A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION POI--Etc(U)

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A THEOREM CONCERNING UNIFORM
SIMPLIFICATION AT A TRANSITION POINT
AND THE PROBLEM OF RESONANCE

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ABSTRACT

Given sectors $S_j = \{ \varepsilon; a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho \} (1 \leq j \leq v)$ and functions $\delta_j (1 \leq j \leq v)$ such that (i) $\bigcup_j S_j = \{ \varepsilon; 0 < |\varepsilon| < \rho \}$, (ii) $\delta_j$ is holomorphic in $S_j$, (iii) $\delta_j$ is asymptotically zero as $\varepsilon \to 0$ in $S_j$, (iv) $|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp(-c_1/|\varepsilon|^\lambda)$ in $S_j \cap S_k$ for some positive numbers $c_0$, $c_1$ and $\lambda$ whenever $S_j \cap S_k \neq \emptyset$, we prove that $|\delta_j(\varepsilon)| \leq c_2 \exp(-c_1/|\varepsilon|^\lambda)$ in $S_j$ for some positive number $c_2$. Then, utilizing this result, we prove that Matkowsky-condition implies the resonance in the sense of N. Kopell under a reasonable assumption. The sufficiency of Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been an open question. This work gives an affirmative answer to this question in a reasonably general case.

AMS (MOS) Subject Classifications: 30B40, 30E15, 33A40, 34E20

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SIGNIFICANCE AND EXPLANATION

The basic question is as follows:

Consider a second-order linear differential equation:

(1) \[ x'' + x' + x = 0 \]

under some reasonable assumptions on \( F \) and \( G \). Let \( v(x,\varepsilon) \) be a solution.

Then, generally speaking, \( \lim_{\varepsilon \to 0} v(x,\varepsilon) \) satisfies the first order equation:

(2) \[ F(x,0)\frac{dv}{dx} + G(x,0)v = 0. \]

The problem of finding the relation between (1) and (2) is called the problem of singular perturbations. "Relation" means that solutions of the equation (2) do not contain as many free parameters as solutions of (1) do. In other words, there is some loss something. This phenomenon is explained by means of a certain ε. Finally, various physical phenomena exhibit similar behaviours. Therefore, the problem of singular perturbations has been studied for many years.

In a certain situation which arises naturally in applications, \( \lim_{\varepsilon \to 0} v(x,\varepsilon) = 0 \) for practically all the solutions \( v \) of (1), except when \( F \) and \( G \) are related in a specific way. This exceptional case is called the case of resonance. It is important to find an effectively computable condition for the resonance. B. J. Matkowsky found such a condition. However, so far, it has been mathematically very difficult to prove that the Matkowsky-condition actually guarantees the resonance. The difficulty is due to the fact that a quantity which is decisive in determining the resonance is so small that any existing mathematical tool has failed to dig this quantity out of the differential equation clearly. In this work, we shall provide such a tool.

The responsibility for the wording and views expressed in this descriptive summary lies with [Author], and not with the author of this report.
A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION POINT AND THE PROBLEM OF RESONANCE

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1. Introduction: The main result of this paper is the following theorem:

Theorem 1.1: Let

\[ S_j = \{ \varepsilon; \ a_j < \text{arg} \varepsilon < b_j, \ 0 < |\varepsilon| < \varrho \} \quad (j = 1, \ldots, v) \]

be sectors in the complex \( \varepsilon \)-plane, where \( \varrho \) is a positive number and the \( a \)'s and the \( b \)'s are real numbers. Let \( \delta_1(\varepsilon), \ldots, \delta_v(\varepsilon) \) be functions of \( \varepsilon \).

Assume that

(i) \( S_1 \cup S_2 \cup \ldots \cup S_v = \{ \varepsilon; \ 0 < |\varepsilon| < \varrho \} \);

(ii) \( \delta_j(\varepsilon) \) is holomorphic in \( S_j \);

(iii) \( \delta_j(\varepsilon) \) is asymptotically zero as \( \varepsilon \to 0 \) in \( S_j \), i.e.

\[ |\delta_j(\varepsilon)| \leq K_N |\varepsilon|^N \quad (N = 0, 1, \ldots) \]

for some positive numbers \( K_N \);

(iv) if \( S_j \cap S_k \neq \emptyset \), we have

\[ |\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp(-c_1|\varepsilon|^\lambda) \]

in \( S_j \cap S_k \),

for some positive numbers \( c_0, c_1 \) and \( \lambda \).

Then, there exists a positive number \( \mu \) such that

\[ |\delta_j(\varepsilon)| \leq \mu \exp(-c_1|\varepsilon|^\lambda) \]

in \( S_j \), \( j = 1, 2, \ldots, v \).

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We shall prove this theorem in Section 8. (For another proof, see J.-P. Ramis [5; Theorem 11-(i), p. 189].) In other sections, utilizing Theorem 1.1, we shall treat the following problem:

We consider a differential equation:

\[ ed^2v/dx^2 + F(x,\varepsilon)dv/dx + G(x,\varepsilon)v = 0 \]

where \( F \) and \( G \) are holomorphic in two complex variables \( x \) and \( \varepsilon \) in a domain:

\[ x \in \mathcal{D}_0, \quad |\varepsilon| < \rho_0, \]

where \( \mathcal{D}_0 \) is a domain in the \( x \)-plane and \( \rho_0 \) is a positive number. We assume that \( \mathcal{D}_0 \) contains a real interval:

\[ I_0 = \{x; -a < \text{Re}(x) < b, \text{Im}(x) = 0\}, \]

where \( a \) and \( b \) are positive numbers. We also assume that

\[ F(x,0) = -2x. \]

We say that the differential equation (1.4) satisfies Matkowsky-condition, if there exists a non-trivial formal power series solution of (1.4):

\[ v = \sum_{m=0}^{\infty} a_m(x)\varepsilon^m \]

such that all the \( a_m(x) \) are bounded on the real interval \( I_0 \). We also say that the differential equation (1.4) exhibits a resonance in the sense of N. Kopell on \( I_0 \) if there exists a solution \( v(x,\varepsilon) \) satisfying \( v(b,\varepsilon) = 1 \), such that \( v(x,\varepsilon) \) converges uniformly on \( I_0 \) as \( \varepsilon \to +0 \) to a non-trivial solution of

\[ F(x,0)dv/dx + G(x,0)v = 0. \]

(Cf. B. J. Matkowsky [4] and N. Kopell [2].)
We shall prove the following theorem:

THEOREM 1.2. If $D_0$ is a disk with the center at $x = 0$, i.e.

$$D_0 = \{x; |x| < r_0 \} \text{ for some } r_0 > 0.$$  

Then, Matkowsky-condition implies the resonance in the sense of N. Kopell.

In our argument, the assumption that $F$ and $G$ are holomorphic in

$$(x,\varepsilon) \text{ in a poly-disk (1.5) is indispensable. In our proof, we follow roughly}$$

the guide-line given by R. McKelvey and R. Bohac [3]. It seems to us that

our results yield a sharp estimate for eigenvalues studied by P. P. N. de Groen

[1]. In Section 2, we discuss a more general case.

Throughout this research, the author enjoyed lively discussions with

N. Kopell, B. J. Matkowsky and P. P. N. de Groen.
2. A standard form: Let $\rho_0$ be a positive number and let $D$ be a domain in the complex $\xi$-plane which contains a real interval

$$I = \{ \xi; -\alpha \leq \text{Re}(\xi) \leq \beta, \text{Im}(\xi) = 0 \},$$

where $\alpha$ and $\beta$ are positive numbers.

We shall consider a linear differential equation:

$$\xi^2 v'' + f(\xi, c)v' + g(\xi, c)v = 0,$$

where $f$ and $g$ are holomorphic in two variables $\xi$ and $c$ in the domain

$$\xi \in D, \quad |c| < \rho_0.$$

Set

$$f_0(\xi) = f(\xi, 0).$$

We assume that

$$f_0(0) = 0, \quad f_0'(0) \neq 0,$$

$$\xi f_0(\xi) < 0 \text{ for } \xi \in I, \text{ if } \xi \neq 0.$$

Under this situation, we can write $f_0$ as

$$f_0(\xi) = \xi h(\xi),$$

where $h(\xi)$ is holomorphic in $D$ and

$$h(\xi) < 0 \text{ for } \xi \in I.$$

Let us change the independent variable by

$$x = \psi(\xi) = \left[ - \int_0^\xi f_0(t) dt \right]^\frac{1}{2}.$$

Then, (2.2) becomes

$$\xi^2 v'' + F(x, \epsilon)v' + G(x, \epsilon)v = 0,$$

where

$$F(x, \epsilon) = (\epsilon')^{-2}(\epsilon f + \epsilon''), \quad G(x, \epsilon) = (\epsilon')^{-2}g.$$

Since $f_0 = -2\epsilon',$ we have

$$F(x, \epsilon) = -2x + \epsilon k(x, \epsilon),$$

-4-
and \( k(x,\varepsilon) \) and \( G(x,\varepsilon) \) are holomorphic in a domain
\[
(2.13) \quad x \in \mathcal{D}_0, \quad |\varepsilon| < \rho_0,
\]
where \( \mathcal{D}_0 \) is a domain in the \( x \)-plane which contains the real interval:
\[
(2.14) \quad I_0 = \{ x; -a \leq \text{Re}(x) \leq b, \text{Im}(x) = 0 \},
\]
where
\[
(2.15) \quad a = \sqrt{-\int_0^{-\infty} f_0(t)\,dt}, \quad b = \sqrt{-\int_0^\infty f_0(t)\,dt}.
\]

Another transformation:
\[
(2.16) \quad v = w \exp\left(-\frac{1}{2\varepsilon} \int_0^x F(t,\varepsilon)\,dt\right)
\]
takes (2.10) to
\[
(2.17) \quad \varepsilon^2 \frac{d^2 w}{dx^2} - \left(\frac{1}{4} F(x,\varepsilon)^2 + \varepsilon \left(\frac{1}{2} \frac{\partial F}{\partial x} (x,\varepsilon) - G(x,\varepsilon)\right)\right) w = 0.
\]

Note that
\[
(2.18) \quad \frac{1}{4} F^2 + \varepsilon \left(\frac{1}{2} \frac{\partial F}{\partial x} - G\right) = x^2 + \varepsilon R(x,\varepsilon),
\]
where \( R \) is holomorphic in (2.13).

Remark: To find the domain \( \mathcal{D}_0 \), we must take into account not only singularities of \( f \) and \( g \), but also singularities of \( \psi \), i.e. the transformation (2.9). In particular, any zeros of \( f_0 \) would yield branch-points with respect to \( x \).
3. **Formal simplification:** It is known that there exist three formal power series in $\varepsilon$:

\begin{align*}
(3.1) \quad A(x,\varepsilon) &= \sum_{m=0}^{\infty} A_m(x) \varepsilon^m , \\
(3.2) \quad B(x,\varepsilon) &= \sum_{m=0}^{\infty} B_m(x) \varepsilon^m , \\
(3.3) \quad C(\varepsilon) &= \sum_{m=0}^{\infty} C_m \varepsilon^m
\end{align*}

such that

(i) $A_m(x)$ and $B_m(x)$ are holomorphic in the domain $D_0$;

(ii) $C_m$ are constants;

(iii) the formal transformation:

\begin{equation}
(3.4) \quad w = A(x,\varepsilon) u + B(x,\varepsilon) (\varepsilon du/dx)
\end{equation}

takes (2.17) to

\begin{equation}
(3.5) \quad \varepsilon^2 \frac{d^2u}{dx^2} - (x^2 + 2\varepsilon C) u = 0 ;
\end{equation}

(iv) we have

\begin{equation}
(3.6) \quad A_0(x)^2 - (xB_0(x))^2 = 1 \text{ identically in } D_0 .
\end{equation}

To effect the transformation (3.4), we differentiate both sides of (3.4) with respect to $x$. Then, we derive

\begin{equation}
(3.7) \quad \varepsilon dw/dx = (\varepsilon A' + (x^2 + \varepsilon C)B)u + (A + \varepsilon B')(\varepsilon du/dx) ,
\end{equation}

and

\begin{equation}
(3.8) \quad \varepsilon^2 d^2w/dx^2 = (\varepsilon (A' + (x^2 + \varepsilon C)B)' + (x^2 + \varepsilon C)(A + \varepsilon B'))u + \left( (\varepsilon A' + (x^2 + \varepsilon C)B) + \varepsilon (A + \varepsilon B')' \right) (\varepsilon du/dx) ,
\end{equation}

where \( \varepsilon \) denotes $\partial/\partial x$. Since $\varepsilon^2 d^2w/dx^2 = (x^2 + \varepsilon K)w$, we derive the following equations on $A$, $B$ and $C$.
\[
\begin{align*}
(x^2 + cR)A &= c(\epsilon A' + (x^2 + cC)B)' + (x^2 + cC)(A + \epsilon B')' \\
(x^2 + cR)B &= (\epsilon A' + (x^2 + cC)B) + \epsilon (A + \epsilon B')'.
\end{align*}
\]

In particular, if we put
\[X = xA_0, \quad Y = xB_0,
\]
we have
\[\frac{dX}{dx} = \frac{R_0(x) - C_0}{2x} Y, \quad \frac{dY}{dx} = \frac{R_0(x) - C_0}{2x} X,
\]
where \(R_0(x) = R(x,0)\). Hence
\[d(x^2 - y^2)/dx = \epsilon \text{ identically.}
\]

Choose \(C_0 = R_0(0)\) and the initial condition: \(X(0) = 1, Y(0) = 0\). Then, we can determine \(A_0, B_0\) and \(C_0\) so that (3.6) is satisfied. Other coefficients \(A_m, B_m\) and \(C_m\) can be determined in a similar way.

By virtue of (3.6), we can solve (3.4) and (3.7) with respect to \(u\) and \(\epsilon du/dx:\)
\[
\begin{align*}
\left\{ \begin{array}{l}
u = E_{11}(x, \epsilon)w + E_{12}(x, \epsilon)(\epsilon dw/dx), \\
\epsilon du/dx = E_{21}(x, \epsilon)w + E_{22}(x, \epsilon)(\epsilon dw/dx),
\end{array} \right.
\end{align*}
\]
where \(E_{jk}\) are formal power series in \(\epsilon\) whose coefficients are holomorphic in \(D_0\). In particular,
\[
\begin{align*}
E_{11}(x,0) &= E_{22}(x,0) = A_0(x), \\
E_{12}(x,0) &= -B_0(x), \quad E_{21}(x,0) = -x^2 B_0(x).
\end{align*}
\]

Note that
\[
C_0 = R_0(0) = -1 + 2 \frac{q(0,0)}{f_0'(0)}.
\]
4. **Outer expansions**: A formal power series in $r$:

$$(4.1) \quad v = \sum_{m=0}^{\infty} a_m(x) r^m$$

is called an outer expansion associated with the differential equation (2.10), if (4.1) formally satisfies (2.10). The power series (4.1) is an outer expansion if and only if

$$(4.2) \begin{cases} -2x \frac{d a_0}{dx} + G_0(x)a_0 = 0, \\ -2x \frac{d a_m}{dx} + G_0(x)a_m = L_m(x) - d^2a_{m-1}/dx^2 (m \geq 1), \end{cases}$$

where $G_0(x) = G(x,0)$ and $L_m(x)$ is linear homogeneous in $a_0, \ldots, a_{m-1}$ and $d a_0/dx, \ldots, d a_{m-1}/dx$ with coefficients holomorphic in $D_0$.

**DEFINITION 4.1**: The differential equation (2.10) is said to satisfy Matkowsky-condition, if there exists a non-trivial outer expansion (4.1) such that all $a_m(x)$ are bounded on the real interval $I_0$ (cf. (2.14)).

**LEMMA 4.2**: The differential equation (2.10) satisfies Matkowsky-condition if and only if $C_0$ is a negative odd integer and

$$(4.3) \quad C_m = 0 \quad (m \geq 1).$$

Proof: The transformation

$$(4.4) \quad u = y \exp\{-x^2/(2C)\}$$

changes (3.5) to

$$(4.5) \quad c d^2y/dx^2 - 2x dy/dx - (1 + C)y = 0.$$

By a straightforward computation, we can prove that the differential equation (4.5) satisfies Matkowsky-condition if and only if $C_0$ is a negative odd integer and $C_m = 0$ for $m \geq 1$.

Note also that, if all the $a_m$ are bounded, then all the $d a_m/dx$ are bounded. Otherwise, $d^2a_m/dx^2$ would have much worse singularities at $x = 0$, and hence $a_{m+1}$ would be unbounded (cf. (4.2)).
Finally, by manipulating with the transformations (2.16), (3.4) and (3.7), and (3.10) together with (4.4), we can show that the differential equation (2.10) satisfies Matkowsky-condition if and only if the differential equation (4.5) satisfies the same condition. This completes the proof of Lemma 4.2.
5. **Uniform simplification:** Hereafter, we shall assume that

\[(5.1)\quad C_0 = -p, \text{ where } p \text{ is a positive odd integer.}\]

\[(5.2)\quad C_m = 0 \text{ for } m \neq 1;\]

\[(5.3)\quad D_0 = \{x: |x| < r_0\} \text{ for some } r_0 > 0.\]

The assumption (5.3) means that \(D_0\) is a disk of radius \(r_0\) with the center at \(x = 0\).

Let us choose two positive numbers \(r_1\) and \(r\) such that

\[(5.4)\quad 0 < r_1 < r < r_0\]

and that the disk

\[(5.5)\quad D_1 = \{x: |x| < r_1\}\]

contains the real interval \(I_0\) (cf. (2.14)).

Let us denote by \(T(x, e)\) the two-by-two matrix:

\[(5.6)\quad \begin{bmatrix}
A(x, e) & B(x, e) \\
(cA'(x, e) + (x^2 - e)p)B(x, e) & A(x, e) + eB'(x, e)
\end{bmatrix}\]

(cf. (3.4) and (3.7)). Set

\[(5.7)\quad U = \begin{bmatrix} u \\ cdu/dx \end{bmatrix}, \quad W = \begin{bmatrix} w \\ cdw/dx \end{bmatrix}.\]

Then, the formal transformation

\[(5.8)\quad W = T(x,e) U\]

takes the system

\[(5.9)\quad \frac{dW}{dx} = \begin{bmatrix} 0 & 1 \\
 X^2 - R(x, 1) & 0 \end{bmatrix} W\]

\[(5.10)\quad \frac{dU}{dx} = \begin{bmatrix} 0 & 1 \\
 X^2 - p & 0 \end{bmatrix} U.\]

The inverse of the matrix \(T(x,e)\) is given by...
\[(5.11) \quad T(x, \varepsilon)^{-1} = \begin{bmatrix} E_{11}(x, \varepsilon) & E_{12}(x, \varepsilon) \\ E_{21}(x, \varepsilon) & E_{22}(x, \varepsilon) \end{bmatrix} \]

(cf. (3.10)).

Set

\[(5.12) \quad \mathcal{D}_2 = \{ x; |x| < r \} . \]

It is known that there exist two positive numbers \( a_1 \) and \( a_2 \), a function \( \delta(\varepsilon) \), and a two-by-two matrix \( P(x, \varepsilon) \) such that

(i) \( \delta(\varepsilon) \) is holomorphic in the sector

\[(5.13) \quad S = \{ \varepsilon; |\text{arg} \ \varepsilon| < a_1, \ 0 < |\varepsilon| < a_2 \} ; \]

(ii) \( \delta(\varepsilon) \) is asymptotically zero as \( \varepsilon \to 0 \) in \( S \), i.e.

\[(5.14) \quad |\delta(\varepsilon)| \leq K_N |\varepsilon|^N \quad (N = 0, 1, 2, \ldots) \text{ in } S \]

for some positive numbers \( K_N \); \n
(iii) entries of \( P \) and \( P^{-1} \) are holomorphic in the domain

\[(5.15) \quad x \in \mathcal{D}_2, \quad \varepsilon \in S ; \]

(iv) \( P \) (resp. \( P^{-1} \)) admits the matrix \( T \) (resp. \( T^{-1} \)) as an asymptotic expansion as \( \varepsilon \to 0 \) in \( S \) which is valid uniformly in \( \mathcal{D}_2 \);

(v) the transformation

\[(5.16) \quad W = P(x, \varepsilon)V \]

takes (5.9) to

\[(5.17) \quad \varepsilon dV/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon (p + \delta(\varepsilon)) & 0 \end{bmatrix} V \]

in the domain (5.15). (Cf. Y. Sibuya [6].)

Utilizing this result and manipulating with rotations of the disk \( \mathcal{D}_2 \), we can prove the following lemma:
Lemma 5.1: There exist sectors
\[
S_j = \{ \varepsilon; a_j < \arg \varepsilon < b_j, \quad 0 < |\varepsilon| < c_3 \} \quad (j = 1, 2, \ldots, k),
\]
where \( a_j \) is a positive number and the \( a \)'s and the \( b \)'s are real
functions \( \delta_1 (\varepsilon), \ldots, \delta_k (\varepsilon) \), and two-by-two matrices \( P_1 (x), \ldots, P_k (x) \)
such that \( S_1 \cup \ldots \cup S_k = \{ \varepsilon; 0 < |\varepsilon| < c_3 \} \) and that

(i) \( \delta_j (\varepsilon) \) is holomorphic in \( S_j \);

(ii) \( \delta_j (\varepsilon) \) is asymptotically zero as \( \varepsilon \to 0 \) in \( S_j \);

(iii) entries of \( \delta_j \) and \( P_j^{-1} \) are holomorphic in the domain
\[
\mathbb{D}_2, \quad \varepsilon \in S_j;
\]

(iv) \( P_j \) (resp. \( P_j^{-1} \)) admits the matrix \( T \) (resp. \( T^{-1} \)) as an
asymptotic expansion as \( \varepsilon \to 0 \) in \( S_j \) which is valid uniformly in \( \mathbb{D}_2 \);

(v) the transformation
\[
W = P_j (x, \varepsilon) V_j
\]
takes \( 5.9 \) to
\[
\frac{dV_j}{dx} = \begin{bmatrix}
0 & 1 \\
\frac{1}{x^2 - \varepsilon (p + \delta_j (\varepsilon))} & 0
\end{bmatrix} V_j
\]
in the domain \( 5.10-j \).
6. An estimate for \( \delta_j(\varepsilon) \). In this section, as an application of our main theorem (cf. Theorem 1.1), we shall derive an estimate

\[
|\delta_j(\varepsilon)| \leq H_j \exp(-r^2/|\varepsilon|) \quad \text{for } \varepsilon \in S_j,
\]

where \( H_j \) is a positive number. To do this, it is sufficient to prove that, if \( S_k \cap S_j \neq \emptyset \), we have

\[
|\delta_k(\varepsilon) - \delta_j(\varepsilon)| \leq M_{kj} \exp(-r^2/|\varepsilon|) \quad \text{for } \varepsilon \in S_k \cap S_j,
\]

where \( M_{kj} \) is a positive number. To derive an estimate (6.2), we need some preparation.

Let us consider the differential equation

\[
d^2z/dt^2 - (t^2 - a)z = 0, \quad \text{where } a \text{ is a parameter.}
\]

This equation admits a solution

\[
z = Z(t,a)
\]

such that

(i) \( Z \) is an entire function in \((t,a)\);

(ii) \[
\lim_{t \to \infty} t^{\frac{1}{2} (1-a)} e^{-\frac{t^2}{2}} Z(t,a) = 1
\]

uniformly in \( a \) if \( a \) is in a compact set in the \( a \)-plane.

The solution \( Z(t,a) \) is uniquely determined by (i) and (ii). The functions

\( Z((-i)t,-a), Z(-t,a), \) and \( Z(it,-a) \) are also solutions of (6.3). Set

\[
\psi_0(t,a) = \begin{bmatrix} Z(t,a) & Z((-i)t,-a) \\ Z'(t,a) & (-i)Z'((-i)t,-a) \end{bmatrix},
\]

\[
\psi_1(t,a) = \begin{bmatrix} Z((-i)t,-a) & Z(-t,a) \\ (-i)Z'((-i)t,-a) & -Z'(-t,a) \end{bmatrix},
\]

\[
\psi_2(t,a) = \begin{bmatrix} Z(-t,a) & Z(it,-a) \\ -Z'(-t,a) & iZ'(-t,a) \end{bmatrix}.
\]
and

$$
\psi_{-1}(t, a) = \begin{bmatrix} Z(t, a) & Z(t, a) \\ iZ'(t, a) & 2'(t, a) \end{bmatrix},
$$

where $'$ denotes $\frac{\partial}{\partial t}$. These four matrices are matrices of independent solutions of (6.3).

Set

$$
\psi_1(a) = e^{-ia} e^{-7ia(a+1)} \frac{\sqrt{2}}{\gamma(1-a)}, \quad \psi_2(a) = (-i)e^{ia-1},
$$

and

$$
C(a) = \begin{bmatrix} \psi_1(a) & 1 \\ \psi_2(a) & 0 \end{bmatrix}.
$$

Then,

$$
\begin{align*}
\psi_0(t, a) &= \psi_1(t, a)C(a), \\
\psi_1(t, a) &= \psi_2(t, a)C(-a), \\
\psi_2(t, a) &= \psi_1(t, a)C(-a), \\
\psi_1(t, a) &= \psi_0(t, a)C(-a).
\end{align*}
$$

Fix $i$ and $j$ so that $S_i \cap S_j \neq \emptyset$. Choose a branch of $\psi$ in the sector $S_i \setminus S_j$. Set

$$
\psi_i(e) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
$$

and

$$
\psi_i(h(x, e)) = \psi_i(e) \cdot \frac{\chi}{\gamma(e)}, \quad p + \ell_i(e),
$$

where $h = -1, 0, 1, 2$. Then, $\psi_i(h(x, e)) = \psi_i(e) \cdot \frac{\chi}{\gamma(e)}, \quad p + \ell_i(e)$, \quad (h = -1, 0, 1, 2).

Thus, $\psi_i(x, e)$, $\psi_{-1}(x, e)$, $\psi_{-2}(x, e)$ and $\psi_{-3}(x, e)$ (resp. $\psi_1(x, e)$, $\psi_{-2}(x, e)$ and $\psi_{-3}(x, e)$) are fundamental matrix solutions of (6.2) (resp. (5.2)-1) such that
\[
\begin{align*}
\phi_{j,0}(x,\varepsilon) &= \phi_{j,1}(x,\varepsilon)C(p + \delta_j(\varepsilon)), \\
\phi_{j,1}(x,\varepsilon) &= \phi_{j,2}(x,\varepsilon)C(-p - \delta_j(\varepsilon)), \\
\phi_{j,2}(x,\varepsilon) &= \phi_{j,-1}(x,\varepsilon)C(p + \delta_j(\varepsilon)), \\
\phi_{j,-1}(x,\varepsilon) &= \phi_{j,0}(x,\varepsilon)C(-p - \delta_j(\varepsilon)),
\end{align*}
\]

(6.11-i)

and

\[
\begin{align*}
\phi_{j,0}(x,\varepsilon) &= \phi_{j,1}(x,\varepsilon)C(p + \delta_j(\varepsilon)), \\
\phi_{j,1}(x,\varepsilon) &= \phi_{j,2}(x,\varepsilon)C(-p - \delta_j(\varepsilon)), \\
\phi_{j,2}(x,\varepsilon) &= \phi_{j,-1}(x,\varepsilon)C(p + \delta_j(\varepsilon)), \\
\phi_{j,-1}(x,\varepsilon) &= \phi_{j,0}(x,\varepsilon)C(-p - \delta_j(\varepsilon)).
\end{align*}
\]

(6.11-j)

Set

\[
J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

and

\[
\begin{align*}
Q_{j,h}(x,\varepsilon) &= \phi_{j,h}(x,\varepsilon)\exp\left(-1h \frac{x^2}{2\varepsilon} J^i\right), \\
Q_{j,h}(x,\varepsilon) &= \phi_{j,h}(x,\varepsilon)\exp\left(-1h \frac{x^2}{2\varepsilon} J^i\right), \\
Q_{j,h}(x,\varepsilon) &= \phi_{j,h}(x,\varepsilon)\exp\left(-1h \frac{x^2}{2\varepsilon} J^i\right),
\end{align*}
\]

(6.13)

It is known that, if \((x,\varepsilon) \in \mathcal{D}\) in a domain

\[
\begin{align*}
x \in \mathcal{D}, \quad \varepsilon \in \mathcal{S}, \quad \left|\arg\left(\frac{x}{\varepsilon}\right) - \frac{1}{4} \pi - \frac{1}{2} h \pi\right| < \frac{1}{2} \pi - \nu,
\end{align*}
\]

where \(\nu\) is a small positive number, we have

\[
\|Q_{j,h}(x,\varepsilon)\| \leq H\varepsilon^q, \quad \|Q_{j,h}(x,\varepsilon)^{-1}\| \leq H\varepsilon^q,
\]

where \(H\) is a positive number depending on \(\nu, q\) is a real number, and \(\|\|\) denotes a usual norm of matrices. Furthermore, the matrix

\[
\begin{align*}
\phi_{j,h}(x,\varepsilon) &= \phi_{j,0}(x,\varepsilon) - \phi_{j,1}(x,\varepsilon)
\]

is asymptotically zero as \(\varepsilon \to 0\) in \(\mathcal{S} \cap \mathcal{S}_j\) uniformly in the domain

(6.14-h). (For those results, see, for example, Y. Sibuya [6,7].)
Let \( P_k(x,\varepsilon) \) and \( P_j(x,\varepsilon) \) be the matrices given in Lemma 5.1. Then, \( P_k(x,\varepsilon) \delta_{j,0}(x,\varepsilon) \) and \( P_j(x,\varepsilon) \delta_{j,0}(x,\varepsilon) \) are two fundamental matrix solutions of (5.9) in the domain \( x \neq D_2, \quad \varepsilon \in S_\downarrow \cap S_\uparrow \).

Moreover, there exists a two-by-two matrix \( L(\varepsilon) \) such that

\[
P_k(x,\varepsilon) \delta_{j,0}(x,\varepsilon) = P_j(x,\varepsilon) \delta_{j,0}(x,\varepsilon) L(\varepsilon) .
\]

Note that \( L(\varepsilon) \) does not depend on \( x \). It follows from (6.11) that

\[
\exp\left( -\frac{x^2}{2\varepsilon} J \right) L(\varepsilon) \exp\left( \frac{x^2}{2\varepsilon} J \right) = L(\varepsilon). 
\]

Hence, the matrix

\[
\exp\left( -\frac{x^2}{2\varepsilon} J \right) L(\varepsilon) \exp\left( \frac{x^2}{2\varepsilon} J \right) - I_2
\]

is asymptotically zero as \( \varepsilon \to 0 \) in \( S_\downarrow \cap S_\uparrow \) uniformly in the domain (6.11-0), where \( I_2 \) is the two-by-two identity matrix.

In the same way (manipulating with the connection formulas (6.11-1) and (6.11-3)), we can prove that the matrix

\[
\exp\left( \frac{x^2}{2\varepsilon} J \right) L_1(\varepsilon) \exp\left( -\frac{x^2}{2\varepsilon} J \right) - I_2
\]

is asymptotically zero as \( \varepsilon \to 0 \) in \( S_\downarrow \cap S_\uparrow \) uniformly in the domain (6.11-1), where

\[
L_1(\varepsilon) = C(\varepsilon + \delta_j(\varepsilon)) L(\varepsilon) C(\varepsilon + \delta_j(\varepsilon))^{-1} .
\]

Similarly, the matrix

\[
\exp\left( \frac{x^2}{2\varepsilon} J \right) L_2(\varepsilon) \exp\left( -\frac{x^2}{2\varepsilon} J \right) - I_2
\]

is asymptotically zero as \( \varepsilon \to 0 \) in \( S_\downarrow \cap S_\uparrow \) uniformly in the domain (6.11-2), where

\[
L_2(\varepsilon) = C(-\varepsilon - \delta_j(\varepsilon)) L(\varepsilon) C(-\varepsilon - \delta_j(\varepsilon))^{-1} .
\]
Set
\[
L(\epsilon) = \begin{bmatrix}
  c_{11}(\epsilon) & c_{12}(\epsilon) \\
  c_{21}(\epsilon) & c_{22}(\epsilon)
\end{bmatrix},
\]
\[
L_1(\epsilon) = \begin{bmatrix}
  \hat{c}_{11}(\epsilon) & \hat{c}_{12}(\epsilon) \\
  \hat{c}_{21}(\epsilon) & \hat{c}_{22}(\epsilon)
\end{bmatrix},
\]
\[
L_2(\epsilon) = \begin{bmatrix}
  \hat{c}_{11}(\epsilon) & \hat{c}_{12}(\epsilon) \\
  \hat{c}_{21}(\epsilon) & \hat{c}_{22}(\epsilon)
\end{bmatrix}.
\]
(6.25)

Then,
\[
\hat{c}_{12}(\epsilon) = \left\{ \lambda_1 (p + \delta_j(\epsilon)) c_{11}(\epsilon) + \frac{c_{21}(\epsilon)}{\lambda_2 (p + \delta_j(\epsilon))} \right\}
\]
\[
- \lambda_1 (p + \delta_j(\epsilon)) \left\{ \lambda_1 (p + \delta_j(\epsilon)) c_{12}(\epsilon) + \frac{c_{22}(\epsilon)}{\lambda_2 (p + \delta_j(\epsilon))} \right\};
\]
and
\[
\hat{c}_{21}(\epsilon) = \lambda_1 (-p + \delta_j(\epsilon)) c_{11}(\epsilon) + \lambda_2 (-p + \delta_j(\epsilon)) c_{12}(\epsilon)
\]
\[
- \frac{\lambda_1 (-p + \delta_j(\epsilon))}{\lambda_2 (-p + \delta_j(\epsilon))} \left\{ \lambda_1 (-p + \delta_j(\epsilon)) c_{21}(\epsilon) + \lambda_2 (-p + \delta_j(\epsilon)) c_{22}(\epsilon) \right\}.
\]
(6.26)

Utilizing the fact that, for any \( \epsilon \in S \cap S \), there exists an \( x \in D \), such that

(i) \((x, r)\) is in the domain (6.14-h),

(ii) \(x/r^i\) takes either a real value or a purely imaginary value,

we derive from (6.20), (6.21) and (6.23) that

(1) \(c_1(\epsilon) - 1\) and \(c_2(\epsilon) - 1\) are asymptotically zero as \( r \to 0 \) in \( S \cap S \);

(2) \(|c_{12}(\epsilon)| \leq c \exp(-2r^2/|r|)\), \(|c_{21}(\epsilon)| \leq c \exp(-2r^2/|r|)\)

for \( \epsilon \in S \cap S \), where \( c \) is a positive constant;

(3) \(|\hat{c}_{12}(\epsilon)| \leq c \exp(-2r^2/|r|)\) for \( \epsilon \in S \cap S \);

(4) \(|\hat{c}_{21}(\epsilon)| \leq c \exp(-2r^2/|r|)\) for \( \epsilon \in S \cap S \).
Set \( \lambda(a) = \frac{\lambda_1(a)}{\lambda_2(a)} \). Then,

\[
|(-p - \delta_j(c))c_{11}(c) - (p - \delta_j(c))c_{22}(c)| \\
\leq |c_{11}(c) - c_{22}(c)| + |(-p - \delta_j(c)) - \mu(-p - \delta_j(c))|c_{22}(c)| \\
\leq c_2 \exp(-r^2/|\epsilon|) \text{ in } S_i \cap S_j
\]

for some \( c_2 > 0 \). Since \( \mu(-p) \neq 0 \), we have

\[
(6.28) \quad c_{11}(c) - c_{22}(c) \leq c_1|\delta_j(c) - \delta_j(c)| + c_2 \exp(-r^2/|\epsilon|)
\]

in \( S_i \cap S_j \) for some \( c_1 > 0 \) and \( c_2 > 0 \). On the other hand,

\[
[(-p + \delta_j(c))c_{11}(c) - (p + \delta_j(c))c_{22}(c)] \\
\leq c \exp(-r^2/|\epsilon|) \text{ in } S_i \cap S_j
\]

for some \( c > 0 \). Since \( \lambda_1(p) = 0 \) and \( \frac{d\lambda_1}{da}(p) \neq 0 \), we have

\[
(6.29) \quad |\delta_j(c) - \delta_j(c)| = c_3|\lambda_1(p + \delta_j(c))|c_{11}(c) - c_{22}(c)| \\
+ c_4 \exp(-r^2/|\epsilon|) \text{ in } S_i \cap S_j
\]

for some \( c_3 > 0 \) and \( c_4 > 0 \). An estimate (6.2) follows from (6.28) and (6.29).
7. Resonance: In this section, we shall prove Theorem 1.2. To do this, we return to Section 5. We proved there that the transformation (5.16) takes the system (5.9) to (5.17) in the domain (5.15). The function \( \delta(\varepsilon) \) satisfies the condition (5.14). We replace (5.14) by

\[
|\delta(\varepsilon)| \leq H \exp(-r^2/|\varepsilon|) \quad \text{in } S
\]

for some positive number \( H \).

Set

\[
\begin{aligned}
\phi_h(x,\varepsilon) &= \Lambda(\varepsilon)\psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta(\varepsilon)), \\
\tilde{\phi}_h(x,\varepsilon) &= \Lambda(\varepsilon)\psi_h(x/\varepsilon^{\frac{1}{2}}, p), \\
\quad (h = -1, 0, 1, 2).
\end{aligned}
\]

Then, \( \phi_h(x,\varepsilon) \) (resp. \( \tilde{\phi}_h(x,\varepsilon) \)) are fundamental matrix solutions of (5.17) (resp. (5.10)) such that

\[
\begin{aligned}
\phi_0(x,\varepsilon) &= \phi_1(x,\varepsilon)C(p + \delta(\varepsilon)), \\
\phi_1(x,\varepsilon) &= \phi_2(x,\varepsilon)C(-p - \delta(\varepsilon)), \\
\phi_2(x,\varepsilon) &= \phi_{-1}(x,\varepsilon)C(p + \delta(\varepsilon)), \\
\phi_{-1}(x,\varepsilon) &= \phi_0(x,\varepsilon)C(-p - \delta(\varepsilon)),
\end{aligned}
\]

and

\[
\begin{aligned}
\tilde{\phi}_0(x,\varepsilon) &= \tilde{\phi}_1(x,\varepsilon)C(p), \\
\tilde{\phi}_1(x,\varepsilon) &= \tilde{\phi}_2(x,\varepsilon)C(-p), \\
\tilde{\phi}_2(x,\varepsilon) &= \tilde{\phi}_{-1}(x,\varepsilon)C(p), \\
\tilde{\phi}_{-1}(x,\varepsilon) &= \tilde{\phi}_0(x,\varepsilon)C(-p).
\end{aligned}
\]

Set

\[
S(x,\varepsilon) = \phi_0(x,\varepsilon)\tilde{\phi}_0(x,\varepsilon)^{-1}.
\]

Then, the transformation

\[
V = S(x,\varepsilon)U
\]

takes (5.17) to (5.10). Hence, the main part of the proof is to show that

\( S(x,\varepsilon) - I_2 \) is asymptotically zero as \( \varepsilon \to 0 \) in \( S \) uniformly in \( D_1 \). Note
that \( r \to r \). To do this we manipulate in a way similar to the argument in Section 6, utilizing the fact that

(i) \( a_n \exp(x^2/n) \) and \( a_n \exp(-x^2/n) \) are asymptotically zero as

(ii) \( C_1: r \to 0 \) uniformly in \( D_1 \);

(iii) \( C_2: r \to 0 \) \( C_2^{-1} - 1_2 = o(\delta(e)) \);

(iv) \( C_3: r \to 0 \) \( C_3^{-1} - 1_2 = o(\delta(e)) \).

The details are left to the reader.
8. Proof of Theorem 1.1: We shall prove Theorem 1.1 in the case when \( v = 3 \). The general case can be treated in the same manner. We shall consider three sectors \( S_1, S_2, S_3 \) as shown in Figure 1.

![Figure 1](image_url)

We denote by \( S_{1,2}, S_{2,3}, S_{3,1}, S_{1} \cap S_{2}, S_{2} \cap S_{3}, S_{3} \cap S_{1} \), respectively.

The three functions \( \delta_1(c), \delta_2(c), \delta_3(c) \) are holomorphic in \( S_1, S_2, S_3 \), respectively. Furthermore,

\[
\delta_j(c) \text{ is asymptotically zero as } c \to 0 \text{ in } S_j,
\]

and

\[
|\delta_j+1(c) - \delta_j(c)| \leq c_0 \exp(-c_1/|c|^\lambda) \text{ in } S_j, j+1,
\]

where \( c_0, c_1, \lambda \) are positive numbers, and \( S_{3,4} = S_{3,2}, \delta_4 = \delta_1 \). We shall denote \( \delta_{j+1}(c) - \delta_j(c) \) by \( \sigma_j(c) \).

We consider a sufficiently small disk:

\[
D = \{ c : |c| \leq \rho_0 \}.
\]

We choose three line-segments \( \ell_1, \ell_2, \ell_3 \) starting from \( c = 0 \) in such a way that

\[
\ell_j \subset S_{j, j+1} \quad \text{(Cf. Figure 2.)}
\]
Three line-segments $\ell_1', \ell_2', \ell_3'$ divide the disk $D$ (cf. (8.3)) into three open sectors $\hat{S}_1', \hat{S}_2', \hat{S}_3'$ (cf. Figure 2). The boundaries of $\hat{S}_1', \hat{S}_2', \hat{S}_3'$ are respectively

$$\ell_{j-1} + \gamma_j - \ell_j', \quad j = 1, 2, 3,$$

where $\ell_0' = \ell_3'$, and the $\gamma$'s are circular arcs such that

$$\gamma_1 + \gamma_2 + \gamma_3 = \mathcal{C} = \{ \varepsilon; |\varepsilon| = \rho_0 \}.$$

The line-segments $\ell_j'$ and the circular arcs $\gamma_j$ are oriented as indicated in Figure 2. We assume that $\rho_0$ is so small that

$$\hat{S}_j \subset S_j,$$

where $\hat{S}_j$ denotes the closure of $S_j$.

Set, for $\varepsilon \in \hat{S}_1' \cup \hat{S}_2' \cup \hat{S}_3'$,

$$\delta(\varepsilon) = \delta_j(\varepsilon) \text{ if } \varepsilon \in \hat{S}_j.$$

Since

$$\frac{1}{2\pi i} \int_{j-1}^{j} \frac{\delta_j(\varepsilon)}{\varepsilon - r} \, d\varepsilon = \begin{cases} \delta_j(r) & r \in \hat{S}_j; \\ 0 & \text{otherwise} \end{cases},$$

we have

Figure 2
\[ \delta(\epsilon) = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\xi_{j-1} + \gamma_{j} - \xi_{j}}^{\xi_{j} - \epsilon} \frac{\delta_{j}(\epsilon)}{\xi - \epsilon} \, d\xi \] in \( \hat{S}_{1} \cup \hat{S}_{2} \cup \hat{S}_{3} \).

Utilizing

\[ \frac{1}{\xi - \epsilon} = \sum_{m=0}^{N} \xi^{-(m+1)} \epsilon_{m} + \frac{\epsilon^{N+1}}{\xi^{N+1}(\xi - \epsilon)}, \]

we derive

\[ \delta(\epsilon) = \frac{1}{2\pi i} \sum_{m=0}^{N} \left\{ \sum_{j=1}^{3} \int_{\xi_{j-1} + \gamma_{j} - \xi_{j}}^{\xi_{j} - \epsilon} \xi^{-(m+1)} \delta_{j}(\epsilon) \, d\xi \right\} \epsilon_{m} \]

\[ + \left\{ \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\xi_{j-1} + \gamma_{j} - \xi_{j}}^{\xi_{j} - \epsilon} \frac{\delta_{j}(\xi)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} \epsilon^{N+1}. \]

Since \( \delta(\epsilon) \) is asymptotically zero as \( \epsilon \to 0 \) in \( \hat{S}_{1} \cup \hat{S}_{2} \cup \hat{S}_{3} \), the first term must be zero, and hence

\[ \delta(\epsilon) = \left\{ \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\xi_{j-1} + \gamma_{j} - \xi_{j}}^{\xi_{j} - \epsilon} \frac{\delta_{j}(\epsilon)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} \epsilon^{N+1}. \]

Thus we arrive at the following formula:

\[ (8.9) \quad \delta(\epsilon) = \frac{1}{2\pi i} \left\{ \sum_{j=1}^{3} \int_{\xi_{j-1} + \gamma_{j} - \xi_{j}}^{\xi_{j} - \epsilon} \frac{\delta_{j}(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi + \int_{C} \frac{\delta(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi \right\} \epsilon^{N} \]

for \( \epsilon \in \hat{S}_{1} \cup \hat{S}_{2} \cup \hat{S}_{3} \) and \( N = 1, 2, 3, \ldots \), where \( \delta_{j} = \delta_{j+1} - \delta_{j} \).

Construct three open sectors \( \hat{S}_{1}', \hat{S}_{2}', \hat{S}_{3}' \) as shown in Figure 3, where \( 0 < \rho_{1} < \rho_{0} \) and \( \theta \) is a small positive number. Then,

\[ \left| \int_{C} \frac{\delta(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi \right| \leq \frac{C_{0}}{\epsilon^{N-1}} \frac{1}{\rho_{0} - \rho_{1}} \]

and

\[ -23 - \]
Figure 3

\[ \left| \int_{S_j} \frac{e_j(\xi)}{r_j^N(\xi - \epsilon)} d\xi \right| \leq \frac{C_0}{\sin \theta} \int_0^{\rho_0} t^{-N-1} \exp(-c_1 t^{-\lambda}) dt \]

\[ \leq \frac{C_0}{\lambda \sin \theta} \int_0^{\infty} r^{(N/\lambda) - 1} \exp(-c_1 r) dr \]

\[ = \frac{C_0}{\lambda \sin \theta} c_1^{-(N/\lambda)} \Gamma(N/\lambda) \]

for \( \epsilon \in \xi_1 \cup \xi_2 \cup \xi_3 \), where \( C_0 \) is a positive number. Since

\[ \Gamma(N/\lambda) \leq C_1 (N/\lambda)^{(N/\lambda)} e^{-(N/\lambda)} \]

for some \( C_1 > 0 \), we have

\[ |\delta(\epsilon)| \leq C_2 \left( \frac{|\xi|}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} \]

for \( \epsilon \in \xi_1 \cup \xi_2 \cup \xi_3 \); \( C_1 \) is a positive number. For a given \( \epsilon \), choose \( N \) so that

\[ \frac{N}{\lambda} < \frac{1}{|\epsilon|} \leq \frac{N + 1}{\lambda} \]

Then, it follows from (8.10) that

\[ |\delta(\epsilon)| \leq C_2 e^{c_1 |\epsilon|} \exp(-c_1/|\epsilon|)^\lambda \]

choosing \( 1', 2', 3' \) in various ways, we can complete the proof of Theorem 1.1.
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Title: A Theorem Concerning Uniform Simplification at a Transition Point and the Problem of Resonance

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Abstract: Given sectors $S_j = \{ \varepsilon; a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho \}$ ($1 \leq j \leq v$) and functions $\delta_j$ ($1 \leq j \leq v$) such that (i) $\bigcup_j S_j = \{ \varepsilon; 0 < |\varepsilon| < \rho \}$, (ii) $\delta_j$ is holomorphic in $S_j$, (iii) $\delta_j$ is asymptotically zero as $\varepsilon \to 0$ in $S_j$, (iv) $|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp(-c_1|\varepsilon|^3)$ in $S_j \cap S_k$ for some positive numbers $c_0$ and $c_1$.
20. ABSTRACT - Cont'd.

c_0, c_1 and λ whenever S_j ∩ S_k ≠ φ, we prove that |δ_j(ε)| ≤ c_2 exp(-c_1/|ε|^2) in S_j for some positive number c_2. Then, utilizing this result, we prove that Matkowsky-condition implies the resonance in the sense of N. Kopell under a reasonable assumption. The sufficiency of Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been an open question. This work gives an affirmative answer to this question in a reasonably general case.