THE M-TRAVELLING SALESMEN PROBLEM: A DUALITY BASED BRANCH-AND-BOUND ETC (U)

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A DUALITY BASED BRANCH-AND-BOUND
ALGORITHM

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ABSTRACT

This paper presents a new model and branch-and-bound algorithm for the m-travelling salesmen problem. The algorithm uses a Lagrangean relaxation, a subgradient algorithm to solve the Lagrangean dual, a greedy algorithm for obtaining minimal m-trees, penalties to strengthen the lower bounds on candidate problems, and a new concept known as staged optimization. Computational experience for both symmetric and asymmetric problems having up to 100 cities is presented.

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1. OVERVIEW

This paper extends the highly successful algorithm of Held and Karp [11, 12] for the travelling salesman problem, to the \( m \)-travelling salesmen problem. The algorithm involves the use of a Lagrangean dual within a branch-and-bound structure. A subgradient procedure is used to solve the dual and it is shown that a greedy algorithm may be used to evaluate points of the dual function.

We now formally state the \( m \)-travelling salesman problem. We define a \textit{tour} beginning at city \( n \) as a sequence of distinct cities \( \{n, i_1, i_2, \ldots, i_k\} \) where \( k \geq 1 \). Given \( n \) cities \((1, \ldots, n)\) and \( m \) salesmen, all based at city \( n \), we wish to find a set of \( m \) tours such that each city other than \( n \) is a member of exactly one tour. Let \( c_{ij} \) denote the distance from city \( i \) to city \( j \) and let \( c_{n1} + c_{1i_2} + \ldots + c_{i_{k-1}i_k} + c_{i_kn} \) denote the distance for the tour \( \{n, i_1, \ldots, i_k\} \). The objective is to select \( m \) tours (one for each salesman) such that the total distance for all tours is a minimum. A related problem has also been discussed in the literature in which the objective is to select at most \( m \) tours with total minimum distance (see [2]). Clearly, for \( m = 1 \) both models are the classical travelling salesman problem. If the distances \( c_{ij} = c_{ji} \) for all city pairs, then this model is called \textit{symmetric}; otherwise, it is called \textit{asymmetric}.

This model was first formulated in integer programming terms by Miller, Tucker, and Zemlin [17]. Svetska and Huckfeldt [18] developed a specialized branch-and-bound algorithm for this problem which uses a linear programming relaxation. Also, Gavish and Srikanth [7] developed
a branch-and-bound algorithm which uses a different relaxation. The Gavish-Srikanth model does not permit the use of a greedy algorithm to solve subproblems whereas our model does. Bellmore and Hong [5] proved that the asymmetric version of this model was equivalent to an asymmetric travelling salesman problem on n + m - 1 cities. The question of whether this problem should be attacked directly as in [7, 18] and the present work or whether it should be converted to its one salesman equivalent and solved using [3] or [12] is an open question.
II. THE MODEL

In this section we present a new model for the m-travelling salesmen problem. The model is developed using the notion of an m-tree, which is a generalization of the Held-Karp 1-tree. The m-tree is defined such that it has the matroidal property so that the integer programming relaxation is solvable via a greedy algorithm. The original generalization proposed by Held and Karp [11] is not matroidal.

Let the decision variable

\[ x_{ij} = \begin{cases} 
1, & \text{if some salesman travels from city } i \text{ to city } j; \\
0, & \text{otherwise.} 
\end{cases} \]

Assuming that each salesman is based at city n, we call any tour on cities 1, ..., n - 1 a subtour. Then the m-travelling salesmen problem may be stated mathematically as follows:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}
\]

s.t. \( \sum_{k=1}^{m} x_{kj} = \begin{cases} 
1, & \text{for } j = 1, \ldots, n-1 \\
m, & \text{for } j = n 
\end{cases} \tag{2} \]

\[
\sum_{k=1}^{m} x_{ik} = \begin{cases} 
1, & \text{for } i = 1, \ldots, n-1 \\
m, & \text{for } i = n 
\end{cases} \tag{3} \]

\[ x_{ij} = 0 \text{ or } 1 \quad (\text{all } i, j) \tag{4} \]

\[ x_{ii} = 0 \quad (\text{all } i) \tag{5} \]

no subtours.
For the classical travelling salesman problem (i.e. \( m = 1 \)), a pair of expository papers by Held and Karp [11, 12] showed that (4) and (5) could be replaced by a constraint that required the graphical structure associated with any feasible vector \( x \) to be a 1-tree. A 1-tree on a graph having \( n \) nodes (cities) is a spanning tree on \( n-1 \) nodes and two distinct edges connecting node \( n \) to two other nodes. Letting \( Y = \{ (x_{11}, \ldots, x_{ln}, \ldots, x_{nl}, \ldots, x_{nn}) : x_{ii} = 0, x_{ij} = 0 \) or 1 and the edges \( ij \) having \( x_{ij} = 1 \) form a 1-tree}, the travelling salesman problem may be stated as (1), (2), (3), \( x \in Y \), and \( m = 1 \).

We now generalize the notion of a 1-tree to an \( m \)-tree.

An \( m \)-tree on a graph having \( n \) nodes is an acyclic graph having \( n-m-1 \) edges on \( n-1 \) nodes and \( 2m \) edges such that the edge \( (n,i) \) never appears more than twice.

Note that every \( m \)-tree has \( n+m-1 \) edges and an \( m \)-tree need not be connected. A set of 2-trees is illustrated in Figure 1 while Figure 2 illustrates some graphs having \( n+m-1 \), (6), edges which are not 2-trees. The graph in Figure 2a is not acyclic on nodes 1 and 2 and the graph in Figure 2b has three copies of the edge (5,2).

Letting \( X = \{ (x_{11}, \ldots, x_{ln}, \ldots, x_{nl}, \ldots, x_{nn}) : x_{ii} = 0, x_{ij} = 0 \) or 1, and the edges \( ij \) with \( x_{ij} = 1 \) form an \( m \)-tree}, the \( m \)-travelling salesmen problem may be stated as follows:

\[
\min \sum_{ij} c_{ij} x_{ij} \tag{6}
\]

\[
\text{s.t. } \sum_{k} x_{kj} = \begin{cases} 
1, \text{ for } j = 1, \ldots, n-1 \\
m, \text{ for } j = n 
\end{cases} \tag{7}
\]
\[ \sum_{k} x_{ik} = \begin{cases} 1, & \text{for } i = 1, \ldots, n-1 \\ m, & \text{for } i = n \end{cases} \]  \tag{8} \\

\[ x \in X \quad \tag{9} \]

A model having only \( n \) constraints in place of (7) and (8) exists for the symmetric \( m \)-travelling salesmen problem (see Ali [1]).

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Figures 1 and 2

About Here

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III. THE ALGORITHM

In this section we present a branch-and-bound algorithm for the m-travelling salesmen problem. The algorithm uses a Lagrangean relaxation, a subgradient algorithm to solve the Lagrangean dual, a greedy algorithm for evaluation of a point of the dual function, penalties to strengthen the lower bounds obtained by the greedy algorithm, and a new concept known as staged optimization. Each of these techniques are explained in the subsections to follow.

3.1 Lagrangean Relaxation

The Lagrangean dual of (6) - (9) selected for this algorithm is as follows:

$$\max_{u,v} \Theta(u,v)$$

where

$$\Theta(u,v) = \min_{x \in X} \left\{ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{ij} x_{ij} + \sum_{i=1}^{n-1} u_i (x_{ij} - 1) + \sum_{j=1}^{n-1} v_j (x_{ij} - 1) + \sum_{i=1}^{n} v_n (x_{in} - m) \right\}.$$ 

After rearranging terms $\Theta(u,v) = \min_{x \in X} \left\{ \sum_{k=1}^{n-1} c_{ij}^* x_{ij} \right\} - \alpha$ where $\alpha = \sum_{k=1}^{n-1} (u_k + v_k) - m(u_n + v_n)$ and $c_{ij}^* = c_{ij} + u_i + v_j$. The Lagrangean dual has been used by both Held and Karp [11, 12] and Bazaraa and Goode [3] in their highly successful work on the travelling salesman problem. Our relaxation is a natural extension of their model. Lagrangean relaxation for general integer programs has been extensively studied by Geoffrion [10].
Consider the following results which relate the Lagrangean dual and the primal.

**Theorem 1.** (Bazaraa and Shetty [4])

Let \( x^* \) solve (6) - (9) and let \((u^*, v^*)\) solve (10), (11). Then

\[
\theta(u^*, v^*) \leq \sum_{ij} c_{ij} x^*_{ij}.
\]

**Theorem 2.** (Bazaraa and Shetty [4])

\( \theta(u, v) \) is concave.

**Theorem 3.** (Geoffrion [8])

Let \((\bar{u}, \bar{v})\) be any vectors, let \( \bar{x} \in X \) solve \( \theta(\bar{u}, \bar{v}) \) and let

\[
\begin{align*}
    r_i &= \begin{cases} 
        \sum_j x_{ij} - 1, & \text{for } i = 1, \ldots, n-1 \\
        \sum_j x_{ij} - m, & \text{for } i = n 
    \end{cases} \\
    s_j &= \begin{cases} 
        \sum_i x_{ij} - 1, & \text{for } j = 1, \ldots, n-1 \\
        \sum_i x_{ij} - m, & \text{for } j = n 
    \end{cases}
\end{align*}
\]

The vector \((r, s)\) is a subgradient of \( \theta(u, v) \) at the point \((\bar{u}, \bar{v})\).

The above three results are well-known and are easily proved. Given the optimal vectors associated with both the primal and dual, the non-negative quantity

\[
\sum_{ij} c_{ij} x^*_{ij} - \theta(u^*, v^*)
\]

is called the duality gap.
For this work, the subgradient algorithm ([13, 15]) is used to solve the dual (10) and (11). Our implementation of this general method is presented below.

**ALG-1: SUBGRADIENT OPTIMIZATION METHOD FOR DUAL**

0. **Parameter Selection and Initialization**
   a. [Select Step Size Parameters] Select $\bar{G}$, $\bar{H}$, $\lambda_1$, ..., $\lambda_\infty$.
   b. [Set Initial Solution] $u + 0$, $v + 0$, $u^* + 0$, $v^* + 0$, $r^* + 0$, $s^* + 0$.
   c. [Set Lower Bound] $L + - \infty$.
   d. [Obtain Over Estimate] Set $\bar{G}$ such that $\bar{G} \geq \max_{u,v} G(u,v)$.
   e. [Initialize Counters] $i + 1$, $h + 0$.

1. **Solve Subproblem**
   a. [Evaluate Dual At $(u,v)$] Let $x^*$ solve $\min \{ \sum_{ij} C_{ij} x_{ij} \}$ and set $\bar{c} = \sum_{ij} C_{ij} x_{ij}^* - \alpha$.
   b. [Determine Subgradient]
      \[
      \begin{align*}
      r_i &= \begin{cases}
      x_{ij}^* - 1, & \text{for } i = 1, \ldots, n-1, \\
      x_{ij}^* - m, & \text{for } i = n
      \end{cases} \\
      s_j &= \begin{cases}
      x_{ij}^* - 1, & \text{for } j = 1, \ldots, n-1, \\
      x_{ij}^* - m, & \text{for } j = n
      \end{cases}
      \end{align*}
      \]
   c. [Test For Optimality] If $r = s = 0$, $(u,v)$ with $(u,v)$ an optimum for the dual and $x^*$ an optimum for the primal.
   d. [Improved Solution?] If $\bar{c} > L$, then $L = \bar{c}$, $u^* = u$, $v^* = v$, $r^* = r$, $s^* = s$, and go to 2.
e. [Test Step Size Counter] $h \leftarrow h + 1$. If $h = \overline{H}$, go to 3.

2. **Move To New Point**

$$(u,v) \leftarrow (u,v) + \left( \frac{|r,s|}{|\langle r,s \rangle|} \right) \langle r,s \rangle,$$ and go to 1.

3. **Change Step Size Or Terminate**

If $i = \overline{G}$, terminate with $(u^*,v^*)$ as an optimum for the dual; otherwise, $h \leftarrow 0$, $i \leftarrow i + 1$, $$(u,v) \leftarrow (u^*,v^*) + \left( \frac{|r,s|}{|\langle r,s \rangle|} \right) \langle r,s \rangle,$$ and go to 1.

An excellent discussion of convergence results for the subgradient algorithm are given in Helgason [14].

### 3.2 Matroidal Structure of $M$-Trees

To implement ALG-1 efficiently, one needs a fast procedure for solving

$$\min_{x \in X} \{ \sum_{i,j} c_{ij} x_{ij} \}. \tag{13}$$

In this section we show that $X$ has a matroidal structure and (13) may be solved by a greedy algorithm.

We now present results from matroid theory which will be used in the development of an efficient algorithm for (13). Recall that a matroid, $M = [Z, \psi]$, is a finite set $Z$ and a set $\psi$ of subsets of $Z$ such that the following axioms hold:

M1. $\emptyset \in \psi$.

M2. If $X \in \psi$ and $Y \subseteq X$, then $Y \in \psi$.

M3. If $U$, $V \in \psi$ with $|U| = |V| + 1$, then there exists an $x \in U - V$ such that $V \cup \{x\} \in \psi$.

Let $2^Z$ denote the power set of $Z$. Then the elements of $\psi \subseteq 2^Z$ are called independent subsets of $2^Z$ and $2^Z - \psi$ are called dependent subsets. A maximal independent subset is called a base.
We now show that a greedy algorithm can be used to find a minimum weight base. Let $\beta \subseteq \psi$ denote the set of bases for some matroid. Let $w : \mathbb{Z} \to \mathbb{R}$ be a weight function and extend this to $w : \mathbb{Z}^2 \to \mathbb{R}$ as follows:

$$w(\mathbf{C}) = \sum_{e \in \mathbf{C}} w(e), \mathbf{C} \subseteq \mathbb{Z}.$$ 

The minimal weight base problem may be described as follows:

Find $X \in \beta$ such that

$$w(X) = \min_{B \in \beta} w(B). \quad (14)$$

An optimal base $X$ may be obtained by employing the following algorithm.

**ALG-2: GREEDY ALGORITHM FOR A MINIMAL WEIGHT BASE**

0. $X_0 \leftarrow \emptyset$, $i = 1$, $Y \leftarrow \mathbb{Z}$.

1. Find $x_i$ such that $w(x_i) = \min\{w(x) : x \in Y, \{x\} \cup X_{i-1} \in \psi\}$. If no such $x_i$ exists, stop, $X_{i-1}$ is an optimum.

2. $X_i = X_{i-1} \cup \{x_i\}$, $Y = Y - \{x_i\}$, $i = i + 1$ and go to 1.


To apply Edmonds result to problem (13) we must show that $X$ (the set of $m$-trees) is the set of bases for some matroid. If this can be shown, then (13) can be solved by the efficient greedy algorithm (ALG-2).

Let $A = \{(i,j) : 1 \leq i \leq n-1, i \leq j \leq n-1, i \neq j\}$, $\tilde{A} = \{(i,n), (n,i) : 1 \leq i \leq n-1\}$, and let $\tilde{A} = \tilde{A} \cup \tilde{A}$. Then we will call a set $X \subseteq \tilde{A}$ independent if the edges of $X$ do not form a cycle. We will call a set $X \subseteq \tilde{A}$ independent if $|X| \leq 2m$. Furthermore, $X \subseteq \tilde{A}$ will
be called independent if \( |X \cap \tilde{A}| \leq n + m - 1 \) and \( X \cap \tilde{A} \) and \( X \cap A \) are independent. Let \( \psi \) be the set of all independent subsets of \( A \).

Clearly a maximal independent subset of \( A \) is an \( m \)-tree. Hence, we need only show that \([A, \psi]\) is a matroid to prove that \( ALG-2 \) solves (13).

Before proving that \([A, \psi]\) is a matroid we develop the following preliminary result.

**Theorem 4.**

Let \( G \) be a graph and let \( \Omega \) be the set of all spanning trees of \( G \). Let \( E_1 \subseteq S_1 \) and \( E_2 \subseteq S_2 \) where \( S_1, S_2 \in \Omega \) and \( |E_1| = m_1 \), \( |E_2| = m_2 \), \( m_1 > m_2 \). Then there exists an arc \( e \in E_1 - E_2 \) such that \( E_2 \cup \{e\} \subseteq S_3 \in \Omega \).

**Proof.** Suppose \( G \) has \( n \) nodes. Then \( E_1 \) has \( n - m_1 \) components and \( E_2 \) has \( n - m_2 \) components. Suppose \( E_1 \) has no arc \( e = (i,j) \) such that \( i \) and \( j \) are in different components of \( E_2 \). This implies that \( E_1 \) has at least \( n - m_2 \) components. Then \( n - m_1 \geq n - m_2 \Rightarrow m_1 \leq m_2 \) which contradicts \( m_1 > m_2 \). Therefore, there exists an \( e \in E_1 - E_2 \) which connects two components of \( E_2 \) and \( E_2 \cup \{e\} \subseteq S_3 \in \Omega \). This completes the proof of Theorem 4.

**Theorem 4** will be used in the proof of the following result.

**Theorem 5.**

\([A, \psi]\) is a matroid.

**Proof.** Axioms M1 and M2 hold trivially. Therefore, we must show that M3 holds. Let \( U_1 \) and \( U_2 \in \psi \) with \( |U_1| = s_1 \), \( |U_1 \cap \tilde{A}| = t_1 \), \( |U_1 \cap A| = s_1 \), \( i = 1, 2 \) and \( u_1 = u_2 + 1 \). Let \( e \in U_1 - U_2 \).

**Case 1:** \((s_1 > s_2)\). If \( s_1 > s_2 \), then there exists an arc \( e \in \{(U_1 \cap \tilde{A}) - U_2\} \) for which \( U_2 \cup \{e\} \in \psi \).
Case 2: \( s_1 \leq s_2 \). If \( s_1 \leq s_2 \), then \( t_1 > t_2 + 1 \). Now \((U_1 \cap A)\) and \((U_2 \cap A)\) have no cycles. By Theorem 4, there exists an \( e \in (U_1 \cap A) - (U_2 \cap A)\) such that \((U_2 \cap A) \cup \{e\}\) has no cycles. Therefore \( U_2 \cup \{e\} \subseteq U_3 \).

This completes the proof of Theorem 5.

Therefore the minimal m-tree problem,

\[
\min \left\{ \sum_{i,j} c_{ij} x_{ij} \right\},
\]

is solvable via ALG-2.

3.3 Separation

In [2] it is shown that the number of feasible solutions for an \( n \) city asymmetric \( m \)-travelling salesmen problem having \( m \) salesmen is

\[
\binom{n-1}{m} \frac{(n-2)!}{(m-1)!}.
\]

The proof of this result is constructive and shows precisely how one may generate an enumeration tree for this problem.

The tree is generated in two phases with phase 1 corresponding to \( n-1 \) in (15) and phase 2 corresponding to \( (n-2)! \). Assume that the tree is constructed from top down and let the single top node correspond to level 0. All nodes at level \( l \) in the enumeration tree will have \( l \) arcs fixed and the tree has \( n-1 \) levels since fixing \( n-1 \) arcs uniquely determines an \( m \)-tree. The first phase corresponds to the levels 1 through \( m \) while the second phase corresponds to levels \( m+1 \) through \( n-1 \). The two phases of the enumeration tree construction are now given.

**Phase 1**: At level 0, construct \( n-m \) new nodes by fixing arcs \((n,1), (n,2), \ldots, (n,n-m)\). For the node with fixed arc \((n,j)\), construct \( n-m+1-j \) new nodes by fixing arcs \((n,j+1), \ldots, (n,n-m+1)\). For any node at level \( \ell \) (\( <m \)) having arcs
(n,j₁), ..., (n,j_k) fixed where j_{k+1} > j_k, construct n-m+2-j_k new nodes by fixing (n,j_{k+1}), ..., (n, n-m-1).

Phase 2: At level \( k > m \), for any partial solution having \( \ell \) arcs fixed, there are \( n-\ell \) nodes which have no fixed arc into them. Choose one of these nodes, say \( q \). Then \( n-\ell+m-2 \) new nodes are constructed by fixing the appropriate arcs whose "to" nodes is \( q \). In developing the enumeration tree, the node (city) selected to create the new nodes is the one whose component in the subgradient differs from 0 the most.

A partial enumeration tree for six cities and two salesmen is illustrated in Figure 3.

---

3.4 Problem Selection

Following the notational conventions of Geoffrion and Marsten [9], we represent the enumeration tree by a set of candidate problems which are maintained in the candidate list. Let \( CP_i \) denote a candidate problem with fixed arcs \((i_1, j_1), \ldots, (i_\ell, j_\ell)\). Define \( Y_i \) corresponding to \( CP_i \) as \( Y_i = \{(x_{i1}, \ldots, x_{in}) : x_{i1}j_1 = \ldots = x_{i\ell}j_\ell = 1\} \). Then \( CP_i \) is the problem.

\[
\min \sum_{ij} c_{ij}x_{ij} \quad \sum_k x_{ij} = \begin{cases} 1, & \text{for } j = 1, \ldots, n-1 \\ m, & \text{for } j = n \end{cases} \\
\sum_k x_{ik} = \begin{cases} 1, & \text{for } i = 1, \ldots, n-1 \\ m, & \text{for } i = n \end{cases} \\
x \in X \cap Y_i.
\]
We make use of two relaxations of CP in the branch-and-bound algorithm. The relaxation $\overline{CP}_1$ is the Lagrangean dual,

$$\max \Theta(u,v)$$

where $\Theta(u,v) = \min_{x \in X \cap Y_1} \{ \sum_{ij} c_{ij} x_{ij} \} - \alpha$, while $\overline{CP}_2$ is simply $\Theta(u,v) = \min_{x \in X \cap Y_1} \{ \sum_{ij} c_{ij} x_{ij} \} - \alpha$. $\overline{CP}_1$ is used only at the initial node in the enumeration tree, and $\overline{CP}_2$ is used at all other nodes where $(u, v)$ is the optimal solution of $\overline{CP}_1$.

Let $v(\overline{CP}_2)$ denote the optimal objective value of $\overline{CP}_2$. Then a lower bound for all nodes constructed from $CP_i$ is $v(\overline{CP}_2)$. Let $CP_{i+1}$ be any descendent of $CP_i$ with $k + 1$ fixed arcs and let the arcs selected by the greedy algorithm for $\overline{CP}_1$ and $\overline{CP}_2$ be given by $(i_1, j_1)$, \ldots, $(i_{k}, j_{k})$ and $(i_1, j_1)$, \ldots, $(i_{k}, j_{k})$, \ldots, $(i_{k+1}, j_{k+1})$. Since the greedy algorithm was used to obtain the solution to $\overline{CP}_1$ and $\overline{CP}_2$, then

$$c_{i_k}^{\hat{}} - c_{i_{k-1}}^{\hat{}} \geq c_{i_{k-1}}^{\hat{}} - c_{i_k}^{\hat{}} \quad \text{for } k = 2, \ldots, n+m. \tag{16}$$

Thus, summing (16) we obtain

$$\sum_{k=\ell+2}^{k} c_{i_k}^{\hat{}} - c_{i_{k-1}}^{\hat{}} \geq \sum_{k=\ell+2}^{k} c_{i_{k-1}}^{\hat{}} - c_{i_k}^{\hat{}}. \tag{17}$$

Adding $\sum_{k=1}^{\ell} c_{i_k}^{\hat{}} + c_{i_{k+1}}^{\hat{}} - c_{i_{k+1}}^{\hat{}} + c_{i_{k+1}}^{\hat{}}$ to both sides of (17), we obtain

$$\sum_{k=\ell+2}^{k} c_{i_k}^{\hat{}} - c_{i_{k-1}}^{\hat{}} \geq \sum_{k=\ell+2}^{k} c_{i_{k-1}}^{\hat{}} - c_{i_k}^{\hat{}} + \sum_{k=\ell+2}^{k} c_{i_{k+1}}^{\hat{}} - c_{i_{k+1}}^{\hat{}}. \tag{18}$$

Then the right-hand-side of (18) provides a lower bound for $CP_{i+1}$.

The candidate problem selected at each iteration is the one with smallest lower bound.
3.5 The Algorithm

Using the ideas of the previous sections, we now summarize the new branch-and-bound algorithm for the asymmetric m-travelling salesmen problem. The algorithm incorporates a new idea which we call **staged optimization**. Following the conventions of [9], let CL denote the candidate list and INC denote the objective value of the incumbent. Let \( \hat{Z}^* \) denote an estimate of the value of the optimal solution. This estimate is based on the observed duality gap and, of course, may be either larger or smaller than the optimal value.

The branch-and-bound algorithm is executed with \( \hat{Z}^* \) playing the usual role of the incumbent, INC. This may substantially aid the fathoming routine with the risk of fathoming an optimum. If a feasible solution is found such that \( \text{INC} \leq \hat{Z}^* \), then the fathoming strategy using \( \hat{Z}^* \) was justified and the algorithm guarantees optimality. If the complete tree is fathomed without obtaining a feasible solution, then the fathoming strategy was not justified and the optimum has objective value greater than \( \hat{Z}^* \). For this case \( \hat{Z}^* \) is increased and the procedure is repeated. A similar idea has been reported by Marsten and Morin [16].

The algorithm incorporating staged optimization follows:

**ALG-3: BRANCH-AND-BOUND METHOD FOR M-TRAVELLING SALESMEN PROBLEM**

0. **Initialization**
   
   Choose \( \theta_1, \theta_2, \theta_3, \ldots \) for staged optimization, set \( t = 1 \), and set \( \text{INC} = \infty \).

1. **Solve Dual**
   
   Use ALG-1 to solve the dual. Let \( (u^*, v^*) \) denote the optimal...
dual variables and let $x^*$ solve $\min \{ \sum_{ij} \bar{c}_{ij} x_{ij} \}$. Set the lower bound $L = C(u^*, v^*)$ and let $(r^*, s^*)$ denote the subgradient of $G(u^*, v^*)$.

2. **Test For Duality Gap**

If $r^* = s^* = 0$, stop there is no duality gap and $x^*$ solves the primal; otherwise, set $Z^* = L(1 + \delta_c)$.

3. **Construct First Level In Enumeration Tree**

Construct the first $n-m$ nodes in the enumeration tree and place them in the candidate list.

4. **Solve New Candidate Problem**

Let $Y \in \CL$ and let $x^Y$ solve $\min \{ \sum_{ij} \bar{c}_{ij} x_{ij} \}$. Let $\Theta_Y = \sum_{ij} \bar{c}_{ij} x^Y_{ij}$, and $(r_Y, s_Y)$ denote the subgradient.

5. **Fathom Test**

If $\Theta_Y > Z^*$ go to 7.

6. **Feasibility Test**

If $r_Y = s_Y = 0$, $x^* = x^Y$, INC + $\Theta_Y$, $Z^* + \Theta_Y$, fathom candidate problems with lower bounds greater than $\Theta_Y$; otherwise go to 9.

7. **Termination Test**

If the candidate list is not empty, then go to 4.

8. **New Stage Required?**

If INC $\neq Z^*$, then set $t \leftarrow t + 1$, $L \leftarrow Z^*$, and go to 3; otherwise, stop with $x^*$ as the optimum.

9. **Separate**

Separate $Y$, update the candidate list, and go to 4.

Note that ALG-3 can be easily converted into a method to find approximate solutions. The interval of uncertainty at any point in the procedure is $[\text{INC}, L]$. An additional test in step 6 is required to make this conversion.
IV. COMPUTATIONAL EXPERIENCE

The branch-and-bound procedure, ALG-3, has been coded in standard FORTRAN for an in-core implementation. The code was initially tested on randomly generated asymmetric problems on a CDC 6600. The random number generator employed for the generation of problems is the one available on the CDC FTN compiler. The range used for generating the distances was \([100, 3400]\). The same range was used by Bazaraa and Goode in their computational work on the travelling salesman problem [3]. Since the code is designed for an in-core implementation, one of the major problems encountered in the computational testing was core storage. The size of the candidate list grows rapidly for the larger problems, \(n > 50\) and the storage requirement soon exceeds the available 200K octal words, even with most of the data packed.

The computational efficiency of the code is sensitive to the selection of parameters. Bazaraa and Goode observed that there is a trade-off in the amount of computational effort expended in the maximization of the dual and the branch-and-bound procedure. The more time spent in ALG-7, the better the lower bound obtained. We employed \(\lambda_1 = 2^{-i}\) in the code and \(\bar{G}\) and \(\bar{H}\) were selected based on the size of the problem. The subgradient procedure increases the lower bound \(L\) rapidly in the initial iterations and minimal increases are obtained for \(\bar{G} > 10\). However, for larger problems, we found it beneficial to use a larger value for \(\bar{G}\), even though the relative increase in the lower bound is minimal. Further, the choice of the duality gap estimates \(\bar{\delta}_1\) is crucial. If they are chosen too large, then the candidate list grows rapidly, while if they are too small,
the number of candidate problems solved increases.

Our initial computational experience centered on evaluating the behavior of the code to parameter selection and to the type of problem being solved. Table 1 summarizes computational results for asymmetric problems for \( n = 30, 40, 50, \) and 60 with \( 1 \leq m \leq n/10. \) The parameter settings for the first seven sets used were \( \bar{H} = 5, \bar{C} = 10 \) and \( \beta_1 = .01, \beta_2 = .02, \) and \( \beta_3 = .03. \) We found that the duality gaps for problems with \( n = 50 \) and 60 were smaller than .005 and further, the number of subproblems generated with lower bounds within one percent of the solution for the Lagrangean dual was large. Problem sets 9 - 12 were solved with the duality gap estimates set to \( \beta_1 = .005, \beta_2 = .01, \beta_3 = .015. \)

The general conclusions which may be drawn from the results of problem sets 1 - 12 is that the duality gap seems to decrease as \( m \) increases. Further, the problems with larger values of \( m \) are easier to solve than those with smaller values of \( m. \) However, as the problem size increases, the number of candidate problems within small percentages of the lower bound obtained from the Lagrangean dual increases. To circumvent the consequential storage problem, we have devised an approximation technique which simply consists of terminating the branch-and-bound procedure as soon as a solution is obtained. The solution so obtained is provable to be within a small percentage of the optimal solution. Since the staged optimization technique makes use of estimates of the duality gap of a problem, it is possible to obtain a suboptimal solution within a known percentage of the true solution by terminating the branch-and-bound procedure as soon as an \( m \)-tour is obtained. This approximate solution technique was tested on problems for \( n = 60. \) The results are summarized
in problem sets 13 - 18 of Table 1. To obtain a tighter lower bound, 
\( e = 12 \) was used with \( \theta_1 = .005, \theta_2 = .01, \) and \( \theta_3 = .02. \)

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Table 1 About Here
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Table 2 reports computational results for the solution of 25 100-city problems for which solutions were obtained. There were 25 other problems which terminated due to storage limitations before locating an m-tour. For the use of the approximate technique, a better selection procedure, which makes use of the subgradient, can be devised so that an m-tour can be located before the number of candidate problems becomes unmanageable (see [1]).

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Table 2 About Here
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To gain insight into the solution of symmetric m-travelling salesmen problems, we attempted to solve 50 100-city problems by randomly generating such problems. The range on the costs used was the same as for asymmetric problems. For such problems, the code was slightly modified so that the subgradient employed had \( n \) rather than \( 2n \) components. However, the enumeration tree was not modified for these problems, nor were other specializations made. Computational results for 15 problems which were successfully solved are given in Table 3. The remaining 35 were terminated due to storage limits. The problems which were solved had negligible duality gaps, and thus solutions were obtained before the candidate list grew. The computational results indicate that the duality gap for large problems is exceedingly small. Thus, even though minimal increases are obtained in the course of final iterations of the
subgradient procedure for maximizing the dual, it is beneficial to employ larger values for \( \delta \).

Table 3 About Here

Table 4 gives results on Euclidean problems obtained by the use of intercity distances which were provided by the Civil Aeronautics Board for 59 cities. Because the code has not been designed for the solution of such problems specifically, the computational experience on these problems is not extensive. Rather, the focus here was to obtain insight into the nature of solutions to such problems. Five networks were chosen arbitrarily and three problems were defined for each network. The first four networks have 30 nodes each. Networks III and IV differ only in the choice of the base node. The solutions to the problem are illustrated in Figures 4-8. The most interesting inference that may be drawn from the solutions is the relationship between the duality gaps for the problems and the solutions. The solutions to the six problems on networks I and II are intuitively obvious, whereas the solutions to the problems on networks III and IV are not. Thus, the harder the problem, the larger the duality gap. Furthermore, note the manner in which solutions to problems on the same network are related. From the first three networks, it would seem that realistic m-travelling salesmen problems would have to be further constrained to ensure that each salesman visit at least some minimum number of cities. Failing this constraint, solutions of the kind illustrated may be expected. Whereas, as the number of salesmen increases, as in Figure 7, the solutions become more meaningful. That is, not all but
one salesman travel to only one other city. The solutions obtained for the network on 59 nodes are illustrated in Figure 8. For the problems on networks III, IV and V, even though the initial estimates of the upper bounds used were quite close to the true duality gap, the number of subproblems examined before verification of optimality is large.

Based on our computational experience using the technology developed in this exposition, we have arrived at the following conclusions:

(i) The duality gap for problems with up to 100 cities is quite small (less than 1%).

(ii) Holding the number of cities constant, the problems become easier to solve as the number of salesmen increases and the duality gap decreases.

(iii) Exact solutions for problems with up to 50 cities can be obtained routinely with the implementation of the algorithm reported.

(iv) An in-core/out-of-core algorithm using this technique could be used to solve problems having 100 cities.
REFERENCES


Figure 1. Examples of 2-trees on 5 nodes
Figure 2. Graphs Having 6 Edges Which Are Not 2-trees
Figure 3. Partial Enumeration Tree For n = 6 m = 2.
Figure 4. Illustration of Solutions to Problems On Network I.
Figure 5. Illustration of Solutions to Problems On Network II.
Figure 6. Illustration of Solutions to Problems On Network III.
Figure 7. Illustration of Solutions to Problems On Network IV.
Figure 8. Illustration of Solutions to Problems on Network V.
### Table 1. Summary of Computational Results for the Asymmetric m-travelling Salesmen Problem.

Each problem set contains 10 problems and all timings are in CPU seconds on a CDC 6600.

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<th>Number of Subproblems</th>
<th>Duality Gap</th>
<th>Solution Time</th>
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* Results are for Suboptimal Solutions. (Interval of Uncertainty = Duality Gap)

† Some problems in the set were not solved due to storage and time limits. (For set 9, 1; for set 13; 2, for set 15, 1).
Table 2. Approximate Solutions For the Asymmetric m-travelling Salesmen Problem on 100 cities.

All timings are in CPU seconds on a CDC 6600 and the interval of uncertainty = Duality Gap.

<table>
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<tr>
<th>Problem No.</th>
<th>Problem Seed</th>
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Parameters: $\bar{n} = 5$, $\bar{c} = 15$, $\vartheta_1 = .005$, $\vartheta_2 = .01$, $\vartheta_3 = .02$

* $\vartheta_1 = .01$
Table 2 continued

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<th>Lagrangean Dual $0(u^<em>,v^</em>)$</th>
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<th>Branch and Bound Solution</th>
<th>Branch and Bound Nodes</th>
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Table 3. Approximate Solutions For the Symmetric m-travelling salesmen problem on 100 cities.
All timings are in CPU seconds on a CDG 6600 and the interval of uncertainty = Duality Gap.

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<th>Problem No.</th>
<th>Problem Seed</th>
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<th>Branch and Bound</th>
<th>Duality Gap</th>
<th>Total Time</th>
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Parameters: \( \bar{H} = 5, \bar{C} = 15, \theta_1 = .005, \theta_2 = .01, \theta_3 = .015 \)
Table 4. Solution of Euclidean m-travelling Salesmen Problems. All timings are in CPU seconds on a CDC 6600.

<table>
<thead>
<tr>
<th>Network</th>
<th>Î</th>
<th>m</th>
<th>( \Theta(0,0) )</th>
<th>( \Theta(u^<em>,v^</em>) )</th>
<th>Time</th>
<th>Branch and Bound</th>
<th>Duality Gap</th>
<th>Total Time</th>
<th>Figure</th>
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<td>9799</td>
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<td>9799</td>
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Parameters: \( \bar{n} = 5, \bar{G} = 20, \bar{a}_1 = .005, \bar{a}_2 = .01, \bar{a}_3 = .015 \)

* \( \bar{a}_1 = .01 \)
**14. CCS-RR-372**

**THE M-TRAVELLING SALESMSN PROBLEM: A DUALITY BASED BRANCH-AND-BOUN D ALGORITHM**

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**ABSTRACT**
This paper presents a new model and branch-and-bound algorithm for the m-travelling salesmen problem. The algorithm uses a Lagrangean relaxation, a subgradient algorithm to solve the Lagrangean dual, a greedy algorithm for obtaining minimal m-trees, penalties to strengthen the lower bounds on candidate problems, and a new concept known as staged optimization. Computational experience for both symmetric and asymmetric problems having up to 100 cities is presented.