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EXTENED VALIDITY OF LINEARIZED KINEMATIC MODEL FOR OPTIMAL MISS--ETC(U)
MAR 80 Y ROTSZTEIN, J SHINAR F49620-79-C-0135

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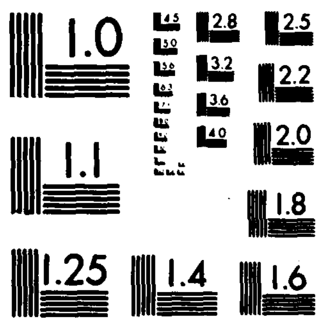
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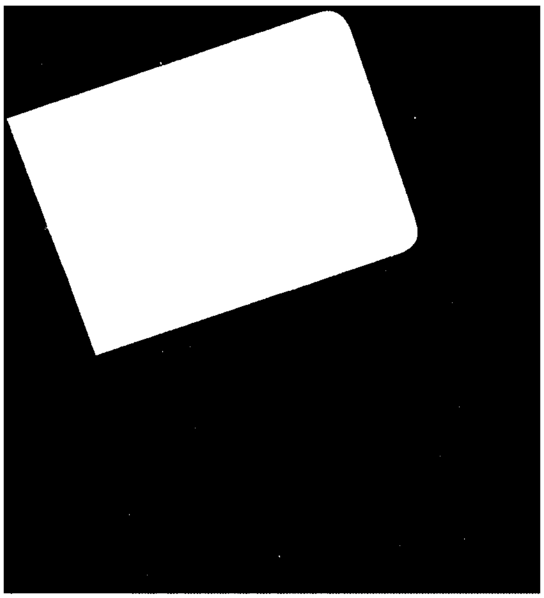

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John F. Zingg

JOHN F. ZINGG
Project Engineer
Fire Control Technology Group

Lester McFawn

LESTER MCFAWN, Chief
Fire Control Technology Group
Fire Control Branch

FOR THE COMMANDER

Marvin Spector

MARVIN SPECTOR, Chief
Fire Control Branch
Reconnaissance & Weapon Delivery Div.

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non-linear simulation.

EXTENDED VALIDITY OF LINEARIZED KINEMATIC MODEL FOR OPTIMAL MISSILE AVOIDANCE*

Y. Rotsztein** and J. Shinar***

Department of Aeronautical Engineering
Technion - Israel Institute of Technology,
Haifa, Israel

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** Research Engineer, Graduate Student

*** Associate Professor

ABSTRACT

Optimal missile avoidance is analyzed with a two-dimensional linearized kinematic model. It is shown that inclusion of a control-effort penalization term in the payoff function leads to extend the domain of validity of the trajectory linearization. The required values of the weighting coefficient of the control penalization and the resulting loss in the optimal miss distance are evaluated. A recursive algorithm for the numerical solution of the modified optimal avoidance problem is presented. The optimal solutions obtained for the linearized kinematical model are compared to results of non-linear simulation.

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I. INTRODUCTION

The problem of optimal missile avoidance was analysed in the past using different types of simplified mathematical models. Three types of simplifying assumptions were used:

a. Neglecting guidance dynamics.[1-3]

b. Restricting the motion to a plane.[3-10]

c. Trajectory linearization.[1-3, 7, 9-12]

It was shown that the attractive assumption, made by neglecting the dynamics of the pursuer, yields seriously misleading results.[1,2,3] Whenever guidance dynamics is considered[4-12] (even if by an approximation of a first order time constant or a pure time delay), optimal evasion can guarantee non-zero miss distance even from a pursuer of unlimited maneuverability[7] or from one of an optimal guidance strategy.[12]

Trajectory linearization, both for two and three dimensional models has been proved to be a useful way to obtain analytical solutions providing an insight to the problem. Recently accomplished studies with a linearized kinematic model[10,11] indicate that the optimal maneuver for missile avoidance is a "bang-bang" type with the continuous use of maximum load factor of the evading airplane. It can be therefore reduced to an optimal roll-position control problem of two consecutive phases: (1) Orienting the airplane lateral acceleration vector into the plane of optimal evasion; (2) Changing the direction of this acceleration, which has to be maximal, by rapid roll maneuvers of 180° in accordance with an optimal switch function.

However, the validity of trajectory linearization is not always obvious. This assumption is valid as long as the evader's trajectory does not deviate much from its initial direction. This requirement can be satisfied if:

a. The dynamic similarity parameter of the problem,[13] defined by direction change of the evader during a period of the pursuer's time constant, is small.

b. The solution does not include excessively long turns in one direction.

The first condition has to be examined before trajectory linearization. The second one, however, can be verified only a-posteriori. Due to the "bang-bang" structure of the optimal evasive maneuver, in most cases this second condition is also satisfied. A recent investigation[14] has shown that there exists a range of parameters (long flight times, small values of effective proportional navigation constants, low missile/target maneuver ratios) for which long turns are predicted by the linearized kinematic model maximizing the miss distance. Moreover, it has been shown[10] that the sensitivity of the miss distance to target maneuver, performed far away from the point of closest approach is relatively small.

The objective of the present paper is to modify slightly the optimal control problem, to enable the extension of the validity of trajectory linearization.

In this new formulation the payoff to be maximized is the square of the miss distance penalized by a quadratic integral term of the control effort. A similar cost function has been used in the past[8] to avoid numerical difficulties in singular control. The optimal miss distance obtained with this new formulation will be, no doubt, smaller than the value predicted in the original problem.

Recently, it has been proven[15] that the difference between the miss distance obtained using different cost functions can be made small by proper choice of the weighting coefficient of the integral term.

In the present work the emphasis is to eliminate the long initial turn of the optimal maneuver sequence, which can be achieved by using a relatively large weighting coefficient for the control penalization.

The analysis is carried out under the following set of assumptions:

1. Missile and target are considered as constant speed point-mass elements.
2. The missile is guided by proportional navigation with constant effective P.N. coefficient.
3. Both missile and . ;et perform lateral accelerations perpendicular to the initial line of sight.
4. The deviation of the trajectory from the reference line of sight can be decomposed in two perpendicular planes. For the sake of simplicity only one of these planes is considered and the gravity component in this plane is neglected.
5. The dynamic response of the guidance system is approximated by a first order transfer function with a time constant τ.

6. The missile has unbounded lateral acceleration.
7. Target dynamics are neglected in first approximation but target lateral acceleration is bounded.

Based on these assumptions the modified optimal missile avoidance problem is formulated in Section II in a nondimensional form, and solved in a closed form (up to a multiplicative constant) in Section III. The criteria for the selection of the proper weighting of control effort penalization term is outlined in Section IV, while the reduction in the optimal miss distance due to the penalization is estimated in Section V. In Section VI a simple recursive algorithm for computing the solution of the modified optimal missile avoidance problem is presented. In the sequel the extension of the solution to cases of limited missile acceleration and target roll rate is discussed. Solution of the linearized modified optimal missile avoidance is compared to results of non-linear simulation in Section VII

II. PROBLEM STATEMENT

A. Mathematical Model

The goemetry of a two-dimensional pursuit-evasion is shown in Fig. 1 defining the parameters of the problem. The equations of motion of optimal missile avoidance can be written, subject to the set of assumptions outlined in the Introduction, as follows:

a. Relative geometry perpendicular to the line of sight

$$y(t) = y_T(t) - y_M(t) \tag{1}$$

and consequently the relative acceleration is

$$\ddot{y}(t) = \ddot{y}_T(t) - \ddot{y}_M(t) \tag{2}$$

b. Missile guidance transfer function is expressed by

$$\tau \dddot{y}_M + \ddot{y}_M = (\ddot{y}_M)_C \tag{3}$$

c. Missile acceleration command is obtained according to the guidance law of Proportional Navigation[10]

$$(\ddot{y}_M)_C = \frac{N'}{(t_f-t)} \left[\dot{y} + \frac{y}{(t_f-t)}\right] \tag{4}$$

The time of of flight of the missile t_f is fixed, determined by

$$t_f = \frac{R(0)}{V_M \cos \gamma_{M(0)} - V_T \cos \gamma_{T(0)}} \tag{5}$$

Target acceleration perpendicular to the line of sight, which is the control function of the problem, is bounded

$$|\ddot{y}_T(t)| \leqslant (a_T)_{max} \tag{6}$$

Introducing nondimensional variables for time and distance[13] by

$$\begin{aligned} \tilde{t} &= t/\tau \\ \tilde{y} &= \frac{y}{\tau^2 (a_T)_{max}} \end{aligned} \tag{7}$$

leads to normalize the velocity components by $\tau(a_T)_{max}$ and accelerations by $(a_T)_{max}$. As a result Eqs.(1), (2), and (4) are transformed to

$$\begin{aligned} \ddot{\tilde{y}}(\tilde{t}) &= \ddot{\tilde{y}}_T(\tilde{t}) - \ddot{\tilde{y}}_M(\tilde{t}) \\ \dddot{\tilde{y}} + \ddot{\tilde{y}}_M &= (\ddot{\tilde{y}}_M)_C \\ (\ddot{\tilde{y}}_M)_C &= \frac{N'}{(\tilde{t}_f - \tilde{t})^2}\, \tilde{y} + \frac{N'}{(\tilde{t}_f - \tilde{t})}\, \dot{\tilde{y}} \end{aligned} \tag{8}$$

with the constraint

$$|\ddot{\tilde{y}}_T(\tilde{t})| \leqslant 1 \tag{9}$$

The non-dimensional state vector of the problem can thus be defined as

$$\underline{X}^T(\tilde{t}) \triangleq [\tilde{y}(\tilde{t}),\tilde{\dot{y}}(\tilde{t}),\tilde{\ddot{y}}_M(\tilde{t})] \qquad (10)$$

If the non-dimensional control vector is defined as

$$\underline{u}^T(\tilde{t}) \triangleq \text{col}[0,u,0] \triangleq \text{col}[0,\tilde{\ddot{y}}_T(\tilde{t}),0] \qquad (11)$$

The normalized state equation can be written as

$$\frac{d\underline{X}}{d\tilde{t}} = A(\tilde{t})\underline{X} + \underline{u} \qquad (12)$$

with

$$A(\tilde{t}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \dfrac{N'}{(\tilde{t}_f-\tilde{t})^2} & \dfrac{N'}{(\tilde{t}_f-\tilde{t})} & -1 \end{bmatrix} \qquad (13)$$

and the constraint

$$|u| \leq 1 \qquad (14)$$

B. Formulation of the Optimal Control Problem

The objective of the missile avoidance is to maximize the survivability of the evading aircraft. Assuming uniformly performing warhead and proximity fuse leads to determine the payoff as the square of the miss distance. For normalized parameters and linearized kinematics it is expressed as

$$J = \tilde{y}^2(\tilde{t}_f) \triangleq m^2 \tag{15}$$

The optimal missile avoidance with the above described mathematical model can be formulated as a fixed duration optimal control problem.

This problem with the payoff (15) was solved in a previous work,[10] yielding a non-singular bang-bang solution. This solution predicts, for long normalized times of flight, long initial maneuvers. Implementation of such a maneuver in the real world (described by non-linear equations) results in significant changes in interception geometry, and may consequently invalidate the assumptions of the linearized kinematic model. It has been also shown[10] that the sensitivity of the miss-distance to target acceleration performed far away from intercept is negligibly small.

In order to avoid excessively long and inefficient maneuvers it is proposed to modify the payoff of Eq.(15) by adding a term of a target maneuver effort penalization. Such a modified payoff for the non-dimensional mathematical model has the form

$$J = \tilde{y}^2(\tilde{t}_f) - K \int_0^{\tilde{t}_f} u^2 \, d\tilde{t} \tag{16}$$

The value of the weighting coefficient K has to be specified later. The modified optimal evasion problem can be thus formulated:

Given the dynamic system described by Eqs.(12) with zero initial conditions ($\underline{X}_0 = 0$) *and unspecified terminal state, find, for a fixed normalized time of flight* $\tilde{t}_f$ *, the optimal control* $\underline{u}^*(\tilde{t})$ *subject to the constraint (14), which maximizes the payoff given in Eq.(16).*

The payoff function in the form of Eq.(16) was used in the past[8] to avoid numerical difficulties. Here it is used for a different purpose and it will be shown that the value of the weighting coefficient K will determine the domain of validity of the linearized kinematic model.

III. SOLUTION OF THE MODIFIED OPTIMAL CONTROL PROBLEM

For the optimal control problem formulated in the previous section, the variational Hamiltonian

$$H(\underline{x},\underline{\lambda},\underline{u},\tilde{t}) = -Ku^2 + \underline{\lambda}^T[A(\tilde{t})\underline{x} + \underline{u}] \qquad (17)$$

can be rewritten, separating the part independent of the control variable u, as

$$H = H_0(\underline{x},\underline{\lambda},\tilde{t}) - Ku^2 + \lambda_2 u \qquad (18)$$

The components of the costate vector $\underline{\lambda}$ are determined by the adjoint equation

$$d\underline{\lambda}/d\tilde{t} = -\partial H/\partial \underline{x} \tag{19}$$

with the terminal conditions

$$\lambda_1(\tilde{t}_f) = -2x_1(\tilde{t}_f) = -2m \tag{20}$$

$$\lambda_2(\tilde{t}_f) = \lambda_3(\tilde{t}_f) = 0 \tag{21}$$

For a linear system as (12) Eq.(19) yields

$$d\underline{\lambda}/d\tilde{t} = -A^T(\tilde{t})\underline{\lambda} \tag{22}$$

which can be transformed by introducing the normalized time-to-go

$$\theta \triangleq \tilde{t}_f - \tilde{t} \tag{23}$$

to

$$d\underline{\lambda}/d\theta = A^T(\theta)\underline{\lambda} \tag{24}$$

with the initial conditions

$$\begin{aligned} &\lambda_1(\theta=0) = -2m \\ &\lambda_2(0) = \lambda_3(0) = 0 \end{aligned} \tag{25}$$

The system of equations (24) can be reduced to a scalar differential equation of the form

$$\theta \left(\frac{d^3\lambda_3}{d\theta^3} + \frac{d^2\lambda_3}{d\theta^2} \right) + N' \frac{d\lambda_3}{d\theta} = 0 \tag{26}$$

which was partially solved in the past.[10] A complete close-form solution for integer values of N', is presented in the Appendix.

The optimal control u^* is obtained by

$$u^* = \arg \max_{|u|\leq 1} H = \arg \max_{|u|\leq 1} (-Ku^2 + \lambda_2 u) \tag{27}$$

yielding

$$u^*(\theta) = \text{sat} \left(\frac{\lambda_2^*(\theta)}{2K} \right) \tag{28}$$

where

$$\lambda_2(\theta) = -2m \; f_2(\theta) \tag{29}$$

and $f_2(\theta)$ is given by Eq.(A-13)

$$f_2(\theta) = e^{-\theta} \sum_{i=0}^{N'-2} (-1)^{N'-2-i} \frac{\binom{N'-2}{i}}{(N'-1-i)!} \theta^{N'-1-i} \tag{30}$$

The saturation function is defined by

$$\text{sat}(a) \triangleq \begin{cases} a & \text{if} \quad |a| < 1 \\ \text{sign}(a) & \text{if} \quad |a| \geq 1 \end{cases} \tag{31}$$

For integer values of N' the value of $\lambda_2(\tilde{t})$, hence the value of $u^*(\tilde{t})$ can be computed in a closed-form up to a multiplicative constant. It can be easily seen that as K approaches zero, $u^*(\tilde{t})$ becomes a "bang-bang" control. The normalized form of λ_2 is depicted in Fig. 2. In guidance analysis this function is well known as the miss distance sensitivity function to a unit target lateral acceleration impulse. The exponential form of Eq.(30) leads to affirm, as it can be seen also in Fig. 2, that

$$\lim_{\theta \to \infty} \lambda_2(\theta) = 0 \tag{32}$$

In Figs. 3 and 4 the optimal control function $u^*(\theta)$ is depicted for different values of N' and K.

From these figures we observe that the optimal control solution, for long normalized time of flight (but using a not excessively large weighting coefficient K) consists of 3 phases:

1. An initial phase of no-maneuver.
2. A phase of gradually increasing maneuver.
3. An almost "bang-bang" terminal phase.

Thus excessively long initial target maneuvers can be avoided by a proper choice of K.

The miss distance using non-zero values of K will be obviously smaller than the miss distance with $K=0$. In a recent study[15] it was shown that the upper bound of the difference in the miss distances can be estimated. Such an estimate will be made in the sequel after the criteria of selecting the proper value of the penalization coefficient K is discussed.

IV. DETERMINATION OF THE WEIGHTING COEFFICIENT K.

In the mathematical model used in this work, trajectory linearization is performed assuming that both vehicles have constant speeds and that they do not perform excessive turns. The first assumption appears inherently in the Eqs.(4) and (5). The second assumption, however has to be verified using the optimal control function.

In this section a method is presented which leads to determine a proper value of the weighting coefficient K of the control effort penalization in Eq.(16), such that the validity of trajectory linearization is guaranteed.

The "bang-bang" structure of the optimal missile avoidance maneuver at the terminal phase leads to conclude that validity of trajectory linearization can be achieved by avoiding excessively long initial maneuvers in a constant direction.

Let us define as t_1 the time at which the first direction change of the evader's lateral acceleration occurs (see Figs. 3 and 4).

The validity of trajectory linearization can be preserved by assuming that in the initial phase of the evasion ($t \leq t_1$) the direction change of the target is bounded by some limiting value $(\Delta\Gamma)_{max}$. This limiting value has to be empirically specified.

The target's turning rate is given by

$$\dot{\gamma}_T = \frac{a_T}{V_T} \tag{33}$$

assuming constant velocity, the target's initial direction change, $(\Delta\gamma_T)_1$ is given by

$$(\Delta\gamma_T)_1 \approx \frac{1}{V_T} \left| \int_0^{t_1} a_T(t)dt \right| \tag{34}$$

and it is required that

$$(\Delta\gamma_T)_1 \leq (\Delta\Gamma)_{max} \tag{35}$$

Using the normalized variables of Eq.(7) and noting that

$$u \triangleq \frac{\ddot{y}_T}{(a_T)_{max}} \simeq \frac{a_T}{(a_T)_{max}} \tag{36}$$

Eq.(34) can be rewritten as

$$(\Delta\gamma_T)_1 \cong \frac{\tau(a_T)_{max}}{V_T} \left| \int_0^{\tilde{t}_1} u(\tilde{t})d\tilde{t} \right| \tag{37}$$

Defining the dynamic similarity parameter, $\tilde{\alpha}_T$,[13] (in radians) by

$$\tilde{\alpha}_T \triangleq \frac{\tau (a_T)_{max}}{V_T} \tag{38}$$

and substituting it into Eq.(37) yields

$$(\Delta\gamma_T)_1 = \tilde{\alpha}_T \int_0^{\tilde{t}_1} u(\tilde{t})d\tilde{t} = \tilde{\alpha}_T(\Delta\dot{\tilde{y}}_T)_1 \tag{39}$$

where $(\Delta\dot{\tilde{y}}_T)_1$, the initial normalized target velocity change perpendicular to the initial line of sight, is defined by

$$(\Delta\dot{\tilde{y}}_T)_1 \triangleq \int_0^{t_1} u(\tilde{t})d\tilde{t} \tag{40}$$

In Fig. 5 $(\Delta\dot{\tilde{y}}_T)_1$ is depicted as a function of the weighting coefficient K for different values of N'. Substitution of (39) into (35) leads to the inequality

$$(\Delta\dot{\tilde{y}}_T)_1 \leq \frac{(\Delta\Gamma)_{max}}{\tilde{\alpha}_T} \tag{41}$$

If the value of $(\Delta\Gamma)_{max}$ is chosen to be in the order of 20-30 degrees the validity of the linearization will not be violated. Thus for any given problem the proper value of K can be determined using (41) and Fig. 6.

In the following an approximate method to evaluate the required value of K is presented.

For a given $(\Delta\Gamma)_{max}$ and $\tilde{\alpha}_T$, a maximum admissible value of $(\Delta\dot{\tilde{v}}_T)_1$ is obtained using (41),

$$(\Delta\dot{\tilde{y}}_{T_1})_{max} = \frac{\Delta\Gamma_{max}}{\tilde{\alpha}_T} \tag{42}$$

We define (see Fig. 3)

$$\theta_1 \triangleq \tilde{t}_f - \tilde{t}_1 \tag{43}$$

and

$$\theta_u \triangleq \theta_1 + (\Delta\dot{\tilde{y}}_{T_1})_{max} \tag{44}$$

Thus a unit step of the control, applied at θ_u, will generate $(\Delta\dot{\tilde{y}}_{T_1})_{max}$ as given in Eq.(42).

Inspection of Figs. 3 and 4 indicates that the actual value of the optimal control function at θ_u

$$|u(\theta_u)| = \eta < 1 \tag{45}$$

Assuming a parabolic approximation for the control function in the unsaturated phase and using Eq.(28) and (29) we obtain ($\eta = 4/9$)

$$|u(\theta_u)| = \frac{\bar{m}^*}{K} \; |f_2(\theta_u)| \cong \text{[illegible]} \tag{46}$$

yielding an approximation for K

$$K \cong \frac{9}{1} \bar{m}^* \; |f_2(\theta_u)| \tag{47}$$

The actual value of η ranges from 0.25 to 0.5.
Using this approximation will eventually lead to a value of $(\Delta\dot{\tilde{y}}_T)_I$ only slightly different than the limit determined in Eq.(42).

V. EFFECT OF CONTROL-EFFORT PENALIZATION ON THE MISS DISTANCE

The use of control-effort penalization term in the modified cost J (Eq.16) reduces the optimal miss distance compared to the one obtained optimizing the original cost function $\bar{J}$ (Eq.15). The bounds of this difference can be calculated using the method presented in a previous work.[15]

We denote the optimal control function maximizing $\bar{J}$ by $\bar{u}^*$, and the resulting miss distance by $\bar{m}^*$, while the miss distance obtained by maximizing J is denoted by m^*.

Obviously we have

$$\bar{m}^{*2} - K \int_{\tilde{t}_0}^{\tilde{t}_f} \bar{u}^{*2} \, d\tilde{t} \leq m^{*2} - K \int_{\tilde{t}_0}^{\tilde{t}_f} u^{*2} \, d\tilde{t} \tag{48}$$

Since

$$m^{*2} \leqslant \bar{m}^{*2} \tag{49}$$

we have

$$0 \leqslant \bar{m}^{*2} - m^{*2} \leqslant K \int_{\tilde{t}_0}^{\tilde{t}_f} (\bar{u}^{*2} - u^{*2})\, d\tilde{t} \tag{50}$$

As can be seen from Figs. (3) and (4), both controls are almost equal except for the initial unsaturated part of u^*. Let us denote by $\tilde{t}_s$ the normalized time when saturation starts. The integral in the right side of (50) can be thus decomposed

$$\int_{\tilde{t}_0}^{\tilde{t}_f} (\bar{u}^{*2} - u^{*2})d\tilde{t} = \int_{\tilde{t}_0}^{\tilde{t}_s} (\bar{u}^{*2} - u^{*2})d\tilde{t} + \int_{\tilde{t}_s}^{\tilde{t}_f} (\bar{u}^{*2} - u^{*2})d\tilde{t} \tag{51}$$

Since the second integral has a negligeable small value (see Figs. 3 and 4) we may write approximately

$$\int_{\tilde{t}_0}^{\tilde{t}_f} (\bar{u}^{*2} - u^2)d\tilde{t} \cong \int_{\tilde{t}_0}^{\tilde{t}_s} (\bar{u}^{*2} - u^{*2})d\tilde{t} \tag{52}$$

We also recall that the optimal control of the original problem is of "bang-bang" type, i.e.,

$$\bar{u}^{*2} = 1 \tag{53}$$

Substituting (52) and (53) into (50) leads to

$$0 \leqslant \bar{m}^{*2} - m^{*2} \leqslant K\left[(\tilde{t}_s - \tilde{t}_0) - \int_{\tilde{t}_0}^{\tilde{t}_s} u^{*2}\, d\tilde{t}\right] \tag{54}$$

or

$$0 \leqslant \bar{m}^{*2} - m^{*2} \leqslant K(\tilde{t}_s - \tilde{t}_0)(1 - \hat{u}^{*2}) \tag{55}$$

where $\hat{u}^*$ is the average value of the optimal control u^* in the interval $\tilde{t}_0 \leqslant \tilde{t} \leqslant \tilde{t}_s$ being defined by

$$\hat{u}^{*2} \triangleq \frac{1}{\tilde{t}_s - \tilde{t}_0} \int_{\tilde{t}_0}^{\tilde{t}_s} u^{*2}\, d\tilde{t} \leqslant 1 \tag{56}$$

Dividing Eq.(55) by $\bar{m}^{*2}$

$$\frac{\bar{m}^{*2} - m^{*2}}{\bar{m}^{*2}} \leqslant \frac{K(\tilde{t}_s - \tilde{t}_0)(1 - \hat{u}^{*2})}{\bar{m}^{*2}} \triangleq \varepsilon \tag{57}$$

an upper bound for the difference between the normalized miss distances,

$$\Delta m^* = (\bar{m}^* - m^*) \geqslant 0 \tag{58}$$

can be obtained Assuming that $\varepsilon << 1$, i.e.,

$$\bar{m}^{*2} - m^{*2} = (\bar{m}^* + m^*)\Delta m^* \approx 2\bar{m}^* \Delta m^* \tag{59}$$

the upper bound of Δm^* (for $\tilde{t}_0 = 0$) becomes

$$\Delta m^* \leq \frac{K\tilde{t}_s(1-\hat{u}^{*2})}{2\bar{m}^*} \tag{60}$$

A rough approximation of this upper bound is obtained by taking the "worst case", i.e., $\tilde{t}_s = \tilde{t}_f$ and $\hat{u}^* = 0$ leading to

$$\Delta m^* < \frac{K\tilde{t}_f}{2\bar{m}^*} \tag{61}$$

This inequality is of course valid only if the product $K\tilde{t}_f$ is sufficiently small.

From Table 1 it can be clearly seen that the difference Δm^* in the optimal miss distances is well within the bounds predicted by Eq.(60) and (61).

			Actual Value		Predicted upper bounds	
N'	$\bar{m}^*$	K	Δm^*	$\frac{\Delta m^*}{\bar{m}^*}$ (%)	Eq.(60)	Eq.(61)
3	0.541	10^{-4}	1.03×10^{-4}	0.02	1.53×10^{-3}	2.8×10^{-3}
		10^{-2}	1.4×10^{-2}	2.58	0.21	0.28
5	0.294	10^{-4}	2.9×10^{-4}	0.1	2.53×10^{-3}	5.1×10^{-3}
		10^{-3}	5.4×10^{-3}	1.83	3.49×10^{-2}	5.1×10^{-2}

TABLE 1. Actual loss in the optimal miss distance and its predicted upper bounds $(\tilde{t}_f = 30)$.

VI RECURSIVE ALGORITHM FOR NUMERICAL SOLUTION

A. Linear Model

The state equation (12) yields a formal solution of the form

$$\underline{X}(\tilde{t}) = \Psi(\tilde{t},\tilde{t}_0)\underline{X}(0) + \int_0^{\tilde{t}} \Psi(\tilde{t},\xi)\underline{u}(\xi)d\xi \tag{62}$$

where $\Psi(\tilde{t},\tilde{t}_0)$ is the state transition matrix of the homogeneous time-varying linear system

$$\dot{\underline{X}} = A(\tilde{t})\underline{X} \tag{63}$$

This solution can be expressed in terms of confluent hypergeometric functions, which are not suitable for explicit analysis. Therefore, a complete solution of the optimal problem requires some numerical aid. The existence of explicit, time-dependent expression for $f_2(\tilde{t})$ which serves as a switch function of the "bang-bang" solution in the original problem · (K = 0) provides a handy algorithm for the solution of the modified optimal evasion.

The procedure is the following:

1. Known the function $f_2(\tilde{t})$ from Eqs. [illegible]) and ([illegible]), solve Eq.(12) and compute the miss distance $\bar{m}^*$, obtained by "bang-bang" control $\bar{u}^* = \mathrm{sign}[f_2(\tilde{t})]$. Pose

$$m_i = \bar{m}^* \qquad i = 1 \tag{64}$$

where i is the iteration index.

2. Solve Eq.(12) with the new control

$$u_{i+1} = \text{sat}\left(\frac{m_i}{K} f_2(\tilde{t})\right) \tag{65}$$

and compute the new miss distance m_{i+1}.

3. Advance the index i by a unit and repeat step 2 until a required convergence criterion for $\bar{m}^*$ is satisfied.

This algorithm has a very fast convergence for values of $K < 10^{-2}$. In most of the practical cases, 1 to 3 iterations are needed to obtain a relative error of 0.1% in the miss distance.

B Extension to Other Cases

In a realistic description of the problem of optimal missile avoidance the assumptions of infinite missile maneuverability and instantaneous target response (implying an infinite roll-rate), enabling to solve a linear problem, have to be abandoned. In previous works[10,11] it was shown that taking into account the limits of missile lateral acceleration and target roll-rate does not change the "bang-bang" structure of the optimal control solution. However, due to the non-linear effects the solution has to be obtained numerically. Assume that the original solution for $K=0$ is obtained and it seems to generate large variation of the interception geometry, which invalidate trajectory linearization. (This solution includes the optimal switch function $\lambda_2(\tilde{t})$ obtained by some numerical method). The appropriate value of $K > 0$ that guarantees validity of linearized kinematics can be estimated using the same equation (47) as in the presented case, giving to $f_2(\tilde{t})$ the correct interpretation.

In order to obtain the optimal solution of the modified missile avoidance problem ($K > 0$) the recursive algorithm presented in the previous subsection can be used. The only difference is that for each iteration a new switch function has to be computed numerically.

VII COMPARISON TO NON-LINEAR SIMULATION

Solutions of modified optimal missile avoidance problems with linearized kinematics were compared to results of non-linear simulations. In the simulation the optimal control functions obtained by the linearized model were used. The comparison was carried out for several values of N', K and $\tilde{t}_f$. The constant parameters used for the comparison were:

$\tau = 0.35$ sec

$V_M = 750$ m/sec

$(a_T)_{max} = 50 \text{ m/sec}^2$

$V_T = 250 \text{ m/sec}$

Thus the value of the dynamic similarity parameter,[13] $\tilde{\alpha}_T$, defined in Eq.(38), is $\tilde{\alpha}_T = 0.07$ rad.

In Table 2 results of the comparison are presented for $\tilde{t}_f = 30$, i.e., predicted flight time of $t_f = 10.5$ sec for the linear model. In the Table, T_f denotes the actual time of flight and M^* is the normalized miss distance obtained by the simulation.

From the results of Table 2 the following conclusions can be drawn:

a. For small (or zero) values of the penalization coefficient K which allow large changes in the geometry $(\Delta\gamma_T > 30°)$ the miss distance obtained by the simulation is smaller than the linearized prediction. Moreover, the actual time of flight T_f also differs from the predicted value t_f. Both observations indicate that assumption of trajectory linearization is not valid in these cases.

b. If the direction change of the target $\Delta\gamma_T$ is kept below 30° the miss distance obtained by non-linear simulation is slightly larger than the value predicted by the linearized model.

c. If $\Delta\gamma_T < 30°$ the actual time of flight is well approximated by the linearized prediction and the resulting miss distance is even larger than the value predicted by the linear model with $K=0$.

N'	K	m*	Simulation		
			M*	$\Delta\gamma_T$(deg)	T_f (sec)
3	0	0.541	0.138	95.4	9.08
	10^{-4}	0.541	0.496	46.0	10.33
	10^{-2}	0.537	0.551	23.9	10.47
5	0	0.294	0.177	73.3	9.41
	10^{-4}	0.294	0.300	30.3	10.41
	10^{-2}	0.243	0.256	1.1	10.50

TABLE 2. Comparison of results of the linearized model to non-linear simulation.

VIII. CONCLUSIONS

The work presented in this paper showed that the validity of linearized kinematical model used in the analysis of optimal missile avoidance[10,11] can be largely extended by adding a control effort penalization term to the original pay-off function. Both linear prediction and simulation indicate that the loss of miss distance due to the control effort penalization is of no practical significance. In the present paper a 2-D case was analysed merely for sake of simplicity. The method is equally applicable for 3-D optimal missile avoidance.

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APPENDIX

CLOSED FORM SOLUTION OF THE COSTATE VECTOR FOR LINEAR FIRST-ORDER DYNAMICS

The costate equations of the problem are given by Eq.(24)

$$d\underline{\lambda}/d\theta = A^T(\theta)\underline{\lambda} \tag{A-1}$$

with the initial conditions of Eq.(25).

Following the solution of Ref.[10] we obtain

$$\lambda_1(\theta) = d\lambda_2/d\theta - \frac{N'}{\theta}\lambda_3(\theta) \tag{A-2}$$

$$\lambda_2(s) = -c\frac{s^{N'-2}}{(s+1)^{N'}} \tag{A-3}$$

$$\lambda_3(s) = c - \frac{s^{N'-2}}{(s+1)^{N+1}} \tag{A-4}$$

and, for integer values of j and ℓ,

$$L^{-1}\left\{\frac{s^j}{(s+1)^{\ell+1}}\right\} = \frac{d^j}{d\theta^j}\left\{e^{-\theta}\frac{\theta^\ell}{\ell!}\right\} \qquad j \neq K \tag{A-5}$$

According to Leibnitz rule for the derivative of a product[16]

$$\frac{d^j}{d\theta^j}\left(f(\theta)\cdot g(\theta)\right) = \left[\frac{d^j}{d\theta^j} f(\theta)\right]\cdot g(\theta) + \binom{j}{1}\left[\frac{d^{j-1}}{d\theta^{j-1}} f(\theta)\right]\frac{d}{d\theta} g(\theta) + \ldots$$

$$+\binom{j}{j-1}\left[\frac{d}{d\theta} f(\theta)\right]\frac{d^{j-1}}{d\theta^{j-1}} g(\theta) + f(\theta)\frac{d^j}{d\theta^j} g(\theta) =$$

$$= \sum_{i=0}^{j} \binom{j}{i}\left[\frac{d^{j-i}}{d\theta^{j-i}} f(\theta)\right]\frac{d^i}{d\ ^i} g(\theta) \qquad \text{(A-6)}$$

where the binomial coefficients $\binom{j}{i}$ are defined by

$$\binom{j}{i} \triangleq \frac{j!}{i!(j-i)!} \qquad \text{(A-7)}$$

the i-th derivative of the expression

$$f(\theta) = e^{-\theta} \qquad \text{(A-8)}$$

yields

$$\frac{d^i}{d\theta^i} e^{-\theta} = (-1)^i e^{-\theta} \qquad \text{(A-9)}$$

and the i-th derivative of the expression

$$g(\theta) = \frac{\theta^\ell}{\ell!} \qquad \text{(A-10)}$$

yields, provided $i \leqslant \ell$,

$$\frac{d^i}{d\theta^i} \frac{\theta^\ell}{\ell!} = \frac{\ell(\ell-1)\ldots(\ell-i+1)}{\ell!} \theta^{\ell-1} = \frac{\theta^{\ell-1}}{(\ell-i)!} \tag{A-11}$$

Substituting (A-9) and (A-11) in (A-6) we obtain

$$L^{-1}\left\{\frac{s^j}{(s+1)^{\ell+1}}\right\} = \frac{d^j}{d\theta^j}\left(e^{-\theta}\frac{\theta^\ell}{\ell!}\right) = e^{-\theta}\sum_{i=0}^{j}(-1)^{j-i}\frac{\binom{j}{i}}{(\ell-i)!}\theta^{\ell-i} \tag{A-12}$$

Equations (A-3) and (A-4) are solved by direct application of Eq.(A-12) yielding

$$\lambda_2(\theta) = -c\, e^{-\theta}\sum_{i=0}^{N'-2}(-1)^{N'-2-i}\frac{\binom{N'-2}{i}}{(N'-1-i)!}\theta^{N'-1-i} \tag{A-13}$$

$$\lambda_3(\theta) = c\, e^{-\theta}\sum_{i=0}^{N'-2}(-1)^{N'-2-i}\frac{\binom{N'-2}{i}}{(N'-i)!}\theta^{N'-i} \tag{A-14}$$

$\dot{\lambda}_2(\theta)$ in Eq.(A-2) can be obtained by using Eq.(A-12) in the expression

$$\dot{\lambda}_2(\theta) = L^{-1}\ [s\lambda_2(s)] = L^{-1}\left[-c\ \frac{s^{N'-1}}{(s+1)^{N'}}\right] =$$

$$= -c\ e^{-\theta} \sum_{i=0}^{N'-1} (-1)^{N'-1-i}\ \frac{\binom{N'-1}{i}}{(N'-1-i)!}\ \theta^{N'-1-i} \qquad \text{(A-15)}$$

by substituting Eqs.(A-15) and (A-14) into Eq.(A-2) and rearranging, we obtain

$$\lambda_1(\theta) = -c\ e^{-\theta} \sum_{i=1}^{N'-1} (-1)^{N'-1-i}\ \frac{(N'-2)!}{(i-1)!(N'-i)!(N'-1-i)!}\ \theta^{N'-1-i} \qquad \text{(A-16)}$$

The initial conditions of Eq.(A-1) are satisfied by Eqs.(A-13), (A-14) and (A-16). The constant c is determined by Eqs.(A-16) and (20)

$$\lambda_1(0) = -c = -2\tilde{y}(\tilde{t}_f) = -2m$$

yielding

$$c = 2m \qquad \text{(A-17)}$$

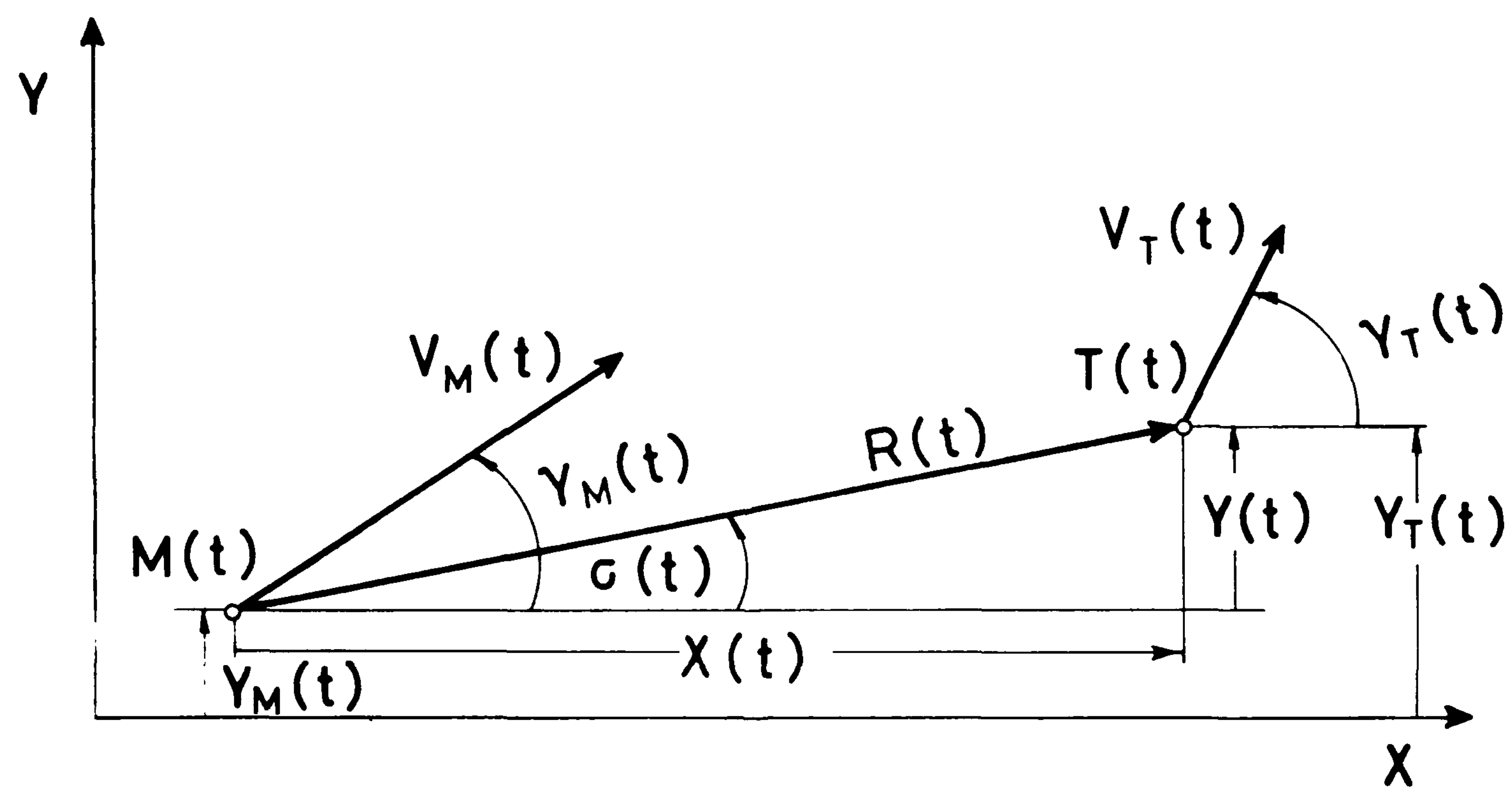

Fig. 1. 2-D pursuit geometry.

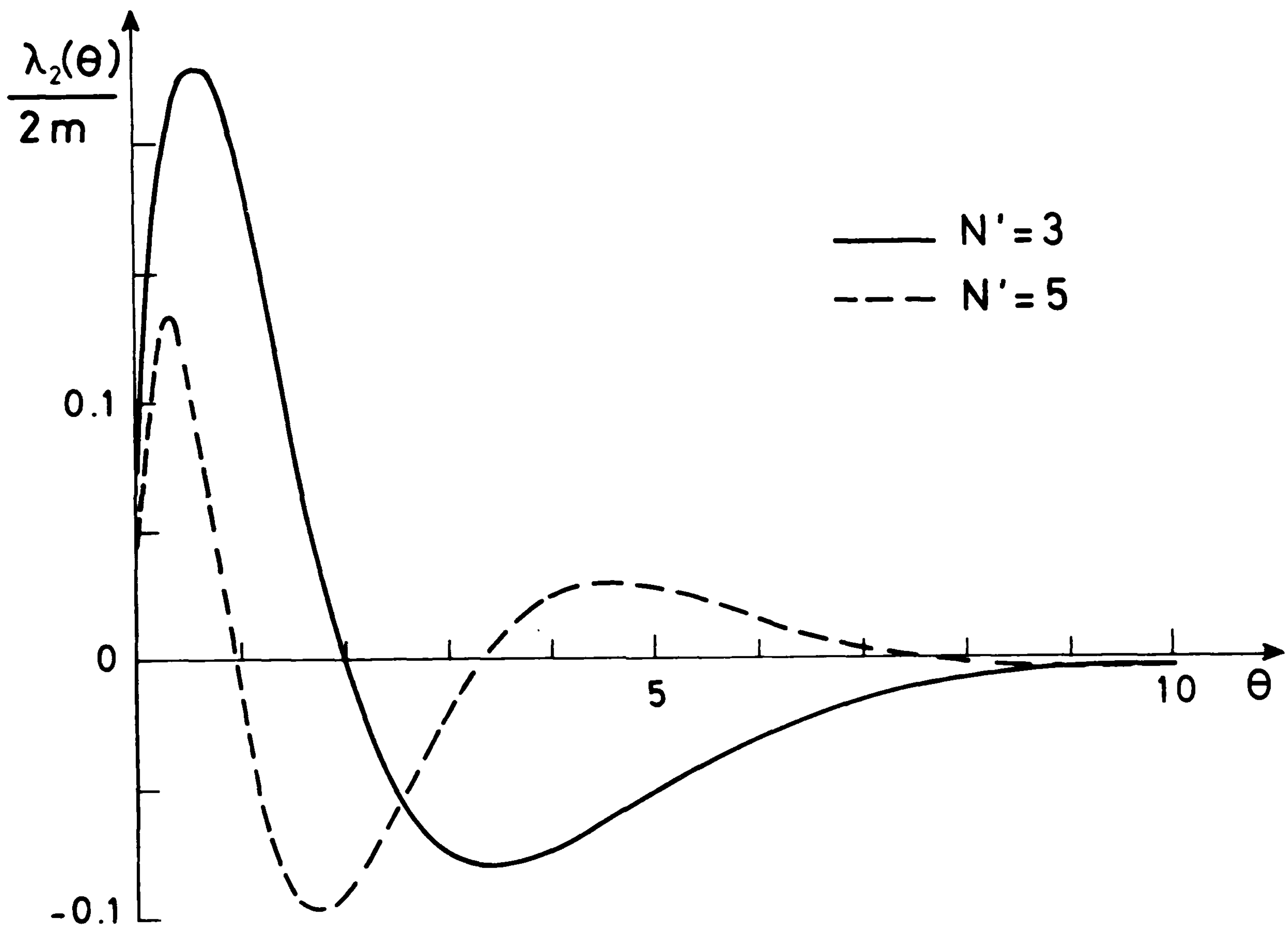

Fig. 2. Normalized miss-distance sensitivity function.

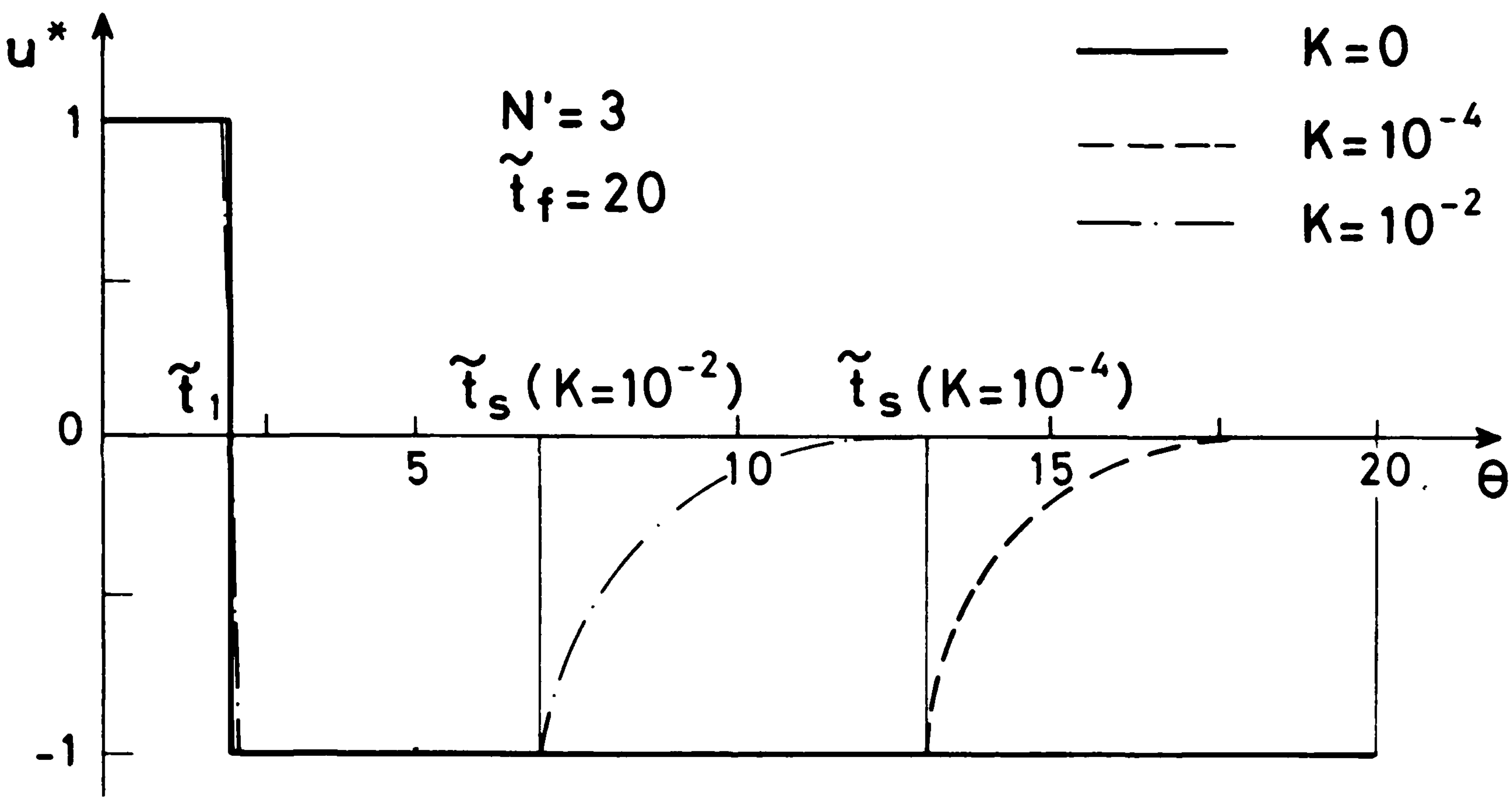

Fig. 3. Optimal avoidance control for different values of penalization coefficient (N' = 3).

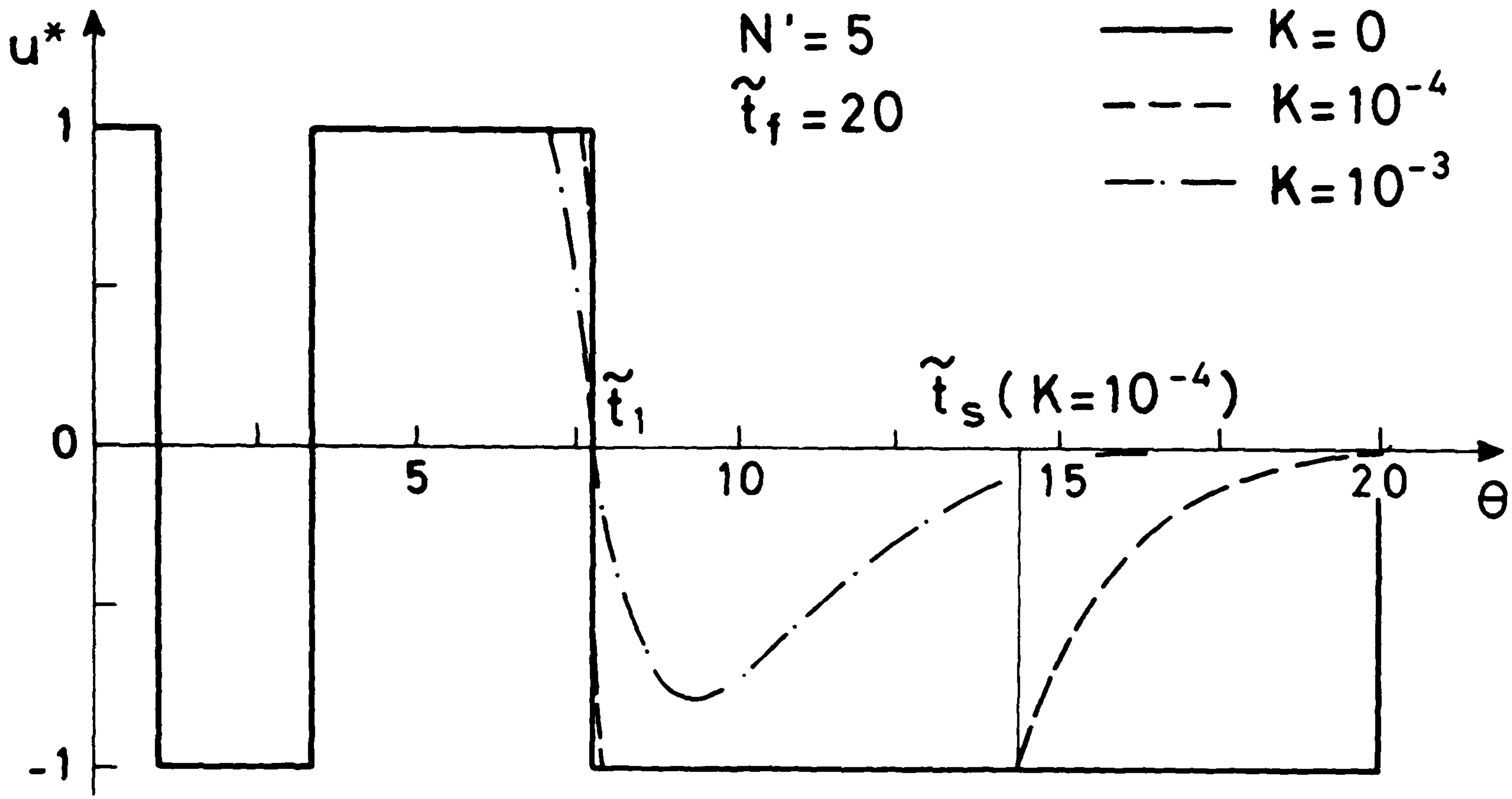

Fig. 4. Optimal avoidance control for different values of penalization coefficient ($N' = 5$).

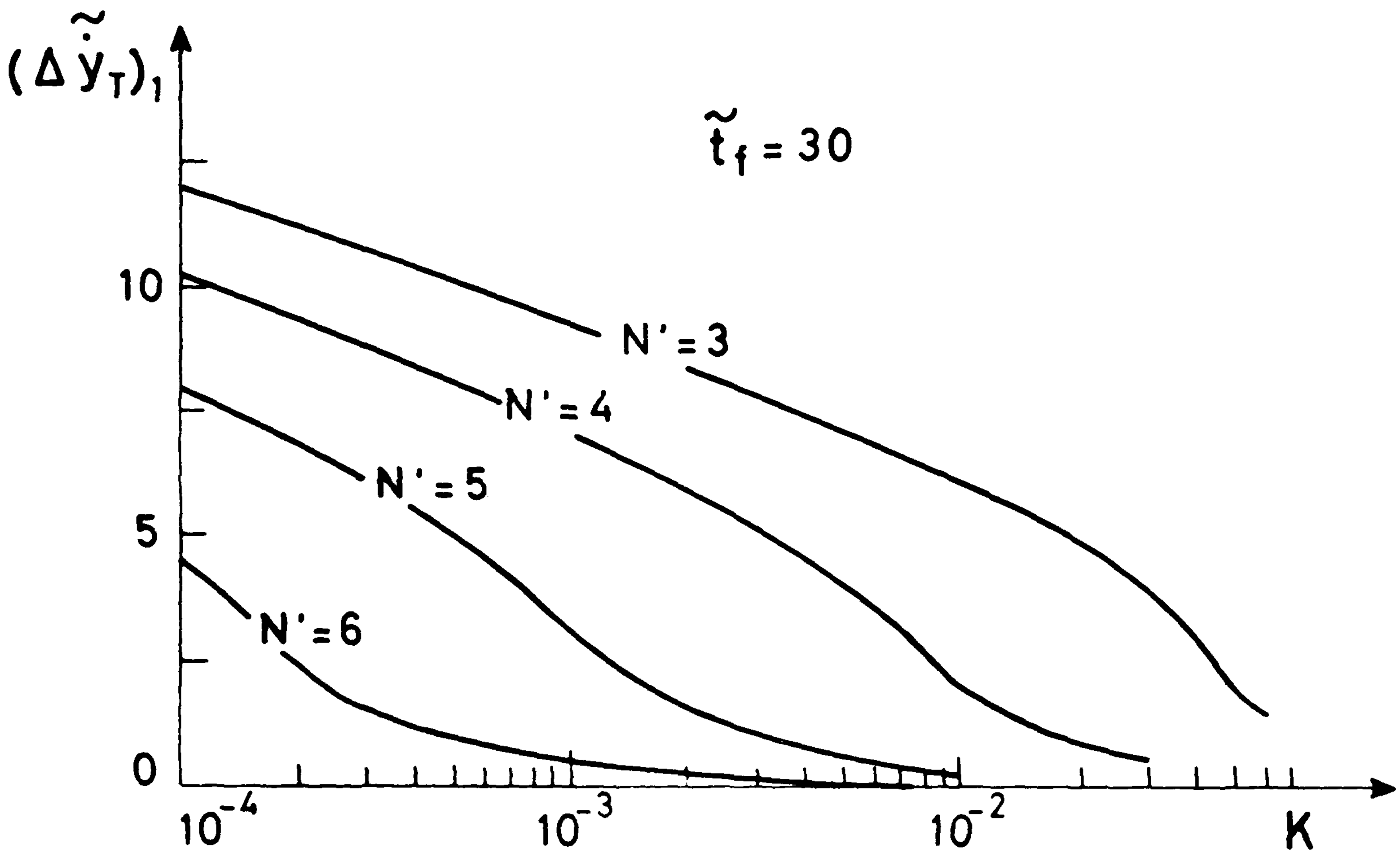

Fig. 5. Effect of the penalization coefficient on the initial target direction change.

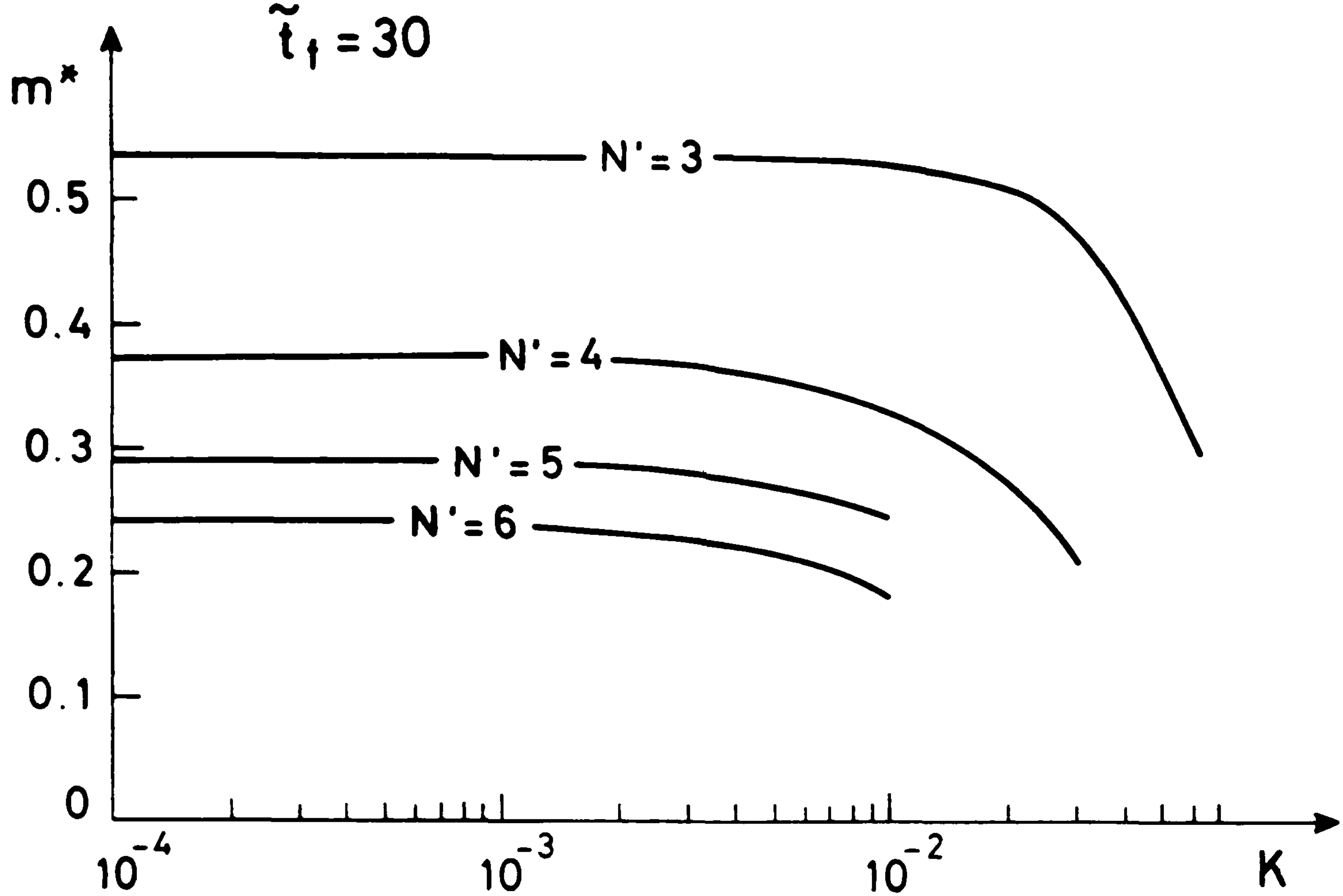

Fig. 6. Effect of the penalization coefficient on the optimal miss distance.