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M.F. McCarthy and H.F. Tiersten

Office of Naval Research
Contract N00014-76-C-0368
Project NR 318-009
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ABSTRACT

A one-dimensional version of a theory of composite materials modeled as interpenetrating solid continua is applied in the analysis of acceleration waves in composites containing two identifiable constituents. As expected, two distinct acceleration waves always propagate except when one of the constituents consists of a chopped fiber. The influence of viscous type damping is included in only the volumetric interaction between the constituents in portions of the treatment. Equations are derived both for the propagation velocities and the varying amplitudes of the disturbance as a function of the state of the material immediately ahead of the wavefront. These rather general results are specialized to the case of a homogeneous steady-state ahead of the fast wave. The various types of behavior possible and the order of the discontinuities occurring across the wavefront are discussed in detail for a number of special cases.

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1. Introduction

Recently, a continuum theory of finitely deformable, heat conducting composite materials was developed by modeling N identifiable constituents as interpenetrating solid continua[1]. In deriving the general system of nonlinear equations governing the behavior, the motion of a point of the combined continuum was permitted to be finite while the relative motion of the individual constituents was constrained to be infinitesimal in order that the solid composite not rupture. The restriction imposed in Ref.1 which demands that the relative motion of the constituents be infinitesimal is one of the features that distinguishes this theory from other work[2-8] on composites. Another important feature distinguishing the description in Ref.1 from that of the other work[2-8] is that no energy of interaction between the constituents is omitted in Ref.1 as it is in the other work.

In this paper we specialize the theory developed in Ref.1 and consider the one-dimensional motion of a two constituent composite material whose identifiable constituents are elastic, as is the interaction between the constituents with the exception of the volumetric part. We examine the behavior of one-dimensional acceleration waves in such media on the assumption that thermodynamic influences may be ignored. In particular we seek to determine how the behavior of such waves is influenced by (i) the mechanical properties of the mean (center of mass) behavior of the combined composite, (ii) the mechanical properties associated with the relative motion of the individual constituents, (iii) the coupling between these two motions, (iv) the relative mass densities of the individual components of the composite, and, finally, (v) the dynamical conditions prevailing ahead of the wavefront; and in certain interesting simplified special cases by (vi) the mechanical properties of the individual constituents of the composite and (vii) the coupling effects arising from the volumetric interaction between the elements of the composite.

Section 2 of this paper is devoted to a brief review of the equations which govern the one-dimensional motion of two constituent composites. After recording the global forms of the equations which govern the balance of linear momentum, we state the constitutive equations which govern the one-dimensional motions of elastic composites which are made up of two identifiable elastic media. Sections 3 and 4 are devoted to the study of the propagation of acceleration waves. In Section 3 we show that the balance laws and constitutive equations set forth in Section 2 imply the existence, in general, of two distinct types of acceleration waves, the fast one of which is associated with the mean elasticity of the combined composite and the slow one with the elasticity associated with the relative motion of the individual constituents. When the effects of coupling between the center of mass motion of the combined composite and the relative motion of the constituents is small, one wave propagates with a velocity which is close to that of the ordinary elastic wave speed of the combined composite while the speed of propagation of the second wave is close to that of the wave of the relative motion of the constituents. The behavior of waves in a number of highly restrictive special types of composite is examined in Section 4. It is shown, in particular, that when one of the components is a chopped fiber, only one acceleration wave may exist in the composite and when the interaction between the constituents is purely volumetric, the two acceleration waves propagate with the respective speeds of those in the individual constituents.

The manner in which the amplitudes of acceleration waves vary as they traverse the composite is examined in Sections 5 and 6. A standard analysis is employed in Section 5 to show that the amplitude of an acceleration wave satisfies an equation of Bernoulli type. The various types of behavior possible in a number of situations, including the possibility of shock formation, are discussed in Section 6. In the general case where two waves may exist in the material the behavior of both the "fast" and "slow" waves is discussed. It is noted that while the medium ahead of the fast wave may be in a steady state before the arrival of the wave, this condition will be unlikely to prevail ahead of the "slow" wave because of the motion induced ahead of this wave by the passage of the precursor. The propagation of a "fast" wave in a composite which is initially at rest in an arbitrary permissible state of deformation is examined in detail and it is shown that the behavior of the amplitude of such a wave is the same as that of an acceleration wave propagating in a single phase elastic material which is in a state of nonhomogeneous deformation ahead of the wave. A similar situation prevails when the center of mass deformation is homogeneous but the deformation fields of the two continua which make up the composite are not. If the material ahead of the wave is in its natural stress-free state then it is found that, as far as "fast" waves are concerned, the material behaves in the same way as would a single phase thermoelastic medium.

The behavior of the amplitudes of acceleration waves in the highly special cases treated in Section 4,as well as the higher order discontinuities induced by some acceleration waves,is also examined in Section 6. Thus, when the composite is such that the interaction between the constituents is purely volumetric and depends only on the relative displacement of the constituents, the composite behaves, as far as acceleration waves are concerned, as an elastic material composed solely of one of the component continua. In this particular situation, the acceleration of only one of the components suffers a discontinuity at either of the waves which may exist and the acceleration of the points of the second continuum are continuous along with the first order time derivative, but the second order time derivative of the acceleration of the second continuum suffers a jump discontinuity at the wavefront. Finally, we examine the behavior of an acceleration wave propagating in a composite one of whose component continua is made up of chopped fibers. Only one acceleration wave may exist in such a material and the acceleration of the chopped fiber continuum is continuous at the wavefront. The order of the discontinuity in the motion of the chopped fiber continuum depends on the nature of the composite and the conditions prevailing ahead of the wave. In general, the first derivative of the acceleration of the chopped fiber continuum suffers a jump discontinuity across the wavefront. On the other hand, if the composite is centrosymmetric and is in a state of equilibrium ahead of the wave then the first derivative of the acceleration of the chopped fiber continuum is continuous everywhere and for all time while the second derivative suffers a jump discontinuity at the acceleration wave.

2. Basic Equations for One-Dimensional Motions

We are interested here in studying the motion in one dimension of a composite consisting of two interpenetrating solid continua. Initially, the two continua occupy the same region of space and hence the location of the identifiable components of the composite may be specified by a single reference coordinate X. It should be noted that X specifies the position of a point of each of the interpenetrating continua at some fixed time $t=0$, say. The subsequent motion of the composite is described by specifying two functions

$$y^{(1)} = y^{(1)}(X,t), \; y^{(2)} = y^{(2)}(X,t) , \tag{2.1}$$

which give the positions at time t of the points of the two interpenetrating continua which were simultaneously located at the point X at time $t=0$. We denote the mass density of the ith constituent in the reference configuration by $\rho_o^{(i)}$ and in the current configuration by $\rho^{(i)}$. The center of mass, at time t, of the particles of the continua which simultaneously occupied the point X at time $t=0$, is given by

$$y = y(X,t) = \frac{\rho^{(1)} y^{(1)}(X,t) + \rho^{(2)} y^{(2)}(X,t)}{\rho^{(1)} + \rho^{(2)}} \tag{2.2}$$

Clearly, Eqs. (2.1) may be written in the form

$$y^{(1)} = y + w^{(1)}(X,t), \; y^{(2)} = y + w^{(2)}(X,t) , \tag{2.3}$$

where $w^{(i)}(X,t)$ is the displacement of the point X of the ith continuum relative to the center of mass of the points originally at X at time $t=0$. As in Ref.1, we place no restriction on the magnitude of y, but the relative displacements $w^{(1)}$, $w^{(2)}$ are taken to be infinitesimal. The deformation gradients at the point X are

$$F = F(X,t) = \partial_X y(X,t), \; \overline{F}^{(i)} = \overline{F}^{(i)}(X,t) = \partial_X y^{(i)}(X,t) = F + F^{(i)} \tag{2.4}$$

where

$$F^{(i)} = \partial_X w^{(i)}(X,t) . \tag{2.5}$$

In (2.4) F is the deformation gradient of the center of mass, $\overline{F}^{(i)}$ is the deformation gradient of the point of the ith constituent which was located at the point X at $t=0$ and $F^{(i)}$ is the relative deformation gradient of this point. Since $w^{(i)}(X,t)$ is an infinitesimal displacement field, $|F^{(i)}| \ll |F|$ and, since mass is conserved separately for each constituent, we have

$$\rho_o^{(i)} = \rho^{(i)}(F + F^{(i)}) \approx \rho^{(i)}F, \tag{2.6}$$

so that we may write

$$\rho = \rho^{(1)} + \rho^{(2)}, \quad \rho F = \rho_o , \tag{2.7}$$
$$\rho_o = \rho_o^{(1)} + \rho_o^{(2)} ,$$

where ρ_o is the total reference mass density of the composite.

Since $y(X,t)$ is the position at time t of the center of mass of the points of the constituent continua which were at X at $t=0$, it follows from (2.2), (2.3), (2.6) and (2.7) that

$$rw^{(1)} + w^{(2)} = 0 , \tag{2.8}$$

where $r = \rho_o^{(1)}/\rho_o^{(2)}$ is assumed to be constant[9]. At this point it should be noted that $\rho_o^{(1)}$ and $\rho_o^{(2)}$ do not represent the actual mass densities of each of the constituents in the composite, but only represent those quantities in each of the interpenetrating continua, which occupy the same region of space and, respectively, represent each constituent in the model. Suppose that at time $t=0$, the ith constituent occupies a fraction λ_i of the volume of the constituent so that $\rho_o^{(i)} = \lambda_i \bar{\rho}_o^{(i)}$, where $\bar{\rho}_o^{(i)}$ is the mass density which a body composed solely of the ith constituent would have. It follows from (2.7) that

$$\rho_o = \lambda_1 \bar{\rho}_o^{(1)} + \lambda_2 \bar{\rho}_o^{(2)}, \quad \lambda_1 + \lambda_2 = 1 , \tag{2.9}$$

while $r = R\lambda_1/\lambda_2$, where $R = \bar{\rho}_o^{(1)}/\bar{\rho}_o^{(2)}$ represents the constant ratio of the actual mass densities of the constituents.

The one-dimensional version of the integral forms of the equations of balance of the composite follows from Eqs. (6.1) and (6.2) of Ref.1 in the form

$$\frac{d}{dt}\int_{X_\alpha}^{X_\beta} \rho_o \dot{y}\, dX = K(X_\beta,t) - K(X_\alpha,t) , \tag{2.10}$$

$$\frac{d}{dt}\int_{X_\alpha}^{X_\beta} \rho_o r\dot{w}^{(1)} dX = \mathcal{D}(X_\beta,t) - \mathcal{D}(X_\alpha,t) + \int_{X_\alpha}^{X_\beta} \mathfrak{F}\, dX , \tag{2.11}$$

where X_α, X_β are two arbitrary points in the reference configuration of the composite and

$$K = \tau^{(1)} + \tau^{(2)} , \tag{2.12}$$

$$\mathcal{D} = \tau^{(1)} - r\tau^{(2)} , \tag{2.13}$$

$$\mathfrak{F} = (1+r)\ {}^{L}F^{12} . \tag{2.14}$$

In (2.12) - (2.14) $K,\mathcal{D}$ represent the total stress and the relative stress for the combined continuum, respectively, $\tau^{(1)}$ and $\tau^{(2)}$ are the stresses for each of the interpenetrating continua, while ${}^{L}F^{12}$ is the force exerted by continuum 2 on continuum 1. In Eqs. (2.10), (2.11) and in what follows a superposed dot denotes material differentiation: $\dot{G} = \partial G(X,t)/\partial t$.

In addition to the foregoing we have the relevant constitutive equations[10], which we take in the form

$$\begin{gathered} K = \hat{K}(F,F^{(1)},\ w^{(1)}),\ \mathcal{D} = \hat{\mathcal{D}}(F,F^{(1)},\ w^{(1)}), \\ \mathfrak{F} = \hat{\mathfrak{F}}(F,F^{(1)},\ w^{(1)}) + \hat{\hat{\mathfrak{F}}}(F,F^{(1)},\ w^{(1)};\ \dot{w}^{(1)}), \end{gathered} \tag{2.15}$$

and we assume that the functions $\hat{K}(.,.,.)$, $\hat{\mathcal{D}}(.,.,.)$, $\hat{\mathfrak{F}}(.,.,.)$ and $\hat{\hat{\mathfrak{F}}}(.,.,.;.)$ are $C^{(2)}$ functions of their arguments. For future reference, we note that it has been shown in Ref.1 that $\hat{K}$, $\hat{\mathcal{D}}$ and $\hat{\mathfrak{F}}$ are related to the stored energy density $\Sigma = \hat{\Sigma}(F, F^{(1)}, w^{(1)})$ by the formulae

$$K = \rho_o \partial_F \hat{\Sigma}(F, F^{(1)}, w^{(1)}), \quad \mathcal{B} = \rho_o \partial_{F^{(1)}} \hat{\Sigma}(F, F^{(1)}, w^{(1)}),$$
$$\hat{\mathfrak{F}} = -\rho_o \partial_{w^{(1)}} \hat{\Sigma}(F, F^{(1)}, w^{(1)}). \tag{2.16}$$

For a positive rate of entropy production $\hat{\hat{\mathfrak{F}}}(.,.,.;\dot{w}^{(1)})$ must be an odd function of $\dot{w}^{(1)}$, i.e.,

$$\hat{\hat{\mathfrak{F}}}(.,.,.;\dot{w}^{(1)}) = -\hat{\hat{\mathfrak{F}}}(.,.,.;-\dot{w}^{(1)}), \tag{2.17}$$

and the condition (2.17) implies that $\hat{\hat{\mathfrak{F}}}(-)$ has the representation

$$\hat{\hat{\mathfrak{F}}} = \hat{g}(.,.,.;\dot{w}^{(1)})\dot{w}^{(1)}, \tag{2.18}$$

where $\hat{g}$ is an even function of $\dot{w}^{(1)}$ which must be strictly negative[1].

Under certain circumstances it turns out to be convenient for interpretive purposes to take the stored energy Σ in the equivalent form $\Sigma = \tilde{\Sigma}(\overline{F}^{(1)}, \overline{F}^{(2)}, w^{(1)})$, then it follows from Eqs. (2.4), (2.8), (2.12), (2.13) and (2.16) that

$$\tau_1 = \rho_o \partial_{\overline{F}^{(1)}} \tilde{\Sigma}, \quad \tau_2 = \rho_o \partial_{\overline{F}^{(2)}} \tilde{\Sigma}. \tag{2.19}$$

It is clear that the stored energy density may be written in the form

$$\rho_o \Sigma = \lambda_1 \overline{\rho}_o^{(1)} \overline{\Sigma}^{(1)}(\overline{F}^{(1)}) + \lambda_2 \overline{\rho}_o^{(2)} \overline{\Sigma}^{(2)}(\overline{F}^{(2)})$$
$$+ \lambda_1 \lambda_2 \rho_o \overline{\Sigma}^{(12)}(\overline{F}^{(1)}, \overline{F}^{(2)}; w^{(1)}), \tag{2.20}$$

where $\overline{\Sigma}^{(i)}(\overline{F}^{(i)})$ is the energy density which the body would have at the point X if it were composed solely of the ith constituent. The third term on the right-hand side of (2.20) may be called the interaction energy density of the constituent continua and it is the presence of this term which causes coupling between the deformation fields of the constituent continua which make up the composite. This term is frequently neglected in the study of composites[6] but has been taken into account in the recent work of McNiven and Mengi[11] in their study of two-phase composites with linear response.

Equations (2.19) and (2.20) together imply that

$$\tau_i = \lambda_i T_i + \lambda_1 \lambda_2 \partial_{\overline{F}} (i)\overline{\Sigma}^{(12)}, \qquad i = 1,2, \tag{2.21}$$

where

$$T_i = \overline{\rho}_o^{(i)} \partial_{\overline{F}} \, (i)\overline{\Sigma}^{(i)} (\overline{F}^{(i)}), \quad i = 1,2, \tag{2.22}$$

are the stresses which would arise at the point X in a single phase medium composed solely of the ith continuum.

3. Propagation of Acceleration Waves

In one dimension, the motion of a nonmaterial surface of discontinuity with respect to the reference coordinates is given by

$$Z = \tilde{Z}(t), \tag{3.1}$$

where Z(t) denotes the position of the surface in the reference configuration at time t. The intrinsic velocity U of the surface of discontinuity is given by

$$U(t) = \frac{d\tilde{Z}(t)}{dt} > 0, \tag{3.2}$$

and this quantity is a measure of the speed of propagation of the discontinuity surface with respect to the reference coordinates of material points.

We use the standard notation to denote the jump in the magnitude of a quantity across the propagating surface of discontinuity; thus, if

$\varphi(X,t)$ is a quantity which suffers a jump discontinuity at the surface $Z=\tilde{Z}(t)$ but is a continuous function of (X,t) jointly elsewhere, we define the jump in φ at time t across the propagating surface of discontinuity to be

$$[\varphi] = [\varphi](t) = \varphi^{-} - \varphi^{+} \tag{3.3}$$

where

$$\varphi^{-+} = \lim_{X \to Z(t)^{-+}} \varphi(X,t) . \tag{3.4}$$

Since $U(t) > 0$, φ^{-} and φ^{+}, respectively, denote the limiting values of φ immediately behind and just in front of the propagating surface. Of course $[\varphi]$ must also obey the kinematical condition of compatibility[12],

$$\frac{d}{dt}[\varphi] = [\dot{\varphi}] + U[\partial_X \varphi] . \tag{3.5}$$

Furthermore, we note the formula

$$[\varphi\psi] = \varphi^{+}[\psi] + \psi^{+}[\varphi] + [\varphi][\psi] . \tag{3.6}$$

A propagating nonmaterial surface of discontinuity is called an acceleration wave if $y(X,t)$ and $w^{(1)}(X,t)$ or, equivalently, $y^{(1)}(X,t)$ and $y^{(2)}(X,t)$ have the properties that while $y(.,.)$, $w^{(1)}(.,.)$, $\dot{y}(.,.)$, $\dot{w}^{(1)}(.,.)$, $F(.,.)$ and $F^{(1)}(.,.)$ or, equivalently, $y^{(1)}(.,.)$, $y^{(2)}(.,.)$, $\dot{y}^{(1)}(.,.)$, $\dot{y}^{(2)}(.,.)$, $\overline{F}^{(1)}(.,.)$ and $\overline{F}^{(2)}(.,.)$ are continuous everywhere, the second and higher order partial derivatives of the fields $y(.,.)$ and $w^{(1)}(.,.)$ or, equivalently, $y^{(1)}(.,.)$ and $y^{(2)}(.,.)$ suffer jump discontinuities across the propagating surface $Z=\tilde{Z}(t)$, but are continuous functions of X and t everywhere else. Thus, at an acceleration wave, we have

$$[\dot{y}] = [\dot{w}^{(1)}] = [F] = [F^{(1)}] = [\dot{y}^{(1)}] = [\dot{y}^{(2)}] = [\overline{F}^{(1)}] = [\overline{F}^{(2)}] = [F^{(2)}] = 0 . \tag{3.7}$$

The integral forms (2.10), (2.11) of the equations of motion imply that for all $X \neq Z(t)$ we have

$$\partial_X K = \rho_o \ddot{y}, \tag{3.8}$$

$$\partial_X \mathcal{B} + \mathcal{F} = r\rho_o \ddot{w}^{(1)},$$

while across the surface of discontinuity we have

$$\underset{\sim}{[K]} + \rho_o U \underset{\sim}{[\dot{y}]} = 0,$$

$$\underset{\sim}{[\mathcal{B}]} + r\rho_o U \underset{\sim}{[\dot{w}^{(1)}]} = 0. \tag{3.9}$$

It follows from (2.15), (3.7) and the assumed continuity of the response functions that Eqs. (3.9) are satisfied identically at an acceleration wave.

When the jumps across the wavefront in Eqs. (3.8) are evaluated we have

$$\underset{\sim}{[\partial_X K]} = \rho_o \underset{\sim}{[\ddot{y}]},$$

$$\underset{\sim}{[\partial_X \mathcal{B}]} = r\rho_o \underset{\sim}{[\ddot{w}^{(1)}]}. \tag{3.10}$$

It follows from (2.16) that

$$\partial_X K = \alpha_1 \partial_X F + \alpha_2 \partial_X F^{(1)} + \alpha_3 F^{(1)},$$

$$\partial_X \mathcal{B} = \alpha_2 \partial_X F + \beta_2 \partial_X F^{(1)} + \beta_3 F^{(1)}, \tag{3.11}$$

where

$$\alpha_1 = \partial_F \hat{K}(F, F^{(1)}, w^{(1)}) = \rho_o \partial_F^2 \hat{\Sigma}(F, F^{(1)}, w^{(1)})$$

$$= \lambda_1 E_1 + \lambda_2 E_2 + \lambda_1 \lambda_2 (a_{11} + 2a_{12} + a_{22}),$$

$$\alpha_2 = \partial_{F^{(1)}} \hat{K}(F, F^{(1)}, w^{(1)}) = \partial_F \mathcal{B}(F, F^{(1)}, w^{(1)}) = \rho_o \partial_F \partial_{F^{(1)}} \hat{\Sigma}(F, F^{(1)}, w^{(1)})$$

$$= \lambda_1 E_1 - r\lambda_2 E_2 + \lambda_1 \lambda_2 \{a_{11} + (1-r) a_{12} - r a_{22}\},$$

$$\alpha_3 = \partial_{w^{(1)}}\hat{K}(F, F^{(1)}, w^{(1)}) = -\partial_F\hat{\mathfrak{F}}(F, F^{(1)}, w^{(1)}) = \rho_o\partial_F\partial_{w^{(1)}}\hat{\Sigma}(F, F^{(1)}, w^{(1)})$$
$$= \lambda_1\lambda_2(a_{13} + a_{23}),$$
$$\beta_2 = \partial_{F^{(1)}}\hat{\mathcal{D}}(F, F^{(1)}, w^{(1)}) = \rho_o\partial^2_{F^{(1)}}\hat{\Sigma}(F, F^{(1)}, w^{(1)}) \tag{3.12}$$
$$= \lambda_1 E_1 + \lambda_2 r^2 E_2 + \lambda_1\lambda_2(a_{11} - 2ra_{12} + r^2 a_{22}),$$
$$\beta_3 = \partial_{w^{(1)}}\hat{\mathcal{D}}(F, F^{(1)}, w^{(1)}) = -\partial_{F^{(1)}}\hat{\mathfrak{F}}(F, F^{(1)}, w^{(1)}) = \rho_o\partial_{F^{(1)}}\partial_{w^{(1)}}\hat{\Sigma}(F, F^{(1)}, w^{(1)})$$
$$= \lambda_1\lambda_2(a_{12} - ra_{23})$$

with

$$E_i = \frac{\partial T_i}{\partial \bar{F}^{(i)}}, \quad \text{no sum on } i, \tag{3.13}$$

and

$$a_{ij} = \frac{\partial^2 \bar{\Sigma}^{(12)}}{\partial \bar{F}^{(i)}\partial \bar{F}^{(j)}}(\bar{F}^{(1)}, \bar{F}^{(2)}; w^{(1)}), \quad i, j = 1, 2, \tag{3.14}$$

which appear in the last line of each equation in (3.12) are in terms of the aforementioned fully equivalent alternate representation. Since the coefficients in (3.12) are continuous functions for all X and t, it follows from (3.11) with the aid of (3.7) that

$$[\partial_X K] = \alpha_1^+[\partial_X F] + \alpha_2^+[\partial_X F^{(1)}],$$
$$[\partial_X \mathcal{D}] = \alpha_2^+[\partial_X F] + \beta_2^+[\partial_X F^{(1)}]. \tag{3.15}$$

If we put $\varphi = \dot{y}$, F, $\dot{w}^{(1)}$ and $F^{(1)}$ successively in (3.5), we find that

$$a = [\ddot{y}] = -U[\dot{F}] = U^2[\partial_X F],$$
$$b = [\ddot{w}^{(1)}] = -U[\dot{F}^{(1)}] = U^2[\partial_X F^{(1)}]. \tag{3.16}$$

We call $a(t) = [\ddot{y}]$ the <u>mean amplitude</u> of the acceleration wave and say that the wave is compressive if $a > 0$, expansive if $a < 0$. Furthermore, from

(2.3), (2.8) and (3.16) we note that

$$[\ddot{\mathbf{y}}^{(1)}] = a + b, \quad [\ddot{\mathbf{y}}^{(2)}] = a - rb \tag{3.17}$$

are the amplitudes of the waves experienced by the two identifiable continua which make up the composite.

The substitution of (3.15) and (3.16) into (3.10) results in the pair of coupled equations

$$(\rho_o U^2 - \alpha_1^+)a - \alpha_2^+ b = 0, \quad -r^{-1}\alpha_2^+ a + (\rho_o U^2 - r^{-1}\beta_2^+)b = 0, \tag{3.18}$$

which admit a nontrivial solution in which $a \neq 0$, $b \neq 0$ provided U^2 is a root of the equation

$$U^4 - (C_1^2 + C_2^2)U^2 + (C_1^2 C_2^2 - \beta) = 0, \tag{3.19}$$

where

$$C_1^2 = \frac{\alpha_1^+}{\rho_o} = \frac{1}{\rho_o}(\partial_F \hat{K})^+, \quad C_2^2 = \frac{\beta_2^+}{\rho_o r} = \frac{1}{\rho_o r}(\partial_{F(1)} \hat{\mathcal{D}})^+,$$
$$\beta = \frac{1}{\rho_o^2 r}(\alpha_2^+)^2 = \frac{1}{\rho_o^2 r}(\partial_F \hat{\mathcal{D}}^+)^2 = \frac{1}{\rho_o^2 r}(\partial_{F(1)} \hat{K}^+)^2, \tag{3.20}$$

and we assume that $C_1 > C_2$. The roots of (3.19) are

$$U^2 = \frac{1}{2}\left\{(C_1^2 + C_2^2) \pm \sqrt{(C_1^2 - C_2^2)^2 + 4\beta}\right\} \tag{3.21}$$

and, since it is clear from $(3.20)_3$ that $\beta > 0$, both of the roots (3.21) will be real. Furthermore, if we assume that

$$C_1^2 C_2^2 > \beta, \tag{3.22}$$

then (3.19) implies the existence of two types of acceleration waves[13] whose speeds of propagation U_F, U_S are given by

$$U_F^2 = \frac{1}{2} \{ (C_1^2 + C_2^2) + \sqrt{(C_1^2 - C_2^2)^2 + 4\beta} \}, \tag{3.23}$$

and

$$U_S^2 = \frac{1}{2} \{ (C_1^2 + C_2^2) - \sqrt{(C_1^2 - C_2^2)^2 + 4\beta} \}, \tag{3.24}$$

respectively. Since $C_1 > C_2$, it follows at once from (3.22) - (3.24) that at any point X and time t

$$U_F \geq C_1 \quad \text{and} \quad U_S \leq C_2, \tag{3.25}$$

with the equalities holding when $\beta = 0$. The term C_1 is the speed one would calculate from the initial slope of the K-F curve, α_1, and thus (3.23) suggests that the "fast" wave is predominantly associated with the mean elasticity of the composite. The fact that U_F may exceed C_1 is a direct consequence of the nonlinear coupling effects which arise when β does not vanish. On the other hand, the "slow" wave always propagates into a deforming composite behind the "fast" wave and, since β is the slope of the $\mathcal{D}$-$F^{(1)}$ curve, the "slow" wave is associated with the relative motion of the constituents.

It has been pointed out by Nunziato and Walsh[16], in a somewhat different context, that the inequality (3.22) is capable of misrepresentation. In order that the physical significance of the inequality may be more fully appreciated we note from Eqs. (2.12), (2.13), (2.16), (2.19) - (2.22) and (3.20) that

$$C_1^2 = \frac{r}{1+r} V_1^2 + \frac{1}{1+r} V_2^2 + \frac{\lambda_1}{(1+r)\bar{\rho}_o^{(2)}} \{a_{11} + 2a_{12} + a_{22}\}, \tag{3.26}$$

and

$$C_2^2 = \frac{1}{1+r} V_1^2 + \frac{r}{1+r} V_2^2 + \frac{\lambda_2}{(1+r)\bar{\rho}_o^{(1)}} \{a_{11} - 2ra_{12} + r^2 a_{22}\}, \tag{3.27}$$

where

$$V_1^2 = \frac{1}{\bar{\rho}_o^{(1)}} \partial_{\bar{F}^{(1)}} T_1 , \quad V_2^2 = - \frac{1}{\bar{\rho}_o^{(2)}} \partial_{\bar{F}^{(2)}} T_2 , \tag{3.28}$$

are the intrinsic velocities which acceleration waves would have in bodies composed solely of either of the interpenetrating continua which make up the composite.

It follows from (3.18) and (3.20) that

$$b = Ha , \tag{3.29}$$

where

$$H^2 = \frac{U^2 - C_1^2}{(U^2 - C_2^2) r} , \tag{3.30}$$

and it is to be noted that in the "fast" wave, (for which $U = U_F$), $H = H_F$ and $\operatorname{sgn} H_F = \operatorname{sgn} \alpha_2^+$ while in the case of the "slow" wave, (for which $U = U_S$), $H = H_S$ and $\operatorname{sgn} H_S = -\operatorname{sgn} \alpha_2^+$. Equations (3.17) and (3.29) together imply that

$$\begin{aligned} [\ddot{\underset{\sim}{y}}^{(1)}] &= \underset{\sim}{a}(1 + H) , \\ [\ddot{\underset{\sim}{y}}^{(2)}] &= \underset{\sim}{a}(1 - rH) , \end{aligned} \tag{3.31}$$

so that we have

$$\begin{aligned} \alpha_2^+ > 0 \rightarrow |[\ddot{\underset{\sim}{y}}^{(1)}]_F| &> |[\ddot{\underset{\sim}{y}}^{(2)}]_F| , \\ |[\ddot{\underset{\sim}{y}}^{(1)}]_S| &< |[\ddot{\underset{\sim}{y}}^{(2)}]_S| \end{aligned} \tag{3.32}$$

while

$$\begin{aligned} \alpha_2^+ < 0 \rightarrow |[\ddot{\underset{\sim}{y}}^{(1)}]_F| &< |[\ddot{\underset{\sim}{y}}^{(2)}]_F| , \\ |[\ddot{\underset{\sim}{y}}^{(1)}]_S| &> |[\ddot{\underset{\sim}{y}}^{(2)}]_S| . \end{aligned} \tag{3.33}$$

11)

4. Acceleration Waves in Some Particular Media

We now examine the propagation properties of acceleration waves in a number of special situations.

First, let us consider the situation which arises when the expression (2.20) reduces to

$$\rho_o \Sigma = \lambda_1 \bar{\rho}_o^{(1)} \bar{\Sigma}^{(1)} (\bar{F}^{(1)}) + \lambda_2 \bar{\rho}_o^{(2)} \bar{\Sigma}^{(2)} (\bar{F}^{(2)}) + \lambda_1 \lambda_2 \rho_o \bar{\Sigma}^{(12)} (w^{(1)}), \tag{4.1}$$

and we note that any stresses $\tau^{(i)}$ in each of the constituents of such a composite depend only on the state of deformation of that constituent and are independent of the state of deformation of the other constituent. Any coupling that may take place between the motion of the constituents occurs because of the existence of the relative body force $\mathfrak{F}$. It follows from (3.26) and (3.27) that in a composite of this type

$$c_1^2 = \frac{r}{1+r} V_1^2 + \frac{1}{1+r} V_2^2, \quad c_2^2 = \frac{1}{1+r} V_1^2 + \frac{r}{1+r} V_2^2, \tag{4.2}$$

while

$$\beta = \left\{ \frac{r}{1+r} V_1^2 - V_2^2 \right\}^2. \tag{4.3}$$

When (4.2) and (4.3) are used in (3.23) and (3.24), it is found that two waves with intrinsic velocities $U_F = V_1$ and $U_S = V_2$, respectively, may propagate. Furthermore, it is easily verified that in these waves

$$[\ddot{\underset{\sim}{y}}^{(2)}]_{\underset{\sim}{F}} = 0, \quad [\ddot{\underset{\sim}{y}}^{(1)}]_{\underset{\sim}{S}} = 0. \tag{4.4}$$

This is precisely the situation which will always arise in theories of the type developed by Bedford and Stern[6,7].

Finally, let us turn our attention to the case of a fiber reinforced composite in which the fiber is not continuous (i.e., chopped fiber). Suppose that continuum 2 represents the chopped fiber continuum. For a composite of this nature $\tau_2 = 0$ and consequently Eq. $(2.19)_2$ implies that

$$\tilde{\Sigma} = \tilde{\Sigma}(\bar{F}^{(1)}, w^{(1)}) , \tag{4.5}$$

which, with (2.12) and $(2.19)_1$ yields

$$K = \rho_o \partial_{\bar{F}^{(1)}} \tilde{\Sigma}(\bar{F}^{(1)}, w^{(1)}) . \tag{4.6}$$

Under this restrictive circumstance Eq. (4.6), with (2.12), (2.13) and (3.20), leads to the relations

$$\begin{aligned} C_1^2 &= rC_2^2 = \partial_{\bar{F}^{(1)}} \tilde{K}(\bar{F}^{(1)}, w^{(1)}) , \\ \beta &= \frac{1}{\rho_o^2 r} (\partial_{\bar{F}^{(1)}} \tilde{K})^2 = C_1^2 C_2^2 , \end{aligned} \tag{4.7}$$

so that Eq. (3.19) has only one root

$$U^2 = C_1^2 + C_2^2 = \frac{1+r}{r} C_1^2 = (1+r)C_2^2 . \tag{4.8}$$

Thus, in this restrictive case only one wave propagates and, since $H = 1/r$ for this wave, from $(3.31)_2$ $[\ddot{\underset{\sim}{y}}^{(2)}] = 0$, so that the acceleration of the chopped fiber is continuous at the wavefront.

5. Variation of the Amplitudes of Acceleration Waves

In this section we derive the equations which govern the evolutionary behavior of the amplitudes of acceleration waves as they propagate in two-constituent composite materials modeled as interpenetrating solid continua. We shall suppose that at each instant both "fast" and "slow" acceleration waves may exist in the body. For the moment there is no need for us to distinguish between the two types of acceleration waves nor do we need to prescribe in detail the conditions which prevail ahead of the waves.

The jumps in the material time derivatives of Eqs. (3.8) across the acceleration wave yield

$$[\partial_X \dot{\underset{\sim}{K}}] = \rho_o [\ddot{\underset{\sim}{y}}],$$

$$[\partial_X \dot{\underset{\sim}{\mathcal{D}}}] + [\dot{\underset{\sim}{\mathfrak{F}}}] = r\rho_o [\ddot{\underset{\sim}{w}}^{(1)}] . \tag{5.1}$$

On setting $\varphi = \ddot{y}$, $\dot{F}$, $\ddot{w}^{(1)}$ and $\dot{F}^{(1)}$, successively, in (3.5) we obtain the relations

$$2\,\frac{da}{dt} = \frac{d}{dt}\,(\ell n U)\,a + [\ddot{\underset{\sim}{y}}] - U^2[\partial_X \dot{\underset{\sim}{F}}],$$

$$2\,\frac{db}{dt} = \frac{d}{dt}\,(\ell n U)\,b + [\ddot{\underset{\sim}{w}}^{(1)}] - U^2[\partial_X \dot{\underset{\sim}{F}}^{(1)}], \tag{5.2}$$

the substitution of which in (5.1) yields

$$\begin{aligned} 2\,\frac{da}{dt} &= \frac{d}{dt}\,(\ell n U)\,a + \frac{1}{\rho_o}\,[\partial_X \dot{\underset{\sim}{K}}] - U^2[\partial_X \dot{\underset{\sim}{F}}], \\ 2\,\frac{db}{dt} &= \frac{d}{dt}\,(\ell n U)\,b + \frac{1}{r\rho_o}\,([\partial_X \dot{\underset{\sim}{\mathcal{D}}}] + [\dot{\underset{\sim}{\mathfrak{F}}}]) - U^2[\partial_X \dot{\underset{\sim}{F}}^{(1)}], \end{aligned} \tag{5.3}$$

which represent a set of coupled differential equations for a(t) and b(t) which hold for each admissible propagating acceleration wave.

In order to further simplify Eqs. (5.3) we need to evaluate the quantities $[\partial_X \dot{\underset{\sim}{K}}]$, $[\partial_X \dot{\underset{\sim}{\mathcal{D}}}]$ and $[\dot{\underset{\sim}{\mathfrak{F}}}]$. Differentiating Eqs. (3.11) with respect to time and evaluating the jumps across the wavefront, with the aid of (3.6), (3.12) and (3.16), we obtain

$$\begin{aligned} [\partial_X \dot{\underset{\sim}{K}}] = \overset{+}{\alpha}_1[\partial_X \dot{\underset{\sim}{F}}] + \overset{+}{\alpha}_2[\partial_X \dot{\underset{\sim}{F}}^{(1)}] + \frac{\overset{+}{\xi}_1}{U}\,a + \frac{1}{U}\,(\overset{+}{\xi}_2 - \overset{+}{\alpha}_3)\,b \\ - \frac{1}{U^3}\,\{\overset{+}{\alpha}_{11}a^2 + 2\overset{+}{\alpha}_{12}ab + \overset{+}{\alpha}_{22}b^2\} \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} [\partial_X \dot{\underset{\sim}{\mathcal{D}}}] = \overset{+}{\alpha}_2[\partial_X \dot{\underset{\sim}{F}}] + \overset{+}{\beta}_2[\partial_X \dot{\underset{\sim}{F}}^{(1)}] + \frac{\overset{+}{\xi}_2}{U}\,a + \frac{1}{U}\,(\overset{+}{\nu}_2 - \overset{+}{\beta}_3)\,b \\ - \frac{1}{U^3}\,\{\overset{+}{\alpha}_{12}a^2 + 2\overset{+}{\alpha}_{22}ab + \overset{+}{\beta}_{22}b^2\}, \end{aligned} \tag{5.5}$$

where

$$\xi_1 = -\left\{\alpha_{11}\left(\partial_X F - \frac{\dot F}{U}\right) + \alpha_{12}\left(\partial_X F^{(1)} - \frac{\dot F^{(1)}}{U}\right) + \alpha_{13}\left(F^{(1)} - \frac{\dot w^{(1)}}{U}\right)\right\} ,$$

$$\xi_2 = -\left\{\alpha_{12}\left(\partial_X F - \frac{\dot F}{U}\right) + \alpha_{22}\left(\partial_X F^{(1)} - \frac{\dot F^{(1)}}{U}\right) + \alpha_{23}\left(F^{(1)} - \frac{\dot w^{(1)}}{U}\right)\right\} \tag{5.6}$$

and

$$\nu_2 = -\left\{\alpha_{22}\left(\partial_X F - \frac{\dot F}{U}\right) + \beta_{22}\left(\partial_X F^{(1)} - \frac{\dot F^{(1)}}{U}\right) + \beta_{23}\left(F^{(1)} - \frac{w^{(1)}}{U}\right)\right\} \tag{5.7}$$

with

$$\alpha_{11} = \partial_F^2 \hat K(F,F^{(1)},w^{(1)}) = \rho_o \partial_F^3 \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1 \tilde E_1 + \lambda_2 \tilde E_2 + \lambda_1\lambda_2(a_{111} + 3a_{112} + 3a_{122} + a_{222}) ,$$

$$\alpha_{12} = \partial_F \partial_{F^{(1)}} \hat K(F,F^{(1)},w^{(1)}) = \rho_o \partial_F^2 \partial_{F^{(1)}} \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1 \tilde E_1 - r\lambda_2 \tilde E_2 + \lambda_1\lambda_2(a_{111} + (2-r)a_{112} + (1-2r)a_{122} - ra_{222}) ,$$

$$\alpha_{13} = \partial_F \partial_w \hat K(F,F^{(1)},w^{(1)}) = \rho_o \partial_F^2 \partial_{w^{(1)}} \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1\lambda_2(a_{113} + 2a_{123} + a_{223}) ,$$

$$\alpha_{22} = \partial^2_{F^{(1)}} \hat K(F,F^{(1)},w^{(1)}) = \rho_o \partial^2_{F^{(1)}} \partial_F \hat\Sigma(F,F^{(1)},w^{(1)}) \tag{5.8}$$
$$= \lambda_1 \tilde E_1 + r^2\lambda_2 \tilde E_2 + \lambda_1\lambda_2(a_{111} + (1-2r)a_{112} - r(2-r)a_{122} + r^2 a_{222}) ,$$

$$\alpha_{23} = \partial_{F^{(1)}} \partial_{w^{(1)}} \hat K(F,F^{(1)},w^{(1)}) = \rho_o \partial_{F^{(1)}} \partial_{w^{(1)}} \partial_F \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1\lambda_2(a_{113} + (1-r)a_{123} - ra_{223}) ,$$

$$\beta_{22} = \partial^2_{F^{(1)}} \hat{\mathcal{D}}(F,F^{(1)},w^{(1)}) = \rho_o \partial^3_{F^{(1)}} \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1 \tilde E_1 - r^3\lambda_2 \tilde E_2 + \lambda_1\lambda_2(a_{111} - 3ra_{112} + 3r^2 a_{122} - r^3 a_{222}) ,$$

$$\beta_{23} = \partial_{F^{(1)}} \partial_{w^{(1)}} \hat{\mathcal{D}}(F,F^{(1)},w^{(1)}) = \rho_o \partial^2_{F^{(1)}} \partial_{w^{(1)}} \hat\Sigma(F,F^{(1)},w^{(1)})$$
$$= \lambda_1\lambda_2(a_{113} - 2ra_{123} + r^2 a_{223}) ,$$

while

$$\tilde E_i = \frac{\partial^2 T_i}{\partial \bar F^{(i)2}} , \quad \text{(no sum on i)} , \tag{5.9}$$

$$a_{ijk} = \frac{\partial^3 \Sigma^{(12)}}{\partial \bar{F}^{(i)} \partial \bar{F}^{(j)} \partial \bar{F}^{(k)}} (\bar{F}^{(1)}, \bar{F}^{(2)}; w^{(1)}), \qquad i,j,k = 1,2 \tag{5.10}$$

$$a_{ij3} = \frac{\partial a_{ij}}{\partial w^{(1)}}, \tag{5.11}$$

which appear in the last line of each equation in (5.8) are in terms of the fully equivalent alternate representation. It follows from Eqs. $(2.15)_3$, $(2.16)_3$ and (2.18) together with the definitions in (3.12) that we have

$$\begin{aligned} \mathfrak{F} = &-(\alpha_3 - \dot{w}^{(1)} \partial_F \hat{g}) \dot{F} - (\beta_3 - \dot{w}^{(1)} \partial_{F^{(1)}} \hat{g}) \dot{F}^{(1)} + (\partial_{w^{(1)}} \hat{\mathfrak{F}} \\ &+ \dot{w}^{(1)} \partial_{w^{(1)}} \hat{g}) \dot{w}^{(1)} + (\hat{g} + \dot{w}^{(1)} \partial_{\dot{w}^{(1)}} \hat{g}) \ddot{w}^{(1)}, \end{aligned} \tag{5.12}$$

the jump in which, with the aid of (3.16), yields

$$[\mathfrak{F}] = (\alpha_3^+ - \dot{w}^{(1)} \partial_F \hat{g}^+) \frac{a}{U} + (\beta_3^+ - \dot{w}^{(1)} \partial_{F^{(1)}} \hat{g}^+) \frac{b}{U} + (\hat{g}^+ + \dot{w}^{(1)} \partial_{\dot{w}^{(1)}} \hat{g}^+) b. \tag{5.13}$$

When the expressions (5.4), (5.5) and (5.13) are substituted in Eqs. (5.3) we arrive at the coupled differential equations

$$\begin{aligned} 2\frac{da}{dt} = &\left\{\frac{d}{dt}(\ell n U) + \frac{\xi_1^+}{\rho_o U}\right\} a + \frac{1}{\rho_o U}(\xi_2^+ - \alpha_3^+) b - \frac{1}{\rho_o U^3}\{\alpha_{11}^+ a^2 \\ &+ 2\alpha_{12}^+ ab + \alpha_{22}^+ b^2\} + (C_1^2 - U^2)[\partial_X \dot{F}] + \frac{\alpha_2^+}{\rho_o}[\partial_X \dot{F}^{(1)}], \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} 2\frac{db}{dt} = &\left\{\frac{d}{dt}(\ell n U) + \frac{1}{r\rho_o U}(\nu_2^+ - \dot{w}^{(1)} \partial_{F^{(1)}} \hat{g}^+) + (\hat{g}^+ + \dot{w}^{(1)} \partial_{\dot{w}^{(1)}} \hat{g}^+)\right\} b \\ &+ \frac{1}{r\rho_o U}\{\xi_2^+ - \alpha_3^+ - \dot{w}^{(1)} \partial_F \hat{g}^+\} a - \frac{1}{r\rho_o U^3}\{\beta_{22}^+ b^2 + 2\alpha_{22}^+ ab + \alpha_{12}^+ a^2\} \\ &+ \frac{\alpha_2^+}{r\rho_o}[\partial_X \dot{F}] + (C_2^2 - U^2)[\partial_X \dot{F}^{(1)}]. \end{aligned} \tag{5.15}$$

Of course, b and a are related through Eq. (3.29) and the intrinsic velocity U is a root of Eq. (3.19). These facts enable us to combine (3.29) with Eqs. (5.14) and (5.15) in order to obtain a single first order differential equation which governs the amplitude a(t) of each admissible acceleration wave. After some tedious algebra, we find that the amplitude of an admissible type of acceleration wave in a two-component composite medium satisfies the equation

$$\frac{da}{dt} + \mu a - \zeta a^2 = 0\,, \tag{5.16}$$

where

$$\begin{aligned} 2(1 + rH^2)\mu(t) = -\{ &(1 + rH^2)\frac{d}{dt}(\ell n U) + 2rH^2 \frac{d}{dt}(\ell n H) + (\xi_1^+ + 2H\xi_2^+ + H^2\nu_2^+)/\rho_o U \\ &- H\dot{w}^{(1)}(\partial_F \hat{g}^+ + H\partial_{F^{(1)}}\hat{g}^+)/\rho_o U - rH^2(\hat{g}^+ + \dot{w}^{(1)}\partial_{\dot{w}^{(1)}}\hat{g}^+)\} \end{aligned} \tag{5.17}$$

$$\zeta(t) = \frac{-\tilde{E}}{2\rho_o U^3}\,, \tag{5.18}$$

and

$$\begin{aligned} 2(1 + rH^2)\tilde{E} = \{\alpha_{11} + 3H\alpha_{12} + 3H^2\alpha_{22} + H^3\beta_{22}\} = (1 + H)\lambda_1\tilde{E}_1 + (1 - rH)^3\lambda_2\tilde{E}_2 \\ + \lambda_1\lambda_2\{(1 + H)^3 a_{111} + 3(1 + H)^2(1 - rH)a_{112} + 3(1 + H)(1 - rH)^2 a_{122} \\ + (1 - rH)^3 a_{222}\} \end{aligned} \tag{5.19}$$

is the effective second order elastic modulus of the composite for the particular wave under consideration, and where the expression after the second equals sign is in terms of the fully equivalent alternate representation.

Equation (5.16) is a differential equation of Bernoulli type. As one might expect, it is similar to the equation recently derived by Nunziato and Walsh[16] in their study of the propagation of acceleration waves in granular media. However, a cursory examination of the coefficients $\mu(t)$ and $\zeta(t)$ shows that the similarity is somewhat superficial. The coefficients $\mu(t)$ and $\zeta(t)$ are determined by the particular type of wave under study, the mechanical properties of the composite and by the conditions prevailing ahead of the wave. We shall study the properties of the solutions of Eq. (5.16) in a number of particular situations in the following section.

6. The Behavior of Some Particular Acceleration Waves

In this section we study the evolutionary behavior of the amplitudes of some particular acceleration waves. In general, at a given instant of time, two acceleration waves will propagate in the body. Suppose that the "fast" wave is located at the poin $X = Z_F(t)$ while the "slow" wave is at $X = Z_S(t)$ where $Z_F(t) > Z_S(t)$. In order to simplify matters we shall assume that the material ahead of the "fast" wave is in a steady state of equilibrium. Even though the deformation field behind a fast acceleration wave may be such that $U_S(t) > U_F(t)$, the "slow" wave can never pass through the "fast" wave[20] (cf. Nunziato and Walsh[16]). Thus, at all points $X > Z_F(t)$ the fields $y(X)$ and $w^{(1)}(X)$, or alternatively, $y^{(1)}(X)$ and $y^{(2)}(X)$ do not depend on t, i.e.,

$$y = y(X),\ w^{(1)} = w^{(1)}(X),\ X > Z_F(t)\ ,$$
$$y^{(1)} = y^{(1)}(X),\ y^{2} = y^{(2)}(X),\ X > Z_F(t)\ . \tag{6.1}$$

Since the fields (6.1) must satisfy the equations of equilibrium, at all points $X > Z_F(t)$ we have

$$\alpha_1 \partial_X F + \alpha_2 \partial_X F^{(1)} + \alpha_3 F^{(1)} = 0\ ,$$
$$\alpha_2 \partial_X F + \beta_2 \partial_X F^{(1)} + \beta_3 F^{(1)} = -\hat{\mathfrak{I}}\ . \tag{6.2}$$

From (5.16) we see that the equation satisfied by the amplitude $a_F(t)$ of a "fast" wave propagating into a region which is in a steady state of equilibrium is

$$\frac{d}{dt}\, a_F + \mu_o(t) a_F - \zeta_o(t) a_F^2 = 0\ , \tag{6.3}$$

where, from (5.17), with (5.6), (5.7) and the fact that from (3.20) and (3.23) now U(X) is independent of t, we have

$$\begin{aligned} 2(1+rH_F^2)\mu_o(t) = &-(1+rH_F^2)\partial_X U_F - rH_F U_F \partial_X H_F + \varphi_1 \partial_X F \\ &+ \varphi_2 \partial_X F^{(1)} + \varphi_3 F^{(1)} + rH_F^2 g^{(0)}\ , \end{aligned} \tag{6.4}$$

with

$$\begin{aligned} \varphi_1 &= (\alpha_{11}^{(0)} + 2H_F\alpha_{12}^{(0)} + H_F^2\alpha_{22}^{(0)})/\rho_o U_F\ , \\ \varphi_2 &= (\alpha_{12}^{(0)} + 2H_F\alpha_{22}^{(0)} + H_F^2\beta_{22}^{(0)})/\rho_o U_F\ , \\ \varphi_3 &= (\alpha_{13}^{(0)} + 2H_F\alpha_{23}^{(0)} + H^2\beta_{23}^{(0)})/\rho_o U_F\ , \\ H_F &= (\rho_o U_F^2 - \alpha_1^{(0)})/\alpha_2^{(0)}\ , \end{aligned} \tag{6.5}$$

and $\zeta_o(t)$ is still given by (5.18). The superscript 0 occurring in $g^{(0)}$ and on the right-hand sides of the expressions in (6.5) denote that these quantities are evaluated for the steady-state deformation fields described by (6.1) and, consequently, these quantities are functions of X only and do not depend on t. Equation (6.3) has the same form as that which governs the evolutionary behavior of the amplitude of one-dimensional acceleration waves in a single phase elastic material which is in a state of nonhomogeneous deformation ahead of the wave (see, e.g., Chen[12], Coleman, Greenberg and Gurtin[21]).

If the material ahead of the "fast" wave is at rest in its natural stress-free state so that

$$y = X, \quad w^{(1)} = 0, \quad X > Z_F(t) ,$$

or

$$y_1(X) = y_2(X) = X , \quad X > Z_F(t) , \tag{6.7}$$

and it follows from (6.4) that

$$\mu_o(t) = rH_F^2 g^{(0)}/2(1 + rH_F^2) = \omega_o , \quad \text{say}, \tag{6.8}$$

for such a wave so that $a_F(t)$ obeys the differential equation

$$\frac{da_F}{dt} = -\omega_o a_F + \zeta_o a_F^2 , \tag{6.9}$$

which admits the solution

$$a_F(t) = \frac{\lambda_o}{\left(\frac{\lambda_o}{a_F^{(0)}} - 1\right)e^{\omega_o t} + 1} \tag{6.10}$$

where

$$\lambda_o = \frac{\omega_o}{\zeta_o} = \frac{-\rho_o r U_F^3 H_F^2}{(1 + rH_F^2)\tilde{E}} g^{(0)} , \tag{6.11}$$

and $a_F^{(0)}$ is the value of the mean amplitude of the acceleration wave at time $t = 0$. Equation (6.10) indicates that the behavior of a "fast" acceleration wave propagating into a two-component composite in its natural state is the same as that of an acceleration wave propagating into a homogeneously deformed material with memory[22] or a piezoelectric semiconductor which is in a steady state ahead of the wave[23].

The properties of the solution (6.10) are well documented (see, e.g., Refs. 12 and 23) and it is not our intention to study them in detail here. The critical mean amplitude for acceleration waves, λ_o, plays a fundamental role in determining whether the amplitude of an acceleration wave will grow or decay as the wave traverses the material. In particular, the sign of λ_o plays a critical role in determining whether the mean amplitude will grow or decay. Since $g^{(0)} > 0$, by assumption, it follows from (6.8) that $\omega_o > 0$ so that sgn $(\lambda_o) = -$ sgn $(\tilde{E})$ and it is clear from (5.8) and (5.19) that $\tilde{E}$ may differ in sign from either $\tilde{E}_1$ or $\tilde{E}_2$. Notice that (6.10) implies that

(i) If $|a_F(0)| < \lambda_o$, then $a_F(t) \to 0$ monotonically as $t \to \infty$.

(ii) If sgn $(a_F(0)) =$ sgn λ_o and $|a_F(0)| > |\lambda_o|$, then $a_F(t) \to \infty$ monotonically within a finite time

$$t_\infty = -\frac{1}{\omega_o} \ln\{1 - (\lambda_o/a_F(0))\}, \tag{6.12}$$

and this is usually taken to indicate shock formation.

Next, let us consider an acceleration wave propagating into a composite material in which $\mathfrak{F}$ is independent of $\dot{w}^{(1)}$ so that $g(0) = 0$. Equation (6.9) now reduces to

$$\frac{da_F}{dt} + \frac{\tilde{E}}{2\rho_o U_F^3} a_F^2 = 0, \tag{6.13}$$

which admits the solution

$$a_F(t) = \frac{a_F(0)}{1 + \dfrac{a_F(0)\tilde{E}}{2\rho_o U_F^3}}. \tag{6.14}$$

If $\operatorname{sgn}(a_{F(0)}\tilde{E}) < 0$, then the solution (6.14) becomes unbounded after a time

$$\bar{t} = \frac{-2\rho_o U_F^3}{a_F(0)\tilde{E}}, \tag{6.15}$$

?,)

and, of course, this is precisely what happens in a perfectly elastic single phase continuum (see, e.g., Green[24]).

If the material ahead of the fast wave is at rest such that the center of mass y is in a state of homogeneous strain, we have

$$y = \lambda X, \quad \lambda = \text{constant}, \quad X > Z_F(t) \, . \tag{6.16}$$

Since $F = \lambda$, a constant, in this case it follows from Eqs. (6.3) that at all points $X > Z_F(t)$ the relative displacement $w^{(1)}(X)$ must satisfy the nonlinear first-order differential equation

$$(\alpha_3\beta_2 - \beta_3\alpha_2) \frac{dw^{(1)}}{dX} = \alpha_2 \, , \tag{6.17}$$

and it is to be noted that both α_1 and the coefficients occurring in (6.17) are (for fixed $F = \lambda$), functions of $\partial w^{(1)}/\partial X$ and $w^{(1)}$. It is to be expected that, even if $g^+(0) = 0$, the solution of (6.17) for $w^{(1)}(X)$ will lead to a nonvanishing expression for $\mu_o(t)$ when substituted into (6.4). Thus, the coefficient $\mu_o(t)$ will be a consequence of the inhomogeneities in the deformation fields of the two continua which make up the composite, as well as the coefficient $g^+(0)$. The solution of Eq. (6.3) in this case is

$$a_F(t) = \frac{a_F(0)\exp\left\{-\int_0^t \mu_o(s)\,ds\right\}}{1 - a_F(0)\int_0^t \zeta_o(s)\exp\left\{-\int_0^s \mu_o(\xi)\,d\xi\right\} ds} \, . \tag{6.18}$$

The properties of the solution have been discussed in detail by a number of authors (Bailey and Chen[25], Nunziato and Walsh[16]) and we refer the reader to these works for details.

The behavior of "slow" waves is always more complicated than that of "fast" waves since "slow" waves propagate into regions which are not in equilibrium. Thus, in the case of the "slow" wave the coefficients $\mu(t)$ and $\zeta(t)$ will contain derivatives with respect to both X and t. The amplitude of the "slow" wave is given by an expression similar to (6.18).

In Section 4 we examined the propagation of waves in a number of special situations. Let us now consider the evolutionary behavior of the amplitudes of these waves.

We saw in Section 4 that in the restrictive case when the internal energy is given by (4.1) two waves may propagate with intrinsic velocities $U_F = V_1$ and $U_S = V_2$, respectively. The equations satisfied by the amplitudes of these waves may be deduced from the results of Section 5, but the properties of the waves in the special highly simplified cases considered here become more transparent when it is noted that, with the aid of (2.3), (2.8), (2.9), (2.12), (2.13), (2.21) and (2.22), Eqs. (3.8) may be transformed to

$$\partial_X T_1 + \frac{1}{\lambda_1(1+r)} \mathfrak{F} = \bar{\rho}_o^{(1)} \ddot{y}^{(1)} ,$$

$$\partial_X T_2 - \frac{1}{\lambda_2(1+r)} \mathfrak{F} = \bar{\rho}_o^{(2)} \ddot{y}^{(2)} . \tag{6.19}$$

It follows from $(2.16)_3$ and (4.1) that

$$\hat{\mathfrak{F}} = -\rho_o \lambda_1 \lambda_2 \partial_{w^{(1)}} \bar{\bar{\Sigma}}^{(12)} (w^{(1)}) , \tag{6.20}$$

and we further assume that $\hat{\hat{\mathfrak{F}}} = g\dot{w}^{(1)}$, where g is a negative constant. If we write $[\ddot{\underset{\sim}{y}}^{(1)}\underset{\sim}{]} = \bar{a}^{(1)}$ and $[\ddot{\underset{\sim}{y}}^{(2)}\underset{\sim}{]} = \bar{a}^{(2)}$, then an elementary calculation leads to the growth equations

$$\frac{d\bar{a}^{(i)}}{dt} + \mu^{(i)} \bar{a}^{(i)} - \zeta^{(i)} \bar{a}^{(i)^2} = 0, \quad i = 1,2 , \tag{6.21}$$

where

$$\mu^{(i)} = \frac{1}{2\rho_o^{(i)}}\left\{V_i \frac{d}{dt}\left(\frac{1}{V_i}\right) - \tilde{E}_i(\dot{\bar{F}}^{(i)} - V_i\partial_X\bar{F}^{(i)})/V_i^2 - g/\lambda_i(1+r)^2\right\}$$

$$\zeta^{(i)} = \tilde{E}_i/\rho_o^{(i)}V_i^3, \quad \tilde{E}_i = \partial_{\bar{F}(i)}T_i, \qquad i = 1,2. \tag{6.22}$$

It follows from $(6.19)_2$, that at the "fast" wave we have

$$[\partial^3 y^{(2)}/\partial t^3]_F = \frac{V_1^2 g}{\lambda_2\bar{\rho}_o^{(2)}(1+r)^2(V_2^2 - V_1^2)}\bar{a}^{(1)}, \tag{6.23}$$

so that the third and higher order derivatives of $y^{(2)}(X,t)$ suffer jump discontinuities across the "fast" wave. Likewise at a "slow" wave $y^{(1)}(X,t)$ and its first and second partial derivatives will be continuous for all X,t, but

$$[\partial^3 y^{(1)}/\partial t^3]_S = \frac{V_2 g}{\lambda_1\bar{\rho}_o^{(1)}(1+r)^2(V_1^2 - V_2^2)}\bar{a}^{(2)}. \tag{6.24}$$

On the other hand, in materials in which $g = 0$, the right-hand sides of the expressions (6.23), (6.24) vanish and the fourth order time derivatives of $y^{(2)}$ and $y^{(1)}$, respectively, suffer jumps across the wavefronts which are given by the expressions

$$[\partial^4 y^{(2)}/\partial t^4]_F = \frac{V_1^2\partial_{w^{(1)}}\hat{\mathfrak{F}}}{\lambda_1\bar{\rho}_o^{(2)}(1+r)(V_1^2 - V_2^2)}\bar{a}^{(1)}, \tag{6.23a}$$

and

$$[\partial^4 y^{(1)}/\partial t^4]_S = \frac{V_2^2\partial_{w^{(1)}}\hat{\mathfrak{F}}}{\lambda_1\bar{\rho}_o^{(1)}(1+r)(V_2^2 - V_1^2)}\bar{a}^{(2)}. \tag{6.24a}$$

The importance of the foregoing is that in a composite of the type characterized by the restrictive expression (4.1) for the internal energy per unit mass, a "fast" wave, moving into a material which is initially at rest in an equilibrium configuration, will induce a motion in both constituents of the composite. Notice from (6.22) that the evolutionary behavior of a particular wave is influenced only by the properties of one of the constituents and the state of this constituent ahead of the wavefront. Thus we note that, in particular, the behavior of a "fast" wave propagating into a composite which is at rest in a homogeneous state before the arrival of the wavefront is qualitatively the same as that of a wave propagating into an equilibrium configuration in a single phase thermoelastic medium. On the other hand the second constituent of the composite will be set in motion through the coupling caused by $\tilde{\mathfrak{J}}$ because of the passage of the "fast" wave and consequently $\mu^{(2)}$ will generally be nonzero even when $g = 0$.

The expressions (6.23), (6.24), (6.23a), (6.24a) are also interesting in that, when the internal energy is given by (4.1), it is evident that the higher order discontinuity induced in the motion of one component of the composite because of a discontinuity in the acceleration of the second component across the wavefront is the result of coupling effects caused by the relative body force $\mathfrak{J}$.

To complete our study we return to the case of a composite in which one of the components is a chopped fiber. The internal energy density is now given by (4.5) and only one wave, across which $[\ddot{\underset{\sim}{y}}^{(1)}] \neq 0$, $[\ddot{\underset{\sim}{y}}^{(2)}] = 0$, may propagate in the composite. When Eqs. (2.3), (2.8), (2.12), (2.13) and (3.8) are combined, it follows that the motion of the chopped fiber (i.e., component 2 of the composite) is given by the formula

$$\rho_o \ddot{y}^{(2)} = -\mathfrak{J}, \qquad (6.25)$$

with

$$\mathfrak{F} = -\rho_o \partial_{w^{(1)}} \tilde{\Sigma}(\tilde{F}^{(1)}, w^{(1)}) + g\dot{w}^{(1)}, \tag{6.26}$$

and once again we assume that g is a positive constant. It follows from (6.25) and (6.26) that

$$[\partial^3 y^{(2)}/\partial t^3] = -\frac{1}{\rho_o}[\dot{\mathfrak{F}}] = \frac{1}{\rho_o rU}\{\partial_{\overline{F}^{(1)}}\hat{\mathfrak{F}} - gU\}a, \tag{6.27}$$

where $a = [\partial^2 y^{(1)}/\partial t^2]$.

The one-dimensional behavior of a centrosymmetric medium of the type under study is characterized by an internal energy density function $\tilde{\Sigma}^{(1)}(\overline{F}^{(1)}, w^{(1)})$ which is an even function of $w^{(1)}$, i.e.,

$$\tilde{\Sigma}(\overline{F}^{(1)}, w^{(1)}) = \tilde{\Sigma}(\overline{F}^{(1)}, -w^{(1)}), \tag{6.28}$$

so that the relative body force $\hat{\mathfrak{F}}(\overline{F}^{(1)}, w^{(1)})$ is an odd function of $w^{(1)}$. It follows that $\partial_{\overline{F}^{(1)}}\hat{\mathfrak{F}}(\overline{F}^{(1)}, 0)_{\overline{F}^{(1)}=1} = 0$ so that if the material ahead of the wave is in equilibrium in its natural stress-free state then

$$[\partial^3 y^{(2)}/\partial t^3] = ga/\rho_o r, \tag{6.29}$$

while if $g = 0$, $\partial^3 y^{(2)}/\partial t^3$ is continuous for all X,t but

$$[\partial^4 y^{(2)}/\partial t^4] = \partial_{w^{(1)}}\hat{\mathfrak{F}}a/\rho_o r. \tag{6.30}$$

Finally, let us consider the behavior of the amplitude of an acceleration wave which is propagating into a composite in which one of the components is a chopped fiber and which is in a steady natural stress-free state before the arrival of the wave. It follows from Section 5 that the amplitude of the wavefront is governed by Eq. (5.16) with

$$\overline{\mu} = \frac{1}{2(1+r)}g, \quad \overline{\zeta} = \frac{1}{2\rho_o U^3}(1 + 1/r)^2(\lambda_1\tilde{E}_1 + \lambda_2 a_{111}), \tag{6.31}$$

so that at anytime t the amplitude of the wave is given by

$$a(t) = \frac{\bar{\lambda}_o}{\left(\frac{\bar{\lambda}_o}{a(0)} - 1\right)\rho^{\bar{\mu}t} + 1}, \tag{6.32}$$

where $\bar{\lambda}_o = \bar{\mu}/\bar{\zeta}$. Thus, in a chopped fiber composite the behavior of the acceleration wave is qualitatively the same as that in a single phase heat conducting elastic medium. Notice in particular that the wave will be undamped if $g = 0$, in which case (6.32) reduces to the expression

$$a(t) = \frac{a(0)}{1 + \bar{\zeta}a(0)t}. \tag{6.33}$$

The influence of the chopped fiber on the behavior of the amplitude is evident from the manner in which the parameters r, λ_2 and g influence the coefficients $\bar{\mu}$ and $\bar{\zeta}$.

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