ATTRACTIVITY PROPERTIES OF \( \alpha \)-CONTRACTIONS

by

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attracts neighborhoods of compact sets uniformly.

Since, in practice, it is much easier to verify that a map is point dissipative rather than compact dissipative, it is desirable to say more about the limiting behavior when $T$ is only assumed to be point dissipative. In this paper, we show, with the addition of only a few general assumptions, that point dissipative and compact dissipative are equivalent. The assumptions seem to be general enough to include almost all of the practical applications. Applications are given, or referenced, to stable neutral functional differential equations, retarded functional differential equations of infinite delay, and strongly damped nonlinear wave equations.
Abstract: It is known that if $T: X \to X$ is completely continuous where $X$ is a Banach space, then point dissipative and compact dissipative are equivalent, and imply the existence of a maximal compact invariant set which is uniformly asymptotically stable and attracts bounded sets uniformly. If $T$ is an $\alpha$-contraction it is not known whether point dissipative and compact dissipative are equivalent. However, $T$ is compact dissipative, then there exists a maximal compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets uniformly.

Since, in practice, it is much easier to verify that a map is point dissipative rather than compact dissipative, it is desirable to say more about the limiting behavior when $T$ is only assumed to be point dissipative. In this paper, we show, with the addition of only a few general assumptions, that point dissipative and compact dissipative are equivalent. The assumptions seem to be general enough to include almost all of the practical applications. Applications are given, or referenced, to stable neutral functional differential equations, retarded functional differential equations of infinite delay, and strongly damped nonlinear wave equations.
It is known that if $T: X \to X$ is completely continuous then point dissipative and compact dissipative are equivalent and imply the existence of a maximal compact invariant set which is uniformly asymptotically stable and attracts bounded sets uniformly. If $T$ is an $\alpha$-contraction it is not known whether point dissipative and compact dissipative are equivalent. However, if $T$ is compact dissipative then there exists a maximum compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets uniformly (see [2], [4]).

If $T$ is an $\alpha$-contraction and point dissipative, less is known. Cooperman [2] has shown that under these assumptions, if there is a maximal compact invariant set, then it is uniformly asymptotically stable and attracts neighborhoods of compact sets. This paper will extend some results of a recent paper of mine [8]. In particular, we will show that point dissipative and compact dissipative are equivalent for a large class of $\alpha$-contractions. In fact, most $\alpha$-contractions which arise in the applications fall into this class.

Before proceeding, it is best to explain some of the terminology. A bounded set $B$ dissipates a set $J$ under $T$ if there exists $n_0$ such that $n \geq n_0$ implies $T^nJ \subseteq B$. $T$ is point dissipative if there is a bounded set $B$ which dissipates all points. $T$ is compact dissipative if there is a bounded set $B$ which dissipates all compact sets. $T$ is local dissipative if there is a bounded set
which dissipates a neighborhood of any point. \( T \) is **local compact dissipative** if there is a bounded set which dissipates a neighborhood of any compact set. Finally, \( T \) is **bounded dissipative** or **ultimately bounded**, if there is a bounded set which dissipates bounded sets.

A set \( J \) is **invariant** if \( TJ = J \). It is **stable** if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( n > 0 \), \( T^n(J + B_\varepsilon(0)) \subseteq J + B_\delta(0) \) where \( B_r(a) \) is a ball of radius \( r \) and center \( a \). \( J \) **attracts** \( B \) if every neighborhood of \( J \) dissipates \( B \). \( J \) is **uniformly asymptotically stable** if it is stable and attracts a neighborhood of itself. The **orbit** of a set \( B \), \( \gamma^+(B) \), is defined by \( \gamma^+(B) = \bigcup_{n=0}^{\infty} T^n(B) \) and the **\( \omega \)-limit set** of \( B \), \( \omega(B) \), is defined by \( \omega(B) = \bigcap_{m=0}^{\infty} \text{Cl}\{\bigcup_{n=m}^{\infty} T^n(B)\} \).

The Kuratowski measure of noncompactness, or \( \alpha \)-measure, is a useful tool in dealing with a large class of operators which are not compact. The **\( \alpha \)-measure** is a map \( \alpha : \mathcal{B} \to [0, \infty) \) where \( \mathcal{B} \) is the collection of bounded sets, defined by \( \alpha(B) = \inf\{r/B \text{ can be covered by a finite collection of sets of diameter less than } r\} \). In a sense it can be considered a measure of the total boundedness of a set. The \( \alpha \)-measure has the following properties:

(i) \( \alpha(B) = 0 \) if and only if \( \text{Cl} \ B \) is compact

(ii) \( \alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\} \)

(iii) \( \alpha(A + B) \leq \alpha(A) + \alpha(B) \)

(iv) \( \alpha(\overline{\mathcal{D}}A) = \alpha(A) \).

\( T \) is called an **\( \alpha \)-contraction** if there exists a \( k \in [0,1) \) such that for all \( B \in \mathcal{B} \) we have \( \alpha(TB) \leq k\alpha(B) \). \( T \) is a
conditional-$\alpha$-contraction if there exists a $k \in [0,1)$ such that for all $B \in \mathcal{B}$ with $TB \in \mathcal{B}$ we have $a(TB) \leq k a(B)$. The conditional-$\alpha$-contraction is a more general case which is often more appropriate for the applications.

The major result of this paper is the following theorem.

**Theorem:** Let $X_1$ and $X_2$ be two Banach spaces with $i: X_1 \hookrightarrow X_2$ a compact imbedding. Let $T: X_1 \to X_2$ be continuous on both spaces. We assume $T$ can be decomposed as $T = C + U$ with $C$ and $U$ also continuous on $X_1$ and $X_2$. Let $C(0) = 0$, $C$ be a contraction on both spaces, and $U$ satisfy the property that for all $B \subset X_1$ with $B$ and $U(B)$ bounded in $X_2$, then $U(B)$ is bounded in $X_1$. Under these assumptions point dissipative and compact dissipative are equivalent in $X_2$, and imply the existence of a maximal compact invariant set in $X_2$ which is uniformly asymptotically stable, attracts neighborhoods of compact sets, and has a fixed point.

Previously, it was known that point dissipative in $X_2$ (or $X_1$) implies bounded dissipative in $X_1$ [8]. These results above imply much stronger stability results for the space $X_2$. In particular, we get that point dissipative and compact dissipative are equivalent for all stable neutral functional differential equations and retarded functional differential equations of infinite delay.

The first section of this paper will summarize known results for when $T$ is assumed to be compact dissipative. The second section gives the results mentioned above.

This paper is a part of my Ph.D. thesis [9] at Brown University. I am also deeply grateful to Professor Jack K. Hale for his help.
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1. Compact Dissipative

In this section, we summarize a few of the known results which apply when $T$ is an $\alpha$-contraction and compact dissipative. The results apply for more general situations than $\alpha$-contractions, but for simplicity we confine our attention to $\alpha$-contractions. These results can be found in [2], [4], [6], [7], and [9].

In the following, let $X$ be a Banach space and $T: X \to X$ be continuous.

Theorem 1: If $T$ is a conditional-$\alpha$-contraction and $B, \gamma^+(B) \subseteq X$ are bounded, then $\omega(B) \neq \emptyset$, is compact, invariant, and attracts $B$.

Theorem 2: If $T$ is a conditional-$\alpha$-contraction and compact dissipative then there exists a maximal compact invariant set which is uniformly asymptotically stable and attracts neighborhoods of compact sets.

Theorem 3: If $T$ is a conditional-$\alpha$-contraction and compact dissipative then $T$ has a fixed point.

We now prove that the maximum compact invariant set in the above is connected. We relax the assumptions somewhat to obtain greater generality.

Theorem 4: If $K$ is a compact invariant set which attracts compact sets, then $K$ is connected.
Proof: Clearly co K is compact and connected. So K attracts co K. Suppose K is not connected. Then there exists open sets U,V with U ∩ K ≠ ∅, V ∩ K ≠ ∅, K ⊂ U ∪ V, and U ∩ V = ∅. By the continuity of T we know T^n(co K) is connected for each n ≥ 0. Furthermore, since K ⊂ T^n(co K) for all n ≥ 0 we have U ∩ T^n(co K) ≠ ∅ and V ∩ T^n(co K) ≠ ∅. The connectedness of T^n(co K) implies there is a sequence \{x_n\} with x_n ∈ T^n(co K) and x_n ∉ U ∪ V. But K attracts \{x_n\} so \{x_n\} is precompact. Thus \{x_n\} has a converging subsequence with some limit point x ∈ K. Clearly x ∉ U ∪ V which is a contradiction.

Corollary: The maximal compact invariant set in Theorem 2 is connected.

2. Main Results

Under the assumption that T is a conditional-α-contraction and compact dissipative we can say pretty much about the limiting behavior of \{T^k\}_{k>0}. In this section, we give some of the known point results when T is a conditional-α-contraction and only known to be/ dissipative. We will also show for a large class of conditional-α-contractions (which includes most applications) that point dissipative and compact dissipative are equivalent.

The first theorem was originally proved by Cooperman [2]. The second theorem was proved by myself [8]. These two theorems are also contained in my thesis [9].
Theorem 1: If $T$ is a conditional-\(\alpha\)-contraction and $K$ is a compact invariant set which attracts points, then the following are equivalent.

(i) $K$ attracts compact sets
(ii) $K$ is stable
(iii) $K$ is a maximal compact invariant set.

One of the important implications of this theorem is given by the following corollary.

Corollary 1: If $T$ is a conditional-\(\alpha\)-contraction, point dissipative, and has a maximal compact invariant set $A$, then $A$ is uniformly asymptotically stable, connected, and attracts neighborhoods of compact sets.

Proof: Theorem 1 implies $A$ attracts compact sets. Hence, $T$ is compact dissipative. Now apply Theorem 2 and Theorem 4 of section 1.

Theorem 2: Let $i: X_1 \subseteq X_2$ be a compact imbedding where $X_j$ are Banach spaces with norm $|| \cdot ||_j$. Let $T, C,$ and $U$ be continuous operators mapping $X_j$ into itself. Let $T = C + U$ with $C(0) = 0$, $C$ a contraction in $X_1$ and $U$ having the property that if $B \subseteq X_1$ and $U(B)$ are bounded in $X_2$ then $U(B)$ is bounded in $X_1$. Then the following are equivalent:

(i) $T$ is point dissipative in $X_1$
(ii) $T$ is bounded dissipative in $X_1$
(iii) there is a bounded set in $X_2$ which dissipates points in $X_1$. 
In the next theorem we combine these results to prove the equivalence of point dissipative and compact dissipative under certain natural hypotheses.

Theorem 3: Under the conditions of Theorem 2, and (*) any compact invariant set in $X_2$ is a subset of the closure in $X_2$ of a bounded set in $X_1$, then point dissipative and compact dissipative are equivalent in $X_2$.

Proof: From Theorem 2, there is a bounded set $B$ in $X_1$ which dissipates bounded sets in $X_1$. Then $Y^+(B)$ is bounded in $X_1$, and hence, precompact in $X_2$. Therefore, its $\omega$-limit set in $X_2$, $\omega_2(B)$, is nonempty, compact, invariant and attracts $B$ in $X_2$. Now let $J$ be any compact invariant set in $X_2$. Then there is a bounded set $A$ in $X_1$ with $J \subseteq \text{Cl}_2(A)$. Now, since $B$ dissipates bounded sets in $X_1$, $B$ dissipates $A$. Hence $\omega_2(B)$ attracts $A$ in $X_2$. Since $T$ is continuous, $\omega_2(B)$ attracts $\text{Cl}_2(A)$ in $X_2$. Hence, $J \subseteq \text{Cl}_2A$ implies $\omega_2(B)$ attracts $J$ in $X_2$. But since $J$ is invariant, $J \subseteq \omega_2(B)$. Hence, $\omega_2(B)$ is a maximal compact invariant set in $X_2$. Now apply Corollary 1.

Corollary 2: Under the conditions of Theorem 2, if $C$ a contraction in both spaces, and if we know any compact invariant set in $X_2$ is a subset of the closure in $X_2$ of a bounded set in $X_1$, then point dissipative implies there exists a maximal compact invariant set which is connected, uniformly asymptotically stable, attracts a neighborhood of any compact set, and has a fixed point.
Proof: Clearly, T is a conditional-α-contraction in $X_2$ since $C$ is a contraction and $U$ is conditionally-completely continuous. Also, Theorem 3 implies $T$ is compact dissipative. Hence, Theorems 2 and 3 imply the result.

The condition (*) may not always be easily verified. A more natural and simple condition to verify is given in the next theorem.

Theorem 4: Assume the conditions of Theorem 2 and let $C$ be a contraction in both spaces. Then point dissipative and compact dissipative are equivalent.

Proof: All we need to do is show that any compact invariant set in $X_2$ is a subset of the closure in $X_2$ of a bounded set in $X_1$ and apply Theorem 3. Let $J$ be a compact invariant set in $X_2$. Let $k$ be a contraction constant for $C$ in both spaces. Let $r = 1/(1-k)$. Let $B$ be a closed ball in $X_1$ with $rU(J) \subset B$. We will show $J \subset \text{Cl}_2(B)$.

Let $d_2(x, B) = \inf_{y \in B} \|x - y\|_2$. If $d_2(x, B) = 0$ then $x \in \text{Cl}_2(B)$. Let $\eta = \sup\{d_2(x, B)/x \in J\}$. If we show $\eta = 0$ then we have $J \subset \text{Cl}_2(B)$. This is our goal.

Let $x \in J$ and $y \in B$. Let $z = Cy + Ux$. Then

$$\|z\|_1 \leq k\|y\|_1 + \|Ux\|_1 \leq k\|B\|_1 + (1/r)\|B\|_1 \leq \|B\|_1.$$  

Hence, $z \in B$. Furthermore, $\|Tx - z\|_2 = \|(Cx + Ux) - (Cy + Ux)\|_2 = \|Cx - Cy\|_2 \leq k\|x - y\|_2$. Hence, $d_2(Tx, B) = \sup_{y \in B} \|Tx - y\|_2 \leq k \sup_{y \in B} \|x - y\|_2 = k d_2(x, B)$. But this implies,
\[ \eta = \sup_{x \in J} d_2(x, B) = \sup_{x \in J} d_2(Tx, B) \leq k \sup_{x \in J} d_2(x, B) = \eta. \] In the second step we use the invariance of \( J \). Thus, we get \( \eta = 0 \) and so the result is proved.

**Corollary 3**: Under the conditions of Theorem 2 and if \( C \) is a contraction in both spaces then point dissipative implies the existence of a maximal compact invariant set which is connected, uniformly asymptotically stable, attracts a neighborhood of any compact set, and has a fixed point.

**Proof**: The proof of Theorem 4 shows (*) is satisfied. Hence, we may apply Corollary 3.

**Remark**: These theorems show that for stable neutral functional differential equations, and for retarded functional differential equations of infinite delay with phase space in a Banach space, point dissipative and compact dissipative are equivalent. To verify the hypotheses, see [8]. For an application to strongly damped nonlinear wave equations, see [10].
REFERENCES


