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HARMONIZABLE PROCESSES: STRUCTURE.(U)

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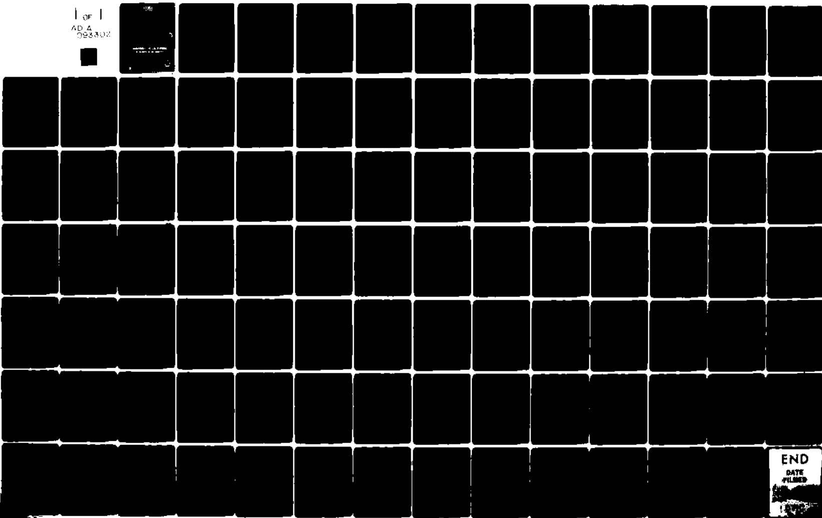

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Harmonizable Processes: Structure

by

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Key words and Phrases: Weakly and strongly harmonizable process, V-boundedness, stationary dilations, domination of vector measures, DS- and MT-integrals, bi measures, filtering, weak and strong solutions, classes (KF) and (C), multidimensional processes, p-absolutely summing operators, associated spectra of processes.

1. Introduction. Recently there have been significant attempts for extending the well-understood theory of stationary processes to classes of nonstationary ones by many writers. These are motivated by real applications. Some of these efforts have been illustrated and analyzed in [31]. A class which has a superficial structural similarity is the harmonizable family. This class was originally introduced by Loève (cf.[21]). A closely related but a more general concept is due to Bochner [2] for essentially the same purpose. Slightly later, Rozanov [34] has also defined a concept, also called "harmonizable", which is weaker than that of Loève's. Each of these notions is inspired by the stationarity of Khintchine's (also termed wide sense or weakly), but each is different from one another. For a systematic study of these classes, it is necessary to determine their interrelations. One of the main purposes of this paper is to present a detailed and unified structural analysis of these processes and obtain characterizations of the respective classes. This involves a free use of some elementary aspects of vector measure theory; and it already raises some interesting problems to be resolved. One finds that Loève's definition is more restrictuve than Rozanov's, and that Bochner's concept is mathematically the most elegant and general. Further in the Hilbert space context, it is shown that Bochner's and Rozanov's concepts coincide. An interesting geometrical feature is that the Bochner class is always a projection of a stationary family. Also Bochner's

concept is based on Fourier vector integration and this identification yields different characterizations, one of which extends a result of Helson's [12] to certain Banach spaces. Further, in many cases, it results that a process of Bochner-Rozanov class is a 'pointwise' limit of a sequence of harmonizable processes in the sense of Loève. A brief account of various contributions, for comparison, is appropriate at this point.

Soon after introduction of the harmonizability concept by Loève in the late 1940's, an abstract generalization of it was considered by Cramér [3]. His is a very general notion, but it only has a superficial contact with Fourier analysis. In a key special case, the first step relating the stationary and Loève harmonizabile concepts was taken by Abreu [1]; and it stimulated much later work. On the other hand in the middle 1950's Kampé de Fériet and Frenkiel (cf. [15], [16]), and independently Parzen and Rozanov, have considered processes which are generalizations of stationary as well as harmonizable types, and which are usually different from those of Cramér noted above. However, they retain some contact with Fourier analysis, and are sometimes referred to as "asymptotically stationary". A detailed analysis of this work and some generalizations are discussed in [31] where further references can be found. The decisive step in extending the stationarity concept is that of Bochner's as the following work will demonstrate. Apparently unaware of this work in [2], Rozanov [34] has initiated an abstract study of spectra for a class labelled "harmonizable" which with a careful study turns out to coincide with Bochner's class for second order processes, and strictly includes that of Loève's. But a systematic study of Bochner and Loève classes in Hilbert space was given only by Niemi in his thesis [26] who showed that they are different types of Fourier transformations of certain Hilbert space valued measures. He also established essentially that the Bochner class in Hilbert space is the projection of a stationary family in [27] and [28]. The latter point was clarified and the same result was obtained by slightly different methods in [22]. Recently an extension of the last work was announced in [33]. I have presented most of the material in Sections 2 - 6 in my graduate seminar lectures in the academic year 1979-80. The following account is a refined version of it.

The key domination inequality, on which the projection or "dilation" results noted above depend, itself is based on an aspect of some results of A. Grothendieck's. The methods of [22], [27] and [28] rest on A. Pietech's rendition of Grothendieck's work, whereas in what follows this is based on some properties of the theory of p-summing operators from [20]. I believe that this yields a better understanding of the problem with additional insight, not afforded by the earlier work. Thus the present paper is devoted to a comprehensive, unified, and extended treatment of the structure of the classes of Bochner and Rozanov. It is of some interest to note that an essentially equivalent characterization of Bochner's Hilbert space version could be obtained from the early paper of Phillips' [29], which seems to have been overlooked by almost all the vector measure theorists and stochastic analysists. It is, in a sense, subsumed under a relatively recent work of Kluvanek's [19]. The relevance of [29] will be noted at appropriate places below. But most of all, Bochner's paper [2] has not been accorded the central place it deserves in the probabilistic treatments on the subject. I hope that the present work will bring the many key ideas of [2] to the forefront.

The structural analysis here is thus developed mostly for scalar processes, but including random fields and some multivariate indications in the last section, in order to lay out a basis for later research on prediction and filtering problems on them. Some of these applications were indicated in [2] and detailed analysis on filtering for Loève type (to be called <u>strong</u> hereafter) harmonizable processes was recently completed in [18] for "polynomial filters". This has to be extended for the Bochner-Rozanov type (termed <u>weak</u> below) harmonizable processes. In this latter work it turns out that the theory of bimeasures and the (non-absolute) integration of Morse and Transue ([23], [24]) will take the center stage. This difference has not been fully appreciated in the literature. (The most comprehensive and precise characterizations are summarized in Theorems 7.3 and 7.4.) For vector valued processes in both (weak and strong) cases, some new technical problems have to be resolved. The same is true of random fields on general locally compact groups. All these aspects are important in applications. Except for some indications in Sections 7 and 8, no specialized applications are detailed at the present time. For accessibility and convenience, the next three sections treat harmonizable processes, and the remaining five consider the more general random fields, with a natural transition. However, an essentially self-contained exposition (modulo some standard measure theory) is presented here.

Let us now introduce the terminology and present precise and analytical details of the preceding discription. Throughout the paper the following notation is used: $\mathbb{R}$ for reals, $\mathbb{C}$ for complex numbers, $\mathbb{Z}$ for integers, $\mathbb{R}^n$ the n-dimensional number space, LCA for locally compact abelian, and E for expectation. Also a <u>step</u> function is a mapping taking finitely many values on disjoint measurable sets and a <u>simple</u> function on a measure space is a step function vanishing outside of a set of finite measure. Usually over bar denotes complex conjugation. Other symbols and terms are explained as they occur.

2. Harmonizability and stationarity. Let (Ω,Σ,P) be a probability space and $L^2(\Omega,\Sigma,P)$, or $L^2(P)$ for short, be the space of (equivalence classes of scalar square integrable functions (= random variables) on Ω . Let $L_0^2(P)$ be the subspace of elements of $L^2(P)$ having zero mean values, for convenience. Then $L_0^2(P)$ (as well as $L^2(P)$) is a Hilbert space under the usual inner product $(f,g) = \int_\Omega f\bar{g}dP$, and a mapping $X:\mathbb{R} \to L_0^2(P)$ is called a stationary time series or process (in the wide or Khintchine sense, and this qualification will be omitted below), if for any s,t in $\mathbb{R}$, the covariance $r(s,t)$ of $X(s)$, $X(t)$ depends only on the difference $s - t$. Thus $r(s,t) = r(s-t)$ where

$$r(s,t) = E(X(s)\overline{X(t)}) = \int_\Omega X(s)\overline{X(t)}\, dP = (X(s),X(t)),\ s,t\in\mathbb{R}. \quad (1)$$

The following analysis is valid in an abstract Hilbert space $\mathfrak{X}$ if the "covariance" is interpreted as its inner product, without reference to an underlying probability space. However, this is not really a generalization since any Hilbert space is isomorphic (and isometric) to an $L^2(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$ on some probability space $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$, (cf., e.g., [32], p. 414). Thus in the following $L_0^2(P)$, or an abstract space $\mathfrak{X}$, may (and will) be considered according to convenience.

Observing that $r(s,t)$ is of positive type (= positive [semi-] definite), assume that $r(\cdot,\cdot)$ is jointly measurable which is implied by the measurability of the random function $\{X(t), t \in \mathbb{R}\}$. In the stationary case by Bochner's classical theorem, there exists a bounded, unique, nondecreasing function

$F:\mathbb{R} \to \mathbb{R}^+$ such that

$$r(h) = \int_{\mathbb{R}} e^{ith}\, dF(t) \ , \ a \cdot a \cdot h \in \mathbb{R} \ , \ (\text{Lebegue}) \ , \qquad (2)$$

and conversely every such F defines by (2), a measurable (even uniformly continuous) covariance r. Then by the classical Kolmogorov existence theorem (cf. e.g., [6], p. 608 ff; [32], Ch. I for this theorem and extensions), one deduces the existence of a probability space (Ω,Σ,P) and a stationary process on it with $r(\cdot)$ as its covariance. It may be remarked that in (2), in the original (1932) version Bochner assumed that $r(\cdot)$ is continuous, but soon after in (1933) F. Riesz showed that measurability itself yields the form (2). This general case was also used in [29].

The function F of (2) is called the <u>spectral</u> <u>distribution</u> and the Baire measure μ it generates (by $\mu:A \mapsto \int_A dF$) is its <u>spectral measure</u>. One verifies that $\{X(t), t \in \mathbb{R}\}$ is mean continuous (i.e., $E(|X(s) - X(t)|^2) \to 0$ as $s \to t$), iff (= if and only if) its covariance $r(\cdot,\cdot)$ is continuous on the diagonal of $\mathbb{R} \times \mathbb{R}$ (cf., e.g., [21], p. 470). Thus the stationarity is such a restriction that its measurability and validity of (2) everywhere implies the mean continuity of the process! So for some applications, it is natural to weaken the hypothesis of stationarity, retaining some representative features. This has been done by Loève under the name "harmonizable". For the reasons noted in the introduction, it will be called strongly harmonizable. This is stated as:

<u>Definition 2.1</u>. A process $X:\mathbb{R} \to L_0^2(P)$ with covariance

$r(s,t) = E(X(s)\overline{X(t)})$, is <u>strongly harmonizable</u> if r is the Fourier transform of some covariance function $\rho: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of bounded variation, so that one has,

$$r(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'} \rho(d\lambda, d\lambda') , \quad s,t \in \mathbb{R} . \tag{3}$$

Evidently if ρ concentrates on the diagonal of $\mathbb{R} \times \mathbb{R}$, then (3) reduces to (2). Note also that r is bounded and uniformly continuous. Eventhough (3) is a natural generalization of (2), one does not have an elegant characterization of an harmonizable covariance. In fact Loève has raised this problem ([21], p. 477). A solution of it was presented in ([30], Thm. 5). It is not effective in that the conditions are not easily verifiable, though the characterization does reduce to Bochner's theorem in the stationary case.

The preceding comment shows that the concept of strong harmonizability, though an apparently natural generalization of stationarity, does not have an efficient procedure which enables its early recognition. There is however another <u>real</u> drawback. Since strong harmonizability is derived from stationarity (so that every stationary process is harmonizable), consider a 'partial' series $\{\tilde{X}(n), n \in \mathbb{Z}\}$ of a stationary series $\{X(n), n \in \mathbb{Z}\}$ so that $r(m-n) = E(X(m)\overline{X(n)})$, $X(n) \in L_0^2(P)$, where $\tilde{X}(n) = X(n)$ for finitely many $n \in \mathbb{Z}$, and $= 0$ for all other $n \in \mathbb{Z}$. Then it is clear that $\{\tilde{X}(n), n \in \mathbb{Z}\}$ is strongly harmonizable. However if $\tilde{X}(n) = X(n)$

for infinitely many n , and $= 0$ for all other n , then $\{\tilde{X}(n), n \in \mathbb{Z}\}$ is <u>not</u> necessarily strongly harmonizable, as the following simple counterexample illustrates: Let $\{f_n, n \in \mathbb{Z}\} \subset L_0^2(P)$ be a complete orthonormal set (assuming P to be a separable measure). Then $r(m,n) = \delta_{m-n} = r(m-n)$. So $\{f_n, n \in \mathbb{Z}\}$ is trivially stationary, and

$$r(m-n) = \int_{-\pi}^{\pi} e^{i(m-n)\lambda} \frac{d\lambda}{2\pi}, \quad m,n \in \mathbb{Z} .$$

Let $\tilde{f}_n = f_n$, $n > 0$, $= 0$ *for* $n \le 0$. Then $\tilde{r}(m,n) = E(\tilde{f}_m \bar{\tilde{f}}_n) = 1$, if $m = -n > 0$, $= 0$ otherwise. But $\tilde{r}$ does not admit the representation (3) for a covariance ρ of the desired kind. Indeed, if (3) is true for some such ρ , then $\tilde{r}(m,n)$ will be its Fourier coefficient such that $\tilde{r}(m,n)$ is only non vanishing *on a* ray $(m = -n > 0)$. It is a consequence of an important two dimensional extension by Bochner of the classical F. and M. Riesz theorem that ρ must then be absolutely continuous relative to the planer Lebesgue measure with density ρ' . Hence

$$\tilde{r}(m,n) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(mx+ny)} \rho'(x,y)\, dxdy .$$

But this implies $\tilde{r}(m,n) \to 0$ as $|m|+|n| \to \infty$ by the Riemann-Lebesgue lemma, and contradicts the fact that $\tilde{r}(m,-n) = 1$ as $m = -n \to \infty$. Hence $\tilde{r}$ can not admit the representation (3) so that $\{\tilde{f}_n, n \in \mathbb{Z}\}$ is <u>not</u> strongly harmonizable. This example is a slight modification of one due to Helson and Lowdenslager ([13], p. 183) who considered it for a similar purpose, and is given in [1] for a related elucidation.

The preceding example and discussion motivate us to look for a weakening of the conditions on the covariance function leading to the representation (3) since it is natural to expect each subset of a stationary series to be included in the generalization, retaining the other properties as far as possible. Such an extension was successfully obtained in two different forms in the works of Bochner [2] and Rozanov [34]. The precise concept can be clearly stated only after some detailed preliminaries.

The measure function ρ of (3) has the following properties:

(i) ρ is positive definite, i.e.

$$\rho(s,t) = \overline{\rho(t,s)}\ , \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j\ \rho(s_i,s_j) \geq 0\ ,\ a_i \in \mathbb{C}, \quad (4)$$

(ii) ρ is of bounded variation, i.e.,

$$\sup \{ \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{A_i} \int_{B_j} |\rho(ds,dt)| : A_i, B_j \in \mathcal{B},\ \text{disjoint}\} < \infty, \quad (5)$$

where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. If $F:\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ is defined by $F(A,B) = \int_A \int_B \rho(ds,dt)$, it follows from (4) and (5) that there exists a complex Radon measure μ on $\mathbb{R}^2$ such that $F(A,B) = \mu(A \otimes B)$, where $A \otimes B \in \mathcal{B} \otimes \mathcal{B}$, and μ is positive definite. On the other hand, the defining equation of F implies that F is positive definite (so (4) holds with $\rho(s_i,s_j)$ replaced by $F(A_i,A_j)$) and (5) becomes

$$V(F) = \sup \{ \sum_{i=1}^{n} \sum_{j=1}^{n} |F(A_i,B_j)| : A_i,B_j \in \mathcal{B}\ ,\ \text{disjoint} \} < \infty. \quad (5')$$

But (3) is meaningful, if ρ is replaced by F under the following weaker conditions.

Let $F : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ be positive definite and be σ-additive in each variable separately. Equivalently, if $\mathfrak{m}(\mathbb{R},\mathcal{B})$ is the vector space of complex measures on $\mathcal{B}$, let $\nu(A) = F(A,\cdot)$, $A \in \mathcal{B}$ so that $\nu:\mathcal{B} \to \mathfrak{m}(\mathbb{R},\mathcal{B})$ is a vector measure. By symmetry, $\tilde{\nu}:B \mapsto F(\cdot,B)$ is also a vector measure on $\mathcal{B} \to \mathfrak{m}(\mathbb{R},\mathcal{B})$. But $\mathfrak{m}(\mathbb{R},\mathcal{B}) = \mathcal{X}$ is a Banach space under the total variation norm, and hence ν (as well as $\tilde{\nu}$) has <u>finite semi variation</u> by a classical result (cf. [8], IV.10.4). This means,

$$\|\nu\|(\mathbb{R}) = \sup\{\|\sum_{i=1}^{n} a_i \nu(A_i)\|_{\mathcal{X}}: |a_i| \le 1, A_i \in \mathcal{B}, \text{disjoint}\} < \infty . \tag{6}$$

Transferred to F, this translates to:

$$\|F\|(\mathbb{R} \times \mathbb{R}) = \sup\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \bar{a}_j F(A_i,A_j) : A_i \in \mathcal{B} \text{, disjoint}, |a_i| \le 1\} < \infty . \tag{7}$$

When (7) holds, $F:\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ will be called a <u>$\mathbb{C}$-bimeasure of finite semi variation</u>. It should be noted that the σ-additivity of $F(\cdot,\cdot)$ in each of its components can be replaced by finite additivity and continuity of F from above at $\emptyset$ in that $|F(A_n,A_n)| \to 0$ as $A_n \downarrow \emptyset$. The desired generalization follows from (7) if it is written in the following form. Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=1}^{n} b_j \chi_{B_j}$, $A_i \in \mathcal{B}$, $B_j \in \mathcal{B}$ and each collection is disjoint. Set

$$I(\varphi,\psi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j F(A_i,B_j) . \tag{8}$$

Clearly I is well-defined, does not depend on the representation of φ or ψ, and $I(\varphi,\varphi) \ge 0$. So $(\varphi,\psi) = I(\varphi,\psi)$ is a semi-inner product on the space of $\mathcal{B}$-step functions. Hence by the generalized Schwarz's inequality one has:

$$|I(\varphi,\psi)|^2 \le I(\varphi,\varphi)\cdot I(\psi,\psi) \ . \tag{9}$$

Taking suprema on all such step functions φ,ψ such that $\|\varphi\|_u \le 1$, $\|\psi\|_u \le 1$ ($\|\cdot\|_u$ is the uniform norm), one deduces from (9) and (7) that

$$\|F\|(\mathbb{R}\times\mathbb{R}) \le \sup\{|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{b}_j F(A_i,B_j)| : |a_i|\le 1, |b_j|\le 1, A_i, B_j \in \mathcal{B},$$

$$\text{disjoint}\} \le \|F\|(\mathbb{R}\times\mathbb{R}) \ , \ (\le V(F)) \ . \tag{10}$$

Thus $\|F\|(\mathbb{R}\times\mathbb{R})$ can be defined either by the middle term (as in [34]) or by (7). For a bimeasure, $\|F\|(\mathbb{R}\times\mathbb{R})$ is also called Fréchet variation of F (cf. [23], p. 292.) and $V(F)$ the Vitali variation, (cf.[23], p. 298).

It should be emphasized that a set function F which is only a bimeasure (even positive definite), need not define a (complex) Radon measure on $\mathbb{R}^2$. In fact such bimeasures do not necessarily admit the Jordan decomposition, as counter examples show. Thus integrals relative to F (even if $\|F\|(\mathbb{R}\times\mathbb{R}) < \infty$) cannot generally be of Lebesgue-Stieltjes type. Treating $\nu: A \mapsto F(A,\cdot), A \in \mathcal{B}$, as a vector measure into $\mathfrak{m}(\mathbb{R},\mathcal{B})$, one can employ the Dunford-Schwartz (or D-S) integral (cf. [8], IV.10), or alternately one can use the theory of bimeasures as developed in ([23], [24]) and [39]. This is the price paid to get the desired weakened concept, but it will be seen that a satisfactory solution of our problem is then obtained, and both these integrations will play key roles.

Let us therefore recall an appropriate integration concept to be used in the following. In ([34], p. 276) Rozanov has indicated a modification without detailing the consequences. (This resulted in a conjucture [34,p. 283] which will be resolved in Section 7 below.) Instead, a different route will be followed; namely the integration theory of Morse and Transue will be used from [24] together with a related result of Thomas ([39], p. 146). However, the Bourbaki set up of these papers is inconvenient here, and they will be converted to the set theoretical (or ensemble) versions and employed.

Let $F:\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ be a bimeasure, i.e. $F(\cdot,B)$, $F(A,\cdot)$ are complex measures on $\mathcal{B}$. Hence one can define as usual ([8],III.6),

$$\tilde{I}_1(f,A) = \int_{\mathbb{R}} f(t)\, F(dt,A) \ . \tag{11}$$

for bounded Borel functions $f:\mathbb{R} \to \mathbb{C}$. Then $\tilde{I}_1(f,\cdot)$ is a complex measure and in fact $\tilde{I}_1:\mathcal{B} \to B(\mathbb{R},\mathcal{B},\mathbb{C})$, the Banach space of bounded complex Borel functions under the uniform norm, is a vector measure. So one can use the D-S integral (recalled at the beginning of the next section), defining

$$I_1(f,g) = \left(\int_{\mathbb{R}} \bar{g}(t)\tilde{I}_1(dt)\right)(f) \in \mathbb{C} \ , \tag{12}$$

where f,g are bounded Borel functions. Similarly starting with $F(A,\cdot)$ one can define $I_2(f,g)$. In general

$$I_1(f,g) \neq I_2(f,g) \ . \tag{13}$$

In fact the Fubini theorem need not hold in this context. For a counterexample, see ([24], §8). If there is equality in (13), then the pair (f,g) is said to be integrable relative to the bimeasure F, and the common value is denoted $I(f,g)$

and symbolically written as:

$$I(f,g) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\overline{g(t)}\ F(ds,dt) \ . \tag{14}$$

This is the Morse-Transue (or MT-) integral. While a characterization of MT-integrable functions is not easy, a good sufficient condition for this can be given as follows, (cf. [24], Thm. 7.1; [39], Théorème in §5.17). If f,g are step functions, so that $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $g = \sum_{j=1}^{n} b_j \chi_{B_j}$, then clearly $I(f,g)$ always exists and

$$I(f,g) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j\ F(A_i,B_j) \ . \tag{15}$$

Next define for any $\varphi \geq 0$, $\psi \geq 0$, Borel functions,

$$\tilde{I}(\varphi,\psi) = \sup\{|I(f,g)| : |f| \leq \varphi\ ,\ |g| \leq \psi\ ,\ f,g \text{ Borel step functions}\},$$

and if u,v are any positive functions,

$$I^*(u,v) = \inf\{\tilde{I}(\varphi,\psi) : \varphi \geq u,\ \psi \geq v,\ \varphi,\psi \text{ are Borel}\}. \tag{16}$$

Now the desired result from the above papers is this: If (f,g) is a pair of complex functions such that $I_1(f,g)$ and $I_2(f,g)$ exist and $I^*(|f|,|g|) < \infty$ then (f,g) is MT-integrable for the $\mathbb{C}$-bimeasure F. In the case that the bimeasure F is also positive definite and has finite semivariation, then each pair (f,g) of bounded complex Borel functions is MT-integrable relative to F. Moreover, using the notations of (7), one has

$$|I(f,g)| \leq \|F\| \cdot \|f\|_u \cdot \|g\|_u \ , \tag{17}$$

where $\|F\| = \|F\|(\mathbb{R} \times \mathbb{R})$. It should be noted, however, that the integrability of (f,g) generally does not imply that of $(|f|,|g|)$, and the MT-integral is not an absolutely continuous functional in contrast to the Lebesgue-Stieltjes theory, as already shown by counterexamples in [23] and [24]. Fortunately a certain dominated convergence theorem ([24], Thm. 3.3) is valid and this implies some density properties which can and will be utilized in our treatment below. Also f is termed F-integrable if (f,f) is MT-integrable. If $f = f_1 - f_2 + i(f_3 - f_4), f_i \geq 0$, then f is strongly F-integrable if each f_i is F-integrable, $i = 1,\cdots 4$. Note that, thus far, no special properties of $\mathbb{R}$ were used in the definition of the MT-integral, and the definition and properties are valid if $\mathbb{R}$ is replaced by an arbitrary locally compact space (group in the present context). This remark will be utilized later on.

With this necessary detour, the second concept is given as:

Definition 2.2. A process $X:\mathbb{R} \to L_0^2(P)$, with $r(\cdot,\cdot)$ as its covariance function, is called weakly harmonizable if

$$r(s,t) = I(e^{is(\cdot)}, e^{it(\cdot)}) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), s,t \in R, \quad (18)$$

relative to some positive definite bimeasure F of finite semi variation where the right side is the MT-integral. [Sometimes F is called the "spectral measure" of r.]

In particular r is continuous and bounded, (by (17)). Moreover, if F is of bounded variation, then the MT-integral reduces to the Lebesgue-Stieltjes integral and (18) goes over to (3). The following work shows that the process of the counterexample following Definition 2.1 is weakly harmonizable. Later several characterizations of weak harmonizability will be given, using the fundamental work of [2] and related ideas.

3. Integral representation of a class of second order processes. In order to introduce and utilize the "V-boundedness" concept of Bochner's, it will be useful to have an integral representation of weakly harmonizable processes. This is done by presenting a comprehensive result for a more general class including the (weakly) harmonizable ones. It is based on a method of Cramér's [3], and the resulting representation yields by specializations both the harmonizable, stationary, Cramér class of [3], as well as the Karhunen class (defined below). This is detailed as follows.

Recall that if $(\Omega_0, \mathcal{G})$ is a measurable space (i.e., $\mathcal{G}$ is a σ-algebra of the set Ω_0) and $\mathfrak{X}$ a Banach space, then a mapping $Z: \mathcal{G} \to \mathfrak{X}$ is called a vector measure if Z is σ-additive, or $Z(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Z(A_i)$, $A_i \in \mathcal{G}$, disjoint, the series converging unconditionally in the norm of $\mathfrak{X}$. If $\mathfrak{X} = L_0^2(P)$ where (Ω, Σ, P) is a probability space, then a vector measure is sometimes termed a stochastic measure. The integration of scalar functions relative to a vector measure Z is needed, and it will be in the sense of Dunford-Schwartz ([8], IV.10). This may be recalled quickly. If $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i \in \mathcal{G}$, disjoint, then as usual

$$\int_A f(s) Z(ds) = \sum_{i=1}^{n} a_i Z(A \cap A_i) \in \mathfrak{X}, \ A \in \mathcal{G}. \tag{19}$$

Now if $g: \Omega_0 \to \mathbb{C}$ is $\mathcal{G}$-measurable, and g_n are $\mathcal{G}$-step functions such that $g_n \to g$ pointwise, then g is said to be D-S integrable if for each $A \in \mathcal{G}$, $\{\int_A g_n(s) Z(ds), n \geq 1\} \subset \mathfrak{X}$

is a Cauchy sequence. Then the limit, denoted g_A, of this sequence is called the integral of g on A, and is denoted as

$$\int_A g(s)\,Z(ds) = \lim_{n\to\infty} \int_A g_n(s)Z(ds),\ A \in \mathcal{G}\ . \tag{20}$$

It is a standard (but non-obvious) matter to show that the integral is well-defined, independent of the sequence used, and the mapping $A \mapsto \int_A g(s)Z(ds)$ is σ-additive on $\mathcal{G}$, and $g \mapsto \int_A g(s)\,Z(ds)$ is linear. Also

$$\|\int_A g(s)Z(ds)\| \le \|g\|_u\ \|Z\|(A),\ f \in B(\Omega_0,\mathcal{G},\mathbb{C})\ , \tag{21}$$

where $\|Z\|(\cdot)$ is the semivariation of Z (cf.(6)) which is always finite on the σ-algebra $\mathcal{G}$. [If $\mathcal{G}$ is only a δ-ring and $\Omega_0 \notin \mathcal{G}$, then Z need not have finite semi variation on $\mathcal{G}$.] The dominated convergence theorem is true for the D-S integral. (See [8], IV.10, for proofs and related results. The latter exposition is very readable and nice.)

The general class noted above is the following:

<u>Definition 3.1</u> A process $X:\mathbb{R} \to L_0^2(P)$, with covariance $r(\cdot,\cdot)$, is said to be <u>weakly of class</u> (C) (C for Cramér) if

(i) there exists a covariance bimeasure F on $\mathbb{R}\times\mathbb{R}$ of locally bounded semi variation in the sense that

$$F(A,B) = \overline{F(B,A)},\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}F(A_i,A_j)\ge 0,\ a_i\in \mathbb{C},\ A_i \text{ bounded},$$

$A_i\in\mathcal{B}, 1\le i\le n$, and for each bounded Borel $A \subset \mathbb{R}$, if $\mathcal{B}(A) = \{A\cap B : B \in \mathcal{B}\}$, then

$$\|F\|(A\times A)=\sup\{|\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{b_j}F(A_i,B_j)| : |a_i|\le 1, |b_j|\le 1,$$
$$A_i,B_j \in \mathcal{B}(A),\ \text{disjoint}\} < \infty\ ,$$

(ii) there exists a family $g_t : \mathbb{R} \to \mathbb{C}$ of Borel functions, $t \in \mathbb{R}$, such that $I(|g_s|, |\overline{g_s}|) < \infty$, $s \in \mathbb{R}$, where I denotes the MT-integral relative to F, such that one has ($g_t(\lambda)$ is often written as $g(t,\lambda)$ and g_t is strongly MT-integrable):

$$r(s,t) = I(g_s, \overline{g}_t) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_s(\lambda)\overline{g_t(\lambda')}F(d\lambda, d\lambda') \ , \ s,t \in \mathbb{R} \ . \quad (22)$$

Remark. Note that in this definition F can be given by a covariance function ρ since then for $A = [a,b)$, and $B = [c,d)$ one defines $(\Delta^2 F)(A,B)$ as the increment $\rho(b,d) - \rho(a,d) - \rho(b,c) + \rho(a,c)$ and extend it to $\mathcal{B} \times \mathcal{B}$. Also in (22) it is possible that $\|F\|(\mathbb{R} \times \mathbb{R}) = \infty$. If F has finite variation on each compact rectangle of $\mathbb{R}^2$, then F determines a locally bounded complex Radon measure, and the above class reduces to the family defined by Cramér in [3], and called class (C) and analyzed in [31]. If $\|F\|(\mathbb{R} \times \mathbb{R}) < \infty$, then one can take $g_t(\lambda) = g(t,\lambda) = e^{it\lambda}$ so that the weakly harmonizable class is included. Again it may be noted that $\mathbb{R}$ can be replaced by a locally compact space or an abelian group in (22) so that $\mathbb{R}^n$ or the n-torus $\mathbb{T}^n$ is included.

To present the general representation, it is necessary also to note the validity of the D-S integration embodied in (20), (21) when the set functions are defined on arbitrary δ-rings instead of σ-algebras, assumed in [8]. Further our measure $Z : \tilde{\mathcal{B}} \to \mathfrak{X}$ has the property that it is Baire regular in the sense that for each $A \in \tilde{\mathcal{B}}$ and $\epsilon > 0$, there exist a compact $C \in \tilde{\mathcal{B}}$, open $U \in \tilde{\mathcal{B}}$ such that $C \subset A \subset U$ and

$\|Z(D)\| < \epsilon$ for each $D \in \tilde{\mathcal{B}}$, $D \subset U - C$, where $\tilde{\mathcal{B}}$ is the Baire (= Borel here) σ-ring of $\mathbb{R}$. Even if $\mathbb{R}$ is replaced by a general locally compact space S, and $\tilde{\mathcal{B}}$ is its Baire σ-ring and $Z:\tilde{\mathcal{B}} \to \mathfrak{X}$ is σ-additive, then Z is Baire regular and has a unique regular extension to the Borel σ-ring $\mathcal{B}$ of S and actually Z concentrates on a σ-compact Baire set $S_o \subset S$. Moreover if Z is weakly regular in that $x^* \circ Z$ is a scalar regular signed measure $x^* \in \mathfrak{X}^*$, then Z is itself regular. (See [19], pp. 262-263 for proofs and simple modifications needed for the results of [8], IV.10.) In each case the vector measure Z has finite semi variation on bounded sets in $\tilde{\mathcal{B}}$, (cf. (6) where $\mathcal{B}$ is replaced by the ring generated by all bounded Baire sets for S). If $\mathcal{B}_o \subset \mathcal{B}$ is the class of all bounded sets (a set is bounded if it is contained in a compact set), then it is a δ-ring, and the D-S integration of a scalar function relative to $Z:\mathcal{B}_o \to \mathfrak{X}$ holds as noted above. With this understanding the following is the desired general result.

Theorem 2.2. Let $X:\mathbb{R} \to L_o^2(P)$ be a process which is weakly of class (C) in the sense of Definition 3.1, relative to a positive definite bimeasure F of locally finite semi variation, and a family $\{g_s, s \in \mathbb{R}\}$ of strongly MT-integrable functions for F. Then there exists a stochastic measure $Z:\mathcal{B}_o \to L_0^2(\tilde{P})$ where $\mathcal{B}_o$ is the δ-ring of bounded Borel sets of $\mathbb{R}$, and $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ is an enlargement of (Ω, Σ, P) so $L_0^2(\tilde{P}) \supset L_o^2(P)$, such that

(i) $\quad E(Z(A)\cdot\overline{Z}(B)) = (Z(A), Z(B)) = F(A,B)$, $A, B \in \mathcal{B}_o$,

(ii) $\quad X(t) = \int_{\mathbb{R}} g(t,\lambda) Z(d\lambda)$, $t \in \mathbb{R}$ $\qquad$ (23)

where the integral is in the D-S sense for the δ-ring $\mathcal{B}_o$.

Conversely, if $\{X(t), t \in \mathbb{R}\}$ is a process defined by (23) relative to a stochastic measure $Z: \mathcal{B}_o \to L_0^2(P)$ and a family $\{g_t, t \in \mathbb{R}\}$, D-S integrable for Z and $\mathcal{B}_o$, then it is weakly of class (C) relative to F defined by $F(A,B) = E(Z(A)\cdot\overline{Z(B)})$, $A,B \in \mathcal{B}_o$, and $\{g_t, t \in \mathbb{R}\}$ is strongly MT-integrable for F. Moreover, if $\mathcal{H}_X = \overline{\text{sp}}\{X(t), t \in \mathbb{R}\}$ and $\mathcal{H}_Z = \overline{\text{sp}}\{Z(A), A \in \mathcal{B}_o\}$ in $L_0^2(P)$, then $\mathcal{H}_X = \mathcal{H}_Z$ when and only when the $\{g_t, t \in \mathbb{R}\}$ has the property that $\int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{g_t}(\lambda')F(d\lambda, d\lambda') = 0$, $t \in \mathbb{R}$, implies $\int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f}(\lambda')F(d\lambda, d\lambda') = 0$ both being MT-integrals.

Proof. The basic lay out is that of [3] where the classical integrals there will have to be replaced by the D-S and MT-integrals appropriately. Since the changes are not immediately obvious, the essential details are spelled out so that in subsequent discussions, such arguments can be compressed.

For the direct part, let the process be weakly of class (C). Then its covariance r admits a representation (with the MT-integration) as:

$$r(s,t) = E(X(s)\overline{X}(t)) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_s(\lambda)\overline{g_t}(\lambda')F(d\lambda, d\lambda') . \qquad (24)$$

Since F is a positive definite bimeasure, if $L_F^2 = \{f: \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f}(\lambda')F(d\lambda, d\lambda') = (f,f)_F < \infty$, f strongly MT-integrable for $F\}$, and since $I_F(f,f) = (f,f)_F \geq 0$, the earlier discussion implies $\{L_F^2, (\cdot,\cdot)_F\}$ is a semi-inner product space, and $g_t \in L_F^2$, $t \in \mathbb{R}$. Let $T: L_F^2 \to \mathcal{H}_X$ be defined by $T: g_s \mapsto X(s)$. Then (24) implies

$$(Tg_s, Tg_t)_{\mathcal{H}_X} = (g_s, g_t)_F \ , \ s,t \in \mathbb{R} \ . \tag{25}$$

Thus T is an isometric mapping of $\Lambda_F^2 = sp\{g_t, t \in \mathbb{R}\} \subset L_F^2$ onto $\mathcal{H}_X$ where T is extended linearly to Λ_F^2 from its generators.

Suppose first that Λ_F^2 is dense in L_F^2. By ([24], Thm. 11.1) every Borel function with $I^*(|f|,|f|) < \infty$ is in L_F^2, so that, in particular $\chi_A \in L_F^2$ for each $A \in \mathcal{B}_0$ since F is locally of finite semi variation. By the density of Λ_F^2 in L_F^2, there exists an element $Z_A \in \mathcal{H}_X$ such that $T\chi_A = Z_A$. If $A,B \in \mathcal{B}_0$, then

$$E(Z_A \cdot \bar{Z}_B) = (T\chi_A, T\chi_B)_{\mathcal{H}_X} = (\chi_A, \chi_B)_F = F(A,B) \ ,$$

and if $A \cap B = \phi$ also holds, then

$$E(|Z_{A\cup B} - A_A - Z_B|^2) = (\chi_{A\cup B} - \chi_A - \chi_B, \chi_{A\cup B} - \chi_A - \chi_B)_F = 0$$

since F is additive in both components. Thus $Z_{(\cdot)}: \mathcal{B}_0 \to \mathcal{H}_X \subset L_0^2(P)$ is additive. If $\{A_n\}_1^\infty \subset \mathcal{B}_0$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_0$, then

$$E(|Z_A - \sum_{i=1}^{n} Z_{A_i}|^2) = E(|Z_{\bigcup_{i=1}^n A_i} + Z_{\bigcup_{i>n} A_i} - \sum_{i=1}^{n} A_{A_i}|^2)$$

$$= E(|Z_{\bigcup_{i>n} A_i}|^2) = F(\bigcup_{i>n} A_i , \bigcup_{i>n} A_i) \to 0$$

as $n \to \infty$, since F is continuous at ϕ from above (cf., discussion after (7)). This Z is σ-additive on $\mathcal{B}_0$ and hence is a stochastic measure there. Clearly $\mathcal{H}_Z \subset \mathcal{H}_X$. Since $\{g_t, t \in \mathbb{R}\}$ is dense in L_F^2, $\chi_A \in L_F^2$, and each g_t is strongly MT-integrable for F, there is a sequence $g_{t_i} \to \chi_A$ in L_F^2 so that $(g_{t_i} - \chi_A, g_{t_i} - \chi_A)_F \to 0$. Hence by the isometry $E(|X_{t_i} - Z_A|^2) \to 0$ so that $\{Z_A, A \in \mathcal{B}_o\}$ is also dense in $\mathcal{H}_X$. Thus $\mathcal{H}_X = \mathcal{H}_Z$, and each element in $\mathcal{H}_Z$ corresponds uniquely to an element of $L^2_{\bar{F}}$, the completion of L^2_F, and where elements $h \in \bar{L}^2_F$ with $(h,h)_F = 0$ and 0 are identified. Let $Y(t)$ be defined as

$$Y(t) = \int_{\mathbb{R}} g_t(\lambda) Z(d\lambda) \in \mathcal{H}_Z = \mathcal{H}_X \quad , \tag{26}$$

where the right side is the D-S integral on the δ-ring $\mathcal{B}_o$. But then

$$\begin{aligned}(Y(s),Y(t)) &= \left(\int_{\mathbb{R}} g_s(\lambda)Z(d\lambda) \ , \int_{\mathbb{R}} g_t(\lambda')Z(d\lambda')\right) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} g_s(\lambda)\overline{g_t(\lambda')} \ F(d\lambda, d\lambda')\end{aligned}$$

which holds if g_s is a $\mathcal{B}_o$-measurable step function and the general case follows by ([24], Thm. 3.3) since g_s is strongly MT-integrable. Thus $Tg_s = Y(s) \in \mathcal{H}_X$. But then T being an isometry and $Tg_s = X(s)$, it follows that $X(s) = Y(s)$ a.e. So (26) implies (23) in the event that Λ^2_F is dense in L^2_F.

For the general case where $\tilde{\Lambda}^2_F = \bar{L}^2_F \ominus \bar{\Lambda}^2_F$ is nontrivial where "bar" again denotes completion, let $\{h_t, t \in \tilde{R}\}$ be a basis of $\tilde{\Lambda}^2_F$. If $\tilde{\tilde{R}} = \mathbb{R} \dotplus \tilde{R}$, is a disjoint sum to give a new index, let $\tilde{g}_s = g_s$ for $s \in \mathbb{R}$, $= h_s$ for $s \in \tilde{R}$, then $\{\tilde{g}_s, s \in \tilde{\tilde{\mathbb{R}}}\}$ is dense in $\bar{L}^2_F$. So by the preceding case, by extending T to τ from $L^2_F \to L^2_o(\tilde{P})$ where $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$ is possibly an enlargement of (Ω,Σ,P) (cf., e.g. [32], p. 82) with $\tau\chi_A = Z_A \in L^2_0(\tilde{P})$, since all the $\tilde{g}_s$ are strongly MT-integrable,

$$\tilde{Y}(s) = \int_{\mathbb{R}} \tilde{g}_s(\lambda)Z(d\lambda) \in L^2_0(\tilde{P}) \ . \tag{27}$$

Then as before $\tilde{Y}(s) = X(s)$ for $s \in \mathbb{R}$, and (23) holds again.

Note that in this case $\mathcal{H}_Z \supset \mathcal{H}_X$ properly.

Conversely, let $\{X(t), t \in \mathbb{R}\}$ be a process defined by (23). Let $F(A,B) = (Z(A), Z(B))$ and $g_n = \sum_{i=1}^{n} a_i \chi_{A_i}$, A_i, A, B in $\mathcal{B}_0$. Then for the D-S integral (23) one has

$$\begin{aligned}\|F\|(A,A) &= \sup\{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j F(A_i, A_j) : A_i \in \mathcal{B}(A), |a_i| \le 1\}\\ &= \sup\{\|\sum_{i=1}^{n} a_i Z(A_i)\|_2^2 : |a_i| \le 1, A_i \in \mathcal{B}(A)\}\\ &\le \|Z\|(A) < \infty, A \in \mathcal{B}_0 .\end{aligned}$$

Thus if $X_{g_n} = \int_{\mathbb{R}} g_n(\lambda) Z(d\lambda)$, one has with h_n another such step function,

$$E(X_{g_n} \bar{X}_{h_n}) = \int_{\mathbb{R}} \int_{\mathbb{R}} g_n(\lambda) \overline{h_n(\lambda')} F(d\lambda, d\lambda') . \tag{28}$$

Now given $g_s \in L_F^2$ which is strongly MT-integrable, by ([24], p. 493) the "Riesz components" of g_s (i.e., $g_s = re(g_s)^+ - re(g_s)^- + i(im(g_s)^+ - im(g_s)^-)$) can be approximated by suitable Borel step functions $\{g_n\}^\infty \subset L_F^2$ such that $g_n \to g_s$ pointwise $|g_n| \le |g_s|$ and similarly with $\tilde{g}_n \to g_t$ such that $I(g_n, \tilde{g}_n) \to I(g_s, g_t)$. Applying this to (28), one obtains

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} g_s(\lambda) \overline{g_t(\lambda')} F(d\lambda, d\lambda') &= \lim_n \int_{\mathbb{R}} \int_{\mathbb{R}} g_n(\lambda) \bar{\tilde{g}}_n(\lambda) F(d\lambda, d\lambda')\\ &= \lim_n (X_{g_n}, X_{\tilde{g}_n})\\ &= \lim_n (\int_{\mathbb{R}} g_n(\lambda) Z(d\lambda), \int_{\mathbb{R}} \tilde{g}_n(\lambda') Z(d\lambda'))\\ &= (\int_{\mathbb{R}} g_s(\lambda) Z(d\lambda), \int_{\mathbb{R}} g_t(\lambda) Z(d\lambda)), \text{ since}\\ &\quad \text{for the D-S integral the dominated}\\ &\quad \text{convergence holds,}\\ &= (X(s), X(t)) = r(s,t). \end{aligned} \tag{29}$$

This shows $\{X(t), t \in \mathbb{R}\}$ is of weakly class (C).

Regarding the last assertion, it is evident that $\{g_s, s \in \mathbb{R}\}$ is a basis in L_F^2 iff $I(f,g_t) = 0$, $t \in \mathbb{R}$ implies $I(f,f) = 0$. This is clearly necessary and sufficient for $\mathcal{H}_Z = \mathcal{H}_X$ since otherwise, (with possibly an enlargement of the underlying probability space) $\mathcal{H}_Z \supset \mathcal{H}_X$ and $\mathcal{H}_Z = \mathcal{H}_{\tilde{Y}}$ in the notation of (27). This completes the proof.

<u>Remarks.</u> 1. If F is of locally finite variation, then it defines a locally finite (i.e. finite on compact sets) complex Borel (= Radon) measure in the plane $\mathbb{R}^2$, and then the MT-integrals for F reduce to the Lebesgue-Stieltjes integrals. Thus $I(g_s,g_s) < \infty$ is equivalent to $I(|g_s|,|g_s|) < \infty$ and the above result reduces to Cramér's theorem of [3]. However, for the general case of bimeasures (as here), this is no longer true (cf. [24], p. 497).

2. The above theorem is true if $\mathbb{R}$ is replaced by a <u>locally compact space</u>, since no special property of $\mathbb{R}$ is used. Only the concept of boundedness is needed.

When $\|F\|(\mathbb{R} \times \mathbb{R}) < \infty$, so that F is of finite semi-variation on $\mathbb{R}^2$, then each bounded Borel function is strongly MT-integrable for F . Taking $g_t(\lambda) = e^{it\lambda}$ in the above theorem, one deduces from this result the important representation stated by Rozanov ([34],p. 279). The last statement is not hard to establish.

<u>Theorem</u> 3.3 <u>Let</u> $X:\mathbb{R} \to L_0^2(P)$ <u>be a process such that</u> $\|X(t)\|_2 \le M < \infty$, $t \in \mathbb{R}$, <u>and be weakly continuous</u>. <u>Then the</u>

process is weakly harmonizable relative to some covariance bimeasure F of finite semivariation (cf. Definition 2.2) iff there is a stochastic measure $Z:\mathcal{B} \to L_0^2(P)$ such that for each A,B in $\mathcal{B}$, $F(A,B) = (Z(A),Z(B))$ and

$$X(t) = \int_{\mathbb{R}} e^{it\lambda}\, Z(d\lambda)\ ,\ t \in \mathbb{R}, \tag{30}$$

the right side symbol being the D-S integral and $\|Z\|(\mathbb{R}) < \infty$. Moreover, X is strongly harmonizable iff the covariance bimeasure F of Z in (30) is of bounded variation in $\mathbb{R}^2$ (cf. Definition 2.1). In either case the harmonizable process X is uniformly continuous, and is represented as in (30).

Suppose that in the representation (23) the Z-process is orthogonally scattered in that $(Z(A),Z(B)) = 0$ whenever $A \cap B = \phi$. Then $F(A,B) = (Z(A),Z(B)) = \tilde{F}(A \cap B)$, where F is the covariance bimeasure and $\tilde{F}$ is a positive locally finite measure on $\mathcal{B}$ so that it is σ-finite there. Then

$$r(s,t) = E(X_s\overline{X}_t) = \int_{\mathbb{R}} g_s(\lambda)\overline{g_t(\lambda)}\ \tilde{F}(d\lambda)\ . \tag{31}$$

A process whose covariance function $\mathbb{R}$ satisfies this condition is called a Karhunen process. Moreover, if $\tilde{F}$ is a finite measure and $g_s(\lambda) = e^{is\lambda}$ the resulting one is the classical (wide sense) stationary process. In both these cases there are no weak type extensions. An interesting analysis of Karhunen processes (with $\tilde{F}$, a finite measure) has been given by Getoor [9] where an operator method and conditions for existence of a shift operator (extending the stationary case) where presented. However, the analysis of [9], together with the example following Definition 2.1, implies that weakly harmonizable processes do not generally admit shift operators on them in contrast to the stationary and many Karhunen processes.

Let us introduce a further generalization of the weak Cramér class to illuminate the above Definition 3.1, and for a future analysis. Let (Ω,Σ,μ) be a measure space and $M(\mu)$ be the space of scalar μ-measurable functions on Ω. Let $N(\cdot):M(\mu) \to \mathbb{R}^+$ be a <u>function</u> <u>norm</u> in that for f, f_n in $M(\mu)$, (i) $N(f) = N(|f|) \geq 0$, (ii) $0 \leq f_n \uparrow \Rightarrow N(f_n) \uparrow$, (iii) $N(af) = |a|N(f)$, $a \in \mathbb{C}$ and (iv) $N(f+g) \leq N(f) + N(g)$. The functional N has the weak Fatou property if $0 \leq f_n \uparrow f$, $\lim\limits_n N(f_n) < \infty$ $\Rightarrow N(f) < \infty$, and has the Fatou property if always $N(f_n) \uparrow N(f)$ $(\leq \infty)$. The associate norm N' of N is defined by:

$$N'(f) = \sup\{|\int_\Omega (fg)(\omega)\mu(d\omega)| : N(g) \leq 1\} . \tag{32}$$

One sees that N' is a function norm with the Fatou property. If $N(\cdot) = \|\cdot\|_p$, $1 \leq p \leq \infty$, then $N'(\cdot) = \|\cdot\|_q$, $p^{-1} + q^{-1} = 1$.

The general concept alluded to above is as follows:

<u>Definition</u> 3.4 (a) If $r:\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a covariance function, it is said to be of <u>class</u>$_N$(C) relative to a function norm N, if there is a covariance bimeasure $F:\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of locally finite N-variation (let N' be the associate norm of N), and there exists a family $\{g_t, t \in \mathbb{R}\}$ of Borel functions which are MT-integrable relative to F, such that

$$r(s,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} g_s(\lambda)\overline{g_t(\lambda')}F(d\lambda,d\lambda'),\ s,t \in \mathbb{R} , \tag{33}$$

and where locally finite N-variation is meant the following:

$$\infty > \|F\|_N (A \times A) = \sup\{|I(f,g)| : N'(f) \leq 1, N'(g) \leq 1\}, \tag{34}$$

Here f,g are Borel step functions, with $\text{supp}(f) \subset A$, $\text{supp}(g) \subset A$, $A \in \mathcal{B}_0$, the δ-ring of bounded Borel sets of $\mathbb{R}$.

(b) A process $X:\mathbb{R} \to L_0^2(P)$ is of <u>class$_N$(C)</u> if its covariance function r is of class$_N$(C) so that it is representable as (33).

It is clear that if $N(\cdot) = \|\cdot\|_1$ so that $N'(\cdot) = \|\cdot\|_\infty$, the N-variation is simply the 1-semivariation of Definition 3.1 so that $\|F\|_N = \|F\|_1 (= \|F\|)$.

<u>Remark</u>. Without further restrictions, class$_N$(C) need not contain the weak or strong harmonizable processes. However if N is restricted so that, letting $L^N(P) = \{f \in M(P):N(f) < \infty\}$, $L^\infty(P) \subset L^N(P) \subset L^1(P)$, where $\mu = P$ is a probability then every class$_N$(C) will contain both the weak and strong harmonizable families, as an easy computation shows. If $N(\cdot) = \|\cdot\|_1$, then class$_1$(C) is the class which corresponds to the covariance bimeasure of <u>finite semi-variation</u>. This includes the classical Loéve and Rozanov processes. Again this definition holds, with only a notational change, if $\mathbb{R}$ is replaced by a locally compact group G. A brief discussion on some analysis of these classes, which extend the present work, is included at the end of the paper.

4. <u>V-boundedness, weak and strong harmonizability</u>. The definition of weak harmonizability is of interest only when an effective characterization of it is found and when its relations with strong harmonizability are made concrete. These points will be clarified and answered here. Now Theorem 3.3 shows that a weakly harmonizable process is the Fourier transform of a stochastic measure and this leads us to a fundamental concept called V-boundedness ('V' for "variation"), introduced much earlier by Bochner [2], which is valid in a more general context. This notion plays a central role in the theory and applications of weakly harmonizable processes (and fields) which are shown to be V-bounded in the context of $L_0^2(P)$. Further this characterization facilitates a use of the powerful tools of Fourier analysis of vector measures. The desired concept is as follows (cf. [2], and also [29]):

<u>Definition</u> 4.1 A process $X:\mathbb{R} \to \mathfrak{X}$, a Banach space, is V-<u>bounded</u> if $X(\mathbb{R})$ lies in a ball of $\mathfrak{X}$, X as an $\mathfrak{X}$-valued function is strongly measurable (i.e., range of X is separable and $X^{-1}(B) \in \mathcal{B}$ for each Borel set $B \subset \mathfrak{X}$), and if the set C is relatively weakly compact in $\mathfrak{X}$, where

$$C = \{\int_{\mathbb{R}} f(t)X(t)dt: \|\hat{f}\|_u \le 1,\ f \in L^1(\mathbb{R})\} \subset \mathfrak{X}\ , \tag{35}$$

and where $\hat{f}(t) = \int_{\mathbb{R}} f(\lambda)e^{it\lambda}d\lambda$, $\int_{\mathbb{R}} f(t)X(t)dt$ being the Bochner integral. If $\mathfrak{X}$ is reflexive then the condition on C may be replaced by its boundedness. (Here if the measurability of X is strengthened to weak continuity, then it actually implies the strong [and even uniform] continuity.)

Let us establish the following basic fact when $\mathcal{X} = L_0^2(P)$:

<u>Theorem</u> 4.2 <u>A process</u> $X:\mathbb{R} \to L_0^2(P)$ <u>is weakly harmonizable iff</u> X <u>is</u> V-<u>bounded</u> (i.e., $\|X(t)\|_2 \le M_0 < \infty$, $t \in \mathbb{R}$ <u>and the set in</u> (35) <u>is bounded</u>) <u>and weakly continuous</u>.

<u>Proof</u>. For the direct part, let X be weakly continuous and V-bounded. Then

$$\left\|\int_{\mathbb{R}} f(t)X(t)dt\right\|_2 \le c\,\|\hat{f}\|_u\ ,\quad f \in L^1(\mathbb{R})\ , \tag{36}$$

by Definition 4.1. Let $\mathcal{Y} = \{\hat{f}: f \in L^1(\mathbb{R})\} \subset C_0(\mathbb{R})$, the space of complex continuous functions vanishing at "∞", the inclusion holding because of the Riemann-Lebesgue lemma. Moreover, $\mathcal{Y}$ is uniformly dense in $C_0(\mathbb{R})$. Let $\mathcal{F}: f \mapsto \int_{\mathbb{R}} f(\lambda)\bar{e}_t(\lambda)d\lambda$, $t \in \mathbb{R}$, where $e_t(\lambda) = e^{it\lambda}$. Then $\mathcal{F}: L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is a one-to-one bounded (contractive!) operator. Consider the mapping

$$T: \mathcal{Y} \to \mathcal{X} = L_0^2(P)\ ,\ \text{by}\ \ T(\hat{f}) = \int_{\mathbb{R}} f(t)X(t)dt \in \mathcal{X}\ .$$

This is well-defined, and the following diagram is commutative:

$$T_1(f) = \int_{\mathbb{R}} f(t)X(t)dt \in \mathcal{X}\ .\qquad \begin{tikzcd} L^1(\mathbb{R}) \arrow[rr, "\mathcal{F}"] \arrow[dr, "T_1"'] & & \mathcal{Y} \arrow[dl, "T"] \\ & \mathcal{X} & \end{tikzcd}$$

By hypothesis T is bounded and by the density of $\mathcal{Y}$ in $C_0(\mathbb{R})$, it has a norm preserving extension $\tilde{T}$ to $C_0(\mathbb{R})$. Now $\tilde{T}$ will be given an integral representation using a classical theorem due to Dunford-Schwartz ([8], VI. 7.3) since $\tilde{T}$ is a weakly compact operator because $\mathcal{X}$ is reflexive.

To invoke the above cited theorem, however, it should first be observed that the result holds even if the space $C(S)$ of continuous (scalar) functions on a compact space S (for which it is proved) is replaced by $C_o(\mathfrak{S})$ with a locally compact space $\mathfrak{S}$. Here $\mathfrak{S} = \mathbb{R}$. Indeed, let $\bar{\mathfrak{S}}$ be the one-point (at "∞") compactification of $\mathfrak{S}$ and consider the space $C(\bar{\mathfrak{S}})$. Now $C_o(\mathfrak{S})$ can be identified with the subspace $\{f \in C(\bar{\mathfrak{S}}): f(\infty) = 0\}$. Since $\tilde{T}: C_o(\mathfrak{S}) \to \mathfrak{X}$ is continuous and $C_o(\mathfrak{S})$ is an "abstract M-space", there is a continuous operator $\bar{T}: C(\bar{\mathfrak{S}}) \to \mathfrak{X}$ such that $\bar{T}|C_o(\mathfrak{S}) = \tilde{T}$. This follows from the fact that for any Banach space Z containing a subspace which is an abstract M-space, there is a projection of norm one on Z onto that subspace, by the well-known Kelley-Nachbin-Goodner theorem (cf. e.g., [8], p. 398), and $\bar{T} = \tilde{T} \circ Q$. Hence by the Dunford-Schwartz theorem noted above, there is a vector measure $\tilde{Z}$ on $\bar{\mathfrak{S}}$ into $\mathfrak{X}$ such that

$$\bar{T}(f) = \int_{\bar{\mathfrak{S}}} f(t)\tilde{Z}(dt) \ , \ f \in C(\bar{\mathfrak{S}}) \ , \tag{37}$$

and $\|\bar{T}\| = \|\tilde{Z}\|(\bar{\mathfrak{S}})$, the integral on the right being in the D-S sense. Define $Z: \mathcal{B}(\mathfrak{S}) \to \mathfrak{X}$ as $Z(A) = \tilde{Z}(\mathfrak{S} \cap A)$, $A \in \mathcal{B}(\mathfrak{S})$. Then Z is a vector measure and $\|Z\| \le \|\tilde{Z}\|$. Moreover, if $f_o = f|_{\mathfrak{S}}$, then

$$\bar{T}(f) = \int_{\mathfrak{S}} f_o(t)Z(dt) + \int_{\text{"}\{\infty\}\text{"}} f(\infty)\tilde{Z}(dt) \ , \ f \in C(\bar{\mathfrak{S}})$$

$$= \tilde{T}(f_o) \ , \qquad \text{since } f(\infty) = 0 \ .$$

Hence $\bar{T}(f) = \tilde{T}(f)$, $f \in C_o(\mathfrak{S})$ with $\|\tilde{T}\| \le \|\bar{T}\| = \|\tilde{T}Q\| \le \|\tilde{T}\|$, and

$$\tilde{T}(f) = \int_{\mathfrak{S}} f(t)Z(dt) \ , \quad f \in C_o(\mathfrak{S}) \ . \tag{38}$$

Thus writing $\mathbb{R}$ for $\mathfrak{S}$ from now on (the above general case is needed later), it follows that $\|\tilde{T}\| = \sup\{\|\int_{\mathbb{R}} f(t)Z(dt)\|:$

$f \in C_o(\mathbb{R})$, $\|f\|_u \le 1\} = \|Z\|(\mathbb{R}) = \|\tilde{Z}\|(\bar{\mathbb{R}})$, and $\tilde{T}$ and Z correspond to each other uniquely. Since $\tilde{T}|\mathcal{Y} = T$, this implies

$$T(\hat{f}) = \int_{\mathbb{R}} \hat{f}(t)Z(dt) = \int_{\mathbb{R}} f(t)X(t)dt \ , \ f \in L^1(\mathbb{R}) \ , \tag{39}$$

and $\|T\| = \|Z\|(\mathbb{R})$.

Let $\ell \in \mathcal{X}^*$. Then (30) becomes (since a continuous operator commutes with the D-S integral, cf. [8], p. 324 and p. 153),

$$\int_{\mathbb{R}} \hat{f}(t)\ell \circ Z(dt) = \int_{\mathbb{R}} f(t)\ell \circ X(t)dt \ . \tag{40}$$

In (40) now both are ordinary Lebesgue integrals, and hence using the Fubini theorem (for signed measures) on the left one has:

$$\int_{\mathbb{R}} f(t)dt \int_{\mathbb{R}} e_t(\lambda)\ell \circ Z(d\lambda) = \int_{\mathbb{R}} f(t)\ell \circ X(t)dt \ .$$

Subtracting and using the same theorem of ([8], p. 324),

$$\int_{\mathbb{R}} f(t)\ell(\int_{\mathbb{R}} e_t(\lambda)Z(d\lambda) - X(t))dt = 0 \ , \quad \ell \in \mathcal{X}^*, \ f \in L^1(\mathbb{R}). \tag{41}$$

It follows that the coefficient of f vanishes a.e., (everywhere as it is continuous). Since $\ell \in \mathcal{X}^*$ is arbitrary it finally results that the quantity inside ℓ is zero, for each $t \in \mathbb{R}$. Thus

$$X(t) = \int_{\mathbb{R}} e_t(\lambda)Z(d\lambda) = \int_{\mathbb{R}} e^{it\lambda}Z(d\lambda), \ t \in \mathbb{R} \ . \tag{42}$$

Hence X is weakly harmonizable by Theorem 3.3.

For the converse, let $X:\mathbb{R} \to L_0^2(P)$ be weakly harmonizable. Then X admits a representation of (42) by Theorem 3.3. Since $\|Z\|(\mathbb{R}) < \infty$, (21) implies $\|X(t)\|_2 \le M_o < \infty$ for all $t \in \mathbb{R}$, and as $\ell \circ X(\cdot)$ is the Fourier transform of $\ell \circ Z, \ell \in \mathfrak{X}^*$, X is weakly continuous. Consider the Bochner integral for $(fX)(\cdot)$ as

$$\ell\left(\int_{\mathbb{R}} f(t)X(t)dt\right) = \int_{\mathbb{R}} f(t)\ell \circ X(t)dt = \int_{\mathbb{R}} f(t)\cdot\int_{\mathbb{R}} e_t(\lambda)(\ell \circ Z)(d\lambda)dt, \tag{43}$$

since $\ell \circ X$ is the Fourier transform of a signed measure

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)e_t(\lambda)\ell \circ Z(d\lambda)dt, \text{ by Fubini's theorem,}$$

$$= \int_{\mathbb{R}} \hat{f}(\lambda)\ \ell \circ Z(d\lambda)$$

$$= \ell\left(\int_{\mathbb{R}} \hat{f}(\lambda)Z(d\lambda)\right), \text{ by([8], p.324) again.} \tag{44}$$

Since $\ell \in \mathfrak{X}^*$ is arbitrary, (44) implies

$$\int_{\mathbb{R}} f(t)X(t)dt = \int_{\mathbb{R}} \hat{f}(\lambda)Z(d\lambda) \in \mathfrak{X}\ . \tag{45}$$

Hence, using (21), one has

$$\left\|\int_{\mathbb{R}} f(t)X(t)dt\right\|_2 \le \|\hat{f}\|_u \|Z\|(\mathbb{R}) = c\|\hat{f}\|_u, f \in L^1(\mathbb{R})\ , \tag{46}$$

where $c = \|Z\|(\mathbb{R}) < \infty$. It therefore follows that the set

$$\left\{\int_{\mathbb{R}} f(t)X(t)dt : \|\hat{f}\|_u \le 1\ ,\ f \in L^1(\mathbb{R})\right\} \subset L_0^2(P),$$

and is bounded. Since $\mathfrak{X}$ is reflexive, X is V-bounded. This completes the proof.

Remarks. 1. Since V-boundedness concept is defined for general Banach spaces (for a treatment of this case, cf.[29]), and its Hilbert space version is equivalent to weak harmonizability, by the above theorem, the latter term will be used in the Hilbert space context. (Using the general definition of V-boundedness, a characterization of a process $X:\mathbb{R} \to \mathfrak{X}$, a reflexive space, which is a Fourier transform of a vector measure is given in Theorem 7.2 below.)

2. The preceding proof is arranged so that if $\mathbb{R}$ is replaced by a locally compact abelian (LCA) group G, the result and proof hold with essentially no change. The functions $\{e_t(\cdot), t \in G\}$ will then be group characters. Thus the result takes care of $G = \mathbb{R}^n$; so the (weakly) harmonizable random fields are included. Precise statements and further results in the general case will be given later.

If $\mathfrak{w}$ is the set of all weakly harmonizable processes on $\mathbb{R} \to L_0^2(P) = \mathfrak{X}$, and $T \in B(\mathfrak{X})$, the algebra of bounded linear operators on $\mathfrak{X}$, then $Y(t) = TX(t)$, $t \in \mathbb{R}$ defines a process which can be written as:

$$Y(t) = T\left(\int_{\mathbb{R}} e^{it\lambda} Z(d\lambda)\right) = \int_{\mathbb{R}} e^{it\lambda} (T \circ Z)(d\lambda)\ , \tag{47}$$

by ([8], p.324), and it is trivial that $\tilde{Z} = T \circ Z : \mathcal{B} \to \mathfrak{X}$ is a stochastic measure, $\|\tilde{Z}\|(\mathbb{R}) \le \|T\| \|Z\|(\mathbb{R}) < \infty$. Hence $Y \in \mathfrak{w}$. Thus one has:

<u>Corollary</u> 4.3 $B(\mathfrak{X}) \cdot \mathfrak{w} = \mathfrak{w}$, <u>or in words</u>, <u>the linear space of weakly harmonizable processes is a module over the class of all bounded linear transformations on</u> $\mathfrak{X} = L_0^2(P)$.

Since each stationary process X is trivially strongly (hence weakly) harmonizable, if $P : \mathfrak{X} \to \mathfrak{X}$ is any orthogonal projection, then $Y = PX \in \mathfrak{w}$, i.e. weakly harmonizable by Corollary 4.3. In particular if $\{X_n, n \in Z\} \subset \mathfrak{X}$ is an orthonormal sequence, $\mathfrak{X}_0 = \overline{sp}(X_n, n > 0)$, let $Q(\mathfrak{X}) = \mathfrak{X}_0$ be the orthogonal projection and $Y_n = QX_n = X_n$ if $n > 0$, $= 0$

if $n \le 0$. The process $\{Y_n, n \in \mathbb{Z}\} \in \mathfrak{W}$, but it is <u>not</u> strongly harmonizable. Thus the class of weakly harmonizable processes is strictly larger than the strongly harmonizable class. (The latter is not a module over $B(\mathcal{X})$.)

In spite of the above comment, each weakly harmonizable process can be approximated "pointwise" by a sequence of strongly harmonizable ones. This observation is essentially due to Niemi [26]. The precise result is as follows:

<u>Theorem</u> 4.4. <u>Let</u> $X: \mathbb{R} \to L_0^2(P)$ <u>be a weakly harmonizable process</u>. <u>Then there exists a sequence of strongly harmonizable processes</u> $X_n: \mathbb{R} \to L_0^2(P)$ <u>such that</u> $X_n(t) \to X(t)$, <u>as</u> $n \to \infty$, <u>in</u> $L_0^2(P)$ <u>uniformly</u> (<u>in</u> t) <u>on compact subsets of</u> $\mathbb{R}$. <u>If</u> $\mathbb{R}$ <u>is replaced by an</u> LCA <u>group</u> G <u>the same result holds with</u> $\{X_n, n \in I\}$ <u>being a net of such process</u>.

<u>Proof</u>. By hypothesis, there is a stochastic measure $Z: \mathcal{B} \to \mathcal{X} = L_0^2(P)$, such that

$$X(t) = \int_{\mathbb{R}} e_t(\lambda) Z(d\lambda) , \quad t \in \mathbb{R} .$$

Thus $X: \mathbb{R} \to \mathcal{X}$ is a continuous mapping. If $\mathcal{H}_X = \overline{sp}\{X(t), t \in \mathbb{R}\} \subset \mathcal{X}$, then the continuity of X and the separability of $\mathbb{R}$ implies $\mathcal{H}_X$ is separable. Hence there exists a sequence $\{\varphi_n, n \ge 1\} \subset \mathcal{H}_X$ which is a complete orthonormal (CON) basis for $\mathcal{H}_X$, so that

$$X(t) = \sum_{n=1}^{\infty} \varphi_n (X(t), \varphi_n) , \quad t \in \mathbb{R} , \tag{48}$$

the series converging in the (norm) topology of $\mathcal{H}_X$ for each t . Define

$$X_n(t) = \sum_{k=1}^{n} \varphi_t (X(t), \varphi_k) , \quad t \in \mathbb{R} . \tag{49}$$

Claim: $\{X_n(t), t \in \mathbb{R}\}, n \geq 1$, is the desired sequence. [In the general LCA group case $\{\varphi_n, n \in I\}$ is a net of CON elements of $\mathcal{H}_X$, since G , hence $\mathcal{H}_X$, need not be separable. Otherwise the same argument works with trivial modifications.]

To verify the claim, it is clear that $X_n(t) \to X(t)$ in $\mathcal{H}_X$ for each $t \in \mathbb{R}$. To see that X_n is strongly harmonizable, let $\ell_k : X \mapsto (X, \varphi_k)$, $x \in \mathcal{H}_X$. Then $\ell_k \in \mathcal{H}_X^*$ for each k . Hence using the standard properties of the D-S integral, one has

$$X_n(t) = \sum_{k=1}^{n} \varphi_k \ell_k(X(t)) = \sum_{k=1}^{n} \varphi_k \cdot \ell_k\left(\int_{\mathbb{R}} e_t(\lambda) Z(d\lambda)\right), \text{ since } X \text{ is weakly harmonizable,}$$

$$= \sum_{k=1}^{n} \varphi_k \int_{\mathbb{R}} e_t(\lambda) \ell_k \circ Z(d\lambda) = \int_{\mathbb{R}} e_t(\lambda) \zeta_n(d\lambda) \ , \tag{50}$$

where $\zeta_n(\cdot) = \sum_{k=1}^{n} \varphi_k \ell_t \circ Z(\cdot)$. Let $G_n(A,B) = (\zeta_n(A), \zeta_n(B))$. Then G is of finite total variation. Indeed, if $\mu_k = \ell_k \circ Z$, which is a signed measure (hence has finite variation) on $\mathbb{R}$, let $\eta_k(A,B) = (\varphi_k \mu_k(A), \varphi_k \mu_k(B)) = \mu_k(A)\overline{\mu_k(B)}$. So $G_n(A,B) = \sum_{k=1}^{n} \mu_k(A)\overline{\mu_k(B)}$. Since $|\mu_k(A)|\ |\mu_k(B)| \leq (|\mu_k|(\mathbb{R}))^2 < \infty$ for each k , it follows that each η_k and hence G_n for each n has finite variation so that each X_n is strongly harmonizable.

It was already noted that X being weakly harmonizable, it is strongly continuous. [This is true even if $\mathbb{R}$ is replaced by an LCA group G(cf. [19], p. 270).] So if $K \subset \mathbb{R}$ is a compact set, then its image $X(K) \subset \mathcal{H}_X \subset L_0^2(P)$ is also (norm) compact. But $\mathcal{H}_X$ being a Hilbert space it has the (metric) approximation property. [This means the identity on $\mathcal{H}_X$ can be uniformly approximated by a sequence (net) of (contractive) degenerate, or finite rank, operators on each compact subset of $\mathcal{H}_X$.] Then $X_n(t) \to X(t)$ in $\mathfrak{X}$ for each $t \in \mathbb{R}$ implies, by a result in Abstract Analysis in the presence of the approximation property, that the convergence holds in $\mathfrak{X}$ uniformly on compact subsets of $\mathfrak{X}$. This and the fact that $X(K)$ is compact implies that $X_n(t) \to X(t)$ uniformly for $t \in K \subset \mathbb{R}$. In the general LCA case, the same holds with nets replacing sequences. This completes the proof.

Remark. Even though the weakly harmonizable process is bounded and weakly (hence strongly here) continuous with some nice closure properties demonstrated above, it does not exhaust the class of all bounded continuous functions in $L_0^2(P)$. This can be seen from Theorem 3.2 by a suitable choice of a vector measure of finite local semivariation but which is not of finite semivariation. The following example demonstrates this point. Let $L^1(\mathbb{R})$ be identified with $\mathfrak{m}(\mathbb{R})$ of regular signed measures on $\mathbb{R}$ by the Radon-Nikodým theorem (i.e. $f \in L^1(\mathbb{R}) \longleftrightarrow \int_{(\cdot)} f(t)dt \in \mathfrak{m}(R))$. Now it is known that there are nontrivial functions in $C_o(\mathbb{R}) - \mathcal{U}_1$ where $\mathcal{U}_1 = \{\hat{\mu}:\mu\in\mathfrak{m}(\mathbb{R})\}$. Let $f \in C_o(\mathbb{R}) - \mathcal{U}$. For instance $f(x) = \mathrm{sgn}(x)((\log|x|)^{-1} \chi_{[|x|\geq e]} + \frac{|x|}{e}\chi_{[|x|<e]})$, $x \in \mathbb{R}$, is known to be such an f. Let $\varphi \in L_0^2(P)$, $\|\varphi\|_2 = 1$. Let $\ell \in (L_0^2(P))^*$ such that $\ell(\varphi) = 1$. Consider the trivial process $X_0: t \mapsto f(t)\varphi$. Then $X_0: \mathbb{R} \to L_0^2(P)$ is bounded and continuous but not weakly harmonizable, since otherwise there exists a stochastic measure Z such that (by Theorem 3.3)

$$X_{\delta}(t) = \int_{\mathbb{R}} e_t(\lambda) Z(d\lambda) \ , \text{ and}$$

$$f(t) = \ell(X_{\delta}(t) = \int_{\mathbb{R}} e_t(\lambda)(\ell \circ Z)(d\lambda) \ .$$

Since $\ell \circ Z \in \mathfrak{m}(\mathbb{R})$, this would contradict the choice of f.

Here is an interesting consequence of the preceding theorems which is based on the classical Helly-Bray theorem.

Theorem 4.5. Let $X:\mathbb{R} \to L_0^2(P)$ be a weakly harmonizable process and let $Z:\mathcal{B} \to L_0^2(P)$ be its representing measure by (30). Then there is (nonuniquely) a finite regular Borel measure $\beta:\mathcal{B} \to \mathbb{R}^+$ such that

$$\left\|\int_{\mathbb{R}} f(t) Z(dt)\right\|_2 \le \|f\|_{2,\beta} \left(=\left[\int_{\mathbb{R}} |f(t)|^2 \beta(dt)\right]^{1/2}\right), f \in C_0(\mathbb{R}). \quad (51)$$

Remark. Eventhough this result is included in Theorem 5.5 below, its proof is elementary and has independent interest. So it will be given here.

Proof. By hypothesis, $X(\cdot)$ is represented by a stochastic measure Z (cf. (30)), and by the preceding theorem there are strongly harmonizable $X_n \to X$, uniformly on compact subsets of $\mathbb{R}$. Let ζ_n be the representing measure of X_n, so that ζ_n, $Z:\mathcal{B} \to L_0^2(P)$, and

$$\int_{\mathbb{R}} f(\lambda) Z(d\lambda) = \lim_{n\to\infty} \int_{\mathbb{R}} f(\lambda) \zeta_n(d\lambda) \ , \quad (52)$$

the limit existing in $L_0^2(P)$ when f is a trigonometric polynomial. Since such polynomials are uniformly dense in $C_0(\mathbb{R})$ and the integrals in (52) define bounded operators from $C_0(\mathbb{R})$ into $L_0^2(P)$, it follows that (52) holds for all $f \in C_0(\mathbb{R})$, by the standard reasoning, (cf. [8], II.3.6). Hence

$$\alpha_0^f = \|\int_{\mathbb{R}} f(\lambda) Z(d\lambda)\|_2^2 = \lim_{n\to\infty} \|\int_{\mathbb{R}} f(\lambda)\zeta_n(d\lambda)\|_2^2 \ , \ f \in C_0(\mathbb{R})$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f(\lambda')}\ F_n(d\lambda, d\lambda')\ , \qquad (53)$$

where $F_n(s,t) = (\zeta_n(-\infty,s), \zeta_n(-\infty,t))$ is the covariance function of bounded variation for each n. Let $|F_n|(\cdot,\cdot)$ be the (Vitali) variation measure of the bimeasure F_n. Then the hermitian property of F_n implies, in an obvious notation, $|F_n|(A,B) = |F_n|(B,A)$. Now define a mapping $\beta_n : \mathcal{B} \to \mathbb{R}^+$ by the equation:

$$\beta_n(A) = |F_n|(A,\mathbb{R}) = \frac{1}{2}\{|F_n|(A,\mathbb{R}) + |F_n|(\mathbb{R},A)\},\ A \in \mathcal{B}\ ,$$

so that β_n is a finite Borel measure, and

$$\int_{\mathbb{R}} f(\lambda)\beta_n(d\lambda) = \frac{1}{2}[\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\cdot|F_n|(ds,dt) + \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)|F_n|(ds,dt)]. \quad (54)$$

Since F_n is positive (semi-) definite,

$$0 \le \int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\overline{f(t)}\ F_n(ds,dt) \le \int_{\mathbb{R}}\int_{\mathbb{R}} |f(s)\overline{f(t)}|\,|F_n|(ds,dt)$$

$$\le \frac{1}{2}[\int_{\mathbb{R}}\int_{\mathbb{R}} |f(s)|^2|F_n|(ds,dt) + \int_{\mathbb{R}}\int_{\mathbb{R}} |f(t)|^2|F_n|(ds,dt)],$$

since $|ab| \le (|a|^2+|b|^2)/2$,

$$= \int_{\mathbb{R}} |f(s)|^2\beta_n(ds)\ ,\ \text{by (54)}.$$

This and (53) yield

$$\alpha_0^f = \|\int_{\mathbb{R}} f(\lambda)Z(d\lambda)\|_2^2 = \lim_n \int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\overline{f(\lambda')}F_n(ds,dt)$$

$$\le \varliminf_{n} \int_{\mathbb{R}} |f(\lambda)|^2\beta_n(d\lambda)\ ,\ f \in C_0(\mathbb{R}). \qquad (55)$$

Thus if α^f is the right side of (55), then $0 \le \alpha_0^f \le \alpha^f < \infty$

for each f, and there is a subsequence $\{\beta_{n_i}\}$ such that $\alpha^f = \lim_{i\to\infty} \int_{\mathbb{R}} |f(\lambda)|^2 \beta_{n_i}(d\lambda)$. It is to be shown that this exists also for $f = 1$. Since $1 \notin C_0(\mathbb{R})$, a nontrivial extension problem intervenes, and this is resolved with a classical trick (cf. [35], p. 32).

Let $\tilde{\mathbb{R}}$ be the Bohr compactification of $\mathbb{R}$. Then by classical results (as in [35] above) each trigonometric polynomial f on $\mathbb{R}$ extends to a similar polynomial on $\tilde{\mathbb{R}}$, by the formula

$$f(\lambda) = \sum_{k=1}^{n} a_k(e_{t_k},\lambda) \ , \ \lambda \in \tilde{\mathbb{R}} \ ,$$

where $(\cdot,\cdot)$ is the duality pairing of the group $\mathbb{R}$ and its dual $\hat{\mathbb{R}}$, and where $\mathbb{R}$ is identified with its image in $\tilde{\mathbb{R}}$. [If $x \in \mathbb{R}$, $y \in \hat{\mathbb{R}}$, then $(x,y) = (y,\gamma(x))$ defines $\gamma:\mathbb{R} \to \tilde{\mathbb{R}}$ as a continuous isomorphism, and $\gamma(\mathbb{R})$ is a dense subgroup of $\tilde{\mathbb{R}}$.] The density of $\mathbb{R}$ in $\tilde{\mathbb{R}}$ implies the map $f:\lambda \mapsto f(\lambda)$ $\lambda \in \tilde{\mathbb{R}}$, has a (uniform) norm preserving extension so that if ℓ_n is defined by $\ell_n(f) = \int_{\mathbb{R}} f(\lambda)\beta_n(d\lambda)$, for each trigonometric polynomial f in $C_0(\mathbb{R})$, then the continuous linear functional ℓ_n has a norm preserving extension $\tilde{\ell}_n$ to $C(\tilde{\mathbb{R}})$. By the Riesz representation theorem, there is a unique regular Borel measure $\tilde{\beta}_n$ on $\tilde{\mathbb{R}}$, such that $\tilde{\ell}_n(f) = \int_{\mathbb{R}} f(\lambda)\tilde{\beta}_n(d\lambda)$, $f \in C(\tilde{\mathbb{R}})$, $\|\ell_n\| = \tilde{\beta}_n(\tilde{\mathbb{R}}) = \|\ell_n\| = \beta_n(\mathbb{R})$. It is clear that $\tilde{\beta}_n$ is an extension of β_n. But now $1 \in C(\tilde{\mathbb{R}})$, and hence $\tilde{\beta}_n(\mathbb{R}) \le \tilde{\ell}_n(1) = \|\tilde{\ell}_n\| = \|\ell_n\| < \infty$, $n \ge 1$. The inequality in (55) is unchanged if β_n is replaced by $\tilde{\beta}_n$, and $f \in C(\tilde{\mathbb{R}})$. It now follows with $f = 1$, that for some subsequence $\{\beta_{n_k}, k \ge 1\}$, $\int_{\mathbb{R}} \tilde{\beta}_{n_k}(d\lambda) \to \alpha_0^1$.

Hence $\{\tilde{\beta}_{n_k}(\mathbb{R})\ k \geq 1\}$ is bounded, and $\tilde{\beta}_{n_k}(\mathbb{R}) \leq M < \infty$ so that $\{\beta_{n_k}, k \geq 1\}$ is a uniformly bounded sequence of Borel measures on $\mathbb{R}$. Thus by the Helly selection theorem there is a further subsequence $\beta_{n_{k'}}$ such that $\beta_{n_{k'}} \to \beta$ weakly (i.e. in the weak-star topology) for a regular Borel measure β. Then by the extended Helly-Bray theorem (cf. [21], p. 181) one has for this subsequence $\int_{\mathbb{R}} |f|^2(\lambda)\beta_{n_{k'}}(d\lambda) \to \int_{\mathbb{R}} |f|^2(\lambda)\beta(d\lambda)$, $f \in C_o(\mathbb{R})$. Thus (55) becomes

$$\left\|\int_{\mathbb{R}} f(\lambda)Z(d\lambda)\right\|_2^2 \leq \int_{\mathbb{R}} |f|^2(\lambda)\beta(d\lambda) = \|f\|_{2,\beta}^2, \quad f \in C_0(\mathbb{R}).$$

This is (51) and the proof of the theorem is complete.

<u>Remark</u>. The construction of β shows that, it is not unique in general. While the Bohr compactification argument is available for LCA groups the Helly theorems and the rest of the argument becomes more involved or inapplicable with sequences replacing nets, and so a different argument is desirable for general random fields. Such a method will be employed in the next section as it has a potential of application to all locally compact groups, and even to [40].

The preceding theorem can be restated abstractly as follows:

<u>Theorem</u> 4.6. <u>Let</u> $\mathcal{B}$ <u>be the Borel</u> σ-<u>algebra of</u> $\mathbb{R}$ <u>and</u> $\nu:\mathcal{B} \to \mathcal{H}$ <u>be a vector measure where</u> $\mathcal{H}$ <u>is a Hilbert space</u>. <u>Then there is a "dominating" Borel measure</u> $\beta:\mathcal{B} \to \mathbb{R}^+$, <u>such that</u>

$$\left\|\int_R f(t)\nu(dt)\right\|_2 \leq \|f\|_{2,\beta}, \quad f \in C_o(\mathbb{R}). \tag{56}$$

If $\|\nu\|_2(\mathbb{R}) = \sup\{\|\int_{\mathbb{R}} f(t)\nu(dt)\|: \|f\|_{2,\beta} \leq 1\} < \infty$, then ν may be said to have 2-<u>semivariation finite</u> with respect to some finite measure β, thus the above result says that every vector measure on $\mathbb{R}$ into a Hilbert space is always of 2-semivariation finite (but not necessarily of finite variation !) relative to some finite Borel measure. The domination problem for other Banach spaces and other base spaces (different from $\mathbb{R}$) is nontrivial, and is unsolved for most of the Banach spaces. This question will be analyzed in more detail in the next section for random fields. In the following section the analog of Theorem 4.5 will be used to prove the existence of a "stationary dilation" for each (weakly) harmonizable random field. Thereafter several characterizations of these processes will be given, as they facilitate various applications and analyses. It may be recalled for definiteness, that a random family $\{X_t, t \in G\}$ is a <u>process</u> of G is a one-dimensional set $(G \subseteq \mathbb{R})$, and it is a <u>field</u> if G is a subset of a higher dimensional group, (e.g., $G \subseteq \mathbb{R}^n$, $n > 1$).

5. Domination problem for harmonizable fields and vector measures. The work of the preceding section indicates that the weakly harmonizable processes are included in the class of functions which are Fourier transformations of vector measures into Banach spaces. A characterization of such functions, based on the V-boundedness concept of [2], has been obtained first in [29]. For probabilistic applications (e.g., filtering theory) the domination problem, indicated in Theorem 4.6, should be solved. The following result illuminates the nature of the general problem under consideration.

Theorem 5.1. *Let (Ω,Σ) be a measurable space, $\mathfrak{X}$ a Banach space and $\nu:\Sigma \to \mathfrak{X}$ be a vector measure. Then there exists a (finite) measure $\mu:\Sigma \to \mathbb{R}^+$, a continuous convex function $\varphi:\mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{\varphi(x)}{x} \nearrow \infty$ as $x \nearrow \infty$, and ν has φ-semivariation finite relative to μ in the sense that*

$$\|\nu\|_\varphi(\Omega) = \sup\{\|\int_\Omega f(\omega)\nu(d\omega)\|_{\mathfrak{X}} : \|f\|_{\varphi,\mu} \le 1\} < \infty , \tag{57}$$

where $\|f\|_{\varphi,\mu} = \inf\{\alpha > 0 : \int_\Omega \varphi(\frac{|f(\omega)|}{\alpha})\mu(d\omega) \le 1\} < \infty$, and the integral relative to ν in (57) is in the Dunford-Schwartz sense.

Proof. Recall that $\nu:\Sigma \to \mathfrak{X}$ is a vector measure iff it is σ-additive in the norm topology of $\mathfrak{X}$. Let S^* be the unit sphere of the adjoint space $\mathfrak{X}^*$ of $\mathfrak{X}$. Then the above statement is equivalent to the uniform σ-additivity of the scalar measures $\{x^*\circ\nu, x^* \in S^*\}$. In fact, if ν is strongly (i.e. in norm) σ-additive then for any disjoint sequence $A_n \in \Sigma$, $n \ge 1$, one has

$$0 = \lim_{n\to\infty} \|\nu(\overset{n}{\underset{i=1}{\cup}} A_i) - \sum_{i=1}^{n} \nu(A_i)\|$$

$$= \lim_{n\to\infty} \sup_{x^*\in S^*} |x^*\circ\nu(\overset{n}{\underset{i=1}{\cup}} A_i) - \sum_{i=1}^{n} x^*\circ\nu(A_i)|$$

so that $\{x^*\circ\nu, x^*\in S^*\}$ is uniformly σ-additive. On the other hand if ν is weakly σ-additive (i.e. the scalar function $x^*\circ\nu$ on the σ-algebra Σ is σ-additive for each $x^* \in S^*$), then by a classical theorem of Pettis (cf. [8], IV. 10.1), ν is strongly σ-additive. Thus weak σ-additivity and the above stated uniform σ-additivity are equivalent. Now by another classical result due to Bartle-Dunford-Schwartz (cf. [8], IV. 10.5) there exists at least one "control measure" $\mu:\Sigma \to \mathbb{R}^+$ such that $x^*\circ\nu$ is μ-continuous for all $x^* \in S^*$. Hence if $g_{x^*} = \frac{d(x^*\circ\nu)}{d\mu}$ (the Radon-Nikodým derivative), then by the first part on uniformity, one has $\lim_{\mu(A)\to 0} \int_A g_{x^*}(\omega)\mu(d\omega) = \lim_{\mu(A)\to 0} |x^*\circ\nu(A)| = 0$, uniformly in $x^* \in S^*$. Hence $\{g_{x^*} : x^* \in S^*\} \subset L^1(\Omega,\Sigma,\mu)$ and it is a uniformly integrable set, by the well-known Dunford-Pettis theorem, ([8], IV. 8.11). Since $\mu(\Omega) < \infty$, by a (1915) theorem of de la Vallée Poussin (same argument as in [32], p. 65) there exists a continuous convex function $\psi:\mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{\psi(s)}{x} \nearrow \infty$ as $x \nearrow \infty$, and $\int_\Omega \psi(|g_{x^*}(\omega)|)\mu(d\omega) \le M < \infty$, all $x^* \in S^*$. Such a ψ is called a Young function, and this statement is equivalent to saying that $\{g_{x^*}, x^* \in S^*\}$ lies in a ball of the Orlicz space $L^\psi(\mu)$. (For the rudiments of Orlicz spaces one may consult [41], p. 173.)

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as: $\varphi(x) = \sup\{|x|y - \psi(y) : y \geq 0\}$. Then it is easily seen that φ is also a continuous convex function with $\frac{\varphi(x)}{x} \uparrow \infty$ on $x \nearrow \infty$ and φ is called the complementary Young function. Considering the $L^{\varphi}(\mu)$, by the Hölder inequality for these spaces (cf. [41], p. 175), it results that

$$\|\nu\|_{\varphi}(\Omega) = \sup\{\|\int_{\Omega} f(\omega)\nu(d\omega)\|_{\mathfrak{X}} : \|f\|_{\psi,\mu} \leq 1\}$$

$$= \sup\{\sup[|\int_{\Omega}(f, g_{x^*})(\omega)\ \mu(d\omega)| : x^* \in S^*] : \|f\|_{\psi,\mu} \leq 1\}$$

$$\leq 2\sup\{\sup[\|f\|_{\psi,\mu}\|g_{x^*}\|_{\varphi,\mu} : x^* \in S^*] : \|f\|_{\psi,\mu} \leq 1\}, \text{ by Hölder's inequality,}$$

$$\leq 2\sup\{\|g_{x^*}\|_{\varphi,\mu} : x^* \in S^*\} \leq 2K_0 < \infty ,$$

where K_0 is the radius of the ball containing the set $\{g_{x^*}, x^* \in S^*\}$. So (57) is true, and this completes the proof.

Remark. If $\mathfrak{X}$ is a Hilbert space, $(\Omega,\Sigma) = (\mathbb{R},\mathcal{B})$, then Theorem 4.6 shows that $\varphi(x) = |x|^2$ and $\mu = \beta$ there. But $\|\nu\|_1(\Omega)$ $(=\|\nu\|(\Omega)) < \infty$ always, by ([8], IV. 10.2) . Since on a finite measure space $L^{\infty}(\mu) \subset L^{\varphi}(\mu) \subset L^1(\mu)$, for any continuous Young function (the first inclusion is obvious, and the second follows from the support line property), it is easily seen that $\|\nu\|_{\varphi}(\Omega) \leq c\|\nu\|_1(\Omega) < \infty$, and the inequality can be strict so that the above result is an improvement on previously known ones. However, φ may grow faster than a polynomial. Thus Theorem 4.6 and 5.1 imply that φ depends on the space $\mathfrak{X}$. An interesting and nontrivial problem is to classify Banach spaces for given φ-functions such as $\varphi(x) = |x|^p$, $p \geq 1$.

The general case is largely unexplored. Some interesting special problems are considered in the rest of this section.

It will be convenient to introduce a definition and to state a fundamental result of Grothendieck and Pietsch, for the work below.

<u>Definition</u> 5.2 Let $\mathfrak{X},\mathfrak{Y}$ be a pair of Banach spaces and, as usual, $B(\mathfrak{X},\mathfrak{Y})$ be the space of bounded linear operators on $\mathfrak{X}$ into $\mathfrak{Y}$. If $1 \le p \le \infty$, $T \in B(\mathfrak{X},\mathfrak{Y})$, then T is called p-<u>absolutely summing</u> if $\alpha_p(T) < \infty$ where

$$\alpha_p(T) = \inf\{c > 0 : [\sum_{i=1}^{n} \|Tx_i\|^p]^{1/p} \le c \sup_{\|x^*\| \le 1} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}} ,\quad x_i \in \mathfrak{X},\ 1 \le i \le n\ ,\ n \ge 1\} \ , \tag{58}$$

where $x^* \in \mathfrak{X}^*$, the adjoint space of $\mathfrak{X}$.

The following result of Grothendieck-Pietsch with a short proof may be found in [20] together with some extensions and applications.

<u>Proposition</u> 5.3. <u>Let</u> $T \in B(\mathfrak{X},\mathfrak{Y})$ <u>be</u> p-<u>absolutely summing</u>, $1 \le p < \infty$. <u>Let</u> K^* <u>be the weak-star closure of the set of extreme points of the unit ball</u> U^* of $\mathfrak{X}^*$. <u>Then there is a regular Borel probability measure</u> μ <u>on the compact space</u> K^* <u>such that</u>

$$\|Tx\|_{\mathfrak{Y}} \le \alpha_p(T)\ [\int_{K^*} |x^*(x)|^p \mu(dx^*)]^{1/p},\ x \in \mathfrak{X} \ . \tag{59}$$

<u>Conversely</u> (<u>and this is simple</u>), <u>if</u> T <u>satisfies</u> (59) <u>for some</u> μ <u>on</u> K^* <u>with a constant</u> γ_0, <u>then</u> T <u>is</u> p-<u>absolutely summing and</u> $\alpha_p(T) \le \gamma_0$. <u>Further any</u> p-<u>absolutely summing operator is weakly compact</u>.

Let us specialize this result in the case that $\mathfrak{X} = C_r(S)[C(S)]$, the space of real [complex] continuous functions on a compact set S. Let K be the set of all extreme points of the unit ball U^* of $(C_r(S))^*$ and $q: S \to (C_r(S))^*$ be the mapping defined by $q(s) = \ell_s$ with $\ell_s(f) = f(s)$, $f \in C_r(S)$ so that ℓ_s is the evaluation functional, $\|\ell_s\| = 1$, and $\ell_s \in K$, $s \in S$. Some other known results needed from Linear Analysis, in the form used here, are as follows. (For details, see [4], Sec. V.3; [8], p. 441.) In this case the spaces S and $q(S)$ are homeomorphic and $q(S)$ is closed since S is compact. By Mil'man's theorem U^* is the weak-star closed convex hull of $q(S) \cup (-q(S))$, and (by the compactness of S again) it is the extreme point-set of U^* and is closed. Further these are of the form $\alpha\ell_s$, $s \in S$ and $|\alpha| = 1$, (cf. [8], V.8.6). Consequently (59) becomes

$$\|Tf\|^p \le (\alpha_p(T))^p \cdot \int_{q(S)\cup(-q(S))} |\ell_s(f)|^p \, \mu(d\ell_s) \, , \quad f \in C_r(S)$$

$$\le 2\,(\alpha_p(T))^p \cdot \int_{q(S)} |\ell_s(f)|^p \mu(d\ell_s) \, ,$$

$$= 2(\alpha_p(T))^p \cdot \int_S |f(s)|^p \mu(ds), \text{ if } S \text{ and } q(S) \text{ are (as they can be) identified.}$$

For the complex case, $C(S) = C_r(S) + iC_r(S)$, and so the same holds if the constants are doubled. Thus

$$\|Tf\|_{\mathfrak{X}} \le C_p\Big[\int_S |f(s)|^p \mu(ds)\Big]^{1/p} = C_p \|f\|_{p,\mu}, \quad f \in C(S) \, , \tag{60}$$

where $C_p^p = 4[\alpha_p(T)]^p$. This form of (59) will be utilized below.

Definition 5.4 Let $\mathfrak{X}$ be a Banach space, $1 \le p \le \infty$ and $1 \le \lambda < \infty$. Then $\mathfrak{X}$ is called an $\mathcal{L}_{p,\lambda}$-space if for each finite dimensional space $E \subset F \subset \mathfrak{X}$, $1 \le n < \infty$, such that $d(F, \ell_p^n) \le \lambda$ where ℓ_p^n is the n-dimensional sequence space with p^{th} power norm and where $d(E_1,E_2) = \inf\{\|T\|\|T^{-1}\|: T \in B(E_1,E_2)\}$ for any pair of normed linear spaces E_1, E_2. A Banach space $\mathfrak{X}$ is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda \ge 1$.

It is known (and easy to verify) that each $L^p(\mu), p \ge 1$, is an $\mathcal{L}_{p,\lambda}$-space for every $\lambda > 1$, and $C(S)$ [indeed each abstract (M)-space] is an $\mathcal{L}_{\infty,\lambda}$-space for every $\lambda > 1$. The class of $\mathcal{L}_2$-spaces coincides with the class of Banach spaces isomorphic to a Hilbert space. For proofs and more on these ideas the reader is referred to the article of Lindenstrauss and Pełczyński [20].

With this set up the following general result can be established at this time on the domination problem for vector measures.

Theorem 5.5. Let S be a locally compact space and $C_o(S)$ be the Banach space of continuous scalar functions on S vanishing at '∞'. If $\mathfrak{Y}$ is an $\mathcal{L}_p$-space, $1 \le p \le 2$, and $T \in B(C_o(S), \mathfrak{Y})$, then there exist a finite positive Borel measure μ on S, and a vector measure Z on S into $\mathfrak{Y}$, such that

$$\left\|\int_S f(s) Z(ds)\right\|_{\mathfrak{Y}} = \|Tf\|_{\mathfrak{Y}} \le \|f\|_{2,\mu}, \quad f \in C_o(S). \tag{61}$$

<u>Proof</u>. Since $\mathfrak{X} = C_o(S)$ is an abstract (M)-space, it is an $\mathcal{L}_\infty$-space by the preceding remarks. But $\mathfrak{Y}$ is an $\mathcal{L}_p$-space $1 \le p \le 2$, and so $T \in B(\mathfrak{X},\mathfrak{Y})$ is 2-absolutely summing by ([20], Thm. 4.3), and therefore (cf. Prop. 5.3 above) it is also weakly compact. By the argument presented for (37), (38) above, one can use the theorem ([8],VI.7.3) even when S is locally compact (and noncompact) to conclude that there is a vector measure Z on the Borel σ-ring of S into $\mathfrak{Y}$ such that

$$Tf = \int_S f(s)Z(ds) \;,\; \text{(D-S integral)} \;.$$

Using the argument of (37), if $\tilde{S}$ is the one point compactification of S, and $\tilde{T} \in B(C(\tilde{S}),\mathfrak{Y})$ is the norm preserving extension, then $\tilde{T}$ is 2-absolutely summing (since $C(\tilde{S})$ is an abstract (M)-space), and weakly compact. So by (60) there exists a finite Borel measure $\tilde{\mu}$ on $\tilde{S}$ such that

$$\|\tilde{T}f\|_{\mathfrak{Y}} \le c_p \|f\|_{2,\tilde{\mu}} \;,\; f \in C(\tilde{S}) \;.$$

Letting $\bar{\mu} = c_p^p \tilde{\mu}$, one has $\|\tilde{T}f\|_{\mathfrak{Y}} \le \|f\|_{2,\bar{\mu}}$, $f \in C(\tilde{S})$. So (61) holds on $\tilde{S}$. Let $\mu(\cdot) = \bar{\mu}(S \cap \cdot)$ so that μ is a finite Borel measure on S. If now one restricts to $C_o(S)$ identified as a subset of $C(\tilde{S})$, so that $T = \tilde{T}|C_o(S)$, it follows from the preceding analysis that $\|Tf\|_{\mathfrak{Y}} \le \|f\|_{2,\mu}$ for all $f \in C_o(S)$. Since the integral representation of T is evidently true, this establishes (61), and completes the proof of the theorem.

If $\mathfrak{Y}$ is a Hilbert space, it is an $\mathcal{L}_2$-space so that the above theorem includes the result of Theorem 4.5. However, that special case did not depend on any results of [20]. But the general case needs all this machinery!

The following statement is actually a consequence of the above result, and it will be used in the last section:

Proposition 5.6. *Let* (Ω,Σ) *be any measurable space, and* $\mathfrak{X} = B(\Omega,\Sigma)$ *be the Banach space (under uniform norm) of scalar measurable functions. If* $\mathcal{Y}$ *is an* $\mathcal{L}_p$*-space*, $1 \le p \le 2$, *as above*, $T \in B(\mathfrak{X},\mathcal{Y})$ *is such that for each* $f_n \in \mathfrak{X}$, $f_n \to f$ *pointwise boundedly implies* $\|Tf_n\| \to \|Tf\|$, *then there exist* σ*-additive functions* $Z:\Sigma \to \mathcal{Y}$, $\mu:\Sigma \to \mathbb{R}^+$, *such that*

$$\left\|\int_\Omega f(\omega)Z(d\omega)\right\|_{\mathcal{Y}} = \|Tf\|_{\mathcal{Y}} \le \|f\|_{2,\mu} \; , \quad t \in \mathfrak{X} \; . \tag{62}$$

Proof. First a reduction of the hypothesis to that of the preceding theorem will be made through use of a basic isomorphism result (cf., [8], IV 6.18), and then with standard measure theory manipulations (62) will be established. These are not difficult, but need care. Here are the details.

Since $\mathfrak{X} = B(\Omega,\Sigma)$ is a closed subalgebra of $B(\Omega)$ $(= B(\Omega,2^\Omega))$, it follows by ([8], IV. 6.18) that there is a compact Hausdorff space S and an isometric algebraic isomorphism I between the algebra $\mathfrak{X}$ and $\mathfrak{X}_1 = C(S)$ which takes real functions into real functions preserving order and complex conjugate functions into complex conjugate functions. Let $\tilde{T} = T \circ I^{-1}:\mathfrak{X}_1 \to \mathcal{Y}$. Then $\tilde{T} \in B(\mathfrak{X},\mathcal{Y})$ and is 2-absolutely summing. Hence by the preceding theorem there is a regular Borel measure μ_1, on S into $\mathbb{R}^+$, such that $\|\tilde{T}\tilde{f}\|_{\mathcal{Y}} \le \|\tilde{f}\|_{2,\mu_1}$, $\tilde{f} \in \mathfrak{X}_1$. But $f \in \mathfrak{X}$ implies $\tilde{f} = I(f) \in \mathfrak{X}_1$ so that

$$\|Tf\|_{\mathfrak{Y}} = \|\tilde{T}\tilde{f}\|_{\mathfrak{Y}} \le \|\tilde{f}\|_{2,\mu_1}, \quad f \in \mathfrak{X} . \tag{63}$$

To simplify the right side, consider

$$\|\tilde{f}\|^2_{2,\mu_1} = \langle \tilde{f}\bar{\tilde{f}}, \mu_1 \rangle \text{ , since } \mu_1 \in C(S)^*, \ \langle \cdot , \cdot \rangle \text{ being the duality pairing,}$$

$$= \langle I(f)I(\bar{f}), \mu_1 \rangle = \langle I(f\bar{f}), \mu_1 \rangle, \ I \text{ being algebraic and conjugate preserving,}$$

$$= \langle f\bar{f}, I^*(\mu_1) \rangle \ , \ I^*: \mathfrak{X}_1^* \to \mathfrak{X}^* \text{ is the adjoint of } I,$$

$$= \int_\Omega |f|^2(\omega)\mu_2(d\omega) \ , \text{ with } \mu_2 = I^*(\mu_1) \in \mathfrak{X}^* = ba(\Omega,\Sigma). \tag{64}$$

In the last line, μ_2 is a bounded additive function on Σ, and the integral relative to such μ_2 is defined in the standard manner (cf. [8], III.2). Letting $Z(A) = T\chi_A$, $A \in \Sigma$, it follows that $Z:\Sigma \to \mathfrak{Y}$ is additive and for each step function $f(= \sum_{i=1}^{n} a_i\chi_{A_i}, A_i \in \Sigma$, disjoint) one has

$$\left\|\int_\Omega f(\omega)Z(d\omega)\right\|_{\mathfrak{Y}} = \|Tf\|_{\mathfrak{Y}} \le \left[\int_\Omega |f(\omega)|^2\mu_2(d\omega)\right]^{1/2}, \text{by (63)-(64).} \tag{65}$$

Now one can use a definition of the integral for finitely additive Z to conclude that (65) holds for all $f \in \mathfrak{X}$ so that (65) reduces to (62) if σ-additivity is replaced by finite additivity. However, the additional hypothesis on T , namely, its bounded sequential continuity, allows us to conclude that Z is σ-additive since $f = \chi_{A_n}$, $A_n \searrow \emptyset$ implies the left side of (65) tends to zero. Then by the D-S integration theory ([8], IV.10.10) the left side holds for all $f \in \mathfrak{X}$. Thus

$$\left\|\int_\Omega f(\omega)Z(d\omega)\right\|^2_{\mathfrak{Y}} \le \int_\Omega |f(\omega)|^2\mu_2(\omega) \ , \ f \in \mathfrak{X} . \tag{66}$$

To replace μ_2 by a σ-additive function, let μ be the Carathéodory generated measure by the pair (Σ,μ_2) . If $\tilde{\Sigma}$ is the class of μ-sets , then the classical real analysis methods imply (cf. eg., [36], p. 67) μ on $\tilde{\Sigma}$ is σ-additive, $\tilde{\Sigma} \supset \Sigma$ and ($\mu|\Sigma$ is μ_2 iff μ_2 is σ-additive) $\mu(A) \leq \mu_2(A)$, $A \in \Sigma$. It is now asserted that (66) holds if μ_2 is replaced by μ , Z being σ-additive. Since step functions are uniformly dense in $\mathfrak{X}$, it suffices to prove (66) with Z,μ for all step functions f . This is accompl·shed with a simple direct computation as follows.

Let $f = \sum_{i=1}^{m} a_i\chi_{A_i}$, $A_i \in \Sigma$. By the finiteness of μ_2 , and the definition of μ , given $\epsilon > 0$, there exist $A_{in}^{\epsilon} \in \Sigma$ such that $A_i \subset \bigcup_{n=1}^{\infty} A_{in}^{\epsilon}$ and

$$\mu(A_i) + \frac{\epsilon}{|a_i|^2 m} > \sum_{n=1}^{\infty}\mu_2(A_{in}^{\epsilon}) , \quad (a_i \neq 0 \text{ may be assumed }). \tag{67}$$

Replacing A_{in}^{ϵ} by $A_{in}^{\epsilon} \cap A_i (\in\Sigma)$, it may also be assumed that $A_i = \bigcup_{n=1}^{\infty} A_{in}^{\epsilon}$ here. Hence if $f_N^{\epsilon} = \sum_{i=1}^{m} a_i \chi_{\bigcup_{k=1}^{N} A_{ik}}$, then $f_N^{\epsilon} \in \mathfrak{X}$ and $f_N^{\epsilon} \to f$ pointwise and boundedly. Thus (66) becomes

$$\|Tf_N^{\epsilon}\|_{\mathfrak{Y}}^2 = \left\|\int_{\Omega} f_N^{\epsilon}(\omega)\right\|_{\mathfrak{Y}}^2 \leq \int_{\Omega} |f_N^{\epsilon}(\omega)|^2 \mu_2(d\omega) = \sum_{i=1}^{m} |a_i|^2 \mu_2\Big(\bigcup_{n=1}^{N} A_{in}^{\epsilon}\Big)$$

$$\leq \sum_{i=1}^{m} |a_i|^2 \sum_{n=1}^{N} \mu_2(A_{in}^{\epsilon}) , \text{ since } 0 \leq \mu_2 \text{ is additive.}$$

Letting $N \to \infty$ and using the bounded sequential continuity of T , one has with ([8], IV.10.10) ,

$$\|Tf\|_{\mathcal{Y}}^2 = \|\int_\Omega f(\omega)Z(d\omega)\|_{\mathcal{Y}}^2 \le \sum_{i=1}^m |a_i|^2[\mu(A_i) + \frac{\varepsilon}{|a_i|^2 m}] \text{ , by (67),}$$

$$= \int_\Omega |f(\omega)|^2 \mu(d\omega) + \varepsilon \text{ .}$$

Since $\varepsilon > 0$ is arbitrary, this implies (62) for any step function f and thus (as noted above) for all f in $\mathcal{X}$. This completes the proof.

Remarks. 1. The preceding results show that the domination problem for vector measures in L^p-spaces, $1 \le p \le 2$, is solved and hence also for harmonizable fields since only the $\mathcal{L}_2$-type spaces are involved in the latter. However, for $p > 2$, such a satisfactory solution of the problem is not available.

2. The isomorphism mapping I of the above proof, with ([8], IV. 6.18), is very handy and plays a key role in other parts of stochastic analysis. Another such application may be found in ([32], p. 130ff).

6. Stationary dilations. The results of the last section play a key role in showing that each weakly harmonizable random field has a stationary dilation. It is a consequence of the preceding work that for any stationary field $Y:G \to L_0^2(P)$ with G an LCA group, and each orthogonal projection $Q:L_0^2(P) \to L_0^2(P)$, the new random field $X(g) = QY(g)$, $g \in G$, giving $X:G \to L_0^2(P)$, is seen to be weakly harmonizable. The dilation result yields the reverse implication. A "concrete" version of this is given by the following theorem and an "operator" version will be obtained later from it.

Theorem 6.1 *Let* G *be an LCA group*, $X:G \to L_0^2(P) = \mathcal{H}$ *a weakly harmonizable random field. Then there is a super (or extension) Hilbert space* $\mathcal{K} \supset \mathcal{H}$, *a probability measure space* $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$ *with* $\mathcal{K} = L_0^2(\tilde{P})$, *and a stationary random field* $Y:G \to L_0^2(\tilde{P})$, *such that* $X(g) = QY(g)$, $g \in G$, *where* $Q:L_0^2(\tilde{P}) \to L_0^2(\tilde{P})$ *is the orthogonal projection with range* $L_0^2(P)$. *If moreover*, $\mathcal{H} = \overline{sp}\{X(g), g \in G\}$, *then* Y *determines* $\mathcal{K}$ *in the sense that* $\mathcal{K} = \overline{sp}\{Y(g), g \in G\}$. [*Thus* $\mathcal{K}$ *is the minimal super space for* $\mathcal{H}$.]

Proof. The remark has the following easy proof. In fact, if $Y:G \to L_0^2(P)$ is stationary, then Theorem 3.3 inplies

$$Y(g) = \int_{\hat{G}} \langle g,s\rangle Z(ds) \ , \ g \in G \ , \qquad (68)$$

for a vector measure Z on $\hat{G}$ into $\mathcal{K} = L_0^2(P')$, with orthogonal increments (also called orthogonally scattered) where $\hat{G}$ is the dual group of the LCA group G, and $\langle \cdot,s\rangle$ is a character of G. If $Q:\mathcal{K} \to \mathcal{K}$ is any orthogonal projection, then $\tilde{Z} = Q\circ Z$ is a stochastic measure on $\tilde{G}$ into $\mathcal{K}$.

Indeed,

$$\begin{aligned}\|\tilde{Z}\|(\hat{G}) &= \sup\{\|\sum_{i=1}^{n} a_i\tilde{Z}(A_i)\|_2^2 : |a_i| \le 1, A_i \subset \hat{G} \text{ disjoint Borel}, n \ge 1\} \\ &= \sup\{\|Q\sum_{i=1}^{n} a_i Z(A_i)\|_2^2 : |a_i| \le 1, A_i \subset \hat{G}, \text{ as above}\} \\ &\le \|Q\|^2 \sup\{\|\sum_{i=1}^{n} a_i Z(A_i)\|_2^2 : |a_i| \le 1, A_i \subset \hat{G}, \text{ as before}\} \\ &= \|Q\|^2 \sup\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{a}_j\, F(A_i \cap A_j) : |a_i| \le 1, A_i \subset \hat{G} \text{ as before}\} \\ &\qquad \text{where } F(A_i \cap A_j) = (Z(A_i), Z(A_j))_{\mathcal{K}}, \\ &= \|Q\|^2 |F|(\hat{G}) \le F(\hat{G}) < \infty,\end{aligned}$$

since F is the spectral measure of Z and so is finite and Q is a contraction. Hence $\tilde{Z}$ has finite semivariation and is clearly σ-additive, so that it is a stochastic measure. By Theorem 3.3, X given by $X(g) = QY(g) = \int_{\hat{G}} \langle g,s\rangle \tilde{Z}(ds)$, $g \in G$, is weakly harmonizable. (Note that the same conclusion holds if Q is replaced by any bounded linear operator on $\mathcal{K}$. If the range of the projection Q is not finite dimensional, then X need <u>not</u> be strongly harmonizable!)

To go in the reverse direction, the (possibly) augmented space $\mathcal{K} \supset \mathcal{H}$ has to be constructed. Consider $X: G \to \mathcal{H} = L_0^2(P)$, the given weakly harmonizable random field. In order to get simultaneously the additional structure demanded in the last part, let $\mathcal{H} = \overline{sp}\{X(g), g \in G\}$ also. Then, as before, there is a stochastic measure on $\hat{G}$ into $\mathcal{H}$ such that

$$X(g) = \int_{\hat{G}} \langle g,s\rangle Z(ds) \in \mathcal{H}, \quad g \in G. \tag{69}$$

By Theorem 5.5, with $\mathcal{Y} = \mathcal{H}$, there exists a finite Radon (= regular Borel) measure μ on $\hat{G}$ such that

$$\left\|\int_{\hat{G}} f(t)Z(dt)\right\|_2^2 \le \int_{\hat{G}} |f(t)|^2 \mu(dt) \ , \ f \in C_o(\hat{G}) \ . \tag{70}$$

Next define a mapping $v:\mathcal{B}(\hat{G}\times\hat{G}) \to \mathbb{R}^+$ by the equation

$$v(A,B) = \mu(A\cap B), \ A,B \in \mathcal{B}(\hat{G}) \ , \tag{71}$$

where $\mathcal{B}(\hat{G})$ is the Borel σ-ring of $\hat{G}$ and similarly $\mathcal{B}(\hat{G}\times\hat{G})$. Then v is a bimeasure of finite Vitali variation on $\mathcal{B}(\hat{G}) \times \mathcal{B}(\hat{G})$ and since this ring generates $\mathcal{B}(\hat{G}\times\hat{G})$, v extends to a Radon measure on the latter σ-ring. Moreover, it is clear that v concentrates on the diagonal of the product space $\hat{G} \times \hat{G}$. If $C_b(\hat{G})$ denotes the Banach space of bounded continuous scalar functions on $\hat{G}$ with uniform norm, then

$$\int_{\hat{G}}\int_{\hat{G}} f(s,t) \ v(ds,dt) = \int_{\hat{G}} f(s,s)\mu(ds), \ f \in C_b(\hat{G}\times\hat{G}) \ . \tag{72}$$

Let $F(A,B) = (Z(A),Z(B))$ so that $F:\mathcal{B}(\hat{G}\times\hat{G}) \to \mathbb{C}$ is a bimeasure of finite semi‿variation, from (69). Thus using the D-S and MT-integration techniques as before,

$$0 \le \left\|\int_{\hat{G}} f(s)Z(ds)\right\|_2^2 = \int_{\hat{G}}\int_{\hat{G}} f(s)\overline{f(t)}F(ds,dt), \ f \in C_b(\hat{G}) \ . \tag{73}$$

Letting $f(s,t) = f(s)\cdot f(t)$ in (72), $\alpha = v - F$ one has from (70) - (73),

$$0 \le \int_{\hat{G}} |f(s)|^2\mu(ds) - \left\|\int_{\hat{G}} f(s)Z(ds)\right\|_2^2 = \int_{\hat{G}}\int_{\hat{G}} f(s)\overline{f(t)}[v(ds,dt] - F(ds,dt)]$$

$$= \int_{\hat{G}}\int_{\hat{G}} f(s)\overline{f(t)} \ \alpha(ds,dt), \ f \in C_b(\hat{G}) \ . \tag{74}$$

So α is positive semi-definite and $\alpha = 0$ iff $v = F$, i.e., if F concentrates on the diagonal. This corresponds to X being staionary itself. Excluding this trivial case, $\alpha \neq 0$, and (74) is strictly positive, if $f = 1$. It follows from (74) that $[\cdot,\cdot]' : C_b(\hat{G}) \times C_b(\hat{G}) \to \mathbb{C}$ defines a nontrivial semi-inner product, where

$$[f,g]' = \int_{\hat{G}}\int_{\hat{G}} f(s)\overline{g(t)}\, \alpha(ds,dt), \quad f,g \in C_b(\hat{G}) .$$

If $\mathfrak{n}_0 = \{f : [f,f'] = 0 , f \in C_b(\hat{G})\}$, and $\mathcal{H}_1 = C_b(\hat{G})/\mathfrak{n}_0$ is the factor space, let $[\cdot,\cdot] : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ be defined by

$$[(f),(g)] = [f,g]' , \quad f \in (f) \in \mathcal{H}_1, \; g \in (g) \in \mathcal{H}_1 . \tag{75}$$

Then $[\cdot,\cdot]$ is an inner product on $\mathcal{H}_1$ and one lets $\mathcal{H}_0$ be its completion in $[\cdot,\cdot]$. Let $\pi_0 : C_b(\hat{G}) \to \mathcal{H}_0$ be the canonical projection. Thus $\mathcal{H}_0$ is nontrivial and need not be separable. Now let us replace $\mathcal{H}_0$ by $L_0^2(P')$ on a probability space (Ω',Σ',P') . This can be done based on the Fubini-Jessen theorem where P' can even be taken to be a Gaussian measure (for the real $\mathcal{H}$, see [32], pp. 414-415). The complex case is *similar. A quick outline can be given:* Let $\{h_i, i \in I\} \subset \mathcal{H}_0$ be a CON set. If (Ω_i,Σ_i,P_i) is a probability space determined by a standard Gaussian variable, so that one can take $\Omega_i = \mathbb{C}$, Σ = Borel σ-algebra of $\mathbb{C}$, and $P_i(A) = (2\pi)^{-1/2}\int_A \exp(-\frac{|t|^2}{2})dt_1dt_2$, $A \in \Sigma_i$, $(t = t_1+it_2)$, let $(\Omega',\Sigma',P') = \underset{i\in P}{\otimes}(\Omega_i,\Sigma_i,P_i)$ the product space given by the Fubini-Jessen theorem. If $X_i(\omega) = \omega(i) = \omega(i)$, $\omega \in \Omega' = \mathbb{C}^I$, the coordinate function, then $E(X_i) = 0$ and $E(|X_i|^2) = 1$. Also $\{X_i, i \in I\}$ forms a CON basis in $L_0^2(P')$. The correspondence $\tau : h_i \to X_i$, extended linearly, sets up an isomorphism of $\mathcal{H}_0$ onto $L_0^2(P')$, and $\|\tau(h_i)\|_2^2 = E(|X_i|^2) = 1 = [h_i,h_i]$, $i \in I$. Then by polarization one has $[h_i,h_j] = E(\tau(h_i)\overline{\tau(h_j)})$, so that τ is an isometric isomorphism of $\mathcal{H}_0$ onto $L_0^2(P')$.

If $\pi = \tau \circ \pi_0 : f \mapsto \tau(\pi_0(f)) \in L_0^2(P') = \mathcal{H}'$, $f \in C_b(\hat{G})$, is the composite (canonical) mapping, let $X_1(t) = \pi(e_t(\cdot)) \in \mathcal{H}'$ where $e_t : s \mapsto (t,s)$, is a character of G at $t \in G$. Note that $e_0 = 1 \notin \mathfrak{h}_0$, so $\pi_0(1)$ can be identified with the constant $1 \in C_b(\hat{G})$. Thus $X_1(0) = \tau(1)$, $E(|\tau(1)|^2) = 1$. Let $\mathcal{H}'' = \overline{sp}\{X_1(t), t \in G\} \subset \mathcal{H}'$. Then there exists a probability space $(\Omega'', \Sigma'', P'')$, as above, such that $\mathcal{H}'' = L^2(P'')$. Finally set $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}''$, the direct sum of Hilbert spaces $L_0^2(P)$ and $L_0^2(P'')$. If $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) = (\Omega, \Sigma, P) \otimes (\Omega'', \Sigma'', P'')$ then one can identify, in a natural way, $\mathcal{K} = L_0^2(\tilde{P})$. Define $Y(t) = X(t) + X_1(t)$, $t \in G$, so that $(X(t), X_1(t)) = 0$ since $\mathcal{H} \perp \mathcal{H}''$ in $\mathcal{K}$. Then $\{Y(t), t \in G\} \subset \mathcal{K} = L_0^2(\tilde{P})$, and if $Q : \mathcal{K} \to \mathcal{H} = \{\mathcal{H} \oplus \{0\}\}$ is the orthogonal projection, one has $X(t) = QY(t)$, $t \in G$. It remains to show that $Y : G \to L_0^2(\tilde{P})$ is stationary. By construction $Y(0) = X(0) + X_1(0)$ and this is $X(0)$ only when $X_1(0) = 0$ which can happen iff $\mathcal{H}'' = \{0\}$, i.e. when no enlargement is needed.

To verify stationarity, consider

$$
\begin{aligned}
r(s,t) = (Y(s),Y(t)) &= (X(s),X(t)) + (X_1(s),X_1(t)) \text{ since } X \perp X_1, \\
&= \int_{\hat{G}}\int_{\hat{G}} (s,\lambda)\overline{(t,\lambda')} F(d\lambda, d\lambda') + \int_{\hat{G}}\int_{\hat{G}} (s,\lambda)\overline{(t,\lambda')} \alpha(d\lambda, d\lambda'), \\
&\qquad \text{by (73) and (75) and these are MT-integrals,} \\
&= \int_{\hat{G}}\int_{\hat{G}} (s,\lambda)\overline{(t,\lambda')}\, \nu(d\lambda, d\lambda'), \text{ since } \alpha = \nu - F \\
&= \int_{\hat{G}} (s,\lambda)\overline{(t,\lambda)}\, \mu(d\lambda), \text{ by (72),} \\
&= \int_{\hat{G}} (s-t,\lambda)\, \mu(d\lambda), \text{ by the composition of characters.}
\end{aligned} \tag{76}
$$

Since μ is a finite positive measure, (76) implies $r(s+h,t+h) = r(s,t) = \tilde{r}(s-t)$, and so the $Y:G \to L_0^2(\tilde{P})$ is stationary. The construction also implies that $\overline{sp}\{Y(t), t \in G\} = \mathcal{K}$ in the case that $\mathcal{H} = \overline{sp}\{X(t), t \in G\}$. This completes the proof.

The following is a useful deduction:

<u>Corollary</u> 6.2 <u>Every vector measure</u> $\nu:\mathcal{B}(G) \to \mathcal{H}$ <u>where</u> G <u>is an LCA group</u>, $\mathcal{B}(G)$ <u>being its Borel algebra</u>, <u>and</u> $\mathcal{H}$ <u>is a Hilbert space, has an orthogonally scattered dilation</u>.

<u>Proof</u>. Since $G = \hat{\hat{G}}$ consider the mapping $X:\hat{G} \to \mathcal{H}$ defined as the D-S integral $X(\hat{g}) = \int_G \langle \hat{g}, \lambda \rangle \, \nu(d\lambda)$. Then X is V-bounded; so it is weakly harmonizable. By the above theorem there are an extension Hilbert space $\mathcal{K} \supset \mathcal{H}$, an orthogonal projection $Q:\mathcal{K} \to \mathcal{K}$, with range $\mathcal{H}$, and a stationary field $Y:\hat{G} \to \mathcal{K}$ such that $X(\hat{g}) = QY(\hat{g})$. Let Z be the stochastic measure representing Y, (cf. Theorem 3.3). Hence for each $h \in \mathcal{H}$ one has $(Z:\mathcal{B}(\hat{G}) \to \mathcal{K})$

$$\int_G (\hat{g},\lambda)(\nu(d\lambda),h) = (X(\hat{g}),h) = (QY(\hat{g}),h) = \int_{\hat{G}} (\hat{g},\lambda)(Q \circ Z\,(d\lambda),h).$$

These are now scalar (Lebesgue-Stieltjes) integrals. By the classical uniqueness theorem of Fourier analysis for such integrals, one has $(\nu(A) - Q \circ Z(A), h) = 0$, $A \in \mathcal{B}(G), h \in \mathcal{H}$. Hence $\nu = Q \circ Z$. Since Z is orthogonally scattered by virtue of the fact that Y is stationary, the result follows.

With the last theorem, a more perspicuous version of the dilation problem for a weakly harmonizable random field can be given. This, however, depends also on an interesting theorem of Sz.-Nagy [37] and wil be presented. Recall from the classical theory of stationary processes ([6], p. 512 and p. 638) every such process $\{Y_t, t \in \mathbb{R}\} \subset L_0^2(P)$, can be expressed as $Y_t = U_t X_0$, where $\{U_t, t \in \mathbb{R}\}$ is a group of unitary operators acting on $L_0^2(P)$ (first on $\overline{sp}\{Y_t, t \in \mathbb{R}\}$ and then for instance, define each U_t as an identity on the orthogonal complement of this subspace). The spectral theory of U_t then yields immediately the corresponding integral representation of Y_t's. The same result holds if $\mathbb{R}$ is replaced by an LCA group G. The corresponding operator representation for harmonizable processes (or fields) is not so simple. Its solution will be presented in the following theorem. Recall that a family $T: G \to B(\mathfrak{X})$, $\mathfrak{X}$ a Hilbert space, is <u>positive definite</u>, if $T(-g) = T(g)^*$ (adjoint operator) and for each finite set $x_{s_1}, \ldots, x_{s_n}$ of elements of $\mathfrak{X}$ indexed by $J = (s_1, s_2, \ldots, s_n) \subset G$, one has

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (T(s_j^{-1} s_i) x_{s_i}, x_{s_j}) \geq 0 .$$

<u>Theorem</u> 6.3 <u>Let</u> G <u>be an LCA group and</u> $X: G \to L_0^2(P) = \mathfrak{X}$, <u>a Hilbert space, be weakly harmonizable. Then there exists a super Hilbert space</u> $\mathcal{K} = L_0^2(\tilde{P}) \supset \mathfrak{X}$ <u>on an enlarged probability space</u> $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, <u>a random variable</u> $Y_0 \in \mathcal{K}$ <u>a weakly continuous family</u> $\{T(g), g \in G\}$ <u>of contractive linear operators from</u> $\mathcal{K}$ <u>to</u> $\mathfrak{X}$ <u>with</u> $T(0)$ <u>as the identity on</u> $\mathfrak{X}$ (0 <u>being the nutral element of</u> G), <u>such that, when its domain is restricted to</u> $\mathfrak{X}$, <u>it is positive definite, in terms of which</u> $X(g) = T(g) Y_0, g \in G$. <u>Conversely every weakly continuous contractive family</u> $\{T(g), g \in G\}$ <u>of the above type from any super Hilbert space</u> $\mathcal{K} \supseteq \mathfrak{X}$, into $\mathfrak{X}$, <u>which</u>

<u>when restricted to</u> $\mathfrak{X}$ <u>is positive definite, defines a weakly harmonizable process</u> $X:G \to \mathfrak{X}$, <u>by the equation</u> $X(g) = T(g)Y_0$ <u>for any</u> $Y_0 \in \mathfrak{X}$.

<u>Proof</u>. The direct part is an operator-theoretic reformulation of Theorem 6.1. Briefly, let $X:G \to L_0^2(P) = \mathfrak{X}$ be weakly harmonizable. Then there exists a $\mathcal{K} = L_0^2(\tilde{P}) \supset \mathfrak{X}$ and a stationary $Y:G \to \mathcal{K}$ such that $X(g) = QY(g), g \in G$, by Theorem 6.1 where Q is the orthogonal projection on $\mathcal{K}$ with range $\mathfrak{X}$. But $Y(g) = U(g)Y(0)$ where $\{U(g), g \in G\}$ is a (strongly) continuous group of unitary operators on $\mathcal{K}$. This is classical for the stationary case. Let $T(g) = QU(g), g \in G$. It is asserted that $\{T(g), g \in G\}$ is the desired family.

Indeed, $T(0) = Q$ (= identity on $\mathfrak{X}$), and $\|T(g)\| \le \|Q\|\|U(g)\| \le 1$. The continuity of $U(g)$ on G clearly implies the weak continuity of $T(g)$'s. To verify the positive definiteness on $\mathfrak{X}$, let $h_{s_1},\dots,h_{s_n}$ be a finite set in $\mathfrak{X}$. Then letting $\tilde{T}(g) = T(g)|_{\mathfrak{X}}$ one has $\tilde{T}(-g) = (\tilde{T}(g))^*$ since

$$\begin{aligned}(\tilde{T}(-g)h_{s_1},h_{s_2}) &= (QU(-g)h_{s_1},h_{s_2}) = (U^*(g)h_{s_1},Qh_{s_2}) \\ &= (h_{s_1},U(g)h_{s_2}), \text{ since } Qh_{s_i} = h_{s_i} \text{ and } U^{**}(g) = U(g) , \\ &= (Qh_{s_1},U(g)h_{s_2}) = (h_{s_1},QU(g)h_{s_2}) \\ &= (h_{s_1},\tilde{T}(g)h_{s_2}) = (\tilde{T}(g)^*h_{s_1},h_{s_2}), \text{ all } h_{s_i} \in \mathfrak{X} , \ i = 1,2 .\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{T}(s_j^{-1}s_i)h_{s_i},h_{s_j}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(QU(-s_j)U(s_i)h_{s_i},h_{s_j})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}(U(s_j)^{*}U(s_i)h_{s_i},h_{s_j})$$

$$= \|\sum_{i=1}^{n}U(s_i)h_{s_i}\|^2 \geq 0 .$$

The converse depends explicitly on an important theorem of Sz.-Nagy ([37], Thm. III; see also [38], p. 25 for a streamlined proof). According to this result if $\tilde{T}(\cdot) = T(\cdot)|_{\mathfrak{X}}$, then there is a super Hilbert space $\mathcal{K}_1 \supset \mathfrak{X}$ ($\mathcal{K}_1$ may be quite different from $\mathcal{K}$) and a weakly (hence strongly) continuous group $\{V(g), g \in G\}$ of unitary operators on $\mathcal{K}_1$ such that $\tilde{T}(g) = Q_1V(g)|_{\mathfrak{X}}$, Q_1 being the orthogonal projection of $\mathcal{K}_1$ onto $\mathfrak{X}$. Here $\mathcal{K}_1$ can be chosen as $\mathcal{K}_1 = \overline{sp}\{V(g)\mathfrak{X}, g \in G\}$. If $x_0 \in \mathfrak{X}$ is arbitrary, then $x_0 \in \mathcal{K}_1 \cap \mathcal{K}$, and $T(g)x_0 = \tilde{T}(g)x_0 = Q_1V(g)x_0 = X(g)$, say, $g \in G$. Then $\{Y(g) = V(g)x_0, g \in G\} \subset \mathcal{K}_1$ is a stationary process so that by the first paragraph of the proof of Theorem 6.1, $\{X(g), g \in G\} \subset \mathfrak{X}$ is weakly harmonizable. Thus for each $x_0 \in \mathfrak{X}$, $\{T(g)x_0, g \in G\}$ is weakly harmonizable, and this completes the proof.

Remark. In the converse direction one can take $\mathcal{K} = \mathfrak{X}$. However in the forward direction, it is not always possible to take Y_0 in $\mathfrak{X}$, so that $X(0) = Y_0$, as the example following Definition 2.1 shows. Thus there is an inherent asymmetry in the statement of this theorem, and the mention of the super Hilbert space $\mathcal{K}$ in the enunciation cannot be avoided. This does not simplify even for strongly harmonizable fields! (Compare with [1].)

7. Characterizations of weak harmonizability. In this section a different type of characterization, based on the V-boundedness concept crucially, of weak harmonizability as well as a comprehensive statement embodying all the other equivalences of this concept are given. The comparison will illuminate the structure of this general class of processes. However, it is interesting and useful to obtain a characterization of V-boundedness for a general Banach space, and then specialize the result for the harmonizable case.

In this context let us say that $X:G \to \mathfrak{X}$, a Banach space, is a generalized (or vector) Fourier transform if G is an LCA group, and if there is a vector measure $\nu:\mathcal{B}(\hat{G}) \to \mathfrak{X}$ such that $X(g) = \int_{\hat{G}} \langle g,s\rangle \nu(ds), g \in G$. In [29], Phillips has extended the fundamental scalar result of Bochner's V-boundedness to certain Banach spaces with $G = \mathbb{R}$. Later but apparently independently, the LCA group case was given by Kluvanek in ([19], p. 269). In the present terminology this can be stated as follows:

Proposition 7.1 Let G be an LCA group and $\mathfrak{X}$ a Banach space. Then a mapping $X:G \to \mathfrak{X}$ is a generalized Fourier transform of a regular vector measure $\nu:\mathcal{B}(\hat{G}) \to \mathfrak{X}$ (i.e. for given $\varepsilon > 0$ and $E \in \mathcal{B}(\hat{G})$, there exist an open set O and a compact set C with $O \supset E \supset C$ such that for each $F \subset O - C$, $F \in \mathcal{B}(\hat{G})$ one has $\|\nu(F)\| < \varepsilon$) iff X is weakly continuous and V-bounded (in the sense of Definition 4.1).

On the other hand, when $\mathfrak{X} = \mathbb{C}$, a different kind of characterization was given by Helson [12]. A vector extension of this is used for the weak harmonizability problem, and will be presented here. Let $L^k(G)$ be the Lebesgue space on G relative to an Haar measure, denoted dg. Similarly $L^k(\hat{G})$ is defined on the dual group $\hat{G}$, and $L^k_{\mathfrak{X}}(\hat{G})$ for $\mathfrak{X}$-valued function space. Let $\hat{L}^1(\hat{G}) = \{\hat{f}:\hat{f}(t) = \int_G \langle t,s\rangle f(s)ds,$ $f \in L^1(G)\}$ and a similar definition for $\hat{L}^1_{\mathfrak{X}}$, the integrals in the latter being in the sense of Bochner.

The following result contains the desired extension:

Theorem 7.2 Let G be an LCA group, $\mathfrak{X}$ a reflexive Banach space, and $X:G \to L^1_{\mathfrak{X}}(G)$ be a mapping. Then X is a generalized Fourier transform of a vector measure ν on $\hat{G}$ into $\mathfrak{X}$ iff for each $p \in \hat{L}^1(\hat{G})$ the mapping $Y_p = (Xp):G \to \hat{L}^1_{\mathfrak{X}}(\hat{\hat{G}}) = \hat{L}^1_{\mathfrak{X}}(\hat{G})$ is well defined, i.e., iff $\hat{L}^1_{\mathfrak{X}}(\hat{G})$ is a module over $\hat{L}^1(\hat{G})$.

Proof. Suppose X is a generalized Fourier transform of ν on $\hat{G}$ to $\mathfrak{X}$, so that

$$X(g) = \int_{\hat{G}} \langle g,s\rangle \nu(ds) \ , \quad g \in G \ . \tag{77}$$

By hypothesis $p \in \hat{L}^1(\hat{G})$ so that $p = \hat{f}$ for a unique $f \in L^1(\hat{G})$. Hence $X(g)p(g)$ is well-defined, and if $\ell \in \mathfrak{X}^*$, then by the scalar theory one has

$$\begin{aligned}\ell(X(g)\cdot p(g)) = p(g)\ell(X(g)) &= \int_{\hat{G}} \langle g,s\rangle f(s)ds \int_{\hat{G}} \langle g,t\rangle \ell\circ\nu(dt)\\ &= \int_{\hat{G}} \langle g,s\rangle (\ell\circ\nu * f)ds, \text{ since } (\ell\circ\nu * f)^{\wedge} = (\ell\circ\nu)^{\wedge}\cdot\hat{f}\\ &\qquad\qquad \text{the "*" denoting convolution,}\\ &= \int_{\hat{G}} \langle g,s\rangle k_{\ell}(s)ds \ ,\end{aligned} \tag{78}$$

where $k_\ell = \ell\circ\nu * f \in L^1(G)$ by the classical theory (cf. [35], Thm. I.3.5 (a)). Also $k_{(\cdot)}(s):\mathfrak{X}^* \to \mathbb{C}$ is additive, and $\|k_\ell(\cdot)\|_1 \le \|f\|_1\cdot\|\ell\|\cdot\|\nu\|(\hat{G}) \to 0$ as $\ell \to 0$ in $\mathfrak{X}^*$. Hence $k_\ell(s) \to 0$ as $\ell \to 0$ for a.a.(s), so that $k_\ell(s) = \tilde{k}(s)(\ell)$ for a $\tilde{k}(s) \in \mathfrak{X}^{**} = \mathfrak{X}$ by reflexivity, and for a.a.(s). Thus $\tilde{k}(\cdot)$ is Pettis integrable on $\hat{G}$, and the mapping $Z_p(\cdot):A \to \int_A \tilde{k}(s)ds$, defines a σ-additive bounded set function into $\mathfrak{X}$, a vector measure, by known results in Abstract Analysis. Consequently,

$$\begin{aligned}\ell(X(g))\cdot p(g) &= \int_{\hat{G}}\langle g,s\rangle\ \ell\circ Z_p(ds)\\ &= \ell\left(\int_{\hat{G}}\langle g,s\rangle Z_p(ds)\right),\quad \ell \in \mathfrak{X}^* . \end{aligned}\tag{79}$$

Since Z_p is a vector measure, $\|Z_p\|(\hat{G}) < \infty$, and $\ell \in \mathfrak{X}^*$ is arbitrary, one has

$$Y_p(g) = (X\cdot p)(g) = \int_{\hat{G}}\langle g,s\rangle Z_p(ds) \in \mathfrak{X},\ g \in G, \tag{80}$$

to be well-defined. Also $|Y_p(g)|_{\mathfrak{X}} = |p(g)|\,|X(g)|_{\mathfrak{X}} \le \|f\|_1\cdot|X(g)|_{\mathfrak{X}}$ so that $\|Y_p\|_1 \le \|f\|_1\|X\|_1 < \infty$ and by (80) Y_p is the Fourier transform of the vector measure Z_p on $\hat{G}$ into $\mathfrak{X}$. Hence $Y_p \in \hat{L}^1_{\mathfrak{X}}(\hat{G})$. This proves the direct part. The sufficiency is slightly more involved.

Thus, for sufficiency, let $Xp = Y_p \in \hat{L}^1_{\mathfrak{X}}(\hat{G})$ for each $p \in \hat{L}^1(\hat{G})$. Since $\mathfrak{X}$ is reflexive, by Proposition 7.1, it is enough to establish that the (weakly continuous) X is V-bounded (cf., Definition 4.1). This is accomplished in two stages.

Let us first define an operator $\tau:L^1(\hat{G}) \to L^1_{\mathfrak{X}}(\hat{G})$ by the equation:

$$(\tau f)^{\wedge} = p \cdot X = Y_p \text{ , } p = \hat{f} \text{ , } f \in L^1(\hat{G}) \text{ .} \tag{81}$$

Then $(\tau f)^{\wedge} \in \hat{L}^1_{\mathfrak{X}}(\hat{G})$ by hypothesis for each $f \in L^1(\hat{G})$. Clearly τ is linear. It is also bounded. To see this, let us show that it is closed so that the desired assertion follows by the closed graph theorem. So let $f_n, f \in L^1(\hat{G})$ $f_n \to f$ in norm, and $h_n = \tau f_n \to h$ in $L^1_{\mathfrak{X}}(\hat{G})$. Then (cf., [19], p. 268)

$$\|\hat{f}_n - \hat{f}\|_u \le \|f_n - f\|_1 \to 0 \text{ and } \|\hat{h}_n - \hat{h}\|_u \le \|h_n - h\|_1 \to 0 \text{ ,}$$

as $n \to \infty$. But then

$$\hat{h}_n = (\tau f_n)^{\wedge} = X \cdot \hat{f}_n \to \hat{h} \text{ and } \hat{f}_n \to \hat{f} \text{ , uniformly.}$$

$$\|X\hat{f} - \hat{h}\|(s) \le \|X(\hat{f}_n - \hat{f})\|(s) + \|X\hat{f}_n - \hat{h}\|(s)$$

$$\le \|X(s)\| \; |\hat{f}_n - \hat{f}|(s) + \|\hat{h}_n - \hat{h}\|(s) \to 0 \text{ , as } n \to \infty, s \in \hat{G}.$$

Hence $X\hat{f} = \hat{h} = (\tau f)^{\wedge}$, and $\tau f = h$ (by uniqueness). So τ is closed.

Next let us verify the key property of V-boundedness for X. Since Y_p is continuous for each $p \in \hat{L}^1(\hat{G})$, it follows that X is weakly continuous. Let $h \in L^1(G)$. Consider the operator $T: L^1(G) \to \mathfrak{X}$ defined by

$$T(h) = \tilde{T}(\hat{h}) = \int_G X(g)h(g)dg \text{ , } \|\hat{h}\|_u \le 1 \text{ .} \tag{82}$$

Since the correspondence $h \leftrightarrow \hat{h}$ is 1-1, $\tilde{T}$ is well-defined on $\hat{L}^1(\hat{G})$, and it is to be shown that $\tilde{T}: \hat{L}^1(G) \to \mathfrak{X}$ is bounded when the former is endowed with the uniform norm.

Let $\{e_\alpha, \alpha \in I\} \subset L^1(\hat{G})$ be an approximate unit (cf., [35], p. 6) so that $\|e_\alpha\|_1 = 1$, $e_\alpha \ge 0$ and $\|e_\alpha - e_\beta\|_1 \to 0$ as $\alpha, \beta \nearrow$ '∞'. Now $(\tau e_\alpha)^{\wedge} = X \cdot \hat{e}_\alpha$ ($= X_\alpha$, say). The hypothesis implies $X_\alpha \in \hat{L}^1_{\mathfrak{X}}(\hat{G})$, $\alpha \in I$, and

$$\|X_\alpha - X_\beta\|(t) = \|(\tau(e_\alpha - e_\beta))^\wedge(t)\| \le \|\tau(e_\alpha - e_\beta)\|_1$$

$$\le \|\tau\| \|e_\alpha - e_\beta\|_1 \to 0 \ , \ t \in \hat{G} \ , \tag{83}$$

since τ was shown to be bounded. Thus $X_\alpha \to X$ uniformly. Since $\|X_\alpha\|_u < \infty$, the operator T_α defined below is bounded:

$$T_\alpha(h) = \int_{\hat{G}} X_\alpha(t)h(t)dt, \quad h \in L^1(\hat{G}). \tag{84}$$

But X is the uniform limit of X_α's so it is also bounded, and hence T of (82) is bounded. Moreover,

$$\|T(h) - T_\alpha(h)\|_{\mathfrak{X}} = \|\int_{\hat{G}} (X - X_\alpha)(t)h(t)dt\|_{\mathfrak{X}}$$

$$\le \|X - X_\alpha\|_u \cdot \int_G |h(t)|dt \to 0 \ , \ \text{by (83)} \ ,$$

as $\alpha \nearrow {}'\infty{}'$. Hence $\|T_\alpha(h)\| \to \|T(h)\|$, and

$$T(h) = \lim_\alpha \int_{\hat{G}} X_\alpha(t)h(t)dt \ (= \int_{\hat{G}} X(t)h(t)dt) \ . \tag{85}$$

If $\ell \in \mathfrak{X}^*$, (85) implies

$$(\ell \circ T)(h) = \lim_\alpha \int_{\hat{G}} \ell(X(t))h(t)dt \ (= \lim_\alpha (\ell \circ T_\alpha)(h)) \ .$$

On the other hand,

$$\begin{aligned}(\ell \circ T_\alpha)(h) &= \int_{\hat{G}} \ell(X_\alpha(t))h(t)dt = \int_{\hat{G}} \ell((\tau e_\alpha)^\wedge)h(t)dt \\ &= \int_{\hat{G}} h(t) \cdot \ell(\int_{\hat{G}} \langle g,t\rangle (\tau e_\alpha)(g)dg)dt \\ &= \int_{\hat{G}} h(t)\langle g,t\rangle dt \int_{\hat{G}} \ell(\tau e_\alpha)(g)dg \ , \text{ by Fubini's theorem,} \\ &= \int_{\hat{G}} \ell(\tau e_\alpha)(g)\hat{h}(g)dg \ , \text{ by Fubini again} \ .\end{aligned}$$

Thus

$$|(\ell \circ T_\alpha)(h)| \le \|\hat{h}\|_u \|\ell(\tau e_\alpha)\|_1 \le \|\hat{h}\|_u \cdot \|\ell\| \|\tau\| \cdot \|e_\alpha\|_1 \ . \tag{86}$$

Taking suprema on $\|\ell\| \le 1$, and noting that $\|e_\alpha\|_1 = 1$, (86) implies

$$\|T_\alpha(h)\| \le \|\hat{h}\|_u \|\tau\| \ . \tag{87}$$

Thus (85) and (87) yield that $\|T(h)\| \le c\|\hat{h}\|_u$ with $c = \|T\| < \infty$. So X is V-bounded. Since $\mathfrak{X}$ is reflexive, Proposition 7.1 now applies and yields (77) for a unique vector measure ν on $\hat{G}$ into $\mathfrak{X}$. This completes the proof.

Remarks. 1. Since $\hat{G}$ is not necessarily compact, $p = 1$ in Y_p is not necessarily possible; so that the result of the theorem cannot be trivialized.

2. The necessity proof also holds (and thus the theorem) if $\hat{L}^1(\hat{G})$ is replaced by $\hat{\mathfrak{m}}(\hat{G}) = \{\hat{\mu}:\hat{\mu}(t) = \int_{\hat{G}}\langle g,t\rangle\mu(dg),\ \mu \in \mathfrak{m}(\hat{G}), t \in G\}$, where $\mathfrak{m}(\hat{G})$ is the space of regular signed Borel measures on $\hat{G}$. In fact let $Y_p = \hat{\mu}X$, where $p = \hat{\mu}$ (is a function!), so that for $\ell \in \mathfrak{X}^*$,

$$\ell(Y_p(t)) = \int_{\hat{G}}\langle g,t\rangle\mu(dg) \cdot \int_{\hat{G}}\langle s,t\rangle \ell\circ Z(ds) = (\hat{\mu}\cdot \widehat{\ell\circ Z})(t)$$

$$= (\mu * \ell\circ Z)\hat{}\,(t) = \ell\left(\int_{\hat{G}}\langle g,t\rangle(\mu * Z)(dg)\right).$$

using the covolution products appropriately (cf., e.g. [19]). Thus $\mu * Z$ is a vector measure on $\hat{G}$ and $\|\mu * Z\|(\hat{G}) \le |\mu|(\hat{G})\,\|Z\|(\hat{G}) < \infty$. Thus Y_p is a Fourier transform of $\mu * Z$. Identifying $L^1(\hat{G}) \hookrightarrow \mathfrak{m}(\hat{G})$ as $\tilde{\mu}: A \mapsto \int_A f(t)dt$, the sufficiency proof of theorem and the above lines show that $\hat{L}^1(\hat{G})$ can be replaced by $\hat{\mathfrak{m}}(\hat{G})$ everywhere in that result.

Taking $\mathfrak{X} = L_0^2(P)$ so that V-boundedness is the same as weak harmonizability, the above theorem together with Theorems 3.3, 6.3, yield the following two summary statements on characterizations of weakly harmonizable random fields.

Theorem 7.3 *Let* G *be an LCA group*, $\mathfrak{X} = L_0^2(P)$ *and* $X: G \to \hat{L}^1_{\mathfrak{X}}(\hat{G})$

be a weakly continuous mapping. Then the following statements are equivalent:

(i) X is weakly harmonizable

(ii) X is bounded and V-bounded

(iii) X is the Fourier transform of a regular vector measure on $\hat{G}$ into $\mathfrak{X}$.

(iv) for each $p \in \hat{L}^1(\hat{G})$, the process $Y_p = Xp : G \to L_0^2(P)$ is weakly harmonizable.

Furthermore, the following implies (i) - (iv):

(v) if $\mathcal{H} = \overline{sp}\{X(g), g \in G\} \subset \mathfrak{X}$, then there exists a weakly continuous contractive positive definite family of operators $\{T(g), g \in G\} \subset B(\mathcal{H})$ such that $T(0) = $ identity, and $X(g) = T(g)X(0)$, $g \in G$.

In order to present a similar description of the dilation results, these individual statements should be couched in terms of classes. Let us therefore define various classes in $\mathfrak{X}$.

$\mathfrak{V}$ = the set of bounded weakly continuous V-bounded random fields on G.

$\mathfrak{W}$ = the set of weakly harmonizable random fields on G.

$\mathfrak{F}$ = the class of all random fields which are Fourier transforms of regular vector measures on $\hat{G} \to \mathfrak{X}$.

$\mathfrak{M}$ = the module over $\hat{L}^1(\hat{G})$ of all functions on $\hat{G} \to \mathfrak{X}$, in $\hat{L}^1_{\mathfrak{X}}(\hat{G})$.

$\mathfrak{P}$ = the class of all random fields on $G \to \mathfrak{X}$ which are projections of stationary fields on $G \to \mathcal{K}$, where $\mathcal{K} \supset \mathfrak{X}$ is some extension (or super) Hilbert space of $\mathfrak{X}$.

Then the following result obtains:

<u>Theorem</u> 7.4 <u>With the above notation, one has</u> $\mathfrak{F} = \mathfrak{m} = \mathfrak{P} = \mathfrak{v} = \mathfrak{w}$.

These two theorems embody all the known as well as new results on the structure of weakly harmonizable processes or fields. Some applications and extensions will be indicated in the rest of the paper.

8. Associated spectra and consequences. For a large class of nonstationary processes, including the (strongly) harmonizable ones, it is possible to associate a genuine (non negative) spectral measure and study some of the key properties of the process through it. One such reasonably large class, isolated by Kampé de Fériet and Frankiel ([15]-[17]), called class (KF) in [31], is the desired family. This was also considered under the name "asymptotical stationarity" by E. Parzen, (cf. [14] with the same name for a sub class), and by Rozanov ([34], p. 283) without a name. All these authors seem to have arrived at the concept independently. But it is Kampé de Fériet and Frankiel who emphasized the importance of this class and made a deep study. This was further analyzed in [31].

If $X:\mathbb{R} \to L_0^2(P)$ is a process with covariance $k(s,t) = E(X(s)\overline{X(t)})$, then it is said to be of class(KF), after its authors [15]-[17], provided the following limit exists for all $h \in \mathbb{R}$:

$$r(h) = \lim_{T\to\infty} \frac{1}{T} \int_0^{T-|h|} k(s,s+|h|)ds = \lim_{T\to\infty} r_T(h) . \tag{88}$$

It is easy to see that $r_T(\cdot)$, hence $r(\cdot)$, is a positive definite function when $X(\cdot)$ is a measurable process. If $X(\cdot)$ is continuous in mean square, the latter is implied. It is clear that stationary processes are in class(KF). By the classical theorem of Bochner (or its extended form by F. Riesz) there is a unique bounded increasing function $F:\mathbb{R} \to \mathbb{R}^+$ such that

$$r(h) = \int_{\mathbb{R}} e^{ith} F(dt) , \quad a\cdot a\cdot(h)\cdot(Leb) . \tag{89}$$

This F is termed the <u>associated spectral function</u> of the process X. Every strongly harmonizable process is of class (KF). This is not obvious, but was shown in ([34], p. 283), and in [31] as a consequence of the membership of a more general class called <u>almost (strongly) harmonizable</u>. The latter is not necessarily V-bounded and so the weakly harmonizable class is not included. (Almost harmonizable need not imply weakly harmonizable.) Since the bimeasure of (30) is not necessarily of bounded variation, the proof of [34] given for the strongly harmonizable process does not extend. Perhaps for this reason, Rozanov (cf. [34], footnote on p. 283) felt that the weakly harmonizable processes may not be in class(KF). However, a positive solution can be obtained as follows:

<u>Theorem</u> 8.1 <u>Let</u> $X:\mathbb{R} \to L_0^2(P)$ <u>be weakly harmonizable</u>. <u>Then</u> $X \in$ class(KF) , <u>so that it has a well-defined associated spectral function</u>.

<u>Proof</u>. Since X is weakly harmonizable,

$$X(t) = \int_{\mathbb{R}} e^{it\lambda} Z(d\lambda) \ , \ t \in \mathbb{R} \ ,$$

for a stochastic measure Z on $\mathbb{R}$ into $L_0^2(P)$, and if $F(A,B) = (Z(A),Z(B))$, then $F:\mathcal{B} \times \mathcal{B} \to \mathbb{C}$ is a bounded bimeasure. Considering (88) for $h \geq 0$ (the case $h < 0$ being similar) one has

$$r_T(h) = \frac{T-h}{T} \cdot \frac{1}{T-h} \int_0^{T-h} k(s,s+h)ds, \ k(s,t) = (X(s),X(t)) = E(X(s)\overline{X(t)}) \ .$$

To show that $\lim_{T\to\infty} r_T(h)$ exists it suffices to consider

$$\frac{1}{T} \int_0^T k(s,s+h)ds = \frac{1}{T} \int_0^T E(X(s)\cdot\overline{X(s+h)})ds$$

$$= E\left(\frac{1}{T}\int_0^T ds \int_{\mathbb{R}} e^{is\lambda} Z(d\lambda) \int_{\mathbb{R}} e^{-i(s+h)\lambda'} Z(d\lambda')\right), \quad (90)$$

and show that the right side has a limit as $T \to \infty$. Let $\mathfrak{X} = \mathfrak{Y} = L_0^2(P)$, and $\mathcal{Z} = L^1(P)$. Since $Z:\mathcal{B} \to \mathfrak{X}$, $\tilde{Z} = Z:\mathcal{B} \to \mathfrak{Y}$ are stochastic measures, one can define a product measure on $\mathbb{R} \times \mathbb{R}$ into $\mathcal{Z}$, using the bilinear mapping $(x,y) \to xy$, of $\mathfrak{X} \times \mathfrak{Y} \to \mathcal{Z}$, as the pointwise product which is continuous in their respective norm topologies. Under these conditions and identifications, the product measure $Z \otimes \tilde{Z}:\mathcal{B} \times \mathcal{B} \to \mathcal{Z}$ is defined and satisfies (D-S integrals):

$$\int_{\mathbb{R}\times\mathbb{R}} f(s,t)(Z\otimes\tilde{Z})(ds,dt) = \int_{\mathbb{R}} Z(ds)\int_{\mathbb{R}} f(s,t)\tilde{Z}(dt) = \int_{\mathbb{R}}\tilde{Z}(dt)\int_{\mathbb{R}} f(s,t)Z(ds), \quad (91)$$

for all $f \in C_b(\mathbb{R}\times\mathbb{R})$, by ([5], p. 388). In most of the work on product vector measures, Dinculeanu assumes that they are "dominated". However, as shown in a separate Remark (cf. [5], p. 388; cf. also [7], Cor. 3), such a product measure as in (91) is well-defined, eventhough it need not be "dominated". It has finite semi_variation, and $\|Z\otimes\tilde{Z}\|(\mathbb{R}\times\mathbb{R}) \le \|Z\|(\mathbb{R})\cdot\|\tilde{Z}\|(\mathbb{R}) = (\|Z\|(\mathbb{R}))^2 < \infty$, so that $Z \otimes Z$ is again a stochastic measure. Letting $f_{s,h}(\lambda,\lambda') = e^{is\lambda}\cdot e^{-i(s+h)\lambda'}$, so $f_{s,h} \in C_b(\mathbb{R}\times\mathbb{R})$, (91) becomes:

$$\int_{\mathbb{R}} e^{is\lambda} Z(d\lambda)\int_{\mathbb{R}} e^{-i(s+h)\lambda'} Z(d\lambda') = \int_{\mathbb{R}\times\mathbb{R}} e^{is(\lambda-\lambda')-ih\lambda'} Z\otimes Z(d\lambda,d\lambda'), \quad (92)$$

the right side being an element of $L^1(P)$. Applying the same calculation to the measures $Z \otimes Z:\mathcal{B}(\mathbb{R}\times\mathbb{R}) \to \mathcal{Z}$ and $\mu:\mathcal{B}([0,T]) \to \mathbb{R}^+$, ($\mu$ is Lebesgue measure), with $(x,a) \to ax$ being the mapping of $\mathcal{Z} \times \mathbb{R} \to \mathcal{Z}$, one can define

$\mu \otimes (Z\otimes Z):\mathcal{B}(0,T) \times \mathcal{B}(\mathbb{R}\times\mathbb{R}) \to Z$ and hence

$$\int_0^T \mu(dt)\int_{\mathbb{R}\times\mathbb{R}} f(t,\underline{\lambda})Z\otimes Z(d\underline{\lambda}) = \int_{\mathbb{R}\times\mathbb{R}} Z\otimes Z(d\lambda)\int_0^T f(t,\underline{\lambda})\mu(dt) . \tag{93}$$

Writing $\mu(dt)$ as dt, (90) - (93) yield:

$$\begin{aligned} E(\tfrac{1}{T}\int_0^T dt\int_{\mathbb{R}\times\mathbb{R}} & e^{is(\lambda-\lambda')-ih\lambda'} Z\otimes Z(d\lambda,d\lambda')) \\ &= E(\int_{\mathbb{R}\times\mathbb{R}} e^{-h\lambda'} Z\otimes Z(d\lambda,d\lambda')\cdot\tfrac{1}{T}\int_0^T e^{is(\lambda-\lambda')}ds) \\ &= E(\int_{\mathbb{R}\times\mathbb{R}} e^{-ih\lambda'}[\tfrac{e^{iT(\lambda-\lambda')}-1}{iT(\lambda-\lambda')}\chi_{[\lambda\neq\lambda']} + \delta_{\lambda\lambda'}]Z\otimes Z(d\lambda,d\lambda')) \end{aligned} \tag{94}$$

But the quantity inside the expectation symbol E is bounded for all T, and since the dominated convergence is valid for the D-S integral ([8], IV.10.10), constants being $Z\otimes Z$-integrable, one can pass the limit as $T\to\infty$ under the expectation as well as the D-S integral in (94). Hence

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{T}\int_0^T k(s,s+h)ds &= E(\int_{\mathbb{R}\times\mathbb{R}} e^{-h\lambda'}\delta_{\lambda\lambda'}\, Z\otimes Z(d\lambda,d\lambda')) \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{-ih\lambda'}\delta_{\lambda\lambda'}E(Z\otimes Z(d\lambda,d\lambda')) \\ &= \int_{[\lambda=\lambda']} e^{-h\lambda}\, F(d\lambda,d\lambda') , \end{aligned}$$

where F is the bimeasure of Z. Hence $\lim_{T\to\infty} r_T(h) = r(h)$ exists and

$$r(h) = \int_{\mathbb{R}} e^{-ih\lambda}G(d\lambda) ,$$

where $G:A\mapsto\int_{\pi^{-1}(A)}\delta_{\lambda\lambda'}F(d\lambda,d\lambda'), A\in\mathcal{B}$, is a positive finite measure which therefore is the associated spectral measure of $X\in$ class(KF). (Here $\pi:\mathbb{R}^2\to\mathbb{R}$ is the projection.) This completes the proof.

The above result implies that several other considerations of [34] hold for weakly harmonizable processes.

As another application of the present work, especially as a consequence of Theorem 6.1, the following precise version of a result stated in ([34], Thm. 3.2) will be deduced from the corresponding classical stationary case.

Theorem 8.2 *Let* $X: \mathbb{R} \to L_0^2(P)$ *be a weakly harmonizable process with* $Z: \mathcal{B} \to L_0^2(P)$ *as its representing stochastic measure. Then for any* $-\infty < \lambda_1 < \lambda_2 < \infty$, *writing* $Z(\lambda)$ *for* $Z((-\infty,\lambda))$, *one has*

$$\underset{T\to\infty}{\mathrm{l\cdot i\cdot m}} \int_{-T}^{T} \frac{e^{-it\lambda_2}-e^{-it\lambda_1}}{-it} X(t)\,dt = \frac{Z(\lambda_2+)+Z(\lambda_2-)}{2} - \frac{Z(\lambda_1+)+Z(\lambda_1-)}{2} \tag{95}$$

where $\mathrm{l\cdot i\cdot m}$ *is the* $L^2(P)$-*limit. Further the covariance bimeasure* F *of* Z *can be obtained for any* $A = (\lambda_1,\lambda_2)$, $B = (\lambda_1',\lambda_2')$ *as:*

$$\lim_{0\le T_1,T_2\to\infty} \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \frac{e^{-\lambda_2 s}-e^{-\lambda_1 s}}{-is} \cdot \frac{e^{i\lambda_2' t}-e^{-\lambda_1' t}}{it}\, r(s,t)\,ds\,dt = F(A,B) , \tag{96}$$

provided A,B *are continuity intervals of* F *in the sense that* $F((-\infty,\lambda_j\pm),(-\infty,\lambda_j'\pm)) = F((-\infty,\lambda_j'))$, $j = 1,2$, *and where* $r(\cdot,\cdot)$ *is the covariance function of the* X-*process. In particular, if* $S: \mathbb{R} \to \mathbb{C}$ *is continuous,* $\frac{1}{T}\int_0^T S(t)\,dt \to a_0$ *exists as* $T \to \infty$, *and* $\lim_{|s|+|t|\to\infty} r(s,t) = 0$, *then for the observed process* $\tilde{Y}(t) = S(t) + X(t)$, *so that* $S(\cdot)$ *is the nonstochastic 'signal' and* $X(\cdot)$ *is the weakly harmonizable 'noise', the estimator* $\hat{S}_T = \frac{1}{T}\int_0^T \tilde{Y}(t)\,dt \to a_0$ *in* $L_0^2(P)$ (i.e. $E(|S_T - a_0|^2) \to 0$) *as* $T \to \infty$. *Thus* $\hat{S}_T$ *is a consistent estimator of* a_0, *and in other terms, both* X- *and* $\tilde{Y}$-*processes obey the law of large numbers.*

Proof. The key idea of the proof is to reduce the result to the classical stationary case through an application of the dilation theorem. Thus by Theorem 6.1 , there exists a probability space $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$, with $L_0^2(\tilde{P}) \supset L_0^2(P)$, and a stationary process $Y:\mathbb{R} \to L_0^2(\tilde{P})$ such that $X(t) = QY(t)$, $t \in \mathbb{R}$ where Q is the orthogonal projection on $L_0^2(\tilde{P})$ with range $L_0^2(P)$. There is an orthogonally scattered stochastic measure $\tilde{Z}:\mathcal{B} \to L_0^2(\tilde{P})$ such that

$$Y(t) = \int_{\mathbb{R}} e^{it\lambda}\,\tilde{Z}(d\lambda)\ , \quad t \in \mathbb{R}, \tag{97}$$

and $Z(A) = Q\tilde{Z}(A)$, $A \in \mathcal{B}$, where $Z:\mathcal{B} \to L_0^2(P)$ represents the given X-process. Since Q is bounded, as is well-known, it commutes with the integral as well as the l·i·m . Thus (95) is true for the Y-process with $\tilde{Z}$ in place of Z there (cf.,e.g.,[6], p. 527). Then the result follows on applying Q to both sides and interchanging the l·i·m· as well as the integral with Q , which is legitimate. Hence (95) is true as stated.

Next consider the left hand side (LHS) of (96). With (95) it can be expressed as:

$$\text{LHS} = \lim_{T_1,T_2\to\infty} E\Big(\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\Big[\frac{e^{-is\lambda_2}-e^{-is\lambda_1}}{-is}X(s)\Big]\cdot\Big[\frac{e^{-it\lambda_2'}-e^{-it\lambda_1'}}{-it}X(t)\Big]^{-}ds\,dt\Big)$$

$$= \lim_{T_1,T_2\to\infty} E\Big[\Big(\int_{-T_1}^{T_1}\frac{e^{-is\lambda_2}-e^{-is\lambda_1}}{-is}X(s)\,ds\Big)\Big(\int_{-T_2}^{T_2}\frac{e^{-it\lambda_2'}-e^{-it\lambda_1'}}{-it}X(t)\Big)^{-}dt\Big]$$

$$= E\Big[\Big(\frac{Z(\lambda_2+)+Z(\lambda_2-)}{2} - \frac{Z(\lambda_1+)+Z(\lambda_1-)}{2}\Big)\Big(\frac{Z(\lambda_2'+)+Z(\lambda_2'-)}{2} - \frac{Z(\lambda_1'+)+Z(\lambda_1'-)}{2}\Big)^{-}\Big]$$

$$= F(A,B)\ ,$$

by the continuity hypothesis on F, after expanding and taking expectations. This proves (96).

Finally, if $\tilde{Y}(t) = S(t) + X(t)$, $t \in \mathbb{R}$, let $a_T = E(\hat{S}_T) = \frac{1}{T}\int_0^T S(t)dt$. Noting that $\tilde{Y} \in \text{class(KF)}$ since X does (cf. Thm 8.1), and $a_T \to a_0$, as $T \to \infty$,

$$
\begin{aligned}
E(|\hat{S}_T - a_0|^2) &= \frac{2}{T^2}\int_0^T\int_0^T r(s,t)ds\,dt + 2|a_T - a_0|^2 \\
&= \frac{1}{2T}\int_{-T}^T r_T(h)dh + 2\,|a_T - a_0|^2 ,
\end{aligned}
\tag{98}
$$

where, as usual, $r_T(\cdot)$ is given by (88). Since $r_T(h) \to r(h)$ due to the fact that $\tilde{Y} \in \text{class(KF)}$, and since $r(s,s+h) \to 0$ as $|s| \to \infty$ by hypothesis together with the fact that $|r(s,t| \le (r(s,s)r(t,t))^{1/2} \le M^2 < \infty$ where $\|X(t)\| \le M < \infty$ (X being V-bounded), one can invoke a classical result on Cesaro summability (cf., [8], IV.13.83(a)). By this result $r(h) = 0$ for each $h \in \mathbb{R}$. Actually $r_T(h) \to r(h)$ $(=0)$, uniformly in h on compact sets of $\mathbb{R}$. It follows that $E(|\hat{S}_T - a_0|^2) \to 0$, and this completes the proof of the theorem.

Remark. The key reduction for (95), which is used in (96), is possible in the above proof since the linear operation of Q on the process mattered. However, for Theorem 8.1, the dilation result itself is not immediately applicable since the problem there is non-linear, and one had to use alternate arguments as was done there. Thus the point of the general theory here is to clarify the structure of these processes, and a reduction to the stationary case need not always be possible or essential.

9. Multivariate extension and related problems. Here a multi-dimensional extension of weakly harmonizable processes and the filtering problem on them will be briefly discussed. Eventhough some results have direct k-dimensional analogs ($k \geq 2$), there are some new and non-trivial problems in this case for a successful application of the theory. The infinite dimensional case will not be considered here since the key finite dimensional problems are not well-understood and resolved.

Let $L_0^2(P,\mathbb{C}^k)(= L_0^2(\Omega,\Sigma,P;\mathbb{C}^k))$ be the space of equivalence classes of measurable functions $f:\Omega \to \mathbb{C}^k$, the complex k-space, such that (i) $|f|^2 = \sum_{i=1}^{k} |f_i|^2$ is P-integrable, and (ii) $E(f) = \int_\Omega f(\omega)P(d\omega) = 0$, or equivalently, $E(f_i) = \int_\Omega f_i(\omega)P(d\omega) = 0$, $i = 1,\dots,k$, where $f = (f_1,\dots,f_k)$, $|f|$ is the Euclidean norm of f in $\mathbb{C}^k$, and (Ω,Σ,P) is a probability space. If $f,g \in L_0^2(P,\mathbb{C}^k)$, define $\|f\|_2^2 = (f,f)$ where the inner product is given by

$$(f,g) = \int_\Omega (f(\omega),g(\omega))P(d\omega) = \sum_{i=1}^{k} \int_\Omega f_i(\omega)\bar{g}_i(\omega)P(d\omega) .$$

Then $\mathcal{X} = L_0^2(P,\mathbb{C}^k)$ becomes a Hilbert space of k-vectors with zero means. If $k = 1$, one has the space considered in the preceding sections $(\mathcal{H} = L_0^2(P,\mathbb{C}))$.

Definition 9.1 Let G be an LCA group. Then a mapping $X:G \to \mathcal{X}$ is a weakly or strongly harmonizable vector (or k-dimentional) random field or 'process' if for each $a = (a_1,\dots,a_k) \in \mathbb{C}^k$, the mapping $Y_a = a\cdot X(= \sum_{i=1}^{k} a_iX_i): G \to \mathcal{H}$ is a (scalar) weakly or strongly harmonizable random field.

Similarly a vector stationary, Karhunen, or class(C), processes are defined by reducing to the scalar cases.

It is immediate from this definition that the component processes are also harmonizably or stationarily etc. correlated according to the class they belong. Thus if r_a is the covariance function of the Y_a-process and R is the covariance matrix of the X-process, so that $r_a(g,h) = E(Y_a(g)\overline{Y_A}(h))$ and $R(g,h) = E(X^t(g)X(h))$ where $X(g)$ is a k^{th} order (row) vector and "t" denotes the conjugate transpose of a vector or matrix, then $r_a(g,h) = aR(g,h)a^t$. With this notation, the integral representations of multivariate weakly and strongly harmonizable random fields can be obtained, using Theorem 3.3, in a straightforward manner.

<u>Theorem</u> 9.2 <u>Let</u> G <u>be an LCA group and</u> $X:G \to \mathfrak{X} = L_0^2(P,\mathbb{C}^k)$, <u>a weakly continuous bounded mapping</u>. <u>Then</u> X <u>is weakly harmonizable iff there is a stochastic measure</u> $\tilde{Z}$ <u>on</u> $\hat{G} \to \mathfrak{X}$ (<u>or if</u> $\tilde{Z}(A) = (Z_1(A),\ldots,Z_k(A))$, $A \subset \hat{G}$ <u>is a Borel set</u>, <u>then each</u> Z_j <u>is a stochastic measure on</u> $\hat{G} \to \mathcal{H}$, $j = 1,\ldots,k$), <u>such that</u>

$$X(g) = \int_{\hat{G}} \langle g,s\rangle \, \tilde{Z}(ds) \;, \quad g \in G \;, \tag{99}$$

<u>where</u> $\hat{G}$ <u>is the dual group of</u> G . <u>The mapping</u> X <u>is strongly harmonizable if further the matrix</u> $F = (F_{j\ell}, j,\ell=1,\ldots,k)$ <u>with</u> $F(A,B) = ((Z_j(A),Z_\ell(B)), j,\ell=1,\ldots,k)$ <u>is of bounded variation on</u> $\hat{G}$, <u>or equivalently each</u> $F_{j\ell}$ <u>is of bounded variation on</u> $\hat{G}$. <u>The covariance matrix</u> R <u>is representable as</u>:

$$R(g,h) = \int_{\hat{G}}\int_{\hat{G}} \langle g,s\rangle\overline{\langle h,t\rangle} \, F(ds,dt) \;, \quad g,h \in G \;, \tag{100}$$

where the right side is the MT-*integral, or the Lebesgue-Stieltjes integral, defined componentwise, accordingly as* X *is weakly or strongly harmonizable, and where* F *is a positive definite matrix of bounded bimeasures or of Lebesgue-Stieltjes measures.* *Conversely, if* $R(\cdot,\cdot)$ *is a positive definite matrix representable as* (100), *then it is the covariance matrix of a multivariate harmonizable random field.*

Sketch of proof. Let $a \in \mathbb{C}^k$ be arbitrarily fixed and consider $Y_a = a\cdot X(=aX^t)$. If X is weakly harmonizable, so that Y_a is also, then by Theorem 3.3 (trivially extended when $\mathbb{R}$ is replaced by G) , there is a stochastic measure Z_a on $\hat{G} \to \mathcal{H}$ such that

$$Y_a(g) = \int_{\hat{G}} \langle g,s\rangle \, Z_a(ds) \ , \quad g \in G \ .$$

From this and the definition of Y_a , it follows that $Z_{(\cdot)}(A):\mathbb{C}^k \to \mathcal{H}$ is linear and continuous. Hence there is a $\tilde{Z}$ on $\hat{G} \to \mathfrak{X}^{**}(=\mathfrak{X}$, by reflexivity) such that $Z_a(A) = a\cdot\tilde{Z}(A)$, and it is evident that $\tilde{Z}$ is σ-additive on $\mathcal{B}(\hat{G}) \to \mathfrak{X}$ so that it is a stochastic measure. It follows from the properties of the D-S integral that:

$$Y_a(g) = a\cdot X(g) = \int_{\tilde{G}} \langle g,s\rangle \, a\cdot\tilde{Z}(ds) = a\cdot\int_{\tilde{G}} \langle g,s\rangle Z(ds) \ , \qquad (101)$$

where the last integral defines an element of $\mathfrak{X}$. This implies (99) since a is arbitrary and $X(\cdot)$ as well as the integral operator are continuous. The converse is similarly deduced from the corresponding part of Theorem 3.3.

If X is strongly harmonizable, then so is Y_a and if F_a is its covariance bimeasure, then $F_a = aFa^t$ where $F(A,B) = ((Z_j(A),Z_\ell(B)),\ j,\ell=1,2,\ldots,k)$. Now taking special values for a in $\mathbb{C}^k$, it follows immediately that each component $F_{j\ell}$ of F is of bounded variation. Interpreting (100) componentwise, the result follows from the scalar case. The same representation holds with the MT-integration for the weakly harmonizable case. All other statements, including the converses, are similarly deduced. This terminates the sketch.

By an analogous reasoning, it is evidently possible to assert that there is a 2-majorant of $\tilde{Z}$, and the X-process has a (vector) stationary dilation. These results are of real interest in the context of the important filtering problem which can be abstractly stated following Bochner [2].

If $X:G \to \mathfrak{X}$ is a random field, a (not necessarily bounded) linear operator $\Lambda:\mathfrak{X} \to \mathfrak{X}$ is called a _filter_ of X, if Λ commutes with the translation operator on X, i.e., if $(\tau_h X)(g) = X(hg)$, then $\tau_h(\Lambda X) = \Lambda(\tau_h X)$, where domain $(\Lambda) \supset \{\tau_h X(g),\ g \in G,\ h \in G\}$. The problem is to find solutions X of the equation:

$$\Lambda X = Y \quad (\in \mathfrak{X}), \tag{102}$$

such that if Y is a given weakly or strongly harmonizable random field so must X be.

For the stationary case, a general concept of filter was discussed by Hannan [11]. If $k = 1$, $\Lambda = \sum_{i=1}^{m} a_i \Delta_i$ is a reverse shift operator with $G = \mathbb{R}$ (so $\Delta_i X(t) = X(t-i)$) and Y is stationary, then this problem was completely solved by Nagabhushanam [25], and by Kelsh [18] in the strongly harmonizable case. In both these studies, the conditions are on the measure function F of (33). If $k > 1$, under the usual assumptions on the random fields, the following new questions arise with (99) and (100). Frequently employed general forms of Λ include the constant coefficient difference, differential, or integral operators, or a mixture of these. For instance, if $\Lambda = \sum_{j=0}^{m} A_j D^j$, where A_j's are k-by-k constant matrices, and $D^j = \frac{d^j}{dt^j}$, $(G=\mathbb{R})$ then (102) takes the following form in order that it admit a (weakly) harmonizable solution for a harmonizable Y where $X^{(j)}$ denotes the mean-square j^{th} derivative (assumed to exist):

$$\begin{aligned}\int_{\mathbb{R}} e^{it\lambda} Z_y(d\lambda) = Y(t) = (\Lambda X)(t) &= \sum_{j=0}^{m} A_j X^{(j)}(k-j) \\ &= \sum_{j=0}^{m} A_j \int_{\mathbb{R}} e^{i(t-j)\lambda} (i\lambda)^j \, Z_x(d\lambda) \\ &= \int_{\mathbb{R}} T(\lambda)\cdot e^{it\lambda} \, Z_x(d\lambda) \, , \end{aligned} \tag{103}$$

where $T(\lambda) = \sum_{j=0}^{m} A_j e^{-ij\lambda}(i\lambda)^j$, called the generator of Λ in [2], and Z_x, Z_y are the representing stochastic measures of X-and Y-processes. Clearly the existence of solutions of (102) depends on the coefficients A_j's determining the analytical properties of the generator $T(\cdot)$. If the process is strongly harmonizable then (103) implies (*-denoting conjugate transpose)

$$\begin{aligned} R_y(s,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda - it\lambda'} F_y(d\lambda, d\lambda') \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{is\lambda} \, T(\lambda) F_x(d\lambda, d\lambda')(T(\lambda')e^{it\lambda'})^* \, , \end{aligned} \tag{104}$$

where F_x and F_y are the k-by-k matrix covariance bimeasures of X- and Y-processes. For a special class of strongly harmonizable k-vector processes, recently Kelsh [18] found sufficient conditions on the generator $T(\cdot)$ for a solution of (102) when differential operators are replaced by difference operators so that $\{\lambda : T(\lambda) = 0\}$ is finite. The solution here hinges on the properties of the structure of the space:

$$L^2(F_x) = \{T:\mathbb{R} \to B(\mathbb{C}^k), \text{ measurable}, \|\int_{\mathbb{R}}\int_{\mathbb{R}} T(\lambda)F_x(d\lambda, d\lambda')T^*(\lambda')\| < \infty\} . \tag{105}$$

Since the integral in (105) defines a positive (semi-) definite matrix, its trace gives a semi-norm. The measure function F being a matrix bimeasure, several new problems arise for an analysis of the $L^2(F_x)$-space. For the weakly harmonizable case, an extension of the MT-integration , to include such integrals, should be established. The resulting theorey can then be utilized for the multivariate filtering problem. Even if $k = 2$, the question is non-trivial, involving the rank questions of F_x . Application of the dilation results to the filtering problem has some novel features, but it does not materially simplify the problem.

Another interesting point is to seek "weak solutions" of the filtering equation (102) in the sense of distribution theory. This idea is introduced in [2]. If $\mathcal{G}$ is a class of functions on $\mathbb{R}$ (e.g., Schwartz space $C^\infty(\mathbb{R})$) with a locally convex topology, then one says that (102) has a (weak) solution iff for each $f \in \mathcal{G}$

$$\int_{\mathbb{R}} f(t)Y(t)dt = \int_{\mathbb{R}} f(t)\Lambda X(t)dt = \int_{\mathbb{R}} (\tilde{\Lambda}f)(t)X(t)dt \ , \tag{106}$$

where $\tilde{\Lambda}: \mathcal{G} \to \mathcal{G}$ is an operator, associated with Λ, defined by the last two integrals above. It is an "adjoint" to Λ. For instance, if Λ is a differentail operator with $T(\cdot)$ as its generator, if $k = 1$ and X,Y are stationary, then $\tilde{\Lambda}$ is given by

$$(\tilde{\Lambda}f)(t) = \int_{\mathbb{R}} T(t-\lambda)f(\lambda)F_x(d\lambda) \ , \quad f \in \mathcal{G} \tag{107}$$

where F_x is the spectral measure function of the X-process. Clearly many other possibilities are available. Thus there are a number of directions to presure the research on these problems, and the paper [2] has a wealth of ideas of great interest here.

This essentially includes what is known about weakly harmonizable random fields and processes, as far as the structure is concerned. Since the class(C) of Cramér and its weak counterpart (cf. Definition 3.1) and the Karhunen class of processes, defined by (31), are natural generalizations of harmonizable and stationary classes, it is reasonable to ask whether the latter is a dilation of the former, i.e. is the analog of Theorem 6.1 true for weakly class(C)? A restricted version can be established by the same methods, but the exact (general) result presents some difficulties. This question will be briefly discussed here in order to include it in the set of problems raised by the present study.

Recall that a mapping $X: \mathbb{R} \to L_0^2(P)$ is a <u>Karhunen process</u>, if its covariance function $r(\cdot,\cdot)$ admits a representation

$$r(s,t) = \int_{\mathbb{R}} g_s(\lambda)\overline{g_t(\lambda)} \ F(d\lambda) \ , \quad s,t \in \mathbb{R} \ ,$$

relative to a family $\{g_s(\cdot),\ s \in \mathbb{R}\}$ of measurable functions and F which defines a locally finite positive regular (or Radon) measure on $\mathbb{R}$ and $g_s \in L^2(F)$, (cf. also [10], p. 241). As an immediate consequence of Theorem 3.2 (cf. Remark 2 following its proof), an integral representation for Karhunen processes can be given:

Proposition 9.3 *Let* S *be a locally compact space and* $X:S \to L_0^2(P)$ *be a process of Karhunen class relative to a locally finite positive regular (or Radon) measure* F *on* S *and a family* $\{g_t, t \in S\} \subset L^2(F)$, *the space of all scalar square integrable functions on* $(S,\mathcal{B},F)$. *Then there is a locally bounded regular (or Radon) stochastic measure* $Z:\mathcal{B}_0 \to L_0^2(P)$ *where* $\mathcal{B}_0 \subset \mathcal{B}$ *is the* δ*-ring of bounded sets, such that* (i) $E(Z(A)\cdot\overline{Z(B)}) = F(A\cap B)$, $A,B \in \mathcal{B}_0$, *so that* Z *is orthogonally scattered, and* (ii) *one has*

$$X(t) = \int_S g_t(\lambda) Z(d\lambda) \ , \quad t \in S \ , \tag{108}$$

where the right side symbol is a D-S *integral*, (cf. *also* [39], § 1.) *Conversely, if* $X:S \to L_0^2(P)$ *is a process defined by* (108) *relative to an orthogonally scattered measure* Z *on* S *and* $\{g_t, t \in S\}$ *satisfies the above conditions, then it is a Karhunen process with respect to the family* $\{g_t, t \in S\}$ *and* F *defined by* $F(A\cap B) = (Z(A),Z(B))$. *Moreover* $\mathcal{H}_X = \overline{sp}\{X(t), t \in S\} \subseteq \mathcal{H}_Z = \overline{sp}\{Z(A), A \in \mathcal{B}_0\} \subset L_0^2(P)$ *and* $\mathcal{H}_X = \mathcal{H}_Z$ *iff* $\{g_t, t \in S\}$ *is dense in* $L^2(F)$.

A proof of this result is essentially given in ([10], p. 242) and is a simplification of that of Theorem 3.2. Even a multidimensional version is not difficult, which in fact is analogous to that of Theorem 9.2 above. Actually, the version in [10] is sketched for the k-dimensional case.

It follows from the arguments of the D-S theory of integration that a bounded linear operator T and the vector integral such as that of (108) commute even if Z is of locally finite semivariation on the locally compact space S. This extension of ([8], IV.10) was proved in ([39], p. 79), and shown to be easy. Thus if $X:S \to L_0^2(P)$ is a Karhunen process, so that it is representable as in (108) and if $T \in B(L_0^2(P))$, then it follows that

$$TX(t) = \int_S g_t(\lambda) T\circ Z(d\lambda) , \tag{109}$$

and it is simple to see that $\tilde{Z} = T\circ Z$ is a stochastic measure of locally finite semivariation, but not necessarily orthogonally scattered. Hence by Theorem 3.2, TX is weakly of class(C).

In the opposite direction, for a process $\{X(s), s \in S\} \in$ weakly class(C), one cannot apply the theory of Section 5 above if only $\{g_t \cdot t \in S\} \subset L^2(F)$, and no further restrictions are imposed, where $L^2(F_x)$ is the space of strong MT-integrable functions relative to the covariance bimeasure F_x representing X:(cf. (105), with k=1). The needed analogs of Theorem 5.5 and Proposition 5.6 are not available. Suppose now that F_x is such that if each g_t is a <u>bounded</u> Borel function and $M(S)$ is the uniformly closed algebra generated by $\{g_t, t \in S\}$ then $M(S) \subset L^2(F_x)$. Let $Tg_t = X(t) = \int_S g_t(\lambda) Z(d\lambda)$ and extend T linearly to $M(S)$. Then $T \in B(M(S), \mathcal{H})$. When

$M(S)$ is given the uniform norm. This forces F_x to be of finite semivariation! Under this assumption T is 2-absolutely summing, and Proposition 5.6 is applicable. Hence

$$\|Tf\|_{\mathcal{H}} \le \|f\|_{2,\mu}, \quad f \in M(S) \tag{110}$$

for a finite measure μ on S. (A similar result seems possible *if* Z is restricted so that $T \in B(L^2(F_x),\mathcal{H})$, defined above is Hilbert-Schmidt by [20], p. 302. But it is not a good assumption here.) Thus one can repeat the proof of Theorem 6.1 essentially verbatim and establish a dilation result. Omitting the details of this computation one obtains the following result.

Theorem 9.4 *Let* S *be a locally compact space and* $X:S \to L_0^2(P)=\mathcal{H}$ *be a Karhunen process relative to a Radon measure* F *and a family* $\{g_t, t \in S\} \subset L^2(F)$. *If* $Q:\mathcal{H} \to \mathcal{H}$ *is any* (*bounded*) *projection, then* $\tilde{X}(t) = QX(t)$, $t \in S$, *is a process in* weakly class(C) . *On the other hand if* $\{X(t), t \in S\}$ *is an element of* weak class(C), *and so is representable in the form* (108) *for some family* $\{g_t, t \in S\} \subset L^2(F_x)$ *where* F_x *is a bounded covariance bimeasure of the process, and if each* g_t *is also bounded, then there exists an extension Hilbert space* $\mathcal{K} \supset \mathcal{H}$, *a probability space* $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$ *with* $\mathcal{K} = L_0^2(\tilde{P})$, *and a Karhunen process* $Y:S \to \mathcal{K}$ *such that* $X(t) = QY(t)$, $t \in S$ *where* Q *is the orthogonal projection on* $\mathcal{K}$ *with range* $\mathcal{H}$.

This result points out clearly the need to consider the domination problem for other Banach spaces than those covered by the results of Section 5. Indeed the associated abstract problem of classifying Banach spaces admitting a positive p-majorizable measure for each vector measure from a probability space into that space is essentially open. Also the preceding theorem and related analysis presumably extend to $\text{class}_N(C)$-processes of Definition 3.4. This will be of independent interest in addition to its use in a treatment of the general filtering theory on these processes. Other problems noted in the main text of the paper are of both methodological and applicational importance for a future study.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A unified and detailed treatment of the structure of harmonizable time series is considered. These are divided into strong (or Loève) and weak (Bochner-Rozanov) classes and their characterizations as well as integral representations are obtained. Both concrete and operator versions of the characterizations of weakly harmonizable processes are given. The work here implies

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that weakly harmonizable class is the largest family of second order processes with continuous covariance for which Fourier analysis applies. The treatment includes random fields. It is shown that both the harmonizable processes have an associated spectrum, and they obey the weak law of large numbers. Multi-dimensional extensions and filtering are briefly discussed. Several open problems and avenues of research growing out of this study are indicated at various places, so that the material presented here will form a firm basis for both the theory and applications to follow.