AD-A091 307 CENTER FOR NAVAL ANALYSES ALEXANDRIA VA NAVAL WARFARE--ETC F/G 12/1
ON THE WKB APPROXIMATION TO THE PROPAGATOR FOR ARBITRARY HAMILT--ETC(U)
AUG 80 M M MIZRAHI
UNCLASSIFIED CNA-PP-286 NL

END
DATE FILMED
DTIC

AD A091307

PROFESSIONAL PAPER 286 / August 1980

LEVEL II

ON THE WKB APPROXIMATION TO THE PROPAGATOR FOR ARBITRARY HAMILTONIANS

Maurice M. Mizrahi

DTIC
ELECTE
NOV 3 1980
D
C

DDC FILE COPY

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

CNA
CENTER FOR NAVAL ANALYSES
2000 North Beauregard Street, Alexandria, Virginia 22311

80 10 14 224

The ideas expressed in this paper are those of the author. The paper does not necessarily represent the views of either the Center for Naval Analyses or the Department of Defense.

(9) PROFESSIONAL PAPER 286 / August 1980 (11)

(1)

(6) ON THE WKB APPROXIMATION TO THE PROPAGATOR FOR ARBITRARY HAMILTONIANS.

(10) Maurice M. Mizrahi

(14) CNA-PP-286

DTIC ELECTE NOV 3 1980 D C

(12) 28

Naval Warfare Analysis Group

CENTER FOR NAVAL ANALYSES

2000 North Beauregard Street, Alexandria, Virginia 22311

252950

ABSTRACT

This paper presents a general expression for the WKB approximation to the propagator corresponding to an arbitrary Hamiltonian operator $\underset{\sim}{H}$. For example, if the correspondence rule used to pass from the classical Hamiltonian H_c to $\underset{\sim}{H}$ is such that it associates $a\underset{\sim}{P}_i\underset{\sim}{Q}^j+(1-a)\underset{\sim}{Q}^j\underset{\sim}{P}_i$ to p_iq^j, then the formula gives

$$K_{WKB} = K_{VV}\exp\{(\tfrac{1}{2}-a)\int_T(\partial^2 H_c/\partial q^i\partial p_i)(q_c(t),p_c(t),t)dt\},$$

where $K_{VV} = (2\pi i\hbar)^{-n/2}\ (\det M)^{\frac{1}{2}}\exp(iS_c/\hbar)$ is Van Vleck's well-known formula, S_c being the action functional evaluated at the classical path (q_c,p_c) and $M_{ij} = -\partial^2 S_c/\partial q_a{}^i\partial q_b{}^j$. More generally, the formula presented here applies to any system described by a function $f(x,t)$ whose time evolution is given by $(\underset{\sim}{H}(x, k\partial/\partial x, t) + k\partial/\partial t)f(x,t) = 0$, regardless of the form of $\underset{\sim}{H}$. The Schrödinger equation of quantum mechanics and the Fokker-Planck equation of diffusion are obvious examples. Many examples are discussed. This generalizes results obtained in a previous publication (J. Math. Phys. 18 (1977), 786-90).

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or Special
A

INTRODUCTION

The purpose of this paper is to obtain a general expression for the WKB approximation to the propagator corresponding to an arbitrary Hamiltonian operator. In an earlier publication[1] we determined a range of validity of Van Vleck's well-known formula[2], which was known not to be universally valid[3]. The approximation derived here, generalizing Van Vleck's formula, is also valid beyond quantum-mechanical applications, as it applies to any system described by a function $f(x,t)$ whose time evolution is dictated by

$$\left[\underset{\sim}{H}(x,\ k\partial/\partial x,\ t) + k\partial/\partial t \right] f(x,t) = 0, \qquad (1)$$

such as the Fokker-Planck equation for diffusion processes.

THE GENERAL WKB APPROXIMATION FORMULAE

We operate in n dimensions and summation over repeated indices is implied. The following theorem summarizes our findings:

Theorem

Let

- $\underset{\sim}{H}(\underset{\sim}{Q},\underset{\sim}{P},t)$ be an arbitrary Hamiltonian operator (the lack of constraints imply that it could be non-Hermitian, time-dependent, non-quadratic in $\underset{\sim}{P}$, etc.)
- $H_c(q,p,t) = \underset{\sim}{H}(\underset{\sim}{Q}\to q,\ \underset{\sim}{P}\to p,\ \hbar = 0)$ be its classical counterpart.
- The correspondence rule used to pass from H_c to $\underset{\sim}{H}$ be such that it makes the following associations:

$$f(q) \leftrightarrow f(\underset{\sim}{Q}), \qquad f(p) \leftrightarrow f(\underset{\sim}{P}), \qquad (2)$$

$$p_i q^j \leftrightarrow a\underset{\sim}{P}_i\underset{\sim}{Q}^j + (1-a)\underset{\sim}{Q}^j\underset{\sim}{P}_i \text{ for a given } a. \qquad (3)$$

(Note that (2) and (3) do not imply Hermiticity of $\underset{\sim}{H}$, even if $a=\frac{1}{2}$).

- $(q_c(t), p_c(t))$ be the classical solution, solving Hamilton's equations for H_c such that $q_c(t_a) = q_a$ and $q_c(t_b) = q_b$,
- $S_c(q_b, t_b, q_a, t_a) \equiv \int_T \left[p_{ci}(t) \dot{q}_c^{\,i}(t) - H_c(q_c(t), p_c(t), t) \right] dt$ (4)
  be the classical action ($T \equiv [t_a, t_b]$, also $T \equiv t_b - t_a$),
- $M_{ij}(q_b, t_b, q_a, t_a) \equiv -\partial^2 S_c / \partial q_a^{\,i} \partial q_b^{\,j}$ (5)
  be the Van Vleck - Morette matrix, with determinant det M,
- $K(q_b, t_b, q_a, t_a)$ be the propagator corresponding to $\underset{\sim}{H}$, defined by

$$\left[\underset{\sim}{H}(\underset{\sim}{Q}, \underset{\sim}{P}, t_b) - i\hbar \frac{\partial}{\partial t_b} \right] K = 0, \tag{6}$$

$$\lim_{t_b \to t_a} K = \delta(q_b - q_a), \tag{7}$$

  where $\underset{\sim}{Q}$ is represented by q_b and $\underset{\sim}{P}$ by $-i\hbar \partial / \partial q_b$,

- $K_{WKB}(q_b, t_b, q_a, t_a)$ be the WKB approximation to the propagator, defined by:

$$K_{WKB}^{-1} \left[\underset{\sim}{H}(\underset{\sim}{Q}, \underset{\sim}{P}, t_b) - i\hbar \partial / \partial t_b \right] K_{WKB} = O(\hbar^2), \tag{8}$$

$$\lim_{t_b \to t_a} K_{WKB} = \delta(q_b - q_a). \tag{9}$$

Then the WKB approximation is given by

$$K_{WKB} = K_{VV} \exp\left[(\tfrac{1}{2} - a) \int_{t_a}^{t_b} \frac{\partial^2 H_c}{\partial q^i \partial p_i} (q_c(t), p_c(t), t) \, dt \right], \tag{10}$$

where K_{VV} is Van Vleck's formula:

$$K_{VV} = (2\pi i\hbar)^{-n/2} (\det M)^{\frac{1}{2}} \exp(iS_c/\hbar). \tag{11}$$

(The case where det M = 0 is not examined here).

More generally, the WKB approximation is given by

$$K_{WKB} = A_o \exp(iS_c/\hbar), \tag{12}$$

where

$$\bullet\ A_o \equiv K_o(0,t_b,0,t_a), \tag{13}$$

• $K_o(q_b',t_b,q_a',t_a)$ is the propagator corresponding to the Hamiltonian operator

$$\underset{\sim}{H}_o \equiv \tfrac{1}{2}g^{ij}(t)\underset{\sim}{U}_{ij} + \tfrac{1}{2}f_{ij}(t)\underset{\sim}{V}^{ij} + k_i{}^j(t)\underset{\sim}{W}^i{}_j\,, \tag{14}$$

$$\bullet\ g^{ij}(t) \equiv \left.\frac{\partial^2 H_c}{\partial p_i \partial p_j}\right|_{\substack{q=q_c(t)\\ p=p_c(t)}},\quad f_{ij}(t) \equiv \left.\frac{\partial^2 H_c}{\partial q^i \partial q^j}\right|_{\substack{q=q_c(t)\\ p=p_c(t)}},\quad k_i{}^j(t) \equiv \left.\frac{\partial^2 H_c}{\partial q^i \partial p_j}\right|_{\substack{q=q_c(t)\\ p=p_c(t)}}, \tag{15}$$

• The correspondence rule used to pass from H_c to $\underset{\sim}{H}$ is such that it makes the following associations:

$$p_i p_j \leftrightarrow \underset{\sim}{U}_{ij}\,,\quad q^i q^j \leftrightarrow \underset{\sim}{V}^{ij}\,,\quad q^i p_j \leftrightarrow \underset{\sim}{W}^i{}_j\,. \tag{16}$$

PROOF OF THE GENERAL WKB APPROXIMATION FORMULAE

Let us begin by giving a simple example illustrating the fact that the Van Vleck formula K_{VV} in (11) is not always equal to the WKB approximation K_{WKB}. Let $\underset{\sim}{H}$ be an operator such that $K_{VV} = K_{WKB}$. The operator

$$\underset{\sim}{H}' \equiv \underset{\sim}{F}^{-1}(\underset{\sim}{Q})\underset{\sim}{H}\underset{\sim}{F}(\underset{\sim}{Q}) \tag{17}$$

has the same H_c (and hence the same K_{VV}) as $\underset{\sim}{H}$. Yet its WKB approximation is not K_{VV} but

$$K'_{WKB} = F(q_a)F^{-1}(q_b)K_{VV}\,, \tag{17a}$$

as can be verified by direct substitution.

We now prove the theorem. Formula (10) will be proved by generalizing the proof given in ref. 1. The more general formula (12) will be proved by generalizing the method used in ref. 4, which involves path integrals. It is presented separately because it is more on the heuristic side.

Proof of first formula

In ref. 1, where we investigated the range of validity of Van Vleck's formula, we assumed that the WKB approximation was of the form:

$$K_{WKB} = a(\hbar)C \exp(iB/\hbar), \tag{18}$$

where C and B are real functions of q_b, t_b, q_a, and t_a, independent of $\hbar$, and found that

$$K_{WKB}^{-1}\left(\underset{\sim}{H} - i\hbar\frac{\partial}{\partial t_b}\right)K_{WKB} = \dot{B} + F(0)H_c(B', q_b, t_b) + \hbar\Big[-i\frac{\dot{C}}{C}$$
$$-\left(\frac{\partial^2}{\partial p_i\,\partial q_b^i}H_c(p, q_b, t_b)\right)_{p=B'}\left(F'(0) + \frac{i}{2}F(0)\right)$$
$$-\frac{iF(0)C_i'}{C}\left(\frac{\partial H_c}{\partial p_i}(p, q_b, t_b)\right)_{p=B'} - i\frac{F(0)}{2}B_{ij}''$$
$$\times\left(\frac{\partial^2 H_c}{\partial p_i\,\partial p_j}(p, q_b, t_b)\right)_{p=B'}\Big] + \hbar^2\Big\{\Big[-\frac{1}{8}F(0)\frac{\partial^2}{\partial q_b^i\,\partial q_b^j}$$
$$-F(0)\frac{C_i'}{2C}\frac{\partial}{\partial q_b^j} - F(0)\frac{C_{ij}''}{2C} - \frac{1}{4}F(0)B_{ij}''\frac{\partial^2}{\partial p_k\,\partial q_b^k}$$
$$-F(0)\frac{C_k'}{2C}B_{ij}''\frac{\partial}{\partial p_k} - \frac{1}{8}F(0)B_{ij}''B_{ks}''\frac{\partial^2}{\partial p_k\,\partial p_s}$$
$$-\frac{1}{6}F(0)B_{ijk}'''\frac{\partial}{\partial p_k} + \frac{i}{2}F'(0)\frac{\partial^2}{\partial q_b^i\,\partial q_b^j} + iF'(0)\frac{C_i'}{C}\frac{\partial}{\partial q_b^j}$$
$$+\frac{i}{2}F'(0)B_{ij}''\frac{\partial^2}{\partial p_k\,\partial q_b^k} + \frac{1}{2}F''(0)\frac{\partial^2}{\partial q_b^i\,\partial q_b^j}\Big]$$
$$\times\frac{\partial^2}{\partial p_i\,\partial p_j}H_c(p, q_b, t_b)\Big\}_{p=B'} + O(\hbar^3), \tag{19}$$

where $B_{ij}'' \equiv \partial^2 B/\partial q_b{}^i\,\partial q_b{}^j$, etc., and F is Cohen's F function[5],

establishing the correspondence between $\underset{\sim}{H}$ and H_c by

$$\underset{\sim}{H} = (2\pi\hbar)^{-2n} \int_{\mathbb{R}^{4n}} dp\, dq\, du\, dv\, F(u,v) H_c(q,p,t) \exp\left\{ (i/\hbar)\left[(q-\underset{\sim}{Q}).u + (p-\underset{\sim}{P}).v\right]\right\} . \qquad (20)$$

F effects the generating-function correspondence

$$F(u,v)\, \exp\left[(-i/\hbar)(\underset{\sim}{Q}u+\underset{\sim}{P}v)\right] \leftrightarrow \exp\left[(-i/\hbar)(qu+pv)\right] , \qquad (21)$$

and a set of Fs effecting a given correspondence $H_c \leftrightarrow \underset{\sim}{H}$ can be found by solving[6]

$$F(u,v) \int_{\mathbb{R}^{2n}} dp\, dq\, H_c(q,p,t) e^{i(qu+pv)/\hbar} = (2\pi\hbar)^n \operatorname{tr}\left(e^{i(\underset{\sim}{Q}u+\underset{\sim}{P}v)/\hbar} \underset{\sim}{H}\right) . \qquad (22)$$

This equation for F, to be understood in the sense of distribution theory, may or may not have a solution, and the solution may or may not be unique if it exists.

In ref. 1, F was assumed to be a function of $u.v/\hbar$ only, and this requirement is maintained here, as $u.v/\hbar$ is the only dimensionless quantity one can form with u, v, and $\hbar$, and it is assumed here that $\underset{\sim}{H}$ contains no constants (other than $\hbar$) that do not appear in H_c. It can be shown[7] that the fact that F depends on the combination $u.v/\hbar$ implies that the operator $\underset{\sim}{H}$ corresponding to $p^m q^n$ in 1 dimension is a linear combination of all the possible ordered arrangements of $\underset{\sim}{P}$ m times and $\underset{\sim}{Q}$ n times (true "factor ordering"). Thus, no true divisors are allowed in Cohen's scheme[8], an important restriction since the Laplacian in curved spaces contains true divisors. We shall return to this point later.

In ref. 1, F was also assumed to be real[9] so as to insure the Hermiticity of $\underset{\sim}{H}$. This requirement is dropped here. Therefore, equating both the real and the imaginary part of the constant term and the term proportional to $\hbar$ to 0 in (19) yields the following 4 equations to be satisfied by B and C:

$$H_c\left(q_b, \frac{\partial B}{\partial q_b}, t_b\right) \mathrm{Re}[F(0)] + \frac{\partial B}{\partial t_b} = 0 \tag{23}$$

$$H_c\left(q_b, \frac{\partial B}{\partial q_b}, t_b\right) \mathrm{Im}[F(0)] = 0 \tag{24}$$

$$\mathrm{Re}[F(0)] \frac{\partial}{\partial q_{bj}}\left[c^2 \frac{\partial H_c}{\partial p_j}(q_b, p, t_b)\Big|_{p=\partial B/\partial q_b}\right] \tag{25}$$

$$+ \frac{\partial}{\partial t_b}(c^2) + 2c^2 \frac{\partial^2 H_c}{\partial p_i \partial q^i}(q, p, t_b)\Big|_{\substack{p=\partial B/\partial q_b \\ q=q_b}} \mathrm{Im}[F'(0)] = 0$$

$$\left\{\mathrm{Re}[F'(0)] - \tfrac{1}{2}\mathrm{Im}[F(0)]\right\}\left[\frac{\partial^2}{\partial p_i \partial q^i} H_c(q, p, t_b)\Big|_{\substack{p=\partial B/\partial q_b \\ q=q_b}}\right. \tag{26}$$

$$\left. + \mathrm{Im}[F(0)]\left[\frac{C'_i}{C}\frac{\partial H_c}{\partial p_i}(q_b, p, t_b) + \tfrac{1}{2} B''_{ij}\frac{\partial^2 H_c}{\partial p_i \partial p_j}\right]_{p=\partial B/\partial q_b} = 0,\right.$$

where the third equation was rewritten in a more compact form.

First, let us translate the conditions (2) and (3) on $\underset{\sim}{H}$ into conditions on F. These are:

$$F(0) = 1, \quad \mathrm{Re}[F'(0)] = 0, \quad \mathrm{Im}[F'(0)] = a - \tfrac{1}{2}. \tag{27}$$

Indeed, (2) readily results from (21) with F(0) = 1. If we differentiate (21) with respect to v, then set v=0, then do the same with u, we obtain

$$pq \longleftrightarrow F(0)\,\underset{\sim}{(QP+PQ)}/2 - \tfrac{i}{2}F'(0), \tag{28}$$

which yields the 2 conditions on F'(0).

It is, in fact, possible to drop the requirement that the correspondence be given by an F function, so that the results are valid for any $\underset{\sim}{H}$ satisfying (2) and (3). This will be seen when we treat the more general formula (12).

Let us now solve (23)-(26) for B and C. Since F(0) = 1, (23) and (24) are recognized as being one of the Hamilton-Jacobi equations, yielding $B = S_c$. Note that since (23) and (24) stem from equating only the constant term to 0 in (19), an approximation to zeroth order in $\hbar$ yields $B = S_c$ and no information on C other than the fact that it must be such that the boundary condition (9) is satisfied (this is not sufficient to fix C).

To solve (25), we note that if $\mathrm{Im}[F'(0)] = 0$ $(a=\frac{1}{2})$ then $C^2 = \det M$, since the (continuity) equation satisfied by the Van-Vleck - Morette determinant is precisely

$$\frac{\partial}{\partial q_b^i}\left[\dot{q}_c^{\,j}(t_b)\,\det M\right] + \frac{\partial}{\partial t_b}\left[\det M\right] = 0. \tag{29}$$

This leads us to write

$$C^2 = N \det M, \tag{30}$$

which, when substituted in (25) and using $\mathrm{Im}[F'(0)] = a-\frac{1}{2}$ yields the equation for $N(q_b,t_b,q_a,t_a)$:

$$\frac{\partial N}{\partial t_b} + \dot{q}_c^{\,j}(t_b)\frac{\partial N}{\partial q_b^i} + N(2a-1)\left.\frac{\partial^2 H_c(q,p,t_b)}{\partial q^i \partial p_i}\right|_{\substack{q=q_b \\ p=p_c(t_b)}} = 0, \tag{31}$$

with $N \to 1$ as $t_b \to t_a$ and $q_b \to q_a$ if $K_{VV} \to \delta(q_b-q_a)$ as $t_b \to t_a$.

The search for a solution of (31) is facilitated by the observation that the first two terms form the convective derivative of N with respect to the final endpoint. Knowing that the convective derivative of any function f of position, momentum, and time (at the classical path) with respect to either the initial or the final endpoint is 0, i.e.,

$$\left[\frac{\partial}{\partial t_b} + \dot{q}_c^{\,j}(t_b)\frac{\partial}{\partial q_b^i}\right] f(q_c(t),p_c(t),t) = 0 \tag{32}$$

(to be proved shortly), we are led to a trial solution of the form

$$N = \exp\left[(1-2a) \int_{t_a}^{t_b} \frac{\partial^2 H_c}{\partial p_i \partial q^i}(q_c(t), p_c(t), t)\ dt \right]. \qquad (33)$$

By direct substitution in (31), (33) is seen to be the correct solution (use $f=q_c$ and $f=p_c$ in (32)).

To prove (32), we first observe that it is sufficient to prove it for $f=q_c$ and $f=p_c$: the chain rule will then extend its validity to any f. Now, derivatives of the classical solution with respect to any parameter introduced by the boundary conditions (here, t_a, t_b, q_a, or q_b) are known to be solutions of the equation of small disturbances, obtained from the second variation of the action functional[10-13]. Since this equation is linear, its general solution is a linear combination of 2n linearly independent solutions. Thus, the only solution vanishing at both t_a and t_b must be zero everywhere. Now, the left-hand-side of (32) with $f=q_c^i(t)$ is, by its very construction, a solution of the small-disturbance equation. It vanishes at t_a because $q_c(t_a) = q_a$ (a constant). It also vanishes at t_b because

$$\left.\frac{\partial q_c^i(t)}{\partial t_b}\right|_{t=t_b} = \int_{t_a}^{t_b} dt\ \frac{\partial \dot{q}_c^i(t)}{\partial t_b} = -\dot{q}_c^i(t_b) + \frac{\partial}{\partial t_b}\int_{t_a}^{t_b} \dot{q}_c^i(t)\,dt \qquad (34)$$

$$= -\dot{q}_c^i(t_b) + \frac{\partial}{\partial t_b}(q_b^i - q_a^i) = -\dot{q}_c^i(t_b),$$

(since $\partial/\partial t_b$ commutes with $\partial/\partial t$ when acting on $q_c(t, t_b, q_b)$). Thus, (32) is true for $f=q_c$. The case $f=p_c$ is proved by observing, by substitution in the equation of small disturbances in phase space ($\underset{\sim}{\sigma} h = 0$), that if u is a parameter introduced by the boundary conditions, then

$$\frac{\partial p_{ci}(t)}{\partial u} = \underset{\sim}{D}_{ij}(t)\,\frac{\partial q_c^j(t)}{\partial u}, \qquad (35)$$

where

$$\underset{\sim}{\mathcal{O}} \equiv -\begin{pmatrix} f & k+d/dt \\ \tilde{k} - d/dt & g \end{pmatrix}, \qquad h \equiv \begin{pmatrix} \partial q_c(t)/\partial u \\ \partial p_c(t)/\partial u \end{pmatrix} \tag{36}$$

$$\underset{\sim}{D} \equiv g^{-1}\left(\underset{\sim}{1}\frac{d}{dt} - \tilde{k}\right),$$

and f,g, and k are defined in (15). This completes the proof of the WKB approximation formula (10). ∎

Note that (19) indicates that when H_c is quadratic in both p and q, the term proportional to $\hbar^2$ is 0 because C is independent of q_a and q_b (S_c being quadratic in q_a and q_b), and higher-order terms are 0 because they involve third and higher derivatives of H_c. Thus, the WKB approximation is exact in that case. This goes beyond the well-known result because $\underset{\sim}{H}$ does not have to be Hermitian, so that the extra exponential term in (10) supplementing the Van Vleck formula is not constant.

Proof of second formula

Formula (12) will be proved using the path-integral approach, generalizing a method presented in Ref. 4. More details on this method and its extension to a WKB expansion of the propagator for arbitrary Hamiltonians will be presented elsewhere.

The propagator K can be written as a phase space path integral as follows:

$$K(q_b, t_b, q_a, t_a) = \int_{\mathcal{P}} \left[\frac{dp\,dq}{h^n}\right] \exp(iS/\hbar), \tag{37}$$

where $S \equiv \int_T [p\dot{q} - H(q,p,t)]\,dt$ is the action functional and $\mathcal{P}$ is the space of paths (q,p) such that $q(t_a) = q_a$ and $q(t_b) = q_b$. If S is expanded around the classical path (q_c, p_c), its first functional derivative vanishes by definition of the classical path and we obtain

$$K = e^{iS_c/\hbar} \int_{\mathcal{P}_o} \left[\frac{dx\,dy}{h^n}\right] e^{iS_c''(x,y)/\hbar}\, e^{-i\Omega_c(x,y)/\hbar}, \tag{38}$$

where $\mathcal{P}_o$ is the space of paths (x,y) such that $x(t_a) = x(t_b) = 0$, Ω_c contains the terms beyond the second functional

derivative, and the second functional derivative S_c'' is:

$$S_c''(x,y) = \int_T dt \Big[y_i(t)\dot{x}^i(t) - \tfrac{1}{2}g^{ij}(t)y_i(t)y_j(t) - \tfrac{1}{2}f_{ij}(t)x^i(t)x^j(t) - k^i{}_j(t)y_i(t)x^j(t)\Big] \qquad (39)$$

with f, g, and k in (15).

We can define a measure w_1 on $\mathcal{P}_o$, normalized to 1 and absorbing the second variation of S by:

$$dw(x,y) \equiv A_o^{-1}\left[\frac{dx\,dy}{h^n}\right]\exp(iS_c''(x,y)/\hbar), \qquad (40)$$

the normalization factor being

$$A_o \equiv \int_{\mathcal{P}_o}\left[\frac{dx\,dy}{h^n}\right]\exp(iS_c''(x,y)/\hbar). \qquad (41)$$

Now, it is observed that the S_c'' term in (39) is in the form of an action functional corresponding to the fictitious Hamiltonian

$$H_o(x,y \equiv \tfrac{1}{2}g^{ij}(t)y_iy_j + \tfrac{1}{2}f_{ij}(t)x^ix^j - k^i{}_j(t)y_ix^j. \qquad (42)$$

Hence, (37) indicates that A_o must be the propagator $K_o(q_b', t_b, q_a', t_a)$ corresponding to H_o, evaluated at $q_a' = q_b' = 0$. But for which Hamiltonian operator $\underset{\sim}{H}_o$? It makes sense[6] that it should be the operator derived from H_o using the same correspondence rule linking H_c and $\underset{\sim}{H}$, i.e., (14) with (16). This leaves us with

$$K = A_o e^{iS_c/\hbar}\int_{\mathcal{P}_o} e^{-i\Omega_c(x,y)/\hbar}\,dw(x,y). \qquad (43)$$

It can be shown in this general case, as was done before for special cases[11-13], that the expansion of the Ω_c term followed by the evaluation of the path integrals (the correspondence rule being taken into account) yields a series in $\hbar$,

$$K = A_o e^{iS_c/\hbar}(1 + \hbar K_1 + \hbar^2 K_2 + \ldots), \qquad (44)$$

which identifies the constant term as the WKB approximation. ■

Let us now retrieve formula (10) from this more general case. The operator $\underset{\sim}{H}_o$ in this case is

$$\underset{\sim}{H}_o \equiv \tfrac{1}{2}g^{ij}(t)\underset{\sim}{P}_i\underset{\sim}{P}_j + \tfrac{1}{2}f_{ij}(t)\underset{\sim}{Q}^i\underset{\sim}{Q}^j + k^i{}_j(t)\left[a\underset{\sim}{P}_i\underset{\sim}{Q}^j + (1-a)\underset{\sim}{Q}^j\underset{\sim}{P}_i\right], \quad (45)$$

which can be rewritten, using $\left[\underset{\sim}{Q}^i, \underset{\sim}{P}_j\right] = i\hbar\,\delta^i{}_j$, as

$$\underset{\sim}{H}_o = \underset{\sim}{H}_{oo} + i\hbar(\tfrac{1}{2}-a)k^i{}_i(t), \quad (46)$$

where

$$\underset{\sim}{H}_{oo} \equiv \tfrac{1}{2}g^{ij}(t)\underset{\sim}{P}_i\underset{\sim}{P}_j + \tfrac{1}{2}f_{ij}(t)\underset{\sim}{Q}^i\underset{\sim}{Q}^j + \tfrac{1}{2}k^i{}_j(t)(\underset{\sim}{P}_i\underset{\sim}{Q}^j + \underset{\sim}{Q}^j\underset{\sim}{P}_i). \quad (47)$$

Now, since $\underset{\sim}{H}_{oo}$ is quadratic and Hermitian, its propagator K_{oo} is given exactly by Van Vleck's formula, (11). In this case, however, the S_c in (11) is zero because $q_a' = q_b' = 0$. (In fact, in general $S_c = q_b'^{\,i}p_{co,i}(t_b) - q_a'^{\,i}p_{co,i}(t_a)$, where (q_{co}, p_{co}) is the classical solution for H_o). Further, the "det M" in (11) is the same as the "det M" for H_c because H_o and H_c share the same equation of small disturbances and M_{ij} is a boundary value of a specific solution of that equation[11-13]. Therefore, $K_{oo} = (2\pi i\hbar)^{-n/2}(\det M)^{\frac{1}{2}}$. Now, if $\underset{\sim}{H}_1 = \underset{\sim}{H}_2 + f(t)$, the propagators K_1 and K_2 are related by $K_1 = K_2 \exp(-(i/\hbar)\int_T f(s)ds)$. $\underset{\sim}{H}_o$ and $\underset{\sim}{H}_{oo}$ are related in this manner. Putting all these results together gives:

$$K_{WKB} = (2\pi i\hbar)^{-n/2}(\det M)^{\frac{1}{2}} \exp\left\{\frac{i}{\hbar}S_c + (\tfrac{1}{2}-a)\int_T k^i{}_i(s)ds\right\}, \quad (48)$$

which is formula (10). Note that this suggests that the normalization factor $(2\pi i)^{-n/2}$ is universal and independent of H_c[14].

Note in passing that it is not always easy to find out what $p_i q^j$ corresponds to given the $H_c \leftrightarrow \underset{\sim}{H}$ correspondence. Scaling tricks (replacing $\underset{\sim}{Q}$ by $\lambda\underset{\sim}{Q}$ in functions of $\underset{\sim}{Q}$, then differentiating with respect to λ and setting λ equal to 0) sometimes help.

SOME EXAMPLES

We begin with an example pointing out that formula (10) is not restricted to the correspondence rule being effected by an F-function. Consider the Hamiltonian

$$\underset{\sim}{H} = \tfrac{1}{2}g^{-\frac{1}{4}}(\underset{\sim}{Q})\left[\underset{\sim}{P}_i - A_i(\underset{\sim}{Q})\right] g^{\frac{1}{2}}(\underset{\sim}{Q})g^{ij}(\underset{\sim}{Q})\left[\underset{\sim}{P}_j - A_j(\underset{\sim}{Q})\right]g^{-\frac{1}{4}}(\underset{\sim}{Q}) + V(\underset{\sim}{Q}) \qquad (49)$$

corresponding to $H_c = \tfrac{1}{2}g^{ij}(q)[p_i - A_i(q)][p_j - A_j(q)] + V(q)$, where (50) $g_{ik}g^{kj} \equiv \delta_i{}^j$ and $g \equiv \det(g_{ij})$. Since it is Hermitian, $a=\frac{1}{2}$. There is no F in general because of the divisors[15]. Nevertheless, a direct substitution shows that Van Vleck's formula applies, and we get:

$$K_{VV}^{-1}\left(\underset{\sim}{H}_b - i\hbar\frac{\partial}{\partial t_b}\right)K_{VV} = \frac{\hbar^2}{4}g^{ij}\left(\Gamma^m_{mi}\frac{M_{,j}}{M} + \Gamma^l_{ij}\frac{M_{,l}}{M} + \frac{M_{,i}M_{,j}}{2M^2}\right.$$
$$\left. - \frac{M_{,ij}}{M} - \frac{1}{2}\Gamma^m_{mi}\Gamma^l_{lj} - \Gamma^l_{ij}\Gamma^m_{ml} + \Gamma^l_{lj,i}\right)(q_b)$$
$$= O(\hbar^2), \qquad (51)$$

where the following properties and definitions were used:

$$M \equiv \det M, \qquad \Gamma^i_{jk} \equiv \tfrac{1}{2}g^{ia}(g_{ja,k} + g_{ka,j} - g_{jk,a}),$$
$$g^{ij}{}_{,k} = -\Gamma^i_{mk}g^{mj} - \Gamma^j_{mk}g^{im}, \qquad g^{\alpha}{}_{,i} = 2\alpha g^{\alpha}\Gamma^l_{li}, \qquad (52)$$
$$\dot{q}_c{}^j(t_b) = g^{ij}(t_b)\left[p_{ci}(t_b) - A_i(q_b)\right], \qquad p_{ci}(t_b) = \partial S_c/\partial q_b{}^i,$$
$$\frac{\partial S_c}{\partial t_b} = -H_c\left(q_b, \frac{\partial S_c}{\partial q_b}\right),$$

as well as (29)[16].

On the other hand, (19) gives the following expression for the miss term:

$$K_{VV}^{-1}(\underset{\sim}{H}_b - i\hbar\,\partial/\partial t_b)K_{VV} = -\tfrac{1}{2}\hbar^2\left[g^{ij}{}_{,ij}(\tfrac{1}{4}F(0) - iF'(0) - F''(0))\right.$$
$$\left. + g^{ij}{}_{,j}(F(0) - 2iF'(0))\frac{C'_i}{C} + g^{ij}F(0)\frac{C''_{ij}}{C}\right], \qquad (53)$$

an expression which cannot be matched with (51) for any F, for $C = M^{\frac{1}{2}}$.

Consider now the Fokker-Planck equation of diffusion processes:

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial q^i \partial q^j}\left[D^{ij}(q,t)P\right] - \frac{\partial}{\partial q^i}\left[v^i(q,t)P\right], \qquad (54)$$

where D is the diffusion matrix and v the drift vector[17]. It formally corresponds to a Hamiltonian $\underset{\sim}{H} \equiv \underset{\sim}{P}_i \underset{\sim}{P}_j D^{ij}(\underset{\sim}{Q})/2i\hbar + \underset{\sim}{P}_i v^i(\underset{\sim}{Q})$ with classical Hamiltonian $H_c = p_i p_j D^{ij}/2i\hbar + p_i v^i$. Since the $\underset{\sim}{P}$ factors precede the $\underset{\sim}{Q}$ factors, a is simply equal to 1. Thus, formula (10) gives:

$$K_{WKB} = (2\pi i\hbar)^{-n/2}(\det M)^{\frac{1}{2}}\exp\left[\frac{i}{\hbar}S_c - (2i\hbar)^{-1}\int_T p_{ci}(t)D^{ij}{}_{,j}(q_c(t))dt - \tfrac{1}{2}\int_T v^i{}_{,i}(q_c(t))dt\right]. \qquad (55)$$

Note that the dynamical equation gives $p_{ci}(t) = i\hbar\dot{q}_c{}^j(t)(D^{-1})_{ji}(q_c(t))$, so that in one dimension part of the integral can be performed, yielding

$$K_{WKB} = K_{VV}\left[D(q_b)/D(q_a)\right]^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}\int_T ((vD'-v'D)/D)\,dt\right). \qquad (56)$$

In the case of the backwards equation ($\underset{\sim}{Q}$ precedes $\underset{\sim}{P}$, a=0), the factors ½ are replaced by -½ in (56) and inside the bracket of (55).

For constant diffusion parameter (D=1) and linear drift $v = -\gamma q$, one retrieves the well-known propagator[18]

$$K_{WKB} = \left[\frac{\gamma}{\pi(1-2e^{-2\gamma T})}\right]^{\gamma_2}\exp\left[-\frac{\gamma(q_b - q_a e^{-\gamma T})^2}{1-e^{-2\gamma T}}\right], \qquad (57)$$

which, $\underset{\sim}{H}$ being quadratic, is also exact (it satisfies (54) exactly).

It can be shown, by direct calculations, that the "miss factor" for (55) is exactly as given by (53) with $F(x) = \exp(ix/2)$.

Another interesting application is the "lognormal" process with Hamiltonian

$$\underset{\sim}{H} = \alpha \underset{\sim}{Q}^2 \underset{\sim}{P}^2 + \beta \underset{\sim}{QP} \tag{58}$$

in one dimension, useful in modeling population growth[19]. Using formula (10) on $H_c = \alpha q^2 p^2 + \beta qp$ one gets:

$$K_{WKB} = (q_b/q_a)^{1-2a} K_{VV} \exp\left[\beta(a-\tfrac{1}{2})T\right], \tag{59}$$

where[20]

$$K_{VV} = (2\pi i\hbar)^{-\frac{1}{2}} (2\alpha T q_a q_b)^{-\frac{1}{2}} \exp\left[\frac{i}{4\hbar\alpha T}\left(\log\frac{q_b}{q_a} - \beta T\right)^2\right]. \tag{60}$$

In our case, a=0. Note that for a=0 the exact propagator[21] is

$$K = K_{WKB} \exp(-i\hbar\alpha T/4), \tag{61}$$

so that the expansion of the exponential gives the terms of a WKB expansion of K, useful for checking general formulas.

Let us also mention the elements of the WKB approximation for Hamiltonians in one dimension of the form

$$H_c = k p^m q^n \tag{62}$$

where k is a constant. The classical equation of motion is $\ddot{q}_c = (n/m)\dot{q}_c^2/q_c$ and the Lagrangian is $L = (\dot{q}/q)^{\frac{m}{m-1}} (mk)^{\frac{-1}{m-1}} (1-m^{-1})$. For $m \neq n$ these elements are:

$$\begin{aligned}
&q_c(t) = A(t-t_o)^{\frac{m}{m-n}}, \quad p_c(t) = \left[k(m-n)\right]^{\frac{1}{m-1}} (t-t_o)^{\frac{n}{n-m}} A^{\frac{1-n}{m-1}}, \\
&A \equiv q_a \left(\frac{1-\gamma}{\gamma T}\right)^{\frac{m}{m-n}}, \quad t_o \equiv \frac{t_a - \gamma t_b}{1-\gamma}, \quad \gamma \equiv (q_a/q_b)^{\frac{m-n}{m}}, \\
&S_c = (m-1)(kT)^{\frac{-1}{m-1}} (m-n)^{\frac{m}{1-m}} \left(q_b^{\frac{m-n}{m}} - q_a^{\frac{m-n}{m}}\right)^{\frac{m}{m-1}}, \\
&M = \left[Tk(m-n)\right]^{\frac{1}{m-1}} \frac{m-n}{m(m-1)} (q_a q_b)^{-n/m} \left(q_b^{\frac{m-n}{m}} - q_a^{\frac{m-n}{m}}\right)^{\frac{2-m}{m-1}}.
\end{aligned} \tag{63}$$

For m = n, these elements are:

$$q_c(t) = \exp(At+B), \qquad q_c(t)p_c(t) = (A/km)^{\frac{1}{m-1}},$$

$$A \equiv T^{-1}\mathrm{Log}(q_b/q_a), \qquad B \equiv \mathrm{Log}\left[q_a(q_a/q_b)^{t_a/T}\right],$$

$$S_c = (mkT)^{-1/(m-1)}\,(1-m^{-1})\left[\mathrm{Log}(q_b/q_a)\right]^{m/(m-1)},$$

$$M = (mkT)^{\frac{-1}{m-1}}\left[(m-1)q_a q_b\right]^{-1}\left[\mathrm{Log}(q_a/q_b)\right]^{\frac{2-m}{m-1}}. \qquad (64)$$

In all cases,

$$K_{WKB} = K_{VV}\,(q_b/q_a)^{n(\frac{1}{2}-a)}. \qquad (65)$$

CONCLUSION

We have produced a formula to approximate the propagator corresponding to any system described by a function whose time evolution is given by a partial differential equation that is of first order in time. This approximation can be supplemented by correction terms that can be generated by path integrals, and this will be the subject of a follow-up study[22].

REFERENCES AND FOOTNOTES

1. Maurice M. Mizrahi, "On the semiclassical expansion in quantum mechanics, for arbitrary Hamiltonians", J. Math. Phys. 18 (1977), 786-90.

2. J. H. Van Vleck, "The correspondence principle in the statistical interpretation of quantum mechanics", Proc. Nat. Acad. Sci. (USA) 14(1928), 178-188.

3. Bryce S. DeWitt, "Dynamical Theory in curved spaces. I. A review of the classical and quantum action principles", Rev. Mod. Phys. 29 (1957), 377-97.

4. Maurice M. Mizrahi, "Phase space path integrals, without limiting procedure", J. Math. Phys. 19 (1978), 293-307, with forthcoming erratum.

5. Leon Cohen, "Generalized phase-space distribution functions", J. Math. Phys. 7 (1966), 781-6.

6. Maurice M. Mizrahi, "Correspondence rules and path integrals", in Feynman Path Integrals (Marseille 1978), Springer Verlag Lecture Notes in Physics 106 (1979), pp. 234-53. (Note that some of the pages were interchanged, as evidenced by the numbering of the equations).

7. I. W. Mayes, Ph.D. Thesis, The University of Manchester, 1971.

8. "True" divisors are those that cannot be eliminated by use of the commutation relation $\underset{\sim}{Q}\underset{\sim}{P}-\underset{\sim}{P}\underset{\sim}{Q} = i\hbar$. For example,

$$\underset{\sim}{H} = \underset{\sim}{Q}^{\frac{1}{2}}\underset{\sim}{P}\underset{\sim}{Q}\underset{\sim}{P}\underset{\sim}{Q}^{\frac{1}{2}}$$

does not contain true divisors because the commutation relation $\underset{\sim}{P}f(\underset{\sim}{Q}) - f(\underset{\sim}{Q})\underset{\sim}{P} = -i\hbar f'(\underset{\sim}{Q})$ enables one to rewrite $\underset{\sim}{H}$ as $\underset{\sim}{P}\underset{\sim}{Q}^2\underset{\sim}{P} - \frac{1}{4}\hbar^2$. We can then rewrite $-\hbar^2$ as $(\underset{\sim}{Q}\underset{\sim}{P}-\underset{\sim}{P}\underset{\sim}{Q})^2$ and conclude that $\underset{\sim}{H}$ is only an ordering of the factors of p^2q^2. However,

$$\underset{\sim}{H}' = \underset{\sim}{Q}^{\frac{1}{4}}\underset{\sim}{P}\underset{\sim}{Q}^{\frac{1}{2}}\underset{\sim}{P}\underset{\sim}{Q}^{\frac{1}{4}}$$

does contain true divisors, because one can rewrite it as $\underset{\sim}{P}\underset{\sim}{Q}\underset{\sim}{P} + \hbar^2\underset{\sim}{Q}^{-1}/16 = \underset{\sim}{P}\underset{\sim}{Q}\underset{\sim}{P} - (\underset{\sim}{Q}\underset{\sim}{P}-\underset{\sim}{P}\underset{\sim}{Q})^2\underset{\sim}{Q}^{-1}/16$, which cannot be obtained simply by ordering the factors of p^2q. Thus, $\underset{\sim}{H}'$ is not covered by one of Cohen's F functions, but $\underset{\sim}{H}$ is (any F(x) such that F(0) = 1, F'(0) = 0, F"(0) = 1/8 would do, for example $F(x) = \exp(x^2/16)$).

9. In fact, the "Conclusion" stated at the bottom of p. 788 in ref. 1 remains valid even if the requirement that F be real is dropped.

10. C. G. Jacobi, "On the theory of the calculus of variations and of differential equations", Crelle's Math. J. 17 (1837), 68-82, referred to in Oskar Bolza, Lectures on the Calculus of Variations (U. Chicago Press, Chicago, 1904; also Chelsea, New York, 1960, 1973).

11. Maurice M. Mizrahi, "An investigation of the Feynman path integral formulation of quantum mechanics", Ph.D. thesis, the University of Texas at Austin, August 1975.

12. Cécile DeWitt-Morette, "The semiclassical expansion", Ann. of Phys. (N.Y.) 97 (1976), 367-99.

13. Maurice M. Mizrahi, "The semiclassical expansion of the anharmonic oscillator propagator", J. Math. Phys. 20 (1979), 844-55.

14. Unlike what was suggested in ref. 1.

15. In fact, direct use of (22) indicates that the correspondence $\frac{1}{2}g^{ij}(q)p_ip_j \leftrightarrow \frac{1}{2}g^{\alpha}(\underset{\sim}{Q})\underset{\sim}{P}_i g^{-2\alpha}(\underset{\sim}{Q})g^{ij}(\underset{\sim}{Q})\underset{\sim}{P}_j g^{\alpha}(\underset{\sim}{Q}) + \hbar^2\varphi(\underset{\sim}{Q})$ is fulfilled by an F function iff $\varphi(q) + \eta(q) = k\,\xi(q)$, where k is a constant, $\xi(q) \equiv \frac{1}{2}g^{ij}{}_{,ij}$, and

$$\eta(q) \equiv \frac{1}{2}g^{ij}\Big[\left(2\alpha+\tfrac{1}{2}\right)^2\Gamma^{\ell}_{\ell i}\Gamma^{m}_{mj} + \left(2\alpha+\tfrac{1}{4}\right)\Gamma^{m}_{m\ell}\Gamma^{\ell}_{ij} + \frac{1}{2}\Gamma^{\ell}_{is}\Gamma^{s}_{\ell j} - \frac{1}{4}\Gamma^{\ell}_{ij,\ell} - \left(2\alpha+\tfrac{1}{4}\right)\Gamma^{m}_{mi,j}\Big].$$

If g^{ij} is such that the relation is satisfied, then any F such that $F(0) = 1$, $F'(0) = 0$, $F''(0) = k$ will do. Weyl's rule ($F=1$) is an example for $k=0$. In particular, there is no F in general for the case $\varphi = 0$, In one dimension, $\eta = k\xi$ for all g^{ij} of the form $g^{11}(q) = (aq+b)^c$, a and b being arbitrary constants and $c \equiv (\alpha + \frac{1}{4} - k)/(\alpha^2 + \alpha + \frac{1}{4} - k)$.

16. The result in (51) matches that of DeWitt in ref. 3 for small T because, using the expansion of M in his equation (6.38) we readily get

$$K_{VV}^{-1}(\underset{\sim}{H}_b - i\hbar\,\partial/\partial t_b)K_{VV} = \hbar^2\left[-R(q_b)/12 + o(t_b-t_a) + o(q_b-q_a)\right],$$

where $R \equiv g^{ij}\left(\Gamma^{\ell}_{\ell j,i} - \Gamma^{\ell}_{ij,\ell} + \Gamma^{\ell}_{mj}\Gamma^{m}_{i\ell} - \Gamma^{\ell}_{ij}\Gamma^{m}_{m\ell}\right)$ is the Ricci scalar curvature tensor. In this case, a small-time approximation is the same as a WKB approximation.

17. See, e.g , Robert Graham, "Covariant formulation of non-equilibrium statistical thermodynamics", Z. Physik B <u>26</u> (1977), 397-405 and references listed there.

18. See, e.g., F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw-Hill, 1965, p. 581.

19. See, e.g., L. M. Ricciardi, Diffusion processes and related topics in biology, Springer Verlag, 1977, p. 102, and B. J. Sheeks, "Some applications of path integration using the prodistribution method", Ph.D. thesis, The University of Texas at Austin, May 1980.

20. Note in passing that K_{VV} in (60) is the exact propagator for an ordering patterned after (49), i.e., $\underset{\sim}{H} = \alpha \underset{\sim}{Q}^{\frac{1}{2}}\underset{\sim}{P}\underset{\sim}{Q}\underset{\sim}{P}\underset{\sim}{Q}^{\frac{1}{2}} + \beta(QP+PQ)/2 = \alpha(\underset{\sim}{P}\underset{\sim}{Q}^{2}\underset{\sim}{P} - \tfrac{1}{4}\hbar^2) + \beta(QP+PQ)/2$, a rare case when the WKB approximation is exact for a non-quadratic $\underset{\sim}{H}$. Formula (53) for $F(0) = 1$, $F'(0) = 0$ and $F''(0) = 1/8$ will confirm this.

21. Exact propagators for Hamiltonians of the form $\underset{\sim}{H} = \sum_{i=1}^{N} a_i \underset{\sim}{Q}^i \underset{\sim}{P}^i$ in one dimension can sometimes be found using the identity

$$\left[\sum_{i=1}^{N} a_i x^i \left(\frac{d}{dx}\right)^i\right]^m x^n = \left[\sum_{i=1}^{N} a_i \frac{n!}{(n-i)!}\theta(n-i)\right]^m x^n,$$

where $\theta(x) = 0$ for $x < 0$ and 1 otherwise. Thus, for $T \equiv \sum_{i=1}^{N} a_i x^i (d/dx)^i$ one obtains

$$e^T f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \exp\left[\sum_{i=1}^{N} a_i \frac{n!\,\theta(n-i)}{(n-i)!}\right].$$

For $N=2$, use of the integral representation

$e^{sn^2} = (4\pi s)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-x^2/4s + nx}\, dx$ gives

$e^T f(x) = (4\pi a_2)^{-\frac{1}{2}} \int_{\mathbb{R}} dy\, e^{-y^2/4a_2} f(xe^{y+a_1-a_2})$. Changing variables to $u = x\exp(y+a_1-a_2)$ gives

$e^T \delta(x-x') = (x')^{-1}(4\pi a_2)^{-\frac{1}{2}} \exp(-(\mathrm{Log}(x'/x) + a_2 - a_1)^2/4a_2)$. Since, it will be recalled, the propagator K is equal to $e^{-iT\underset{\sim}{H}/\hbar}\,\delta(q_b - q_a)$ when $\underset{\sim}{H}$ is time-independent, the method can be applied here.

22. A recent study of the correction terms in the case of the Hamiltonian in (49) can be found in F. Langouche, D. Roekaerts and E. Tirapegui, "Higher-order corrections through functional integrals to the WKB expansion in curved spaces", preprint KUL-TF-79/008, 1979.

CNA Professional Papers – 1976 to Present*

PP 141
Mizrahi, Maurice M., "Generalized Hermite Polynomials,"* 5 pp., Feb 1976 (Reprinted from the Journal of Computational and Applied Mathematics, Vol. 1, No. 4 (1975), 273-277).
*Research supported by the National Science Foundation

PP 143
Horowitz, Stanley and Sherman, Allan (LCdr., USN), "Maintenance Personnel Effectiveness in the Navy," 33 pp., Jan 1976 (Presented at the RAND Conference on Defense Manpower, Feb 1976) AD A021 581

PP 144
Durch, William J., "The Navy of the Republic of China – History, Problems, and Prospects," 66 pp., Aug 1976 (Published in "A Guide to Asiatic Fleets," ed. by Barry M. Blechman and Robert Berman, Naval Institute Press) AD A030 460

PP 145
Kelly, Anne M., "Port Visits and the "Internationalist Mission" of the Soviet Navy," 36 pp., Apr 1976, AD A023 436

PP 147
Kessler, J. Christian, "Legal Issues in Protecting Offshore Structures," 33 pp., Jun 1976 (Prepared under task order N00014-68-A-0091-0023 for ONR) AD A028 389

PP 149
Squires, Michael L., "Counterforce Effectiveness: A Comparison of the Tsipis "K" Measure and a Computer Simulation," 24 pp., Mar 1976 (Presented at the International Study Association Meetings, 27 Feb 1976) AD A022 591

PP 150
Kelly, Anne M. and Petersen, Charles, "Recent Changes in Soviet Naval Policy: Prospects for Arms Limitations in the Mediterranean and Indian Ocean," 28 pp., Apr 1976, AD A 023 723

PP 151
Horowitz, Stanley A., "The Economic Consequences of Political Philosophy," 8 pp., Apr 1976 (Reprinted from Economic Inquiry, Vol. XIV, No. 1, Mar 1976)

PP 152
Mizrahi, Maurice M., "On Path Integral Solutions of the Schrodinger Equation, Without Limiting Procedure,"* 10 pp., Apr 1976 (Reprinted from Journal of Mathematical Physics, Vol. 17, No. 4 (Apr 1976), 566-576).
*Research supported by the National Science Foundation

PP 153
Mizrahi, Maurice M., "WKB Expansions by Path Integrals, With Applications to the Anharmonic Oscillator,"* 137 pp., May 1976, AD A025 440
*Research supported by the National Science Foundation

PP 154
Mizrahi, Maurice M., "On the Semi-Classical Expansion in Quantum Mechanics for Arbitrary Hamiltonians," 19 pp., May 1976 (Published in Journal of Mathematical Physics, Vol. 18, No. 4, pp. 789-790, Apr 1977), AD A025 441

PP 155
Squires, Michael L., "Soviet Foreign Policy and Third World Nations," 26 pp., Jun 1976 (Prepared for presentation at the Midwest Political Science Association meetings, Apr 30, 1976) AD A028 388

PP 156
Stallings, William, "Approaches to Chinese Character Recognition," 12 pp., Jun 1976 (Reprinted from Pattern Recognition (Pergamon Press), Vol. 8, pp. 87-98, 1976) AD A028 692

PP 157
Morgan, William F., "Unemployment and the Pentagon Budget: Is There Anything in the Empty Pork Barrel?" 20 pp., Aug 1976 AD A030 455

PP 158
Haskell, LCdr. Richard D. (USN), "Experimental Validation of Probability Predictions," 25 pp., Aug 1976 (Presented at the Military Operations Research Society Meeting, Fall 1976) AD A030 458

PP 159
McConnell, James M., "The Gorshkov Articles, The New Gorshkov Book and Their Relation to Policy," 93 pp., Jul 1976 (Published in Soviet Naval Influence: Domestic and Foreign Dimensions, ed. by M. MccGwire and J. McDonnell. New York: Praeger, 1977) AD A029 227

PP 160
Wilson, Desmond P., Jr., "The U.S. Sixth Fleet and the Conventional Defense of Europe," 50 pp., Sep 1976, AD A030 457

PP 161
Melich, Michael E. and Peet, Vice Adm. Ray (USN, Retired), "Fleet Commanders: Afloat or Ashore?" 9 pp., Aug 1976 (Reprinted from U.S. Naval Institute Proceedings, Jun 1976) AD A030 456

PP 162
Friedheim, Robert L., "Parliamentary Diplomacy," 106 pp. Sep 1976 AD A033 306

PP 163
Lockman, Robert F., "A Model for Predicting Recruit Losses," 9 pp., Sep 1976 (Presented at the 84th annual convention of the American Psychological Association, Washington, D.C., 4 Sep 1976) (Published in Defense Manpower Policy (Richard V. L. Cooper, ed.), The Rand Corporation, 1979), AD A030 459

PP 164
Mahoney, Robert B., Jr., "An Assessment of Public and Elite Perceptions in France, The United Kingdom, and the Federal Republic of Germany," 31 pp., Feb 1977 (Presented at Conference "Perception of the U.S. – Soviet Balance and the Political Uses of *Military Power*" *sponsored by Director*, Advanced Research Projects Agency, April 1976) AD A036 599

PP 165
Jondrow, James M. "Effects of Trade Restrictions on Imports of Steel," 67 pp., November 1976, (Delivered at ILAB Conference in Dec 1976)

PP 166 – Revised
Feldman, Paul, "Why It's Difficult to Change Regulation," Oct 1976, AD A037 682

PP 167
Kleinman, Samuel, "ROTC Service Commitments: a Comment," 4 pp., Nov 1976, (Published in Public Choice, Vol. XXIV, Fall 1976) AD A033 305

PP 168
Lockman, Robert F., "Revalidation of CNA Support Personnel Selection Measures," 36 pp., Nov 1976

PP 169
Jacobson, Louis S., "Earnings Losses of Workers Displaced from Manufacturing Industries," 38 pp., Nov 1976, (Delivered at ILAB Conference in Dec 1976), AD A039 809

PP 170
Brechling, Frank P., "A Time Series Analysis of Labor Turnover," Nov 1976. (Delivered at ILAB Conference in Dec 1976)

PP 171
Jordan, A. S.* and Ralston, J. M., "A Diffusion Model for GaP Red LED Degradation," 10 pp., Nov 1976, (Published in Journal of Applied Physics, Vol. 47, pp. 4518-4527, Oct 1976)
*Bell Laboratories

PP 172
Classen, Kathleen P., "Unemployment Insurance and the Length of Unemployment," Dec 1976, (Presented at the University of Rochester Labor Workshop on 16 Nov 1976)

PP 173
Kleinman, Samuel D., "A Note on Racial Differences in the Added-Worker/Discouraged-Worker Controversy," 2 pp., Dec 1976, (Published in the American Economist, Vol. XX, No. 1, Spring 1976)

PP 174
Mahoney, Robert B., Jr., "A Comparison of the Brookings and International Incidents Projects," 12 pp. Feb 1977 AD A037 206

PP 175
Levine, Daniel; Stoloff, Peter and Spruill, Nancy, "Public Drug Treatment and Addict Crime," June 1976, (Published in Journal of Legal Studies, Vol 5, No. 2)

PP 176
Felix, Wendi, "Correlates of Retention and Promotion for USNA Graduates," 38 pp., Mar 1977, AD A039 040

PP 177
Lockman, Robert F. and Warner, John T., "Predicting Attrition: A Test of Alternative Approaches," 33 pp. Mar 1977. (Presented at the OSD/ONR Conference on Enlisted Attrition, Xerox International Training Center, Leesburg, Virginia, 4-7 April 1977), AD A039 047

PP 178
Kleinman, Samuel D., "An Evaluation of Navy Unrestricted Line Officer Accession Programs," 23 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 048

***CNA Professional Papers with an AD number may be obtained from the National Technical Information Service, U.S. Department of Commerce, Springfield, Virginia 22151. Other papers are available from the Management Information Office, Center for Naval Analyses, 2000 North Beauregard Street, Alexandria, Virginia 22311. An Index of Selected Publications is also available on request. The index includes a Listing of Professional Papers, with abstracts, issued from 1969 to February 1978.**

PP 179
Stoloff, Peter H. and Balut, Stephen J., "Vacate: A Model for Personnel Inventory Planning Under Changing Management Policy," 14 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 049

PP 180
Horowitz, Stanley A. and Sherman, Allan, "The Characteristics of Naval Personnel and Personnel Performance," 16 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 050

PP 181
Balut, Stephen J. and Stoloff, Peter, "An Inventory Planning Model for Navy Enlisted Personnel," 35 pp., May 1977 (Prepared for presentation at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Science, 9 May 1977, San Francisco, California), AD A042 221

PP 182
Murray, Russell, 2nd, "The Quest for the Perfect Study or My First 1138 Days at CNA," 57 pp., April 1977

PP 183
Kassing, David, "Changes in Soviet Naval Forces," 33 pp., November, 1976, (Published as part of Chapter 3, "General Purpose Forces: Navy and Marine Corps," in Arms, Men, and Military Budgets, Francis P. Hoeber and William Schneider, Jr. (eds.), (Crane, Russak & Company, Inc.: New York), 1977), AD A040 106

PP 184
Lockman, Robert F., "An Overview of the OSD/ONR Conference on First Term Enlisted Attrition," 22 pp., June 1977, (Presented to the 39th MORS Working Group on Manpower and Personnel Planning, Annapolis, Md., 28-30 Jun 1977), AD A043 618

PP 185
Kassing, David, "New Technology and Naval Forces in the South Atlantic," 22 pp. (This paper was the basis for a presentation made at the Institute for Foreign Policy Analyses, Cambridge, Mass., 28 April 1977), AD A043 619

PP 186
Mizrahi, Maurice M., "Phase Space Integrals, Without Limiting Procedure," 31 pp., May 1977, (Invited paper presented at the 1977 NATO Institute on Path Integrals and Their Application in Quantum Statistical, and Solid State, Physics, Antwerp, Belgium, July 17-30, 1977) (Published in Journal of Mathematical Physics 19(1) pp. 298-307, Jan 1978), AD A040 107

PP 187
Coile, Russell C., "Nomography for Operations Research," 35 pp., April 1977 (Presented at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Services, San Francisco, California, 9 May 1977), AD A043 620

PP 188
Durch, William J., "Information Processing and Outcome Forecasting for Multilateral Negotiations: Testing One Approach," 53 pp., May 1977 (Prepared for presentation to the 18th Annual Convention of the International Studies Association, Chase-Park Plaza Hotel, St. Louis, Missouri, March 16-20, 1977), AD A042 222

PP 189
Coile, Russell C., "Error Detection in Computerized Information Retrieval Data Bases," July, 1977, 13 pp. (Presented at the Sixth Cranfield International Conference on Mechanized Information Storage and Retrieval Systems, Cranfield Institute of Technology, Cranfield, Bedford, England, 26-29 July 1977), AD A043 580

PP 190
Mahoney, Robert B., Jr., "European Perceptions and East-West Competition," 96 pp., July 1977 (Prepared for presentation at the annual meeting of the International Studies Association, St. Louis, Mo., March, 1977), AD A043 661

PP 191
Sawyer, Ronald, "The Independent Field Assignment: One Man's View," August 1977, 25 pp.

PP 192
Holen, Arlene, "Effects of Unemployment Insurance Entitlement on Duration and Job Search Outcome," August 1977, 6 pp., (Reprinted from Industrial and Labor Relations Review, Vol., 30, No. 4, Jul 1977)

PP 193
Horowitz, Stanley A., "A Model of Unemployment Insurance and the Work Test," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

PP 194
Classen, Kathleen P., "The Effects of Unemployment Insurance on the Duration of Unemployment and Subsequent Earnings," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

PP 195
Brechling, Frank, "Unemployment Insurance Taxes and Labor Turnover: Summary of Theoretical Findings," 12 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

PP 196
Ralston, J. M. and Lorimor, O. G., "Degradation of Bulk Electroluminescent Efficiency in Zn, O-Doped GaP LED's," July 1977, 3 pp. (Reprinted from IEEE Transactions on Electron Devices, Vol. ED-24, No. 7, July 1977)

PP 197
Wells, Anthony R., "The Centre for Naval Analyses," 14 pp., Dec 1977, AD A049 107

PP 198
Classen, Kathleen P., "The Distributional Effects of Unemployment Insurance," 25 pp., Sept. 1977 (Presented at a Hoover Institution Conference on Income Distribution, Oct 7-8, 1977), AD A054 423

PP 199
Durch, William J., "Revolution From A F.A.R. – The Cuban Armed Forces in Africa and the Middle East," Sep 1977, 16 pp., AD A048 268

PP 200
Powers, Bruce F., "The United States Navy," 40 pp. Dec 1977 (Published as a chapter in The U.S. War Machine by Salamander Books, England,1978), AD A049 108

PP 201
Durch, William J., "The Cuban Military in Africa and The Middle East: From Algeria to Angola," Sep 1977, 67 pp., AD A045 675

PP 202
Feldman, Paul, "Why Regulation Doesn't Work," (Reprinted from Technological Change and Welfare in the Regulated Industries, Brookings Reprint 219, 1971, and Review of Social Economy, Vol. XXIX, March, 1971, No. 1.) Sep 1977, 8 pp.

PP 203
Feldman, Paul, "Efficiency, Distribution, and the Role of Government in a Market Economy," (Reprinted from The Journal of Political Economy, Vol. 79, No. 3, May/June 1971.) Sep 1977, 19 pp., AD A045 675

PP 204
Wells, Anthony R., "The 1967 June War: Soviet Naval Diplomacy and The Sixth Fleet – A Reappraisal," Oct 1977, 36 pp., AD A047 236

PP 205
Coile, Russell C., "A Bibliometric Examination of the Square Root Theory of Scientific Publication Productivity," (Presented at the annual meeting of the American Society for Information Science, Chicago, Illinois, 29 September 1977.) Oct 1977, 6 pp., AD A047 237

PP 206
McConnell, James M., "Strategy and Missions of the Soviet Navy in the Year 2000," 48 pp., Nov 1977 (Presented at a Conference on Problems of Sea Power as we Approach the 21st Century, sponsored by the American Enterprise Institute for Public Policy Research, 6 October 1977, and subsequently published in a collection of papers by the Institute), AD A047 244

PP 207
Goldberg, Lawrence, "Cost-Effectiveness of Potential Federal Policies Affecting Research & Development Expenditures in the Auto, Steel and Food Industries," 36 pp., Oct 1977, (Presented at Southern Economic Association Meetings beginning 2 November 1977), AD A046 269

PP 208
Roberts, Stephen S., "The Decline of the Overseas Station Fleets: The United States Asiatic Fleet and the Shanghai Crisis, 1932," 18 pp., Nov 1977 (Reprinted from The American Neptune, Vol. XXXVII, No. 3, July 1977), AD A047 245

PP 209 – Classified.

PP 210
Kassing, David, "Protecting The Fleet," 40 pp., Dec 1977 (Prepared for the American Enterprise Institute Conference on Problems of Sea Power as We Approach the 21st Century, October 6-7, 1977), AD A049 109

PP 211
Mizrahi, Maurice M., "On Approximating the Circular Coverage Function," 14 pp., Feb 1978, AD A054 429

PP 212
Mangel, Marc, "On Singular Characteristic Initial Value Problems with Unique Solutions," 20 pp., Jun 1978, AD A058 535

PP 213
Mangel, Marc, "Fluctuations in Systems with Multiple Steady States. Application to Lanchester Equations," 12 pp., Feb 78, (Presented at the First Annual Workshop on the Information Linkage Between Applied Mathematics and Industry, Naval PG School, Feb 23-25, 1978), AD A071 472

PP 214
Weinland, Robert G., "A Somewhat Different View of The Optimal Naval Posture," 37 pp., Jun 1978 (Presented at the 1976 Convention of the American Political Science Association (APSA/IUS Panel on "Changing Strategic Requirements and Military Posture"), Chicago, Ill., September 2, 1976), AD A056 228

PP 215
Coile, Russell C., "Comments on: Principles of Information Retrieval by Manfred Kochen," 10 pp., Mar 78, (Published as a Letter to the Editor, Journal of Documentation, Vol. 31, No. 4, pages 298-301, December 1975), AD A054 426

PP 216
Coile, Russell C., "Lotka's Frequency Distribution of Scientific Productivity," 18 pp., Feb 1978, (Published in the Journal of the American Society for Information Science, Vol. 28, No. 6, pp. 366-370, November 1977), AD A054 425

PP 217
Coile, Russell C., "Bibliometric Studies of Scientific Productivity," 17 pp., Mar 78, (Presented at the Annual meeting of the American Society for Information Science held in San Francisco, California, October 1976), AD A054 442

PP 218 – Classified.

PP 219
Huntzinger, R. LaVar, "Market Analysis with Rational Expectations: Theory and Estimation," 60 pp., Apr 78, AD A054 422

PP 220
Maurer, Donald E., "Diagonalization by Group Matrices," 26 pp., Apr 78, AD A054 443

PP 221
Weinland, Robert G., "Superpower Naval Diplomacy in the October 1973 Arab-Israeli War," 76 pp., Jun 1978 (Published in Seapower in the Mediterranean: Political Utility and Military Constraints, The Washington Papers No. 61, Beverly Hills and London: Sage Publications, 1979) AD A055 564

PP 222
Mizrahi, Maurice M., "Correspondence Rules and Path Integrals," 30 pp., Jun 1978 (Invited paper presented at the CNRS meeting on "Mathematical Problems in Feynman's Path Integrals," Marseille, France, May 22-26, 1978) (Published in Springer Verlag Lecture Notes in Physics, 106, (1979), 234-253) AD A055 536

PP 223
Mangel, Marc, "Stochastic Mechanics of Molecule-Ion Molecule Reactions," 21 pp., Jun 1978, AD A056 227

PP 224
Mangel, Marc, "Aggregation, Bifurcation, and Extinction In Exploited Animal Populations*," 48 pp., Mar 1978, AD A058 536
*Portions of this work were started at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B.C., Canada

PP 225
Mangel, Marc, "Oscillations, Fluctuations, and the Hopf Bifurcation*," 43 pp., Jun 1978, AD A058 537
*Portions of this work were completed at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada.

PP 226
Ralston, J. M. and J. W. Mann*, "Temperature and Current Dependence of Degradation in Red Emitting GaP LEDs," 34 pp., Jun 1978 (Published in Journal of Applied Physics, 50, 3630, May 1979) AD A058 538
*Bell Telephone Laboratories, Inc.

PP 227
Mangel, Marc, "Uniform Treatment of Fluctuations at Critical Points," 50 pp., May 1978, AD A058 539

PP 228
Mangel, Marc, "Relaxation at Critical Points: Deterministic and Stochastic Theory," 54 pp., Jun 1978, AD A058 540

PP 229
Mangel, Marc, "Diffusion Theory of Reaction Rates, I: Formulation and Einstein-Smoluchowski Approximation," 50 pp., Jan 1978, AD A058 541

PP 230
Mangel, Marc, "Diffusion Theory of Reaction Rates, II Ornstein-Uhlenbeck Approximation," 34 pp., Feb 1978, AD A058 542

PP 231
Wilson, Desmond P., Jr., "Naval Projection Forces: The Case for a Responsive MAF," Aug 1978, AD A058 543

PP 232
Jacobson, Louis, "Can Policy Changes Be Made Acceptable to Labor?" Aug 1978 (Submitted for publication in Industrial and Labor Relations Review), AD A061 528

PP 233
Jacobson, Louis, "An Alternative Explanation of the Cyclical Pattern of Quites," 23 pp., Sep 1978

PP 234 – Revised
Jondrow, James and Levy, Robert A., "Does Federal Expenditure Displace State and Local Expenditure: The Case of Construction Grants," 25 pp., Oct 1979, AD A061 529

PP 235
Mizrahi, Maurice M., "The Semiclassical Expansion of the Anharmonic-Oscillator Propagator," 41 pp., Oct 1978 (Published in Journal of Mathematical Physics 20 (1979), pp. 844-855), AD A061 538

PP 237
Maurer, Donald, "A Matrix Criterion for Normal Integral Bases," 10 pp., Jan 1979 (Published in the Illinois Journal of Mathematics, Vol. 22 (1978), pp. 672-681

PP 238
Utgoff, Kathleen Classen, "Unemployment Insurance and The Employment Rate," 20 pp., Oct 1978 (Presented at the Conference on Economic Indicators and Performance: The Current Dilemma Facing Government and Business Leaders, presented by Indiana University Graduate School of Business), AD A061 527

PP 239
Trost, R. P. and Warner, J. T., "The Effects of Military Occupational Training on Civilian Earnings: An Income Selectivity Approach," 38 pp., Nov 1979, AD A077 831

PP 240
Powers, Bruce, "Goals of the Center for Naval Analyses," 13 pp., Dec 1978, AD A063 758

PP 241
Mangel, Marc, "Fluctuations at Chemical Instabilities," 24 pp., Dec 1978 (Published in Journal of Chemical Physics, Vol. 69, No 8, Oct 15, 1978), AD A063 787

PP 242
Simpson, William R., "The Analysis of Dynamically Interactive Systems (Air Combat by the Numbers)," 160 pp., Dec 1978, AD A063 760

PP 243
Simpson, William R., "A Probabilistic Formulation of Murphy Dynamics as Applied to the Analysis of Operational Research Problems," 18 pp., Dec 1978, AD A063 761

PP 244
Sherman, Allan and Horowitz, Stanley A., "Maintenance Costs of Complex Equipment," 20 pp., Dec 1978 (Published By The American Society of Naval Engineers, Naval Engineers Journal, Vol. 91, No. 6, Dec 1979) AD A071 473

PP 245
Simpson, William R., "The Accelerometer Methods of Obtaining Aircraft Performance from Flight Test Data (Dynamic Performance Testing)," 403 pp., Jun 1979, AD A075 226

PP 246
Brechling, Frank, "Layoffs and Unemployment Insurance," 35 pp., Feb 1979 (Presented at the NBER Conference on "Low Income Labor Markets," Chicago, Jun 1978)

PP 248
Thomas, James A., Jr., "The Transport Properties of Dilute Gases in Applied Fields," 183 pp., Mar 1979

PP 249
Glasser, Kenneth S., "A Secretary Problem with a Random Number of Choices," 23 pp., Mar 1979

PP 250
Mangel, Marc, "Modeling Fluctuations in Macroscopic Systems," 26 pp., Jun 1979

PP 251
Trost, Robert P., "The Estimation and Interpretation of Several Selectivity Models," 37 pp., Jun 1979, AD A075 941

PP 252
Nunn, Walter R., "Position Finding with Prior Knowledge of Covariance Parameters," 5 pp., Jun 1979 (Published in IEEE Transactions on Aerospace & Electronic Systems, Vol. AES-15, No. 3, March 1979

PP 253
Glasser, Kenneth S., "The d-Choice Secretary Problem," 32 pp., Jun 1979, AD A075 225

PP 254
Mangel, Marc and Quanbeck, David B., "Integration of a Bivariate Normal Over an Offset Circle," 14 pp., Jun 1979

PP 255 – Classified, AD B051 441L

PP 256
Maurer, Donald E., "Using Personnel Distribution Models," 27 pp., Feb 1980, AD A082 218

PP 257
Thaler, R., "Discounting and Fiscal Constraints Why Discounting is Always Right," 10 pp., Aug 1979, AD A075 224

PP 258
Mangel, Marc S. and Thomas, James A., Jr. "Analytical Methods in Search Theory," 86 pp., Nov 1979, AD A077 832

PP 259
Glass, David V.; Hsu, Ih-Ching; Nunn, Walter R. and Perin, David A., "A Class of Commutative Markov Matrices," 17 pp., Nov 1979, AD A077 833

PP 260
Mangel, Marc S. and Cope, Davis K., "Detection Rate and Sweep Width in Visual Search," 14 pp., Nov 1979, AD A077 834

PP 261
Vila, Carlos L.; Zvijac, David J. and Ross, John, "Franck-Condon Theory of Chemical Dynamics. VI. Angular Distributions of Reaction Products," 14 pp., Nov 1979 (Reprinted from Journal Chem. Phys. 70(12), 15 Jun 1979), AD A076 287

PP 262
Petersen, Charles C., "Third World Military Elites in Soviet Perspective," 50 pp., Nov 1979, AD A077 835

PP 263
Robinson, Kathy I., "Using Commercial Tankers and Containerships for Navy Underway Replenishment," 25 pp., Nov 1979, AD A077 836

PP 264
Weinland, Robert G., "The U.S. Navy in the Pacific: Past, Present, and Glimpses of the Future," 31 pp., Nov 1979 (Delivered at the International Symposium on the Sea, sponsored by the International Institute for Strategic Studies, The Brookings Institution and the Yomiuri Shimbun, Tokyo, 16-20 Oct 1978) AD A066 837

PP 265
Weinland, Robert G., "War and Peace in the North: Some Political Implications of the Changing Military Situation in Northern Europe," 18 pp., Nov 1979 (Prepared for presentation to the "Conference of the Nordic Balance in Perspective: The Changing Military and Political Situation," Center for Strategic and International Studies, Georgetown University, Jun 15-16, 1978) AD A077 838

PP 266
Utgoff, Kathy Classen, and Brechling, Frank, "Taxes and Inflation," 25 pp., Nov 1979, AD A081 194

PP 267
Trost, Robert P., and Vogel, Robert C., "The Response of State Government Receipts to Economic Fluctuations and the Allocation of Counter-Cyclical Revenue Sharing Grants," 12 pp., Dec 1979 (Reprinted from the Review of Economics and Statistics, Vol. LXI, No. 3, August 1979)

PP 268
Thomason, James S., "Seaport Dependence and Inter-State Cooperation: The Case of Sub-Saharan Africa," 141 pp., Jan 1980, AD A081 193

PP 269
Weiss, Kenneth G., "The Soviet Involvement in the Ogaden War," 42 pp., Jan 1980 (Presented at the Southern Conference on Slavic Studies in October, 1979), AD A082 219

PP 270
Remnek, Richard, "Soviet Policy in the Horn of Africa: The Decision to Intervene," 52 pp., Jan 1980 (To be published in "The Soviet Union in the Third World: Success or Failure," ed. by Robert H. Donaldson, Westview Press, Boulder, Co., Summer 1980), AD A081 195

PP 271
McConnell, James, "Soviet and American Strategic Doctrines: One More Time," 43 pp., Jan 1980, AD A081 192

PP 272
Weiss, Kenneth G., "The Azores in Diplomacy and Strategy, 1940-1945, 46 pp., Mar 1980, AD A085 094

PP 273
Nakada, Michael K., "Labor Supply of Wives with Husbands Employed Either Full Time or Part Time," 39 pp., Mar 1980, AD A082 220

PP 275
Goldberg, Lawrence, "Recruiters Advertising and Navy Enlistments," 34 pp., Mar 1980, AD A082 221

PP 276
Goldberg, Lawrence, "Delaying an Overhaul and Ship's Equipment," 40 pp., May 1980, AD A085 095

PP 277
Mangel, Marc, "Small Fluctuations in Systems with Multiple Limit Cycles," 19 pp., Mar 1980 (Published in SIAM J. Appl. Math., Vol. 38, No. 1, Feb 1980)

PP 278
Mizrahi, Maurice, "A Targeting Problem: Exact vs. Expected-Value Approaches," 23 pp., Apr 1980, AD A085 096

PP 279
Walt, Stephen M., "Causal Inferences and the Use of Force: A Critique of Force Without War," 50 pp., May 1980, AD A085 097

PP 280
Goldberg, Lawrence, "Estimation of the Effects of A Ship's Steaming on the Failure Rate of its Equipment: An Application of Econometric Analysis," 25 pp., April 1980, AD A085 098

PP 283
Dismukes, Bradford, "Expected Demand for the U.S. Navy to Serve as An Instrument of U.S. Foreign Policy: Thinking About Political and Military Environmental Factors," 30 pp., April 1980, AD A085 099

PP 284
J. Keilson,* W. Nunn, and U. Sumita,** "The Laguerre Transform," 119 pp., May 1980
*The Graduate School of Management, University of Rochester and the Center for Naval Analyses
**The Graduate School of Management, University of Rochester, AD A085 100

PP 285
Remnek, Richard B., "Superpower Security Interests in the Indian Ocean Area," 26 pp., Jun 1980

PP 286
Mizrahi, Maurice M., "On the WKB Approximation to the Propagator for Arbitrary Hamiltonians," 25 pp., Aug 1980

PP 287
Cope, Davis, "Limit Cycle Solutions of Reaction Diffusion Equations," 325 pp., June 1980