theta$  and the a priori bound of  $\pi/2$  for  $\theta$  follow immediately.

Let  $\tilde{\theta}$  be the function which is harmonic on  $R_\lambda$  mentioned in Theorem 1.4(ii), and let  $\tilde{T}$  denote the unique function which is harmonic on  $R_\lambda$  and conjugate to  $\tilde{\theta}$  (that is,  $\tilde{T} - i\tilde{\theta}$  is analytic in  $R_\lambda$ ) such that

$$\frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} \exp(\tilde{T}(x - ih)) dx = 1 \quad (1.39)$$

If  $\tilde{\theta}$  denotes the harmonic function on  $\mathcal{D}$  with boundary values  $\theta$ , then the real-analyticity of  $\theta$  ensures that  $\tilde{\theta}$  is real-analytic on  $\bar{\mathcal{D}}$ . Since  $\tilde{\theta}(z) = -\tilde{\theta}(\bar{p}_\lambda(z))$ ,  $z \in \bar{R}_\lambda$ , the analyticity of  $\tilde{p}_\lambda$  on  $\bar{R}_\lambda$  ensures that  $\tilde{\theta}$  is real-analytic on  $\bar{R}_\lambda$ , and hence that  $\tilde{T}$  is real-analytic by the Cauchy-Riemann equations. Because  $\tilde{T} - i\tilde{\theta}$  is analytic on  $\bar{R}_\lambda$ , we can use it to define an analytic function  $\tilde{m}$  on  $\bar{R}_\lambda$  by putting

$$\tilde{m}(z) = \int_{-ih}^z \exp\{\tilde{T}(\xi) - i\tilde{\theta}(\xi)\} d\xi \quad (1.40)$$

The function  $\tilde{m}$  is injective on  $\bar{R}_\lambda$ ; for otherwise there exists

$\zeta_1, \zeta_2 \in \bar{R}_\lambda$  with

$$\int_{\zeta_1}^{\zeta_2} \exp(\tilde{T}(\xi)) \cos \tilde{\theta}(\xi) d\xi = 0 ,$$

and this contradicts the fact that  $|\tilde{\theta}| < \pi/2$  on  $\bar{R}_\lambda$  by the maximum principle. Since  $\tilde{m}'(\zeta) \neq 0$  in  $\bar{R}_\lambda$ , it follows that  $R_\lambda$  is mapped conformally onto a region  $S_\lambda$  by  $\tilde{m}$ , and that  $\tilde{m}$  is invertible there.

Because  $\tilde{\theta}$  is odd on  $A_\lambda$ , and zero on the rest of  $\partial R_\lambda$ , it follows that  $\tilde{\theta}(\zeta) = -\tilde{\theta}(-\bar{\zeta})$  and  $\tilde{T}(\zeta) = \tilde{T}(-\bar{\zeta})$ ,  $\zeta \in \bar{R}_\lambda$ . From this observation and (1.40) there results that  $\overline{\tilde{m}(\zeta)} = \tilde{m}(-\bar{\zeta})$ ,  $\zeta \in \bar{R}_\lambda$ . Combining this with (1.39) and the fact that  $\tilde{\theta} = 0$  on  $\partial R_\lambda \setminus A_\lambda$  yields that  $S_\lambda$  is bounded by the lines  $y = 0$ ,  $x = \pm \lambda/2$  and the curve  $\Gamma_\lambda = \tilde{m}(A_\lambda)$ . If  $\zeta$  lies on the line  $A_\lambda$ , then

$$\frac{d}{dx} \operatorname{Re} \tilde{m}(\zeta) = \exp(\tilde{T}(\zeta)) \cos \tilde{\theta}(\zeta) > 0 ,$$

and so, for some even function  $H_\lambda$ , we have

$$\Gamma_\lambda = \{x + iH_\lambda(x) : x \in [-\lambda/2, \lambda/2]\} .$$

A further calculation based on (1.40) yields that

$$H'_\lambda(x) = -\tan \theta(\tilde{m}^{-1}(x + iH_\lambda(x))) .$$

Since  $\tilde{m}$  is invertible,  $\tilde{m}^{-1}$  is analytic on  $\bar{S}_\lambda$ . We shall now show that if a complex potential  $\omega$  is defined on  $\bar{S}_\lambda$  by putting

$$\omega(z) = \phi(z) + i\psi(z) = c\tilde{m}^{-1}(z) \tag{1.41}$$

with  $c$  given by (1.34), then all of the conditions (1.1) - (1.10) are satisfied, and the proof of the theorem will be complete.

The velocity field  $(u, v)$  generated in  $\bar{S}_\lambda$  by  $\omega$  is given by

$$\begin{aligned} u(z) - iv(z) &= -\frac{dw}{dz} \\ &= -c \exp(-\tilde{T}(\tilde{m}^{-1}(z))) \{ \cos \tilde{\theta}(\tilde{m}^{-1}(z)) + i \sin \tilde{\theta}(\tilde{m}^{-1}(z)) \} \end{aligned} \quad (1.42)$$

whence  $-\tilde{\theta}(\tilde{m}^{-1}(z))$  is the angle which the negative velocity vector makes with the  $x$ -axis, and  $c \exp(-\tilde{T}(\tilde{m}^{-1}(z)))$  is the speed of the flow at  $z \in \bar{S}_\lambda$ . Since  $-\overline{\tilde{m}(\zeta)} = \tilde{m}(-\bar{\zeta})$ ,  $\zeta \in \bar{R}_\lambda$ , it follows that equations (1.2) - (1.4) are satisfied. Since  $\tilde{\theta} = 0$  on  $\partial R_\lambda \setminus A_\lambda$ , it is immediate from (1.40) that (1.5) holds. To show that (1.6) is satisfied we note that, by (1.39),

$$\begin{aligned} \frac{1}{\lambda} \{ \phi(-\frac{\lambda}{2}) - \phi(\frac{\lambda}{2}) \} &= \frac{c}{\lambda} \operatorname{Real} \{ \tilde{m}^{-1}(-\frac{\lambda}{2}) - \tilde{m}^{-1}(\frac{\lambda}{2}) \} \\ &= \frac{c}{\lambda} \{-\frac{\lambda}{2} - \frac{\lambda}{2}\} = -c. \end{aligned} \quad (1.43)$$

Next, if  $z \in \Gamma_\lambda$ , then  $\tilde{m}^{-1}(z) \in A_\lambda$ , whence  $\phi(z) = 0$ . If  $z \in \bar{S}_\lambda$  and  $\operatorname{Im} z = 0$ , then  $\operatorname{Im} \tilde{m}^{-1}(z) = -h$ , and so  $\phi(z) = -ch$ . It follows that (1.7) - (1.9) hold.

Let  $\tau : [-\lambda/2, \lambda/2] \rightarrow \mathbb{R}$  denote the restriction of  $\tilde{T}$  to  $A_\lambda$ . Then, since  $\tilde{\theta} = 0$  on  $\partial R_\lambda \setminus A_\lambda$ , it follows, by Cauchy's Theorem and (1.39), that

$$\int_{-\lambda/2}^{\lambda/2} \exp(\tau(x)) \cos \theta(x) dx = \int_{-\lambda/2}^{\lambda/2} \exp(\tilde{T}(x - ih)) dx = \lambda. \quad (1.44)$$

However, from (1.33) and the Cauchy-Riemann equations,

$$\tau(x) = \tau(0) - \frac{1}{3} \ln(1 + (w/\lambda)) \int_0^x \sin \theta(w) dw \quad (1.45)$$

for all  $x \in [-\lambda/2, \lambda/2]$ . Substituting this expression for  $\tau$  into (1.44) gives

$$\begin{aligned}
\exp(-T(0)) &= \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} \frac{\cos \theta(x)}{(1 + (\mu/\lambda) \int_0^x \sin \theta(w) dw)^{1/3}} dx \\
&= \frac{1}{\lambda} \int_{-\pi}^{\pi} \frac{\Lambda f_{\lambda}(t) \cos \theta(t)}{(1 + \mu \int_0^t f_{\lambda}(w) \sin \theta(w) dw)^{1/3}} dt \\
&= \left( \frac{3g\lambda}{\mu c^2} \right)^{1/3}, \tag{1.46}
\end{aligned}$$

by (1.34), whence

$$q_c^3 = c^3 \exp(-3T(0)) = \frac{3g\lambda c}{\mu},$$

and so (1.36) is satisfied.

In order to prove (1.10), we must show that  $\frac{1}{2}(u^2(z) + v^2(z)) + g \operatorname{Im} z$  is constant on  $\Gamma_{\lambda}$ , or, equivalently, that  $\frac{1}{2}c^2 \exp(-2T(x)) + g \operatorname{Im} \bar{m}(x + i0)$  is constant for  $x \in [-\lambda/2, \lambda/2]$ . A calculation now gives

$$\begin{aligned}
&\frac{d}{dx} \left[ \frac{1}{2} c^2 \exp(-2T(x)) + g \operatorname{Im} \bar{m}(x + i0) \right] \\
&= -c^2 \exp(-2T(x)) T'(x) - g \exp(T(x)) \sin \theta(x) \\
&= \exp(T(x)) \{ -c^2 \exp(-3T(x)) T'(x) - g \sin \theta(x) \} \\
&= 0
\end{aligned}$$

by (1.45) and (1.46).

Finally, to calculate the wave profile we proceed as follows. At a point  $x + iy \in \Gamma_{\lambda}$ , the free surface is given by

$$y = \eta_{\lambda}(x),$$

where  $\eta_{\lambda}(x) = -\tan \theta(\bar{m}^{-1}(x + i\eta_{\lambda}(x)))$ . Hence

$$\begin{aligned}
H_\lambda(x) - H_\lambda(0) &= \int_0^x H'_\lambda(w) dw \\
&= - \int_0^{\tilde{a}^{-1}(x)} \tan^{-1}(\lambda) \cos \theta(\lambda) \exp(\Gamma(\lambda)) d\lambda \\
&= - \left( \frac{\mu c^2}{3g\lambda} \right)^{1/3} \int_0^{\tilde{a}^{-1}(x)} \frac{\sin^{-1}(\lambda)}{\left(1 + (\mu/\lambda) \int_0^\lambda \sin \theta(w) dw\right)^{1/3}} d\lambda
\end{aligned}$$

where  $\tilde{a} : [-\lambda/2, \lambda/2] \rightarrow \mathbb{R}$  is given by

$$\begin{aligned}
\tilde{a}(\lambda) &= \text{Real } \tilde{m}(\lambda + i0) \\
&= \left( \frac{\mu c^2}{3g\lambda} \right)^{1/3} \int_0^\lambda \frac{\cos \theta(\lambda')}{\left(1 + (\mu/\lambda') \int_0^{\lambda'} \sin \theta(w) dw\right)^{1/3}} d\lambda'
\end{aligned}$$

Hence if  $x \in [0, \lambda/2]$ ,

$$H_\lambda(x) - H_\lambda(0) = \left( \frac{\mu c^2}{3g} \right)^{1/3} \int_{\tilde{a}^{-1}(x)}^0 \frac{f_\lambda(t) \sin \theta(t)}{\left(\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw\right)^{1/3}} dt$$

where

$$\tilde{a}^{-1}(x) = (\tilde{a} \circ q_\lambda)^{-1}(x) = p_\lambda(\tilde{a}^{-1}(x)) < 0,$$

and so, for  $s \in [-\pi, 0]$ ,

$$\begin{aligned}
\alpha(s) = \tilde{a} \circ q_\lambda(s) &= \left( \frac{\mu c^2}{3g} \right)^{1/3} \int_{q_\lambda(s)}^0 \frac{\cos^{-1}(\lambda')}{\left(1 + (\mu/\lambda') \int_0^{\lambda'} \sin \theta(w) dw\right)^{1/3}} d\lambda' \\
&= \left( \frac{\mu c^2}{3g} \right)^{1/3} \int_s^0 \frac{f_\lambda(t) \cos^{-1}(t)}{\left(\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw\right)^{1/3}} dt
\end{aligned}$$

This completes the proof of the theorem.

□

THEOREM 1.6. Suppose that  $\theta$  is an odd, continuous function on  $[-\pi, \pi]$  with  $0 < \theta(s) < \pi$  on  $(0, \pi)$  and  $\theta(-) = 0$ , which satisfies the integral equation

$$\theta(s) = \frac{1}{6} \int_{-\pi}^{\pi} \frac{1}{\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \frac{f(t) \sin \theta(t)}{\frac{1}{\mu} + \int_0^t f(w) \sin \theta(w) dw} dt \quad (1.47)$$

for all  $s \in [-\pi, \pi]$ , where  $\mu > 0$  and  $f(t) = \frac{1}{2} \sec \frac{t}{2}$  for  $t \in (-\pi, \pi)$ . Then  $\theta f \in L_1(-\pi, \pi)$ ,  $\theta$  is real-analytic on  $(-\pi, \pi)$ , and  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ . Moreover, if  $h$  and  $c$  are any positive real numbers which satisfy

$$\frac{6gh}{\pi c^2} \left( \frac{1}{\mu} + \int_0^{\pi} f(w) \sin \theta(w) dw \right) = 1,$$

then there exists a steady solitary wave flow whose mean velocity and asymptotic height (see section 1.2) are  $-c$  and  $h$  respectively. The speed  $q_c$  of the flow at the wave crest may be calculated from the expression

$$\frac{\pi q_c^3}{6ghc} = \frac{1}{\mu}.$$

Moreover, the solitary wave profile  $\Gamma$  is given by  $\{(x, H(x)) : x \in \mathbb{R}\}$  where for  $x > 0$ ,

$$H(x) - H(0) = \frac{h}{3} \left( \frac{36c^2}{\pi^2 gh} \right)^{1/3} \int_{\alpha^{-1}(x)}^0 \frac{f(t) \sin \theta(t)}{\left( \frac{1}{\mu} + \int_0^t f(w) \sin \theta(w) dw \right)^{1/3}} dt \quad (1.48)$$

and for  $s \in (-\pi, 0)$ ,

$$\eta(s) = \frac{h}{3} \left( \frac{36c^2}{-2gh} \right)^{1/3} \int_s^0 \frac{f(t) \cos \Theta(t)}{\left( \frac{1}{3} + \int_0^t f(w) \sin \Theta(w) dw \right)^{1/3}} dt \quad (1.49)$$

Remark. In this case, we can assert that the value of  $H(0)$  is

$$h \left\{ 1 + \frac{1}{2} \frac{c^2}{gh} - \frac{1}{2} \left( \frac{36c^2}{-2gh} \right)^{1/3} \right\},$$

because the asymptotic height is known (see [1; Theorem 4.6]).

Proof. While this theorem is formally the limiting case of Theorem 1.5 as  $\lambda \rightarrow \infty$ , it needs a separate proof. This may be done by modifying the method of proof of Theorem 1.5, using the mapping  $p$  from  $R_\infty$  onto  $D'$  introduced in section 1.3. The function  $\Theta$  is then required to be in  $L_1(-\infty, \infty)$ , odd, positive on  $(0, \infty)$ , and to satisfy

$$\Theta(\lambda) = \frac{1}{6} \int_{-\infty}^{\infty} \frac{1}{\tau} \ln \left| \frac{\tanh(\pi(\lambda + \epsilon)/4h)}{\tanh(\pi(\lambda - \epsilon)/4h)} \right| \frac{\sin \Theta(\epsilon)}{\frac{2h}{\pi\epsilon} + \int_0^\epsilon \sin \Theta(w) dw} d\epsilon \quad (1.50)$$

an equation which may be obtained from equation (1.47) by putting

$$\lambda = \frac{-2h}{\pi} \ln(\sec \frac{s}{2} + \tan \frac{s}{2}), \text{ and } \epsilon = \frac{-2h}{\pi} \ln(\sec \frac{t}{2} + \tan \frac{t}{2}), \quad s, t \in (-\pi, \pi).$$

An alternative proof is to be found in [1; Theorems 1.1, 4.1, 4.3 and 4.6].

(The function  $\Theta$  in [1; Theorem 1.1] differs from that which arises in the method suggested by the proof of Theorem 1.5 by a change of sign.) *q.e.d.*

For the sake of giving a complete description of Nekrasov's integral equations, we include in the Appendix the equation for periodic waves on a flow which is infinitely deep. The derivation there is slightly different from those already in the literature, and emphasizes the dependence of the flow parameters on a given solution of the equation. It is shown how this equation can be written in an alternative form which involves the conjugate

operator from the  $L_2$ -theory of Fourier series. While a similar formulation might be adopted in the case of finite depth (see [12]), we avoid this approach because the normalization requirement ([12; p. 1002, (1.19)] and (1.39) above) means that when the depth is finite the conjugate operator is nonlinear. In any case, (1.31) and (1.32) are preferred, since the dependence of the integrand on  $\theta$  and  $\Theta$  is given explicitly.

## 2. THE GLOBAL THEORY

### 2.1. Background.

The first proof of the existence of large amplitude, periodic water-waves is due to Krasovskii [12], and is based on an adaptation of the monotone minorant theorem [11] to a particular version of Nekrasov's integral equation. Among his results on the existence of periodic water-waves in a channel with a wave-like bottom is included the special case when the bottom is flat. In this case, the conclusion is that for each positive  $h$  and  $\lambda$ , and for each  $\beta \in (0, \pi/6)$ , there exists a wave of wavelength  $\lambda$ , on a flow whose mean depth is  $h$ , which is such that the maximum angle of inclination of the free surface to the horizontal is  $\beta$  and the mean velocity of all such waves is bounded away from zero and infinity. Though this result is highly suggestive, it does not amount to a global bifurcation theorem since neither the question of bifurcation, nor the question of the existence of a connected set of solutions is considered. The first result of this kind is due to Keady and Norbury [9], who regard Nekrasov's integral equation as an example in the general theory of global bifurcation [7], [23], [30]. Their result is the following: if  $L$  and  $Q$  are fixed, positive real numbers, then there exists a connected set of periodic water-waves which bifurcates from the set of horizontal, uniform flows, each of which is of flux  $Q$ , and each of which has wavelength  $2L$  with respect to the velocity potential. This set contains a wave whose speed at the crest is  $q_c$  for any value of  $q_c$  in the interval  $(0, (\frac{gL}{\pi} \tanh(\frac{\pi Q}{L}))^{1/3})$ .

Since the mathematical theory of steady water-waves still lacks any global uniqueness result, it is not possible to assert that the solutions

obtained by Krasovskii are included in the connected set which Keady and Norbury obtain. (In principle, Krasovskii's method may yield solutions lying off the bifurcating set, if such exist.) Nevertheless, it can be shown [29] (independently of the work of Krasovskii) that this bifurcating set contains waves with maximum angle of inclination to the horizontal  $\beta$ , for all values of  $\beta$  in the interval  $(0, \pi/6)$ . Indeed, it has been shown by McLeod [19] that this connected set of water waves contains a wave whose maximum angle of inclination to the horizontal is  $\beta$ , for all  $\beta \in (0, \pi/6 + \epsilon]$  for some  $\epsilon > 0$ .

In the next section, we shall summarize the global bifurcation theory for periodic water-waves of spatial wavelength  $\lambda$  on a flow of mean depth  $h$ . Because of our declared intention to deduce from these results the corresponding theorems for solitary waves on a flow of mean depth  $h$ , we state theorems about the periodic problem in terms of the integral equation (1.31) rather than the equivalent equation (1.32). In section 2.3, we shall see how the use of (1.32) leads to new results about the bifurcation of periodic waves, which are obscured by the formulation of the problem as (1.31).

## 2.2. The bifurcation of periodic waves of wavelength $\lambda$ on a flow of mean depth $h$ .

Throughout this section, we consider waves of wavelength  $\lambda$  on a flow of fixed mean depth  $h$ . Accordingly, we are interested in solutions  $(\mu, \theta)$  of (1.31) with  $\mu > 0$  and  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ . Since all solutions of (1.31) are odd, it suffices instead to consider the eigenvalue problem

$$\theta(s) = \frac{2}{3} \int_0^\pi G(s,t) \frac{f_\lambda(t) \sin \theta(t)}{\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw} dt \quad (2.1)$$

where the kernel  $G$  is defined in (1.27). Let  $C_0[a, b]$  denote the Banach space of continuous functions on  $[a, b]$  which vanish at  $a$  and  $b$ , and let  $K_0[a, b]$  denote the closed, reproducing cone of non-negative functions in  $C_0[a, b]$ . For any  $[a, b] \subset [0, \pi]$ ,  $C[a, b]$  denotes the usual Banach space of continuous functions on  $[a, b]$  with the supremum norm. For convenience with notation, we will abbreviate  $K_0[0, \pi]$  as  $K_0$ . Since  $G$  is non-negative almost everywhere on  $[0, \pi] \times [0, \pi]$  and is the kernel of a compact, linear Hammerstein operator on  $C_0[0, \pi]$  [1; Theorem 2.5(a), (b)], it follows that this linear operator leaves  $K_0$  invariant. The linearization of (2.1) about  $\theta = 0$  is given by

$$\theta(s) = \frac{2\mu}{3} \int_0^\pi G(s,t) f_\lambda(t) \theta(t) dt, \quad (2.2)$$

and from Theorem 1.3 it follows that the characteristic value with smallest absolute value is  $6/\lambda \pi^{-1} \coth(2^{-n}h/\lambda) \rightarrow \frac{6}{\pi}$  as  $\lambda \rightarrow \infty$ , and the corresponding eigenvector is  $\sin(2\pi q_j(s)/\lambda)$ . Before the global bifurcation result may be stated, one further observation is necessary.

LEMMA 2.1. Let  $\mu > 0$ , and let  $\varphi \in K_0$  be such that, for all  $s \in [0, \pi]$ ,

$$\theta(s) = \frac{2}{3} \int_0^\pi G(s,t) \frac{f_\lambda(t) \sin(J\theta(t))}{\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin(J\theta(w)) dw} dt, \quad (2.3)$$

where

$$Jx = (\operatorname{sgn} x) \min\{|x|, \pi\}, \quad \text{for all } x \in \mathbb{R}.$$

Then (i)  $0 < \theta(s) < \pi$  on  $(0, \pi)$ , and

$$(ii) \mu > \mu_\lambda = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda).$$

Proof. The proof of this result is an easy consequence of the maximum principle, and is proved by the method used to establish [1; Theorem 3.3(a), (c)]. No modifications are required. q.e.d.

The next result is a summary of the global existence theory for solutions of equation (1.31). Throughout the discussion, the mean depth is fixed. Let  $S_\lambda = \{(\mu, \theta) \in (0, \infty) \times K_0 : (\mu, \theta) \text{ satisfies (2.1) and } \theta \neq 0\} - \{(\mu_\lambda, 0)\}$  where  $\mu_\lambda = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$ . Section 2.3 gives more sophisticated properties of  $C_\lambda$ ; in particular, the upper bound of  $\pi/2$  in (ii) and (vi) may be replaced by  $\pi/3$ .

**THEOREM 2.2.** ([9], [12], [15], [19], [29]) Let  $C_\lambda$  denote the maximal connected subset of  $S_\lambda$  in  $\mathbb{R} \times C_0[0, \pi]$  which contains  $(\mu_\lambda, 0)$ . Then

- (i)  $C_\lambda$  is closed and unbounded;
- (ii) if  $(\mu, \theta) \in C_\lambda \setminus \{(\mu_\lambda, 0)\}$ , then  $\mu > \mu_\lambda$  and  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ , whence  $\{\mu : (\mu, \theta) \in C_\lambda\} = [\mu_\lambda, \infty)$ .
- (iii)  $\theta$  is a real-analytic function on  $[0, \pi]$ .
- (iv) For each  $\lambda, \delta > 0$ , there exists a constant  $B_{\lambda, \delta} > 0$  such that

$$\theta(s) \geq B_{\lambda, \delta} \sin s \tag{2.4}$$

if  $\mu > \mu_\lambda + \delta$  and  $(\mu, \theta) \in C_\lambda$ .

(v) If  $(\mu, \theta) \in C_\lambda$ , then the mean velocity of the wave,  $c(\mu, \theta)$ , is given by the formula (1.34). For each  $\lambda > 0$ , there exists a closed interval  $[a_\lambda, b_\lambda] \subset (0, \infty)$  such that

$$\{c(\mu, \theta) : (\mu, \theta) \in C_\lambda\} \subset [a_\lambda, b_\lambda] ,$$

and

$$\bigcup_{\lambda > 0} [a_\lambda, b_\lambda] = (0, M]$$

for some  $M > 0$ .

Let the speed of the corresponding flow at the wave crest, calculated from (1.36), be denoted by  $q_c(\mu, \theta)$ .

(vi) If  $\{(\mu_n, \theta_n)\} \subset C_\lambda$  and  $\mu_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then  $q_c(\mu_n, \theta_n) \rightarrow 0$ , and there exists a subsequence  $\{\theta_{n(k)}\}$  of  $\{\theta_n\}$  such that  $\theta_{n(k)} \rightarrow \theta$  uniformly on  $[\delta, \pi]$  for each  $\delta > 0$ , where  $\theta$  is a non-trivial solution of the equation

$$\theta(s) = \frac{2}{3} \int_0^\pi G(s, t) \frac{f_\lambda(t) \sin \theta(t)}{\int_0^\pi f_\lambda(w) \sin \theta(w) dw} dt \quad \text{for } s \in (0, \pi] \quad (2.5)$$

The function  $\theta$  is real-analytic on  $(0, \pi]$  and  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ .

Furthermore,  $\liminf_{s \rightarrow 0+} \theta(s) = a > 0$  and the following dichotomy holds: either

$$\lim_{s \rightarrow 0+} \theta(s) = \pi/6, \quad \text{or}$$

$$\liminf_{s \rightarrow 0+} \theta(s) < \pi/6 < \limsup_{s \rightarrow 0+} \theta(s) \quad .$$

The periodic wave corresponding to a solution of (2.5) has a stagnation point at its crest (i.e.  $q_c = 0$ ).

(vii) Let  $\{(\mu_n, \theta_n)\} \subset C_\lambda$  denote the subsequence in (vi). Since  $(\mu_n, \theta_n)$  satisfies equation (2.1), it follows that the function  $\theta_n^*$  defined on  $[0, \mu_n \pi]$  by

$$\theta_n^*(x) = \theta_n(x/\mu_n)$$

satisfies the equation

$$\theta_n^*(x) = \frac{2}{3} \int_0^{\mu_n \pi} G(x/\mu_n, y/\mu_n) \frac{f_\lambda(y/\mu_n) \sin \theta_n^*(y)}{1 + \int_0^x f_\lambda(w/\mu_n) \sin \theta_n^*(w) dw} dy$$

for all  $x \in [0, \mu_n \pi]$ .

Moreover, as  $n \rightarrow \infty$ ,  $\{\theta_n^*\}$  converges uniformly on compact subsets of  $(0, \infty)$  to a function  $\theta^*$  which satisfies the boundary-layer equation

$$\theta^*(x) = \frac{2}{3} \int_0^\infty \frac{1}{2\pi} \ln \left| \frac{x+y}{x-y} \right| \frac{\frac{1}{2} \sin \theta^*(y)}{1 + \frac{1}{2} \int_0^y \sin \theta^*(w) dw} dy,$$

and  $\sup_{x \in (0, \infty)} \theta^*(x) > \pi/6$ . It follows that there exists an  $\epsilon > 0$  such that,

for all  $n$  sufficiently large,  $|\theta_n|_{C_0[0, \pi]} \geq \pi/6 + \epsilon$ . Hence, for each  $\beta \in [0, \pi/6 + \epsilon]$ , there exists a periodic water wave of any specified mean depth and wavelength, the free surface of which subtends a maximum angle to the horizontal of  $\beta$ .

(viii) For each  $N > 0$ , the set  $\{(\mu, \theta) \in C_\lambda : \mu \leq N\}$  is relatively compact in the topology of  $\mathbb{R} \times C^\ell$  for each integer  $\ell \geq 0$ , where  $C^\ell$  is the Banach space of  $\ell$ -th order continuously differentiable functions on  $[0, \pi]$ .

Proof. (i) The proof of this is a simple application of [7; Theorem 2] to equation (2.3), once the a priori bound of Lemma 2.1 has been noted (see [9; Lemma 4.1] for a similar treatment of equation (1.32)).

(ii) That  $\mu > \mu_\lambda$  follows after multiplying equation (2.1) by  $f_\lambda$  and by the eigenfunction of the linear equation (2.2), which corresponds to the characteristic value  $\mu_\lambda$ , and integrating over  $(0, \pi)$ .

A slight modification of [1; Theorem 3.3(d)] yields that  $\theta(s) < \pi/2$  on  $(0, \pi)$ . In this case the crucial observation is that the function  $P$  defined on  $\mathcal{D}'$  by putting

$$P(\zeta) = -\frac{1}{2} \exp(-2\tilde{\rho}(\zeta)) - Y(\zeta)$$

is a super-harmonic function on  $\mathcal{D}'$  which attains its minimum at every point of the boundary portion  $\{e^{it} : t \in (-\pi, \pi)\}$ . (The use of the super-harmonic pressure function  $P$  to show that the free surface of periodic water-waves have no vertical tangents was introduced by Spielvogel [25], and used again in [9].) Here  $\tilde{\rho}$  and  $Y$  are defined as follows. If  $(\mu, \theta) \in C_\lambda$ , then suppose that

$$\theta(s) = \sum_{l=1}^{\infty} a_l \sin ls \quad ,$$

and put

$$\rho(t) = -\frac{1}{3} \ln\left(\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w) dw\right)$$

for  $t \in [-\pi, \pi]$ . If  $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t) dt$ , then it follows that

$$F(re^{it}) = a_0 + \sum_{l=1}^{\infty} a_l r^l e^{ilt}$$

for  $r \in [0, 1)$ ,  $t \in (-\pi, \pi]$  defines an analytic function on  $\mathcal{D}$ . Then put

$$\tilde{\rho}(\zeta) = \text{Real } F(\zeta) \quad ,$$

and

$$Y(\zeta) = \text{Imag} \frac{1}{3\lambda} \int_0^\zeta \exp(F(\hat{\zeta})) \tilde{q}_\lambda'(\hat{\zeta}) d\hat{\zeta}$$

for  $\zeta \in \mathcal{D}'$  where  $\tilde{q}_\lambda$  is the inverse of the conformal mapping  $\tilde{p}_\lambda$

introduced in section 1.2 (and prime denotes differentiation). With this definition of  $P$ , the proof that  $\theta < \pi/2$  follows exactly as in [1; Theorem 3.3(d)].

(iii) Levy's theorem [15] ensures that  $\theta$  is real-analytic on  $[0, \pi]$ .

(iv) If this result is false, then for some  $\delta > 0$  and for each  $n$ , there exists  $(\mu_n, \theta_n) \in C_\lambda \cap \{(\mu_\lambda + \delta, \infty) \times K_0\}$  and  $s_n \in (0, \pi)$  such that  $\theta_n(s_n) \leq n^{-1} \sin s_n$ . Now for each closed interval  $[a, b] \subset (0, \pi)$ , there exists  $E > 0$  (depending on  $[a, b]$ ) such that if  $t \in [a, b]$ , then

$$G(s, t) \geq E \sin s$$

for all  $s \in [0, \pi]$  (see [1; Theorem 2.5(c)]). Hence

$$\begin{aligned} n^{-1} \sin s_n \geq \theta_n(s_n) &= \frac{2}{3} \int_0^\pi G(s_n, t) \frac{f_\lambda(t) \sin \theta_n(t)}{t \left( \frac{1}{\mu_n} + \int_0^\pi f_\lambda(w) \sin \theta_n(w) dw \right)} dt \\ &\geq \left\{ \frac{2E}{3} \int_a^b \frac{f_\lambda(t) \sin \theta_n(t)}{t \left( \frac{1}{\mu_n} + \int_0^\pi f_\lambda(w) \sin \theta_n(w) dw \right)} dt \right\} \sin s_n. \end{aligned}$$

Since  $[a, b]$  is chosen arbitrarily in  $(0, \pi)$ , there results that

$$\frac{f_\lambda(t) \sin \theta_n(t)}{t \left( \frac{1}{\mu_n} + \int_0^\pi f_\lambda(w) \sin \theta_n(w) dw \right)} \rightarrow 0$$

almost everywhere in  $[0, \pi]$ . From the a priori bound established in (ii), it follows that  $\theta_n \rightarrow 0$  in  $L_1(0, \pi)$ . However, an integration of (2.1) over  $(0, \pi)$  after multiplication by  $\sin s$  yields that

$$\int_0^{\pi} \theta_n(s) \sin s \, ds = \frac{1}{3} \int_0^{\pi} \frac{f_{\lambda}(t) \sin \theta_n(t) \sin t}{\frac{1}{\mu_n} + \int_0^t f_{\lambda}(w) \sin \theta_n(w) \, dw} \, dt$$

$$\geq \frac{1}{3\pi} \frac{\int_0^{\pi} \theta_n(t) \sin t \, dt}{\frac{1}{\mu_n} + \int_0^{\pi} f_{\lambda}(w) \sin \theta_n(w) \, dw}$$

whence  $\{\mu_n\}$  is bounded, since  $f_{\lambda} \sin \theta_n \rightarrow 0$  in  $L_1(0, \pi)$ . Because  $G$  is the kernel of a compact linear operator on  $C_0[0, \pi]$ , and because  $\{\mu_n\}$  is bounded, it follows that  $\theta_n$  converges to 0 in  $C_0[0, \pi]$ . But  $C_{\lambda}$  is closed, from which there follows the contradiction that the sequence  $\{\mu_n\} \subset [\mu_{\lambda} + \delta, \infty)$  converges to  $\mu_{\lambda}$ .

(v) If  $(\mu, \theta) \in C_{\lambda}$ ,  $\lambda > 0$ , then by (1.23) and (1.34) there results that

$$c(\mu, \theta) \leq \text{const. } K_{\lambda} \lambda^{1/2} \left\{ \int_0^{\pi} \frac{f_{\lambda}(t) \cos \theta(t)}{\left(\frac{1}{\mu} + \int_0^t f_{\lambda}(w) \sin \theta(w) \, dw\right)^{1/3}} \, dt \right\}^{-3/2}$$

Hence, for any  $N > 0$ , the set  $\{c(\mu, \theta) : (\mu, \theta) \in C_{\lambda}, \lambda \in (0, N]\}$  is bounded above, or else there exists a sequence  $(\mu_n, \theta_n) \in C_{\lambda_n}$ ,  $\lambda_n \in (0, N]$ , such that

$$\int_0^{\pi} \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) \, dw\right)^{1/3}} \, dt \rightarrow 0$$

as  $n \rightarrow \infty$ .

In the latter case it follows, by the bounds in (ii) above, that  $\theta_n \rightarrow \pi/2$  in  $L_1(0, \pi)$ , and  $\sin \theta_n \rightarrow 1$  in  $L_1(0, \pi)$ . Without loss of generality, suppose that  $1/\mu_n \rightarrow \alpha \in [0, \pi/6]$  and  $\lambda_n \rightarrow \lambda \in [0, \infty]$  as  $n \rightarrow \infty$ . Hence, for any interval  $[a, b] \subset (0, \pi)$ , it follows by [1; Theorem 2.5(c)] that

$$\begin{aligned} \theta_n(s) &\geq \frac{2}{3} \int_a^b G(s,t) \frac{f_{\lambda_n}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_{\lambda_n} \sin \theta_n} dt \\ &\geq \text{const.} \left\{ \int_a^b \frac{f_{\lambda_n}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_{\lambda_n} \sin \theta_n} dt \right\} \sin s \geq \text{const.} \sin s, \end{aligned}$$

where the constants are independent of sufficiently large  $n$ . Hence, by the Dominated Convergence Theorem

$$\ln(1/\mu_n + \int_0^t f_{\lambda_n} \sin \theta_n) \rightarrow \ln(\alpha + \int_0^t f_{\lambda})$$

in  $L_2(0, \pi)$ , as  $n \rightarrow \infty$ , and so for any  $l \geq 1$ ,

$$\begin{aligned} \int_0^{\pi} \sin ls \theta_n(s) ds &= \frac{1}{3l} \int_0^{\pi} \frac{\sin lt f_{\lambda_n}(t) \sin \theta_n(t)}{1/\mu_n + \int_0^t f_{\lambda_n} \sin \theta_n} dt \\ &= -\frac{1}{3} \int_0^{\pi} \cos lt \ln(1/\mu_n + \int_0^t f_{\lambda_n} \sin \theta_n) dt \rightarrow \\ &\rightarrow -\frac{1}{3} \int_0^{\pi} \cos lt \ln(\alpha + \int_0^t f_{\lambda}) dt \end{aligned} \tag{2.6}$$

as  $n \rightarrow \infty$ .

Therefore, for each integer  $\lambda \geq 1$ , equation (2.6) gives

$$\frac{\pi}{2} \int_0^{\pi} \sin \theta(s) ds = -\frac{1}{3} \int_0^{\pi} \cos \theta(t) \ln\left(\alpha + \int_0^t f_{\lambda}\right) dt .$$

It is immediate by Fulini's theorem that

$$\frac{\pi}{2} = \frac{2}{3} \int_0^{\pi} G(s,t) \frac{f_{\lambda}(t)}{\alpha + \int_0^t f_{\lambda}} dt . \quad (2.7)$$

But this is a contradiction since, if  $\alpha$  is non-zero, the right-hand side is in  $C_0[0, \pi]$  whereas the left-hand side is not; whereas when  $\alpha = 0$ , the right-hand side is continuous on  $(0, \pi]$  and vanishes at  $\pi$  (by [1; Theorem 2.5(f)]) whereas the left-hand side does not.

Hence the set  $\{c(\mu, \theta) : (\mu, \theta) \in C_{\lambda}, \lambda \in (0, N]\}$  is bounded above. In order to show that an upper bound may be found which is independent of  $N$ , we proceed as before by seeking a contradiction. If the result is false, then  $c(\mu_n, \theta_n) \rightarrow \infty$  for some sequence  $\{(\mu_n, \theta_n)\}$ , where  $(\mu_n, \theta_n) \in C_{\lambda_n}$  and  $\lambda_n \rightarrow \infty$ . However, a slight modification of the proof of Theorem 3.1 (iv) yields that there must therefore exist a subsequence  $\{(\mu_{n(k)}, \theta_{n(k)})\}$  such that  $(1/\mu_{n(k)}, \theta_{n(k)}) \rightarrow (\alpha, \theta) \in [0, \infty) \times L_2(0, \pi)$ , and  $c(\mu_{n(k)}, \theta_{n(k)}) \rightarrow \left\{ \frac{6gh}{\pi} \left( \alpha + \int_0^{\pi} f(t) \sin \theta(t) dt \right) \right\}^{1/2} \in [\sqrt{gh}, 2\sqrt{gh}]$ . This is a contradiction.

Finally, to show that, for fixed  $\lambda$ , the set  $c(\mu, \theta) : (\mu, \theta) \in C_{\lambda}$  is bounded below by a positive constant, it suffices to observe that

$$c(\mu, \theta) \geq \text{const.} \left\{ \int_0^{\pi} \left( \frac{1}{\mu} + \int_0^t f_{\lambda}(w) \sin \theta(w) \right)^{-1/3} dt \right\}^{-3/2}$$

$$\geq \text{const.} \quad (\text{by (iv)})$$

where both constants are independent of  $(\mu, \theta) \in C_\lambda$ . To complete the proof, we observe that  $c(\mu_\lambda, 0) = \left\{ \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right) \right\}^{1/2} \rightarrow 0$  as  $\lambda \rightarrow 0$ .

(vi) Since  $c(\mu_n, \theta_n) \leq M$ , it follows from (1.36) that  $q_c(\mu_n, \theta_n) \rightarrow 0$  as  $n \rightarrow \infty$ . The asymptotic behaviour of  $\{\theta_n\}$  as  $n \rightarrow \infty$  is established by a slight modification of the arguments in [1; section 5], using (iv) to obtain the appropriate estimates. The behaviour of the limiting function  $\theta$  may be analyzed by precisely the method used to establish [1; Theorem 5.2(d) - (g)].

(vii) This is the main result of [19] reformulated in terms of the equation (2.1). The proof for equation (2.1) is identical (with certain obvious modifications), and there is no need to repeat it here. Since  $C_\lambda$  is a connected set in  $\mathbb{R} \times C_0[0, \pi]$  which contains  $(\mu_\lambda, 0)$  and a point  $(\mu, \theta)$  with  $\sup_{s \in [0, \pi]} \theta(s) \geq \frac{\pi}{6} + \varepsilon$ , it is immediate that for each  $\beta \in [0, \frac{\pi}{6} + \varepsilon]$  there exists an element  $(\mu, \theta) \in C_\lambda$  with  $\sup_{s \in [0, \pi]} \theta(s) = \beta$ .

(viii) We sketch the proof for  $\ell = 1$ ; for general  $\ell$  the result follows by induction. Let  $(\mu, \theta) \in C_\lambda$  with  $\mu \leq N$ . Then the odd extension of  $\theta$  to  $[-\pi, \pi]$  is the conjugate of the even function  $\rho$  defined in the proof of (ii) (for the  $L_2$ -definition of the conjugate operator, which is sufficient for our purposes here, see Appendix). Standard theory [31; p. 121] then gives that

$$\|\rho\|_{C^\alpha} \leq \text{const.} \|\theta\|_{C^\alpha},$$

where  $C^\alpha$  denotes the Banach space of Hölder continuous functions on  $[-\pi, \pi]$  with exponent  $\alpha \in (0, 1)$ , and the constant depends only on  $\alpha$ . For  $\theta_1, \theta_2 \in [-\pi, \pi]$ ,

$$|\rho(s_1) - \rho(s_2)| = \frac{1}{3} \left| \ln \left[ \frac{1 + \nu \int_0^{s_1} f_\lambda \sin \theta}{1 + \nu \int_0^{s_2} f_\lambda \sin \theta} \right] \right|$$

$$\leq \text{const.} \cdot \frac{1}{3} |s_1 - s_2| \leq \text{const.} \cdot \frac{N}{3} |s_1 - s_2|$$

Thus  $|\rho|_{C^\alpha} \leq \text{const.}$  and hence  $|\theta|_{C^\alpha} \leq \text{const.}$ , where the constant depends only on  $N, \alpha$  and  $\lambda$ .

However, the function  $\frac{d\theta}{ds}$  is the conjugate of  $\frac{d\rho}{ds} = -\frac{1}{3} \frac{f_\lambda(s) \sin \theta(s)}{1/\nu + \int_0^s f_\lambda \sin \theta}$

and so  $|\frac{d\rho}{ds}|_{C^\alpha} \leq \text{const.}$ , whence  $|\frac{d\theta}{ds}|_{C^\alpha} \leq \text{const.}$ , and once again the constant depends only upon  $N, \alpha$  and  $\lambda$ . The result for  $\ell = 1$  then follows by the Ascoli-Arzelà theorem. The proof in the general case follows once one has taken into account that the conjugate of the  $\ell$ -th derivative of  $\rho$  is the  $\ell$ -th derivative of  $\theta$ .

q.e.d.

Remark. The proof of (ii) - (viii) in Theorem 2.2 did not use the connectedness of  $C_\lambda$ , and so all of these results hold with  $C_\lambda$  replaced by  $S_\lambda$ .

### 2.3. Properties of periodic waves.

In section 2.2 the global nature of the solution set of the periodic water-wave problem was studied through its formulation as the integral equation (1.31). This equation bears a striking resemblance to the approximation used in [1; section 3.2] to prove the existence of large-amplitude solitary waves. In section 3 we shall adapt the proofs in [1] to prove that as  $\lambda \rightarrow \infty$  the unbounded,

closed connected sets  $C_\lambda$  converge, in a certain sense, to a global 'branch' of solutions of the solitary wave problem.

In this section, we exploit the integral equation (1.32) to gain further insight into the nature of periodic waves which lie on the bifurcating set  $C_\lambda$ . These results are not accessible directly from (1.31), and are new.

In this section, our interest is restricted to solutions  $(\mu, \theta)$  of equation (1.32). Since  $\theta$  is an odd function on  $[-\lambda/2, \lambda/2]$ , it suffices to consider the equation

$$\theta(x) = A(\mu, \theta)(x) \equiv \frac{1}{3\pi} \int_0^{\lambda/2} \ln \left| \frac{s_\lambda(2K_\lambda(x+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(x-\epsilon)/\lambda)} \right| \frac{\sin \theta(\epsilon)}{\Lambda/\mu + \int_0^\epsilon \sin \theta(w) dw} d\epsilon, \quad (2.8)$$

$x \in [0, \lambda/2]$ . Here  $\Lambda$ ,  $s_\lambda$  and  $K_\lambda$  are defined in section 1.3; the positive parameters  $h$  and  $\lambda$  upon which they depend are chosen arbitrarily but are then fixed.

Since the domain  $R_\lambda = \{(x, \eta) : x \in (-\lambda/2, \lambda/2), \eta \in (-h, 0)\}$  is mapped conformally onto the cut unit disc  $\mathcal{D}' = \{re^{it} : t \in (-\pi, \pi), r \in (0, 1)\}$  by  $\tilde{p}_\lambda$ , the results of Theorem 2.2 have implications for the solution set of (2.8), some of which are set out below. Let  $T_\lambda = \{(\mu, \theta) \in (0, \infty) \times K_0[0, \lambda/2] : \theta \neq 0 \text{ and } (\mu, \theta) \text{ satisfies (2.8)}\} \cup \{(\mu_\lambda, 0)\}$ , where  $\mu_\lambda = 6\Lambda\pi\lambda^{-1} \coth(2\pi h/\lambda)$  is given in Theorem 1.3. Where necessary, we shall identify  $\theta \in K_0[0, \lambda/2]$  with its odd extension to  $[-\lambda/2, \lambda/2]$ .

**THEOREM 2.3.** Let  $E_\lambda$  denote the maximal connected subset of  $T_\lambda$  in  $(0, \infty) \times C_0[0, \lambda/2]$  which contains  $(\mu_\lambda, 0)$ . Then

(i)  $E_\lambda = \{(\mu, \theta) : \theta(x) = -\theta(p_\lambda(x)), \quad x \in [0, \lambda/2], \text{ where } (\mu, \theta) \in C_\lambda\}$ .

(ii)  $E_\lambda$  is closed and unbounded.

(iii) If  $\tilde{\theta}$  denotes the harmonic function on  $R_\lambda$  with  $\tilde{\theta}(x + i0) = \theta(x)$ ,  $x \in [-\lambda/2, \lambda/2]$ , and  $\tilde{\theta} = 0$  elsewhere on  $\partial R_\lambda$ , then

$$\left. \frac{\partial \tilde{\theta}}{\partial n} \right|_{x+i0} = \frac{1}{3} \frac{\sin \theta(x)}{\lambda/\mu + \int_0^x \sin \theta(w) dw}, \quad (2.9)$$

$x \in [-\lambda/2, \lambda/2]$ . Furthermore,  $\tilde{\theta}$  is real-analytic on  $\bar{R}_\lambda$ ; in particular,  $\theta$  is real-analytic on  $[0, \lambda/2]$ . If  $\theta$  is non-trivial, then

(iv)  $\tilde{\theta}_x(0, \eta) > 0$ ,  $\tilde{\theta}_x(\lambda/2, \eta) < 0$ , for all  $\eta \in (-h, 0)$ .

(v)  $\tilde{\theta}_\eta(x, \eta) > 0$  for all  $(x, \eta) \in (0, \lambda/2) \times [-h, 0]$ .

(vi)  $\tilde{\theta}_{\eta x}(0, \eta) > 0$ ,  $\tilde{\theta}_{\eta x}(\lambda/2, \eta) < 0$  for all  $\eta \in (-h, 0)$ .

Proof. Theorem 1.4(i) and the maximality of  $C_\lambda$  and  $E_\lambda$  in  $S_\lambda$  and  $T_\lambda$ , respectively, together prove (i). Parts (ii) and (iii) follow immediately from Theorem 2.2. By Theorem 2.2(iii),  $\tilde{\theta}$  is real-analytic on  $\bar{D}$ , and hence  $\tilde{\theta}$  is real-analytic on  $\bar{R}_\lambda$  since  $\tilde{p}_\lambda$  is analytic there and  $\tilde{\theta}(\zeta) = -\tilde{\theta}(\tilde{p}_\lambda(\zeta))$ . Equation (2.9) is a restatement of (1.33).

To prove (iv) - (vi), we use the maximum principle. By the maximum principle for a harmonic function  $u$  on a rectangle  $R$  we mean the fact that  $\min_{\partial R} u < u(\zeta) < \max_{\partial R} u$ , for all  $\zeta \in R$ ; while the strong maximum principle refers to the fact that at every point of  $\partial R$ , other than corners, where the maximum (minimum) of  $u$  is attained, the outward normal derivative is positive (negative). Let  $R = (0, \lambda/2) \times (-h, 0)$ . Since  $\tilde{\theta}(x, 0) = \theta(x) > 0$  on  $(0, \lambda/2)$  and vanishes elsewhere on  $\partial R$ , the strong maximum principle gives (iv) and the result  $\tilde{\theta}_\eta(x, -h) > 0$ ,  $x \in (0, \lambda/2)$ . Since (2.9) ensures that

$\tilde{\theta}_\eta(x, 0) > 0$  for all  $x \in (0, \lambda/2)$ , and since  $\tilde{\theta}_\eta$  vanishes on the lines  $\{(x, \eta) : x = 0, \lambda/2, \eta \in [-h, 0]\}$ , part (v) follows from the maximum principle. The strong maximum principle for  $\tilde{\theta}_\eta$  then gives (vi). q.e.d.

Theorem 2.3(i) ensures that any properties proved for elements of  $E_\lambda$  may be translated into corresponding results for  $C_\lambda$ . The next three theorems concern solutions of (2.8) with  $0 < \theta(x) \leq \pi$  on  $(0, \lambda/2)$ , and note that the results hold for all elements of  $E_\lambda \setminus \{(\mu_\lambda, 0)\}$ , since such elements satisfy  $0 < \theta(x) < \pi/2$  on  $(0, \lambda/2)$  by Theorem 2.2(ii).

The following theorem ensures that non-trivial elements of  $E_\lambda$  satisfy  $x\theta'(x) < \theta(x)$  on  $(0, \lambda/2)$ , and, equivalently, that  $x^{-1}\theta(x)$  is monotone decreasing on  $(0, \lambda/2)$ . This property implies that  $\theta(x) < \pi/3$ ,  $x \in [0, \lambda/2]$ , for all elements of  $E_\lambda$ , and, equivalently, that  $\theta(s) < \pi/3$ ,  $s \in [0, \pi]$ , for all elements of  $C_\lambda$ .

**THEOREM 2.4.**<sup>†</sup> Assume that  $(\mu, \theta)$  satisfies (2.8) and  $0 < \theta(x) \leq \pi$  on  $(0, \lambda/2)$ . Then

(i)  $x\theta'(x) < \theta(x)$  on  $(0, \lambda/2)$

and

(ii)  $0 < \theta(x) < \pi/3$  on  $(0, \lambda/2)$ .

**Proof.** (i) Assume that (i) is false, and let  $\tilde{\theta}$  be as in Theorem 2.3(iii). Since  $\tilde{\theta}(x, -h) = 0$ , there follows  $\tilde{\theta}_x(x, -h) = 0$ ,  $x \in [0, \lambda/2]$ , and the use of

<sup>†</sup>That this result might hold for periodic waves was suggested to us by J. B. McLeod, who attributed it to Prof. T. B. Benjamin in the case of solitary waves. Note that, in the periodic case, an even finer estimate may be established by the same method, namely:

$$\sin\left(\frac{\pi x}{\lambda}\right)\theta'(x) < \frac{\pi}{\lambda} \cos\left(\frac{\pi x}{\lambda}\right)\theta(x), \quad x \in (0, \lambda/2).$$

As  $\lambda \rightarrow \infty$ , this reduces to Benjamin's result for solitary waves.

this with Theorem 2.3(vi) ensures that

$$\tilde{\theta}_x(0, 0) = \theta'(0) > c \quad \text{and} \quad \tilde{\theta}_x(\lambda/2, 0) = \theta'(\lambda/2) < 0 . \quad (2.10)$$

Hence, there exists  $\chi \in (0, \lambda/2)$  such that  $\chi \tilde{\theta}_x(\chi, 0) \geq \tilde{\theta}(\chi, 0)$ . The use of this with (2.10) ensures that for some constant  $d \geq 1$

$$\chi \tilde{\theta}_x(\chi, 0) \leq d \tilde{\theta}(\chi, 0) \quad \text{for all } \chi \in [0, \lambda/2] , \quad (2.11a)$$

and

$$\hat{\chi} \tilde{\theta}_x(\hat{\chi}, 0) = d \tilde{\theta}(\hat{\chi}, 0) \quad \text{for some } \hat{\chi} \in (0, \lambda/2) . \quad (2.11b)$$

Define a function  $W$  on  $R = (0, \lambda/2) \times (-h, 0)$  by

$$W(\chi, \eta) = \chi \tilde{\theta}_x(\chi, \eta) - d \tilde{\theta}(\chi, \eta) .$$

It follows that  $W$  vanishes on the lines  $\{(0, \eta) : \eta \in (-h, 0)\}$  and  $\{(\chi, -h) : \chi \in (0, \lambda/2)\}$ ; that  $W$  is negative on the line  $\{(\lambda/2, \eta) : \eta \in (-h, 0)\}$  by Theorem 2.3(iv); and that  $W$  is non-positive on the line  $\{(\chi, 0) : \chi \in (0, \lambda/2)\}$  by (2.11a). Hence,  $W \leq 0$  on  $\partial R$ . A calculation yields

$$\Delta W - \frac{2}{x} W_x - \frac{2}{x^2} (d-1)W = \frac{2d}{x^2} (d-1)\tilde{\theta} \geq 0 \quad \text{in } R . \quad (2.12)$$

Standard theory [22; p. 64, 67] applied to (2.12) ensures that  $W < 0$  in  $R$  and that the normal derivative of  $W$  is positive at any point on  $\partial R$  (other than the line  $\chi = 0$  since (2.12) is singular there) where  $W$  equals zero. By (2.11b),  $W(\hat{\chi}, 0) = 0$  for some  $\hat{\chi} \in (0, \lambda/2)$ , and so  $W_\eta(\hat{\chi}, 0)$  must be positive.

A calculation yields

$$0 < W_\eta(\hat{\chi}, 0) = \hat{\chi} \tilde{\theta}_{x\eta}(\hat{\chi}, 0) - d \tilde{\theta}_\eta(\hat{\chi}, 0)$$

$$\begin{aligned}
&= \hat{x} \frac{d}{dx} \left\{ \frac{1}{3} \frac{\sin \theta(x)}{\Lambda/\mu + \int_0^x \sin \theta} \right\} \Big|_{x=\hat{x}} - \frac{d}{3} \left\{ \frac{\sin \theta(x)}{\Lambda/\mu + \int_0^x \sin \theta} \right\} \Big|_{x=\hat{x}} \\
&< \frac{d}{3} \left\{ \frac{\theta(\hat{x}) \cos \theta(\hat{x}) - \sin \theta(\hat{x})}{\Lambda/\mu + \int_0^{\hat{x}} \sin \theta} \right\} \tag{2.13}
\end{aligned}$$

where we have used the relation  $\hat{x} \theta'(\hat{x}) = d\theta(\hat{x})$  from (2.11b). Since  $\theta \leq \pi$ , it follows that the right-hand side of (2.13) is negative, and this is the desired contradiction.

(ii) The arguments for Theorem 2.2(ii) show that  $\theta < \pi/2$ , and the use of this with (i) ensures that  $x^{-1} \sin \theta(x)$  is monotone decreasing on  $(0, \lambda/2)$ .

Hence,

$$\begin{aligned}
\frac{\sin \theta(\epsilon)}{\Lambda/\mu + \int_0^\epsilon \sin \theta} &< \frac{\sin \theta(\epsilon)}{\int_0^\epsilon \sin \theta} = \frac{\sin \theta(\epsilon)}{\int_0^\epsilon \frac{\sin \theta(w)}{w} w dw} \\
&< \frac{\sin \theta(\epsilon)}{\frac{\sin \theta(\epsilon)}{\epsilon} \int_0^\epsilon w dw} = \frac{2}{\epsilon} \text{ for all } \epsilon \in (0, \lambda/2) .
\end{aligned}$$

The use of this estimate in (2.8) yields

$$\theta(x) < \frac{2}{3\pi} \int_0^{\lambda/2} \ln \left| \frac{s_\lambda(2K_\lambda(x+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(x-\epsilon)/\lambda)} \right| \frac{d\epsilon}{\epsilon} \text{ for all } x \in (0, \lambda/2) ,$$

and making the transformation  $\epsilon = q_\lambda(t)$  and  $x = q_\lambda(s)$ ,  $s, t \in (-\pi, 0)$ , gives in the notation of Theorem 1.1,

$$\theta(x) = -\theta(s) < -\frac{4}{3} \int_{-\pi}^0 G(s,t) \frac{q_\lambda'(t)}{q_\lambda(t)} dt , \quad s \in (-\pi, 0) .$$

Since  $\theta$  is an odd function, there results that

$$\theta(s) < \frac{4}{3} \int_0^\pi G(s,t) \frac{q_\lambda'(t)}{q_\lambda(t)} dt, \quad s \in (0, \pi). \quad (2.14)$$

Since  $q_\lambda'(t) = -\lambda f_\lambda(t)$ , we have

$$\frac{q_\lambda'(t)}{q_\lambda(t)} = \frac{f_\lambda(t)}{\int_0^t f_\lambda(w) dw}.$$

It is noted in the proof of Lemma 3.2 that  $f_\lambda(t)/f(t)$  is monotone decreasing on  $(0, \pi)$ , where  $f(t) = \frac{1}{2} \sec \frac{t}{2}$ . It follows that

$$\frac{q_\lambda'(t)}{q_\lambda(t)} = \frac{f_\lambda(t)}{\int_0^t \frac{f_\lambda(w)}{f(w)} f(w) dw} < \frac{f_\lambda(t)}{f(t) \int_0^t f(w) dw} = \frac{f(t)}{\int_0^t f(w) dw},$$

and so

$$\theta(s) < \frac{4}{3} \int_0^\pi G(s,t) \frac{\frac{1}{2} \sec \frac{t}{2}}{\ln(\sec \frac{t}{2} + \tan \frac{t}{2})} dt, \quad s \in (0, \pi).$$

A simple calculation gives

$$\frac{\frac{1}{2} \sec \frac{t}{2}}{\ln(\sec \frac{t}{2} + \tan \frac{t}{2})} < \frac{1}{2} (\tan \frac{t}{2} + \cot \frac{t}{2}), \quad t \in (0, \pi),$$

and so

$$\theta(s) < \frac{2}{3} \int_0^\pi G(s,t) (\tan \frac{t}{2} + \cot \frac{t}{2}) dt = \frac{\pi - t}{3} + \frac{t}{3} = \frac{\pi}{3}.$$

The evaluation of the integral in the expression above is given by [1; Theorem 2.5(d), (e)]. q.e.d.

Remark. A more precise estimate using the right-hand side of (2.14) is not possible since one can show that this quantity approaches  $\pi/3$  as  $s \rightarrow 0$ .

Obviously, part (ii) of Theorem 2.3 follows from the abstract global bifurcation theory for positive operators [7] using the reproducing cone  $K_0[0, \lambda/2]$  (see [9]). The next results of this section (Theorems 2.5 and 2.6) are a consequence of the observation that a smaller cone  $\hat{K}$  is more appropriate in the study of equation (2.8). Here  $\hat{K} = \{u \in K_0[0, \lambda/2]: \text{for all } \hat{\chi} \in [\lambda/4, \lambda/2] \text{ and for all } \chi^* \in [\lambda/2 - \hat{\chi}, \hat{\chi}], u(\hat{\chi}) \leq u(\chi^*)\}$ . Note that if  $u \in \hat{K}$ , then  $u$  is non-increasing on  $[\lambda/4, \lambda/2]$ , and hence  $\hat{K}$  is not reproducing in  $C_0[0, \lambda/2]$ . Our ultimate aim is to show that  $E_\lambda \subset (0, \infty) \times \hat{K}$ , and hence that  $\theta'(x) < 0$  on  $[\lambda/4, \lambda/2]$  for all non-trivial  $(u, \theta) \in E_\lambda$ .

**THEOREM 2.5.** If  $(u, \theta)$  is as in Theorem 2.4, then  $\theta \in \hat{K}$ .

**Proof.** Let  $\tilde{\theta}$  be as in Theorem 2.3(iii). Let  $\hat{\chi} \in (\lambda/4, \lambda/2]$  and  $\chi^* \in (\lambda/2 - \hat{\chi}, \hat{\chi})$ , and define  $\alpha \in (\lambda/4, \lambda/2)$  by  $\alpha = (\hat{\chi} + \chi^*)/2$ . To prove the theorem, we claim that it suffices to show that for all  $\alpha \in (\lambda/4, \lambda/2)$

$$\tilde{\theta}(x, 0) \geq \tilde{\theta}(2\alpha - x, 0) \quad \text{for all } x \in [2\alpha - \lambda/2, \alpha] ; \quad (2.15)$$

indeed, since  $\chi^* \in [2\alpha - \lambda/2, \alpha]$ , it follows from (2.15) that  $\theta(\chi^*) = \tilde{\theta}(\chi^*, 0) \geq \tilde{\theta}(2\alpha - \chi^*, 0) = \tilde{\theta}(\hat{\chi}, 0) = \theta(\hat{\chi})$ .

Assume that (2.15) is false for some  $\alpha \in (\lambda/4, \lambda/2)$ . For each number  $d \geq 1$ , define the continuous function  $g$  by

$$g(d) = \min_{[2\alpha - \lambda/2, \alpha]} \{d\tilde{\theta}(x, 0) - \tilde{\theta}(2\alpha - x, 0)\} .$$

The function  $\tilde{\theta}(x, 0)$  is strictly positive on  $[2\alpha - \lambda/2, \alpha]$  since this closed interval is contained in  $(0, \lambda/2)$ . Hence,  $g(d)$  is positive for all sufficiently large  $d$ , and since  $g(1) < 0$ , there exists  $D > 1$  such that  $g(D) = 0$ . It follows that

$$D\tilde{\theta}(x, 0) \geq \tilde{\theta}(2\alpha - x, 0) \quad \text{for all } x \in [2\alpha - \lambda/2, \alpha] , \quad (2.16a)$$

and

$$D\tilde{\theta}(\tilde{\chi}, 0) = \tilde{\theta}(2\alpha - \tilde{\chi}, 0) \quad \text{for some } \tilde{\chi} \in (2\alpha - \lambda/2, \alpha) \quad (2.16b)$$

Let  $R^\alpha$  denote the region  $(2\alpha - \lambda/2, \alpha) \times (-h, 0)$ , and define a harmonic function  $V^\alpha$  on  $R^\alpha$  by

$$V^\alpha(X, \eta) = D\tilde{\theta}(X, \eta) - \tilde{\theta}(2\alpha - X, \eta)$$

for  $(X, \eta) \in R^\alpha$ . It follows with the use of (2.16a) that  $V^\alpha \geq 0$  on  $\partial R^\alpha$ , and the maximum principle then ensures that  $V^\alpha > 0$  in  $R^\alpha$ . Since  $V^\alpha(\tilde{\chi}, 0) = 0$  by (2.16b), the strong maximum principle gives

$$\begin{aligned} 0 > V^\alpha_{\eta}(\tilde{\chi}, 0) &= D\tilde{\theta}_{\eta}(\tilde{\chi}, 0) - \tilde{\theta}_{\eta}(2\alpha - \tilde{\chi}, 0) \\ &= \frac{D \sin \theta(\tilde{\chi})}{3(\Lambda/\mu + \int_0^{\tilde{\chi}} \sin \theta)} - \frac{\sin \theta(2\alpha - \tilde{\chi})}{3(\Lambda/\mu + \int_0^{2\alpha - \tilde{\chi}} \sin \theta)} \\ &> \frac{D \sin \theta(\tilde{\chi}) - \sin \theta(2\alpha - \tilde{\chi})}{3(\Lambda/\mu + \int_0^{2\alpha - \tilde{\chi}} \sin \theta)} \quad (2.17) \end{aligned}$$

Equation (2.16b) yields

$$D \sin \theta(\tilde{\chi}) = D \sin(\theta(2\alpha - \tilde{\chi})/D) > \sin \theta(2\alpha - \tilde{\chi})$$

since  $D > 1$  and  $\theta \leq \pi$  (indeed,  $\theta < \pi/3$  by Theorem 2.4(i)). The use of this inequality in (2.17) yields a contradiction, and so we conclude that

(2.15) holds.

q.e.d.

The following theorem gives various properties of  $\theta$  implied by membership in  $\hat{K}$ .

**THEOREM 2.6.** Let  $(\mu, \theta)$  be as in Theorem 2.4, and let  $\hat{\theta}$  be as in Theorem 2.3(iii). If, in addition,  $\hat{\chi} \in (\lambda/4, \lambda/2)$  and  $\chi^* \in [\lambda/2 - \hat{\chi}, \hat{\chi})$ , then

(i)  $\tilde{\theta}(\hat{\chi}, \eta) < \tilde{\theta}(\chi^*, \eta)$  for all  $\eta \in (-h, 0]$  ,  
and, in particular,  $\theta(\hat{\chi}) < \theta(\chi^*)$ .

(ii)  $\tilde{\theta}_\eta(\hat{\chi}, \eta) < \tilde{\theta}_\eta(\chi^*, \eta)$ , for all  $\eta \in [-h, 0]$  .

Moreover,

(iii)  $\tilde{\theta}_\chi(\chi, \eta) < 0$  and  $\tilde{\theta}_{\chi\eta}(\chi, \eta) < 0$  for all  $(\chi, \eta) \in [\lambda/4, \lambda/2] \times (-h, 0]$  ;  
in particular,  $\theta'(\chi) < 0$  for  $\chi \in [\lambda/4, \lambda/2]$  .

(iv)  $\tilde{\theta}_\chi(0, \eta) + \tilde{\theta}_\chi(\lambda/2, \eta) > 0$ , and  $\tilde{\theta}_{\chi\eta}(0, \eta) + \tilde{\theta}_{\chi\eta}(\lambda/2, \eta) > 0$  for  
all  $\eta \in (-h, 0]$ ; and, in particular,  $\theta'(0) + \theta'(\lambda/2) > 0$  .

Proof. (i) Since  $\theta \neq 0$ , we know from Theorem 2.4(i) that  $0 < \theta(\chi) < \pi/3$   
on  $(0, \lambda/2)$ . Combining this with the fact that  $\theta \in \hat{K}$ , yields that for  
 $\chi_1 \in (\lambda/4, \lambda/2]$  and  $\chi_2 \in [\lambda/2 - \chi_1, \chi_1)$ ,

$$\frac{\sin \theta(\chi_1)}{N\mu + \int_0^{\chi_1} \sin \theta} < \frac{\sin \theta(\chi_2)}{N\mu + \int_0^{\chi_2} \sin \theta} \quad (2.18)$$

Now suppose  $\hat{\chi} \in (\lambda/4, \lambda/2)$  and  $\chi^* \in [\lambda/2 - \hat{\chi}, \hat{\chi})$ , and put  $\alpha = (\chi^* + \hat{\chi})/2$ .  
Define a harmonic function  $W^\alpha$  on  $R^\alpha$  by putting

$$W^\alpha(\chi, \eta) = \tilde{\theta}(\chi, \eta) - \tilde{\theta}(2\alpha - \chi, \eta)$$

for all  $(\chi, \eta) \in R^\alpha = (2\alpha - \lambda/2, \alpha) \times (-h, 0)$ . Then

$$W^\alpha(2\alpha - \lambda/2, \eta) > 0, \quad \eta \in (-h, 0] ;$$

$$W^\alpha_\eta(\chi, 0) > 0, \quad \chi \in (2\alpha - \lambda/2, \alpha) ,$$

by (2.9) and (2.18); and  $W^\alpha = 0$  elsewhere on  $\partial R^\alpha$ . By the maximum principle

$W^\alpha > 0$  on  $R^\alpha$ , and by the strong maximum principle  $W^\alpha(x, 0) > 0$  for all  $x \in (2\alpha - \lambda/2, \alpha)$ . In particular, for  $x = x^* \in (2\alpha - \lambda/2, \alpha)$ , there results that

$$\tilde{\Theta}(x^*, \eta) - \tilde{\Theta}(\hat{x}, \eta) > 0$$

for all  $\eta \in (-h, 0]$ , and (i) has been established.

(ii) We first prove (ii) for  $\eta = -h$  and  $\eta = 0$ . Since  $W^\alpha > 0$  on  $R^\alpha$  and is zero on the line  $\{(x, -h) : x \in [2\alpha - \lambda/2, \alpha]\}$ , the strong maximum principle for  $W^\alpha$  gives  $W_\eta^\alpha(x, -h) > 0$ ,  $x \in (2\alpha - \lambda/2, \alpha)$ . If we set  $x = x^* \in (2\alpha - \lambda/2, \alpha)$ , then the case  $\eta = -h$  is proved. It was shown in the proof of (i) that  $W_\eta^\alpha(x, 0) > 0$ ,  $x \in (2\alpha - \lambda/2, \alpha)$ , and so the result for  $\eta = 0$  follows upon setting  $x = x^*$ .

We now show that  $W_\eta^\alpha(x, \eta) > 0$  on  $R^\alpha$ , so that the result for  $\eta \in (-h, 0)$  in (ii) will follow upon setting  $x = x^*$ . Because of the maximum principle for  $W_\eta^\alpha$ , it suffices to show that  $W_\eta^\alpha \geq 0$  on  $\partial R^\alpha$ ; note that this has already been done for the horizontal portions of the boundary. For  $\eta \in (-h, 0)$ , we have

$$W_\eta^\alpha(\alpha, \eta) = \tilde{\Theta}_\eta(\alpha, \eta) - \tilde{\Theta}_\eta(\alpha, \eta) = 0,$$

and

$$W_\eta^\alpha(2\alpha - \lambda/2, \eta) = \tilde{\Theta}_\eta(2\alpha - \lambda/2, \eta) - \tilde{\Theta}_\eta(\lambda/2, \eta) = \tilde{\Theta}_\eta(2\alpha - \lambda/2, \eta) \geq 0$$

by Theorem 2.3(v).

(iii) If  $\alpha \in [\lambda/4, \lambda/2)$ , and  $W^\alpha$  is the harmonic function defined on the region  $R^\alpha$  as above, then it follows by the strong maximum principle that

$$W_x^\alpha(\alpha, \eta) < 0 \tag{2.19}$$

for all  $\eta \in (-h, 0)$ , whence, putting  $\alpha = \chi$ ,

$$\tilde{\Theta}_\chi(\lambda, \eta) + \tilde{\Theta}_\chi(\chi, \eta) < 0, \quad (\lambda, \eta) \in [\lambda/4, \lambda/2] \times (-h, 0). \quad (2.20)$$

(Although (2.19) only proves (2.20) for  $\chi < \lambda/2$ , the result for  $\chi = \lambda/2$  is due to Theorem 2.3(iv).) Differentiating (2.9) with respect to  $\chi$  yields that

$$\tilde{\Theta}_{\chi\eta} \Big|_{(\chi, 0)} = \frac{\Theta_\chi \cos \Theta}{M/\mu + \int_0^\chi \sin \Theta} - \frac{\sin^2 \Theta}{(M/\mu + \int_0^\chi \sin \Theta)^2}, \quad (2.21)$$

which, combined with (i) above, yields

$$\tilde{\Theta}_{\chi\eta} \Big|_{(\chi, 0)} < 0, \quad \chi \in [\lambda/4, \lambda/2]. \quad (2.22)$$

Hence,  $\tilde{\Theta}_\chi$  does not attain its maximum on the line segment  $\{(\chi, 0) : \chi \in (\lambda/4, \lambda/2)\}$ , by the strong maximum principle. Combining (2.20), (2.22) and Theorem 2.3(vi) yields that  $\tilde{\Theta}_\chi(\lambda/4, 0), \tilde{\Theta}_\chi(\lambda/2, 0) < 0$ , and the first part of (iii) has been established.

We now prove the second part of (iii). Since  $\tilde{\Theta}_\chi(\lambda/2, 0) < 0$ , equation (2.21) ensures that  $\tilde{\Theta}_{\chi\eta}(\lambda/2, 0) < 0$ , and the use of this with (2.22) proves the case  $\eta = 0$ . It was shown in the proof of (ii) that  $W_\eta^\alpha > 0$  on  $R^\alpha$  and that  $W_\eta^\alpha(\alpha, \eta) = 0$ ,  $\eta \in (-h, 0)$ , for all  $\alpha \in [\lambda/4, \lambda/2]$ . By the strong maximum principle,

$$0 > W_{\eta\chi}^\alpha(\alpha, \eta) = \tilde{\Theta}_{\eta\chi}(\alpha, \eta) + \tilde{\Theta}_{\eta\chi}(\alpha, \eta),$$

whence  $\tilde{\Theta}_{\chi\eta}(\chi, \eta) < 0$  for all  $(\chi, \eta) \in [\lambda/4, \lambda/2] \times (-h, 0)$ . This, along with Theorem 2.3(vi) establishes (iii).

(iv) The function  $W^{\lambda/4}$  is a positive, harmonic function on  $R^{\lambda/4}$ , and is zero on the line  $\{(0, \eta) : \eta \in (-h, 0)\}$ . Hence, by the strong maximum

principle,

$$W_x(0, \eta) > 0, \quad \eta \in (-h, 0).$$

Therefore

$$\tilde{\Theta}_x(0, \eta) + \tilde{\Theta}_x(\lambda/2, \eta) > 0, \quad \eta \in (-h, 0),$$

whence, by (iii),

$$\tilde{\Theta}_x(0, 0) \geq -\tilde{\Theta}_x(\lambda/2, 0) > 0. \quad (2.21)$$

However, by (2.21),

$$\begin{aligned} \tilde{\Theta}_{x\eta}(0, 0) + \tilde{\Theta}_{x\eta}(\lambda/2, 0) &= \frac{\tilde{\Theta}_x(0, 0)}{\Lambda/\mu} + \frac{\tilde{\Theta}_x(\lambda/2, 0)}{\Lambda/\mu + \int_0^{\lambda/2} \sin \sigma} \\ &> 0, \quad \text{by (2.23)}, \end{aligned} \quad (2.24)$$

and the first part of (iv) has been established.

It was shown in the proof of (ii) that  $w_\eta^\alpha > 0$  on  $R^\alpha$  for all  $\alpha \in [\lambda/4, \lambda/2)$ . For the case  $\alpha = \lambda/4$ , we have  $w_\eta^{\lambda/4} = 0$  on the lines  $\{(x, \eta) : x = 0, \lambda/4, \eta \in (-h, 0)\}$ . By the strong maximum principle,

$$0 < w_x^{\lambda/4}(0, \eta) = \tilde{\Theta}_{x\eta}(0, \eta) + \tilde{\Theta}_{x\eta}(\lambda/2, \eta), \quad \eta \in (-h, 0),$$

which, together with (2.24), yields

$$\tilde{\Theta}_{x\eta}(0, \eta) + \tilde{\Theta}_{x\eta}(\lambda/2, \eta) > 0, \quad \eta \in (-h, 0]. \quad \text{q.e.d.}$$

Our aim at the outset was to design a cone which was invariant under the operator in equation (2.8), and which was sufficiently sophisticated in its structure to give information about the shape of solutions  $W$ , at least :

large  $\mu$ . The ultimate goal is to reach a firm conclusion about the veracity of Stokes' conjecture (that  $\theta(0+) = \pi/6$  when  $1/\mu = 0$  in (2.8)), through this limiting process.

Our motivation comes from various numerical results [28; pp. 146-147], [5; p. 215], [24; p. 572] which make it seem plausible that  $\theta'$  should have a unique zero in  $(0, \lambda/2)$ , if  $(\mu, \theta)$  is in  $E_\lambda$ . Physically, all this says is that the wave has only one inflection point between crest and trough; Theorem 2.6 says that there are none between  $\lambda/4$  and  $\lambda/2$ , but this does not appear to help. The Serrin-Lavrentiev comparison theorems have been suggested as a possible way to tackle the problem [10; pp. 356-357], but there are difficulties in applying them in this case [29; p. 484]. (However, there are other indications [19; p. 19] which suggest that the number of zeros of  $\theta'$  approaches infinity as  $\mu \rightarrow \infty$ .)

From the point of view of this section, a natural approach is to let  $N$  be the set of all solutions in  $E_\lambda$  which have the property that  $\theta'$  vanishes only once in  $(0, \lambda/2)$ . One can use the local bifurcation theory to show that  $N$  is not empty and non-trivial, and it is clearly closed. We have been unable to show it to be open, but remark that it suffices to show that  $\theta'$  and  $\theta''$  cannot vanish simultaneously on  $[0, \lambda/2]$ . For the analogous problem in the theory of non-linear Sturm-Liouville problems [23; pp. 500-503] this method works, because there  $\theta'$  and  $\theta''$  cannot vanish simultaneously (because of the uniqueness theorem for differential equations).

Finally, we remark that numerical evidence suggests that the zeros of  $\theta'$  approach 0 as  $\mu$  approaches infinity;  $\theta'$  being negative on  $(0, \lambda/2)$  in the limiting case of  $1/\mu = 0$ , which means that the limiting wave is convex [28; p. 147], [24; p. 576].

3. ON THE CONVERGENCE OF PERIODIC WAVES  
TO SOLITARY WAVES IN THE LONG-WAVE LIMIT

Throughout this section the mean depth  $h$  is fixed. The purpose here is to show the sense in which the sets  $C_\lambda$  of periodic water-waves converge to a set  $C'$  of solitary waves as the wavelength increases indefinitely. Recall from section 1.2, that each set  $C_\lambda$  contains exactly one point corresponding to a uniform horizontal flow of depth  $h$ , and that this point  $(\mu_\lambda, 0)$  is the point at which periodic waves of wavelength  $\lambda$  and mean depth  $h$  bifurcate. In other words, on a flow of depth  $h$ , periodic waves of wavelength  $\lambda$  bifurcate from the horizontal flow when the mean velocity of the flow is  $\{(g\lambda/2\pi) \tanh(2\pi h/\lambda)\}^{1/2}$ . Moreover, the value of  $\mu_\lambda$  converges to  $6/\pi$  as  $\lambda \rightarrow \infty$  (Theorem 1.3).

Let  $U$  be any bounded, open set in  $\mathbb{R} \times C_0[0, \pi]$  such that  $(6/\pi, 0) \in U$ . Then, for all  $\lambda$  sufficiently large,  $C_\lambda \cap \partial U \neq \emptyset$ . The next theorem is the main result of this paper. (Further properties of the function  $\theta$  constructed below are given in the remarks following Theorem 3.5; in particular, part (i) may be improved to  $0 < \theta(s) < \pi/3$  on  $(0, \pi)$ .)

THEOREM 3.1. Suppose  $\{\lambda_n\} \subset \mathbb{R}$ , and  $\lambda_n \uparrow \infty$  as  $n \rightarrow \infty$ , and suppose that  $C_{\lambda_n} \cap \partial U \neq \emptyset$  for each  $n$ . If  $\{(\mu_n, \theta_n)\} \subset (0, \infty) \times K_0$  is a sequence such that  $(\mu_n, \theta_n) \in C_{\lambda_n} \cap \partial U$  for each  $n$ , then the sequence  $\{(\mu_n, \theta_n)\}$  is relatively compact in  $[6/\pi, \infty) \times K_0$ . If  $\{(\mu_{n(k)}, \theta_{n(k)})\}$  is a subsequence of  $\{(\mu_n, \theta_n)\}$  such that

$$(\mu_{n(k)}, \theta_{n(k)}) \rightarrow (\mu, \theta) \in [6/\pi, \infty) \times K_0, \quad (3.1)$$

then (i)  $\mu > 6/\pi$ ,  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$  and  $(\mu, \theta) \in \partial U$ ;

(ii)  $(\mu, \theta)$  is a solution of the equation for solitary waves (1.47).

(iii) The sequence  $\{f_{\lambda_{n(k)}}^{\theta_{n(k)}}\}$  converges in  $L_1(0, \pi)$  to  $f\theta$ , as  $k \rightarrow \infty$ .

(iv) If  $c(\mu_{n(k)}, \theta_{n(k)})$  is calculated using  $\lambda_{n(k)}$  instead of  $\lambda$  in expression (1.34), then

$$c(\mu_{n(k)}, \theta_{n(k)}) \rightarrow \left\{ \frac{6gh}{\pi} \left( \frac{1}{\mu} + \int_0^{\pi} f(t) \sin \theta(t) dt \right) \right\}^{1/2}$$

$$\in (\sqrt{gh}, 2\sqrt{gh}) .$$

(v) For each  $k$ , the free surface may be denoted by  $\{(x, H_k(x)) : x \in (-\lambda_{n(k)}/2, \lambda_{n(k)}/2)\}$  where  $H_k$  depends on  $\lambda_{n(k)}, \mu_{n(k)}$  and  $\theta_{n(k)}$  according to the formulae (1.37) and (1.38). As  $k \rightarrow \infty$ ,

$$H_k(x) - H_k(0) \rightarrow H(x) - H(0) ,$$

uniformly on compact intervals, where  $\{(x, H(x)) : x \in \mathbb{R}\}$  is the profile of the solitary wave corresponding to the solution  $(\mu, \theta)$  of (1.47). The function  $H$  may be calculated from  $(\mu, \theta)$  by the formulae (1.48), (1.49).

A proof of this theorem may be obtained by modifying the arguments of [1; Theorem 3.8]. The following lemmas facilitate this procedure.

LEMMA 3.2. For any non-negative, bounded function  $u$  on  $[0, \pi]$ , whose support has full measure, and for any  $\alpha \geq 0$ ,

$$\frac{f_{\lambda}(t)u(t)}{\alpha + \int_0^t f_{\lambda}(w)u(w)dw} \geq \frac{f_{\nu}(t)u(t)}{\alpha + \int_0^t f_{\nu}(w)u(w)dw}$$

if  $\lambda \geq \nu > 0$ , and  $f_{\lambda}, f_{\nu}$  are defined by the expression (1.22).

Proof. Since  $f_{\lambda}(t) \geq f_{\nu}(t)$  for all  $t \in [0, \pi]$  when  $\lambda \geq \nu$ , it will suffice to show that

$$f_{\lambda}(t) \int_0^t f_{\nu}(w)u(w)dw \geq f_{\nu}(t) \int_0^t f_{\lambda}(w)u(w)dw$$

for all  $t \in [0, \pi]$ . In other words, it will suffice to show that

$$\begin{aligned} 0 &\leq \int_0^t (f_\lambda(t)f_\nu(w) - f_\nu(t)f_\lambda(w))u(w)dw \\ &= \int_0^t f_\nu(t)f_\nu(w) \left( \frac{f_\lambda(t)}{f_\nu(t)} - \frac{f_\lambda(w)}{f_\nu(w)} \right) u(w)dw . \end{aligned}$$

However, a simple calculation yields that  $f_\lambda/f_\nu$  is increasing on  $(0, \pi)$ , and the proof is complete.

q.e.d.

LEMMA 3.3. For each  $\lambda > 1/\pi$ , let  $g_\lambda$  denote the function defined on  $[0, \pi]$  by putting

$$g_\lambda(s) = \begin{cases} f_\lambda(s), & s \in [0, \pi - 1/\lambda] , \\ 0, & s \in (\pi - 1/\lambda, \pi] . \end{cases} \quad (3.2)$$

Then there exists a unique solution  $(\gamma_\lambda, \psi_\lambda)$  of

$$\psi(s) = \frac{2\gamma}{3} \int_0^\pi G(s,t) g_\lambda(t) \psi(t) dt ,$$

with  $(\gamma, \psi) \in [0, \infty) \times K_0$  and  $|\psi|_{C_0[0, \pi]} = 1$ . Moreover  $\gamma_\lambda \downarrow 6/\pi$  as  $\lambda \rightarrow \infty$ .

Proof. The proof is similar to that of [1; Theorem 3.2]. Existence and uniqueness follow immediately from the general theory of  $u_0$ -positive linear operators, and that  $\gamma_\lambda \downarrow 6/\pi$  follows by exactly the same argument as was used to show that  $\gamma_n \downarrow 6/\pi$  in [1; Theorem 3.2].

q.e.d.

Proof of Theorem 3.1. Because of the obvious similarity between the problem here and that of proving [1; Theorem 3.9], we shall limit ourselves to

giving an outline of the proof. The letters (A'), (B'), (C'), etc., when used below, refer to those points of the proof of [1; Theorem 3.9] so labelled.

Since  $\{(\mu_n, \theta_n)\} \subset \mathcal{W} \subset \mathbb{R} \times C_0[0, \pi]$  is a bounded sequence, there exists a subsequence  $\{(\mu_{n(k)}, \theta_{n(k)})\}$  and a corresponding sequence  $\{\lambda_{n(k)}\} \subset \mathbb{R}$ , such that

$$\mu_{n(k)} \rightarrow \mu \text{ in } \mathbb{R} , \quad (3.3a)$$

$$\theta_{n(k)} \rightharpoonup \theta \text{ weakly in } L_2(0, \pi) , \quad (3.3b)$$

$$\sin \theta_{n(k)} \rightharpoonup \sigma \text{ weakly in } L_2(0, \pi) \quad (3.3c)$$

and

$$\lambda_{n(k)} \uparrow \infty \text{ in } \mathbb{R}$$

as  $k \rightarrow \infty$ . We shall show that the conclusions (i) - (v) of the theorem hold for this subsequence. For the sake of having a convenient notation, we shall henceforth use  $\{\mu_n\}$ ,  $\{\theta_n\}$ ,  $\{\lambda_n\}$  to denote the subsequence for which (3.3) holds.

(i), (ii), (iii) An obvious adaptation of (A') - (D') yields that  $\theta_n \rightarrow \theta$  and  $\sin \theta_n \rightarrow \sin \theta$  in  $L_2(0, \pi)$  as  $n \rightarrow \infty$ ; that  $\theta_n \rightarrow \theta$  in  $C[0, \delta]$  for each  $\delta \in (0, \pi)$ ; and that  $(\mu, \theta) \in [6/\pi, \infty) \times K_0$  is a solution of (1.47). The next step is to prove that  $\theta$  is non-trivial. To do this we first show that if  $\theta = 0$ , then  $\mu = 6/\pi$ .

Now for each  $n$ ,

$$\begin{aligned} \theta_n(s) &= \frac{2}{3} \int_0^\pi G(s,t) \frac{f_{\lambda_n}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_n}(w) \sin \theta_n(w) dw} dt \\ &\geq \frac{2}{3} \int_0^\pi G(s,t) \frac{f_{\lambda_\ell}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w) dw} \end{aligned}$$

for all  $n \geq \ell$ , by Lemma 3.2 and the fact that  $\lambda_n \rightarrow \infty$ ,

$$\geq \frac{2}{3} \int_0^\pi G(s,t) \frac{g_{\lambda_\ell}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w)} dt,$$

where  $g_{\lambda_\ell}$  is defined by (3.2). Therefore

$$\theta_n(s) \geq \frac{2}{3} A_{n,\ell} \left\{ \frac{\int_0^\pi G(s,t) g_{\lambda_\ell}(t) \theta_n(t) dt}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w) dw} \right\}, \quad (3.4)$$

for all  $s \in [0, \pi]$ , where

$$A_{n,\ell} = \inf_{s \in [0, \pi-1/\lambda_\ell]} \frac{\sin \theta_n(s)}{\theta_n(s)}.$$

Now multiplying this inequality by  $g_{\lambda_\ell} \psi_{\lambda_\ell}$ , whose existence is guaranteed by Lemma 3.3, and integrating gives

$$\gamma_{\lambda_\ell} \int_0^\pi \theta_n(s) \psi_{\lambda_\ell}(s) g_{\lambda_\ell}(s) ds \geq A_{n,\ell} \left\{ \frac{\int_0^\pi g_{\lambda_\ell}(t) \psi_{\lambda_\ell}(t) \theta_n(t) dt}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w) dw} \right\}.$$

Thus

$$\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w) dw \geq A_{n,\ell} / \gamma_{\lambda_\ell}$$

for all  $n \geq \ell$ . If  $\theta_n \rightarrow 0$  in  $L_2(0, \pi)$  as  $n \rightarrow \infty$ , then  $\theta_n \rightarrow 0$  in  $C[0, \delta]$  for each  $\delta \in (0, \pi)$ , and so  $A_{n,\ell} \rightarrow 1$  as  $n \rightarrow \infty$  for each fixed  $\ell$ . There results that

$$1/\mu = \lim_{n \rightarrow \infty} 1/\mu_n \geq \gamma_{\lambda_\ell}^{-1}$$

for each  $\ell$ . Since  $\gamma_{\lambda_\ell} \downarrow 6/\pi$ , as  $\ell \uparrow \infty$ , it follows that  $\mu \leq 6/\pi$ . But  $(\mu, \theta)$  is a solution of (1.47) and so, by [1; Theorem 3.7],  $\mu \geq 6/\pi$ . We have shown that if  $(\mu_n, \theta_n) \rightarrow (\mu, 0)$  in  $\mathbb{R} \times L_2(0, \pi)$ , then  $\mu = 6/\pi$ .

From this observation, the method of (F') yields that  $f_{\lambda_n} \theta_n$  converges to  $f\theta$  in  $L_1(0, \pi)$ , and then the method of (G') may be used to prove that  $\theta_n \rightarrow \theta$  in  $C_0[0, \pi]$ . The function  $\theta$  must therefore be non-zero, for otherwise, as we have seen,  $(\mu_n, \theta_n) \rightarrow (6/\pi, 0)$  in  $\mathbb{R} \times C_0[0, \pi]$ . This contradicts the fact that  $U$  is an open set which contains  $(6/\pi, 0)$  in its interior. That  $0 < \theta(s) < \pi/2$  on  $(0, \pi)$ , and  $\mu > 6/\pi$  is proved in [1; Theorem 3.7]. This completes the proof of (i), (ii), (iii).

(iv) By (1.23) and (1.34)

$$c(\mu_n, \theta_n) = \frac{2\sqrt{3g} K_{\lambda_n} (1 + k_{\lambda_n})}{\lambda_n} \left( \frac{2}{\lambda_n} \int_0^\pi \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{t \left( \frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw \right)^{1/3}} dt \right)^{-3/2}.$$

From (1.24) it follows that

$$\frac{2\sqrt{3g} K_{\lambda_n} (1 + k_{\lambda_n})}{\lambda_n} \rightarrow \frac{\sqrt{3g} \pi}{2h} \quad (3.5)$$

as  $n \rightarrow \infty$ . Now for any  $\alpha \in (0, \pi)$ ,

$$\frac{2}{\lambda_n} \int_0^\pi \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{t \left( \frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw \right)^{1/3}} dt$$

$$\begin{aligned}
&= \frac{2}{\lambda_n} \int_0^\alpha \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3}} dt + \\
&\quad \frac{2}{\lambda_n} \int_\alpha^\pi \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3}} dt \quad (3.6)
\end{aligned}$$

For any  $\varepsilon > 0$ , choose  $\alpha(\varepsilon) \in (0, \pi)$  such that for all  $n$  sufficiently large  $|\cos \theta_n(t) - 1| \leq \varepsilon$  for all  $t \in [\alpha(\varepsilon), \pi]$ . This can be done since  $\theta_n \rightarrow \theta \in K_0$  uniformly on  $[0, \pi]$ . Moreover, by (3.3a) and (iii),  $\alpha(\varepsilon)$  can be chosen so that

$$\left| \left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3} - \left(\frac{1}{\mu} + \int_0^t f(w) \sin \theta(w) dw\right)^{1/3} \right| \leq \varepsilon$$

for all  $t \in [\alpha(\varepsilon), \pi]$  and for all  $n$  sufficiently large. From (3.6) it follows that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_0^\pi \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3}} dt \\
&= \lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_{\alpha(\varepsilon)}^\pi \frac{f_{\lambda_n}(t) \cos \theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3}} dt \\
&\left[ \begin{aligned}
&\geq \lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_{\alpha(\varepsilon)}^\pi \frac{f_{\lambda_n}(t) (1 - \varepsilon)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w) \sin \theta_n(w) dw\right)^{1/3}} dt \\
&\leq \lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_{\alpha(\varepsilon)}^\pi \frac{f_{\lambda_n}(t)}{\left(\frac{1}{\mu} + \int_0^t f(w) \sin \theta(w) dw\right)^{1/3} - \varepsilon} dt \quad (3.7)
\end{aligned} \right.
\end{aligned}$$

However, by (1.24) and (1.25),

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_{\alpha(\epsilon)}^{\pi} f_{\lambda_n}(t) dt &= \lim_{n \rightarrow \infty} \frac{2}{\lambda_n} \int_0^{\pi} f_{\lambda_n}(t) dt \\ &= \lim_{n \rightarrow \infty} \frac{-2}{\lambda_n \Lambda_n} \int_0^{\pi} q'_{\lambda_n}(t) dt = \lim_{n \rightarrow \infty} \Lambda_n^{-1} = \frac{\pi}{2h} . \end{aligned} \quad (3.8)$$

Also, from (iii),

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{1}{\mu_n} + \int_0^{\pi} f_{\lambda_n}(w) \sin \theta_n(w) dw \right)^{1/3} \\ = \left( \frac{1}{\mu} + \int_0^{\pi} f(w) \sin \theta(w) dw \right)^{1/3} . \end{aligned} \quad (3.9)$$

Collecting (3.5) - (3.9), we find that

$$\begin{aligned} \frac{\pi\sqrt{3g}}{2h} \left( \frac{2h}{\pi} \right)^{3/2} \left[ \left( \frac{1}{\mu} + \int_0^{\pi} f(w) \sin \theta(w) dw \right)^{1/3} - \epsilon \right]^{3/2} \\ \leq \lim_{n \rightarrow \infty} c(\mu_n, \theta_n) \\ \leq \frac{\pi\sqrt{3g}}{2h} \left( \frac{2h}{\pi} \right)^{3/2} (1 - \epsilon)^{-3/2} \left( \frac{1}{\mu} + \int_0^{\pi} f(w) \sin \theta(w) dw \right)^{1/2} , \end{aligned}$$

and since  $\epsilon$  is arbitrary, it follows that

$$\lim_{n \rightarrow \infty} c(\mu_n, \theta_n) = \sqrt{\frac{6gh}{\pi}} \left( \frac{1}{\mu} + \int_0^{\pi} f(w) \sin \theta(w) dw \right)^{1/2} .$$

That this last quantity lies in an interval  $(\sqrt{gh}, 2\sqrt{gh})$  has been established in [1; Theorems 3.9, 4.12, and the footnote to Theorem 3.7(c)].

(v) An analogous calculation to that just given yields (v).

q.e.d.

COROLLARY 3.4. The statement of this corollary is given in section 1.1.

Proof. By Theorem 2.2, there exists  $(\mu_n, \theta_n) \in C_{\lambda_n}$  such that

$|\theta|_{C_0[0, \pi]} = \beta$ , for any  $\beta \in [0, \pi/6)$ . The result will follow by the method used in the proof of Theorem 3.1, once it is established that the sequence  $\{\mu_n\}$  is bounded. However

$$\begin{aligned} \theta_n(s) &= \frac{2}{3} \int_0^\pi G(s,t) \frac{f_{\lambda_n}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_n}(w) \sin \theta_n(w) dw} dt \\ &\geq \frac{2}{3} \int_0^\pi G(s,t) \frac{f_{\lambda_m}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_m}(w) \sin \theta_n(w) dw} dt, \end{aligned}$$

if  $n \geq m$ , by Lemma 3.2. Without loss of generality suppose that  $\mu_n \rightarrow \infty$ . Then, as in the proof of Theorem 2.2(iv), it can be shown that there exists  $B > 0$  such that  $\theta_n(s) \geq B \sin s$ , for all  $s \in [0, \pi]$ . But this estimate is enough to guarantee (by a routine adaptation of the methods of [1; section 5]) that a subsequence  $\{\theta_{n(k)}\}$  of  $\{\theta_n\}$  converges in  $C[\delta, \pi]$ , for each  $\delta \in (0, \pi)$ , to a non-trivial solution  $\theta$  of the equation

$$\theta(s) = \frac{2}{3} \int_0^\pi G(s,t) \frac{f(t) \sin \theta(t)}{\int_0^\pi f(w) \sin \theta(w) dw} dt.$$

However, we know from [1; Theorem 5.2] that for such a function  $\theta$ ,

$\lim_{s \rightarrow 0^+} \theta(s) \geq \pi/6$ . This contradicts the fact that  $|\theta_n|_{C_0[0, \pi]} = \beta < \frac{\pi}{6}$ .

q.e.d.

Finally, we have the following result. Let  $S = \{(\mu, \theta) \in (0, \infty) \times K_0 : (\mu, \theta) \text{ solves (1.47) and } \theta \neq 0\}$ . For all  $(\mu, \theta) \in S$ , the product  $f\theta \in L_1(0, \pi)$  ([1; Theorem 4.1]). Let  $S' = \{(\mu, \theta) \in S : (\mu, \theta) \text{ is the limit, as } \lambda \rightarrow \infty, \text{ in } \mathbb{R} \times C_0[0, \pi] \text{ of a sequence } (\mu_\lambda, \theta_\lambda), \text{ where } (\mu_\lambda, \theta_\lambda) \in C_\lambda\}$ .

**THEOREM 3.5.** If  $C'$  is the maximal connected subset of  $S'$  which contains  $(6/\pi, 0)$ , then  $C'$  is closed, unbounded, and has all the properties attributed to  $C$  in [1; Theorem 3.9]. Clearly  $C' \subset C$ .

**Proof.** This is immediate, since it has been shown that the boundary  $\partial U$  of every bounded, open set  $U \subset \mathbb{R} \times C_0[0, \pi]$  which contains  $(6/\pi, 0)$ , contains a point of  $C'$ . Since the set  $S'$  is obviously a closed subset of  $S$ , and it has the property that bounded subsets of it are relatively compact, [1; Theorem 3.8], the result is immediate from [1; Theorem A6].

q.e.d.

**Remarks.** (a) Section 4 of [1] gives further properties of the elements of  $C$ . In particular, the function  $\theta$  is real-analytic on  $[0, \pi)$ , and so the wave profile is an analytic curve in  $\mathbb{R}^2$ , and the rate at which the free-surface approaches its asymptotic level is estimated. In section 5 of [1], it is shown that if  $\{(\mu_n, \theta_n)\} \subset C'$  and  $\mu_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then a subsequence converges to a non-trivial 'solitary wave of greatest height' which satisfies (1.47) with  $\mu = \infty$ . The behaviour of this wave at its crest is similar to that given in Theorem 2.2(vi).

Clearly the results of [19], quoted in Theorem 2.2(vii) for periodic waves, hold also for solitary waves corresponding to  $C'$  or  $C$ . This agrees with numerical results [18; p. 738].

(b) Since periodic waves converge to solitary waves on compact sets as the wavelength goes to infinity, it is reasonable to hope that the limiting solitary wave will inherit some of the properties of periodic waves given in section 2.3. Unfortunately, this has not been proved for the conclusions of Theorem 2.6; only some parts of Theorem 2.3 hold in the solitary wave case. The difficulty lies in the fact that  $R_\lambda \rightarrow R_\infty$ , while the uniform convergence of periodic waves to solitary waves is only on compact intervals. If, however, the plan outlined in the remark following Theorem 2.6 could be implemented, then conclusions would follow which would be compatible with the numerical results on solitary waves [4; p. 185]; on the convergence of periodic waves to solitary waves [5], and on the solitary wave of greatest height [16 p. 10].

The results of Theorem 2.4 do go over in the limit as  $\lambda \rightarrow \infty$ , and one can show that elements of  $C'$  satisfy  $\chi\theta'(x) < \theta(x)$  and  $\theta(x) < \pi/3$  on  $(0, \infty)$ .

## REFERENCES

- [1] Amick, C. J. and Toland, J. F., On solitary water waves of finite amplitude. To appear in Arch. Rational Mech. Anal. .
- [2] Bary, N. K., A treatise on trigonometric series, Vols. I & II. Pergamon, New York, 1964.
- [3] Bona, J. L., Bose, D. K. and Turner, R. E. L., Finite amplitude steady waves in stratified fluids. In preparation.
- [4] Byatt-Smith, J. G. B. and Longuet-Higgins, M.S., On the speed and profile of steep solitary waves. Proc. R. Soc. Lond. A., 350 (1976), 175-189.
- [5] Cokelet, E. D., Steep gravity waves in water of arbitrary uniform depth. Phil. Trans. R. Soc. Lond., 286 (1335) (1977), 183-230.
- [6] Copson, E. T., An introduction to the theory of functions of a complex variable. Oxford University Press, Oxford (1972).
- [7] Dancer, E. N., Global solution branches for positive mappings. Arch. Rational Mech. Anal., 52 (1973), 181-192.
- [8] Gradshteyn, I. S. and Ryzhik, I. M., A table of integrals, series and products. Academic Press, London, 1965.
- [9] Keady, G. and Norbury, J., On the existence theory for irrotational water waves. Math. Proc. Camb. Phil. Soc., 83 (1978), 137-157.
- [10] Keady, G. and Pritchard, W. G., Bounds for surface solitary waves. Proc. Camb. Phil. Soc., 76 (1974), 345-358.
- [11] Krasnosel'skii, M. A., Positive solutions of operator equations. Noordhoff, Groningen, 1964.
- [12] Krasovskii, Yu, P., On the theory of steady state waves of large amplitude. U.S.S.R. Computational Maths. and Math. Phys., 1 (1961), 996-1018.

- [13] Krasovskii, Yu, P., The existence of aperiodic flows with free boundaries in fluid mechanics. Doklady Akad. Nauk., Vol. 133, 4 (1960), 768-770.
- [14] Lavrentiev, M. A., On the theory of long waves (1943); a contribution to the theory of long waves (1947). Amer. Math. Soc. Translation No. 102 (1954).
- [15] Lewy, H., A note on harmonic functions and a hydrodynamical application. Proc. Amer. Math. Soc., 3 (1952), 111-113.
- [16] Longuet-Higgins, M. S., On the mass, momentum, energy and circulation of a solitary wave. Proc. R. Soc. Lond. A., 337 (1974), 1-13.
- [17] Longuet-Higgins, M. S. and Fenton, J. D., On the mass, momentum, energy and circulation of a solitary wave. II. Proc. R. Soc. Lond. A., 340 (1974), 471-493.
- [18] Longuet-Higgins, M. S. and Fox, M. J. H., Theory of the almost-highest wave: the inner solution. J. Fluid Mech., 80 (1977), 721-742.
- [19] McLeod, J. B., Stokes's and Krasovskii's conjecture for the wave of greatest height, Mathematics Research Center TSR #2041, February 1980.
- [20] Milne-Thomson, L. M., Theoretical hydrodynamics. MacMillan, London, 1968.
- [21] Nekrasov, A. I., The exact theory of steady state waves on the surface of a heavy liquid. Mathematics Research Center TSR #813 (1967).
- [22] Protter, M. H. and Weinberger, H. F., Maximum principles in differential equations. Prentice-Hall, Englewood Cliffs, NJ, 1967.
- [23] Rabinowitz, P. H., Some global results for nonlinear eigenvalue problems. J. Functional Anal., 7 (1971), 487-513.
- [24] Schwartz, L. W., Computer extension and analytic continuation of Stokes' expansion for gravity waves. J. Fluid Mech., 62 (1974), 553-570.

- [25] Spielvogel, E. R., A variational principle for waves of infinite depth. Arch. Rational Mech. Anal., 39 (1970), 189-205.
- [26] Ter Krikerov, A. M., The existence of periodic waves which degenerate into a solitary wave. J. Appl. Maths. Mech., 24 (1960), 930-949.
- [27] Ter Krikerov, A. M., Théorie exacté des ondes longues stationnaires dans un liquide hétérogene. J. Mécanique 2 (1963), 351-376.
- [28] Thomas, J. W., Irrotational gravity waves of finite height: a numerical study. Mathematika, 15 (1968), 139-148.
- [29] Toland, J. F., On the existence of a wave of greatest height and Stokes's conjecture. Proc. R. Soc. Lond. A., 363 (1978), 469-485.
- [30] Turner, R. E. L., Transversality and cone maps. Arch. Rational Mech. Anal., 58 (1975), 151-179.
- [31] Zygmund, A., Trigonometric Series I & II. Cambridge University Press, Cambridge, 1977.

CJA/JFT/ck/jvs

APPENDIX

Periodic flows of infinite depth

THEOREM. Suppose that  $\theta$  is an odd, continuous function on  $[-\pi, \pi]$  with  $0 < \theta(s) < \pi$  on  $(0, \pi)$ , which satisfies the integral equation

$$\theta(s) = \frac{1}{6} \int_{-\pi}^{\pi} \frac{1}{\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \frac{\sin \theta(t)}{t} dt + \int_0^t \frac{\sin \theta(w) dw}{\mu} \quad (A1)$$

for some  $\mu > 0$ . Then  $\theta$  is real-analytic on  $[-\pi, \pi]$  and  $0 < \theta(s) < \pi/3$  on  $(0, \pi)$ . Moreover,  $\mu > 3$ , and if  $\lambda$  and  $c$  are positive real numbers such that

$$\left( \frac{3g\lambda}{2\pi c^2} \right)^{1/3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \theta(t)}{\left( \frac{1}{\mu} + \int_0^t \sin \theta(w) dw \right)^{1/3}} dt, \quad (A2)$$

then there exists a periodic wave of wavelength  $\lambda$  on a flow of infinite depth. The velocity of the flow at infinite depth is then  $c$ , and its speed at the crest is given by

$$q_c = \left( \frac{3g\lambda c}{2\pi\mu} \right)^{1/3},$$

The free surface may be parametrized by  $(x, H_\lambda(x))$ , where  $x \in (-\lambda/2, \lambda/2)$  and for  $x \in [0, \lambda/2]$

$$H_\lambda(x) - H_\lambda(0) = \left( \frac{\lambda^2 c^2}{3g\pi^2} \right)^{1/3} \int_{\alpha^{-1}(x)}^0 \frac{\frac{1}{2} \sin \theta(t)}{\left( \frac{1}{2\mu} + \int_0^t \frac{1}{2} \sin \theta(w) dw \right)^{1/3}} dt, \quad (A3)$$

where for  $s \in [-\pi, 0]$

$$\alpha(s) = \left( \frac{\lambda^2 c^2}{3g\pi^2} \right) \int_s^0 \frac{\frac{1}{2} \cos \theta(t)}{\left( \frac{1}{2\mu} + \int_0^t \frac{1}{2} \sin \theta(w) \right)^{1/3}} dt . \quad (A4)$$

Proof. The proof that  $\theta$  is real-analytic and bounded by  $\pi/3$  follows as in Theorems 2.2 and 2.4. To show that  $\mu > 3$ , multiply (A1) by  $\sin s$  and integrate over  $(-\pi, \pi)$ , using (1.27).

As before, there exists a harmonic function  $\tilde{\theta}$  on the unit disc such that  $\tilde{\theta}(e^{is}) = \theta(s)$  for all  $s \in (-\pi, \pi]$ , and

$$\frac{\partial \tilde{\theta}}{\partial r} \Big|_{e^{is}} = \frac{1}{3} \frac{\sin \theta(s)}{\frac{1}{\mu} + \int_0^s \sin \theta(w) dw} . \quad (A5)$$

Using the expansion of  $G$  given in (1.27), it follows from (A1) that for all  $s \in (-\pi, \pi]$ ,

$$\theta(s) = \frac{1}{3} \int_{-\pi}^{\pi} \left\{ \frac{1}{\pi} \sum_{\ell=1}^{\infty} \frac{\sin \ell s \sin \ell t}{\ell} \right\} \frac{\sin \theta(t)}{\frac{1}{\mu} + \int_0^t \sin \theta(w) dw} dt .$$

From this and Fubini's theorem there results that the Fourier series for  $\theta$  is

$$\sum_{\ell=1}^{\infty} a_{\ell} \sin \ell s \quad (A6)$$

where

$$a_{\ell} = -\frac{1}{3\pi} \int_{-\pi}^{\pi} \cos \ell t \ln \left( \frac{1}{\mu} + \int_0^t \sin \theta(w) dw \right) dt . \quad (A7)$$

It follows that putting

$$\tilde{\tau}(\xi) + i\tilde{\theta}(\xi) = \sum_{\ell=1}^{\infty} a_{\ell} \xi^{\ell} \quad (A8)$$

defines an analytic function on the unit disc, and

$$\tilde{\tau}(e^{is}) + i\tilde{\theta}(e^{is}) = a_0 - \frac{1}{3} \ln\left(\frac{1}{\mu} + \int_0^s \sin \theta(w) dw\right) + i\theta(s) \quad (A9)$$

for all  $s \in [-\pi, \pi]$ , where  $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{3} \ln\left(\frac{1}{\mu} + \int_0^t \sin \theta(w) dw\right) dt$ .

Let  $c$  and  $\lambda$  be positive real numbers chosen so that (A2) holds. Then an analytic function  $\tilde{T} - i\tilde{\theta}$  can be defined on  $R_\lambda = \{\chi + i\eta : -\lambda/2 < \chi < \lambda/2, \eta < 0\}$  by putting

$$\tilde{T}(\zeta) - i\tilde{\theta}(\zeta) = \tilde{\tau}(\exp(-2\pi i \zeta / \lambda)) + i\tilde{\theta}(\exp(-2\pi i \zeta / \lambda)).$$

Hence  $\tilde{\theta}(\chi + i0) = -\theta(-2\pi\chi/\lambda)$  and so

$$\begin{aligned} \left. \frac{\partial \tilde{\theta}}{\partial \eta} \right|_{\chi+i0} &= \frac{2\pi}{3\lambda} \frac{-\sin \theta(-2\pi\chi/\lambda)}{\frac{1}{\mu} + \int_0^{-2\pi\chi/\lambda} \sin \theta(w) dw} \\ &= \frac{1}{3} \frac{\sin \theta(\chi)}{\frac{\lambda}{2\pi\mu} + \int_0^\chi \sin \theta(w) dw}, \end{aligned} \quad (A10)$$

where  $\theta(x) = \tilde{\theta}(x + i0)$ . Since  $|\theta| < \pi/3$  on  $[-\pi, \pi]$ , it follows by the maximum principle that  $|\tilde{\theta}| < \pi/3$  in  $R_\lambda$ .

Now define an analytic function  $\tilde{m}$  on  $R_\lambda$  by putting

$$\tilde{m}(\zeta) = \int_0^\zeta \exp(\tilde{T}(\zeta') - i\tilde{\theta}(\zeta')) d\zeta'.$$

Since  $\tilde{\theta}(\pm \lambda/2 + i\eta) = 0$  for all  $\eta < 0$ , and since  $|\tilde{T}(\zeta) - i\tilde{\theta}(\zeta)| \rightarrow 0$  as  $|\zeta| \rightarrow \infty$ ,  $\zeta \in R_\lambda$ , it follows that  $\tilde{m}$  is a conformal mapping from  $R_\lambda$  onto an infinite region in the  $z$ -plane of the form  $S_\lambda = \{x + iy : -\lambda/2 < x < \lambda/2, y < H_\lambda(x)\}$ , and  $\frac{d}{dx} H_\lambda(x) = -\tan \theta(\tilde{m}^{-1}(x + iH_\lambda(x)))$ . Since  $\tilde{m}$  is invertible,

we can define a complex potential  $\tilde{\omega} = \tilde{\phi} + i\tilde{\psi}$  on  $S_\lambda$  by putting

$$\tilde{\omega}(z) = c \tilde{m}^{-1}(z) ,$$

where  $c$  was chosen when  $\lambda$  was chosen so that (A2) holds. Then for  $z \in S_\lambda$ ,

$$\begin{aligned} u(z) - iv(z) &= - \frac{d\tilde{\omega}}{dz} \\ &= -c \exp(-\tilde{T}(\tilde{m}^{-1}(z))) (\cos \tilde{\Theta}(\tilde{m}^{-1}(z)) + i \sin \tilde{\Theta}(\tilde{m}^{-1}(z))) \end{aligned}$$

and it follows that  $c \exp(-\tilde{T}(\tilde{m}^{-1}(z)))$  is the speed of the flow and  $-\Theta(\tilde{m}^{-1}(z))$  is the angle which the negative velocity vector makes with the  $x$ -axis at a point  $z \in S_\lambda$ . Moreover  $u(z) - iv(z) \rightarrow -c$  as  $|z| \rightarrow \infty$ ,  $z \in S_\lambda$ . From the definition of  $\tilde{\omega}$ , it follows that  $\tilde{\psi} \rightarrow -\infty$  as  $|z| \rightarrow \infty$ ,  $z \in S_\lambda$ , and  $\tilde{\psi} = 0$  on the free surface  $\Gamma_\lambda = \{(x, H_\lambda(x)) : x \in (-\lambda/2, \lambda/2)\}$ .

Finally to show that the free surface condition is satisfied, we proceed as follows. By (A8), (A9) and Cauchy's formula there results that

$$\begin{aligned} 1 = \exp(0) &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\exp(\tilde{\tau}(e^{it}) + i\tilde{\theta}(e^{it}))ie^{it}}{e^{it}} dt \\ &= \frac{\exp(a_0)}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \theta(t)}{\left(\frac{1}{\mu} + \int_0^t \sin \theta(w)dw\right)^{1/3}} dt , \end{aligned}$$

and so, by our choice of  $\lambda$  and  $c$ ,

$$\exp(a_0) = \left(\frac{2\pi c^2}{3g\lambda}\right)^{1/3} . \quad (A11)$$

Hence

$$\tilde{\tau}(e^{is}) = \frac{1}{3} \ln\left(\frac{2\pi c^2}{3g\lambda}\right) - \frac{1}{3} \ln\left(\frac{1}{\mu} + \int_0^s \sin \theta(w)dw\right) ,$$

and so

$$\tilde{T}(\chi + i0) = \frac{1}{3} \ln\left(\frac{2\pi c^2}{3g\lambda}\right) - \frac{1}{3} \ln\left(\frac{1}{\mu} + \frac{2\pi}{\lambda} \int_0^\chi \sin \Theta(w) dw\right) \quad . \quad (A12)$$

Therefore

$$\begin{aligned} & \frac{d}{d\chi} \left\{ \left(\frac{c^2}{2}\right) \exp(-2\tilde{T}(\chi + i0)) + g \operatorname{Imag} \tilde{m}(\chi + i0) \right\} \\ &= \frac{2\pi}{3\lambda} \frac{c^2 \exp(-2\tilde{T}(\chi + i0)) \sin \Theta(\chi)}{\frac{1}{\mu} + \frac{2\pi}{\lambda} \int_0^\chi \sin \Theta(w) dw} - g \exp(\tilde{T}(\chi + i0)) \sin \tilde{\Theta}(\chi + i0) \\ &= \exp(\tilde{T}(\chi + i0)) \left\{ \frac{2\pi c^2}{3\lambda} \frac{3g\lambda}{2\pi c^2} \sin \Theta(\chi) - g \sin \Theta(\chi) \right\} \\ &= 0 \quad . \end{aligned}$$

To complete the proof of the theorem we must verify that (A3), (A4) give the wave profile. This is a routine calculation based on the method used in the proof of Theorem 1.3. q.e.d.

Results similar to those in Theorems 2.4 - 2.6 hold if one replaces  $\Theta$  by  $\theta$  and  $\lambda/2$  by  $\pi$ .

Though the proof of this last theorem is in many respects similar to that of Theorem 1.5, we have included it in order to obtain the following corollary. We need the notion of a conjugate function which is defined as follows. If  $u$  is an  $L_2$ -function whose Fourier series is

$$a_0 + \sum_{\ell=1}^{\infty} (a_\ell \cos \ell s + b_\ell \sin \ell s), \quad \text{then the function conjugate to } u \text{ is}$$

denoted by  $\bar{C}u$  and is the  $L_2$ -function whose Fourier series is

$$\sum_{\ell=1}^{\infty} (a_\ell \sin \ell s - b_\ell \cos \ell s) \quad [2].$$

COROLLARY. If  $\theta$  satisfies (A1) for some  $\mu > 0$ , then  $\theta$  satisfies  
the equation

$$\theta(s) = \frac{\nu}{6} \int_{-\pi}^{\pi} \frac{1}{\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \exp(-3\zeta\theta(t)) \sin \theta(t) dt$$

where  $\nu = \frac{3g\lambda}{2\pi c^2}$ .

Proof. By (A6) - (A9)

$$\begin{aligned} -\zeta\theta(t) &= \tilde{\tau}(e^{it}) \\ &= a_0 - \frac{1}{3} \ln\left(\frac{1}{\mu} + \int_0^t \sin \theta(w) dw\right), \end{aligned}$$

whence by (A1)

$$\exp(-3\zeta\theta(t)) = \frac{2\pi c^2}{3g\lambda} \left\{ \frac{1}{\frac{1}{\mu} + \int_0^t \sin \theta(w) dw} \right\}.$$

Substituting this last expression into (A1) gives the required result.

q.e.d.

Remark. In the previous sections, the mean depth was held fixed as  $\lambda \rightarrow \infty$ . If we now fix  $\lambda$ , and let  $h \rightarrow \infty$ , then one can prove a result analogous to Theorem 3.1, but the proof is essentially simpler, because the limiting equation is non-singular. A word of caution is necessary however; if  $\{(\mu_n, \theta_n)\}$  is a sequence of solutions of (1.31) corresponding to waves of the same fixed wavelength  $\lambda$ , but of different mean depths  $h_n \rightarrow \infty$ , and if  $\{(\mu_n, \theta_n)\} \subset \partial U$ , where  $U$  is an open set in  $\mathbb{R} \times C_0[0, \lambda/2]$  containing  $(6, 0)$ , then a subsequence converges to  $(\mu, \theta)$ , where  $(\mu/2, \theta)$  is a solution of (A1). This may be seen from (1.31), since (1.17) and (1.22) together give  $f_{\lambda}(x) \rightarrow 1/2$  uniformly for  $t \in [-\pi, \pi]$ , as  $h \rightarrow \infty$ .

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER #2127	2. GOVT ACCESSION NO. AD-A093	3. RECIPIENT'S CATALOG NUMBER 620	
4. TITLE (and Subtitle) ON PERIODIC WATER-WAVES AND THEIR CONVERGENCE TO SOLITARY WAVES IN THE LONG-WAVE LIMIT		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period	
		6. PERFORMING ORG. REPORT NUMBER	
7. AUTHOR(s) C. J. Amick and J. F. Toland		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 ✓	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis	
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1980	
		13. NUMBER OF PAGES 73	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  UNCLASSIFIED	
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Nekrasov's Integral Equation; asymptotic convergence; long-wave limit; solitary wave; periodic wave; global bifurcation; a priori bounds			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A detailed disucssion of Nekrasov's approach to the steady water-wave prob- lems leads to a new integral equation formulation of the periodic problem. This development allows the adaptation of the methods of [1] to show the global con- vergence of periodic waves to solitary waves in the long-wave limit. In addition, it is shown how the classical integral equation formulation due to Nekrasov leads, via the Maximum Principle, to new results about qualitative features of periodic waves, for which there has long been a global existence theory ([9], [12]). ←			