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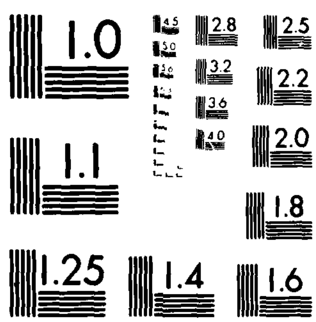

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

CONVERGENCE OF ITERATIVE NONEXPANSIVE SIGNAL RECONSTRUCTION ALGORITHMS

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ABSTRACT

Iterative algorithms for signal reconstruction from partial time- and frequency-domain knowledge have proven useful in a number of application areas. In this report, a general convergence proof, applicable to one class of such iterative reconstruction algorithms, is presented. The proof relies on the concept of a nonexpansive mapping in both the time and frequency domains.

Two examples studied in detail are timelimited extrapolation (equivalently, bandlimited extrapolation) and phase-only signal reconstruction. The proof of convergence for the phase-only iteration is a new result which illustrates this method of proof. The generality of the approach allows the incorporation of nonlinear constraints such as time- (or space-) domain positivity or minimum and maximum value constraints. Finally, the under-relaxed form of these iterations is also shown to converge when the solution is not guaranteed to be unique.

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CONTENTS

1.0 Introduction

Recently, considerable attention has been focused on a class of iterative signal reconstruction algorithms that assume partial knowledge of the signal in both the time and frequency domains. Perhaps the most widely known algorithm of this type is Gerchberg's algorithm [1] for bandlimited extrapolation, which has been considered by several authors including Papoulis [2], Cadzow [3], and Youla [4]. More recently, Quatieri and Oppenheim [5] and Hayes et. al [6] have investigated iterative procedures for reconstructing a signal from the phase of its Fourier transform. A related problem involves reconstruction from the magnitude of the Fourier transform [5,7,8].

The conditions under which such signal reconstruction problems have a unique answer are known. Often these solutions are closed form expressions in terms of the given partial knowledge, but they still may be computationally intractable. Thus, iterative solutions have been proposed to generate the reconstruction.

In this paper, we investigate the convergence of a particular set of iterative signal reconstruction algorithms. These iterative solutions involve repeated transformation between the time and frequency domains where, in each domain, the known information about the signal is incorporated into the current estimate of the desired signal. Although our discussion centers on two specific examples, timelimited extrapolation (and consequently, bandlimited extrapolation) and phase-only reconstruction, our approach is general and may be applied to other iterative algorithms that satisfy the same assumptions. Since the convergence of the iteration for bandlimited extrapolation has been demonstrated by others [2,3,4], the present paper offers an alternative approach into which nonlinear constraints, such as positivity [8,9] can be incorporated. The generality of our approach also yields the first proof of convergence for the phase-only reconstruction problem. Finally, it is straightforward to generalize our convergence proofs to multi-dimensional signals.

We begin in Section 2 with a brief review of the required mathematical notation and terminology and state three theorems that are relevant to our discussion of convergence. In Section 3, we define some specific mappings and establish the nonexpansive property of these mappings. In Section 4, we apply the results of the previous sections to demonstrate the convergence of the iterative solutions to the time-limited extrapolation and phase-only signal reconstruction problems. Finally, in Section 5, we discuss some extensions to the iterative algorithms presented in Section 4.

2.0 Mathematical Preliminaries

In this section, we first establish some notation and terminology related to mappings from one metric space into another*. We also define what is meant by a fixed point of a mapping and review a few results related to the

*We assume that the reader is familiar with the basic concepts and terminology of metric spaces. A detailed treatment of the material presented in this section may be found in [10,11].

existence and uniqueness of fixed points. We then define a special class of mappings which are referred to as nonexpansive mappings. Finally, three theorems are presented which address the existence and uniqueness of fixed points and the convergence of iterative solutions for finding a fixed point of a mapping.

2.1 Metric Spaces, Mappings and Fixed Points

A metric space consists of a set X along with a metric or distance function d defined on X. Frequently, we will simply refer to X as a metric space and assume that an underlying metric d has been defined on X. Throughout most of this paper, we will limit our discussions to the metric space R^N with the Euclidean metric

$$d(x,y) = \left\{ \sum_{n=0}^{N-1} [x(n) - y(n)]^2 \right\}^{1/2} \tag{1}$$

A point $x \epsilon R^N$ is an N-dimensional vector but we will also refer to it as an N-point sequence x(n). In this context, we consider that x(n) is defined for all n with x(n) equal to zero outside the interval [0,N-1].

A mapping F from a subspace A of a metric space X into X will be represented notationally as $F:A \subset X \rightarrow X$. If the image of A under F, F(A), is a subset of A, we say that F maps A into itself. A <u>fixed point</u> of a mapping

$F:A\subset X\to X$ is a point $x^*\varepsilon A$ which is invariant under F, i.e., $F(x^*)=x^*$. Since not all mappings have a fixed point (e.g., $F:R\to R$ with $F(x)=x+1$), conditions for the existence or uniqueness of fixed points are important in many practical applications. Whenever it is desired to numerically determine a fixed point of a mapping which is known to have at least one fixed point, an iterative procedure is often employed. A common iterative approach based on the method of successive approximation is defined by

$$x_{k+1} = F(x_k) \tag{2}$$

where x_k is the kth approximation to the fixed point x^* and x_0 is some initial estimate of x^*. Using the notation $F^k(x)=F(F^{k-1}(x))$, (2) may be expressed as

$$x_k = F^k(x_0) \tag{3}$$

Unfortunately, this iteration need not converge to a fixed point of F even if the fixed point is known to be unique. For example, although the mapping $F:R\to R$ defined by $F(x)=-x$ has a unique fixed point, $x^*=0$, the iteration defined by (2) will not converge unless $x_0=0$.

In order to guarantee that the iteration in (2) converges to a unique fixed point of F, constraints must be imposed on the mapping F and on the underlying metric spaces A and X. Since the convergence of the iteration (2) to a unique fixed point of F plays a central role in this paper, several theorems which address this issue of convergence are presented in the following section.

2.2 Contractions and Nonexpansive Mappings

There are many different types of constraints which may be imposed on a mapping $F:A\subset X\to X$ and on the metric spaces A and X to insure the existence or uniqueness of a fixed point of F or to guarantee the convergence of the iteration defined in (2). Perhaps the most familiar set of constraints is that contained in the Contraction Mapping Theorem. This and two other theorems are presented here in a general form, even though we will confine our subsequent discussions to R^N. First, we define what is meant by a contraction mapping.

Let A be a subset of a metric space X and let F be a mapping which maps A into itself. Then F is said to be a contraction mapping if there is a constant α, $0 \leq \alpha < 1$, such that for all $x,y\in A$.

$$d(Fx,Fy) \leq \alpha\, d(x,y) \tag{4}$$

We then have the following

THEOREM 1 [10, p. 120] (Contraction Mapping Theorem): If F is a contraction mapping on a closed subset A of a complete* metric space X, then there is a unique fixed point $x^*\varepsilon A$. Furthermore, the sequence $x_k = F^k(x_0)$ converges to x^* for any initial point $x_0 \varepsilon A$ and

$$d(x_n, x^*) \lesssim \frac{\alpha^n}{1-\alpha} d(x_1, x_0) \tag{5}$$

Although the Contraction Mapping Theorem is useful in many applications, not all iterations which converge to a unique fixed point are characterized by a contraction mapping.

A wider class of mappings results if we allow α to equal one in the definition of a contraction mapping. In this case, the mapping $F:A \subset X \to X$ is said to be nonexpansive if

$$d(Fx, Fy) \leq d(x,y) \tag{6}$$

for all $x,y\varepsilon A$. Unlike contractions, nonexpansive mappings may have any number of fixed points.

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* We point out that the metric space R^N is complete.

The mapping F is said to be <u>strictly nonexpansive</u> if strict inequality holds in (6) whenever $x \neq y$. Although it is straightforward to show that a strictly nonexpansive mapping has at most one fixed point, strict nonexpansiveness is not sufficient to guarantee that a fixed point exists. Nevertheless, if an additional constraint is imposed on the image of A under F, strictly nonexpansive mappings may be shown to have a unique fixed point. Specifically, we have the following theorem which will play an important role in Section 4.

<u>THEOREM 2</u> [10, p. 404]: Let $F: A \subset X \to X$ be a strictly nonexpansive mapping which maps a subspace A of a complete metric space X into itself. If the image of A under F is compact*, then F has a unique fixed point, $x^* \varepsilon A$. Furthermore, the sequence $x_k = F^k(x_0)$ converges to x^* for any $x_0 \varepsilon A$.

Note that the nonexpansiveness of F implies that F is continuous. Therefore, compactness of F(A) may be replaced by the stronger condition that A be compact.

Unfortunately, Theorem 2 does not hold if the nonexpansiveness of F is not strict. Therefore, the following theorem, which will be needed in Section 5, is useful since F is required only to be nonexpansive.

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* A subset of A of R^N is compact if and only if it is closed and bounded.

THEOREM 3 [10, p. 404]: If F is nonexpansive and maps a convex* compact subset A of the metric space R^N into itself, then F has a fixed point in A. Furthermore, for any $\omega \varepsilon (0,1)$ and $x_0 \varepsilon A$, the sequence

$$x_{k+1} = (1-\omega)x_k + \omega F(x_k) \tag{7}$$

converges to a fixed point of F in A.

(7) is often referred to as the relaxed form of the iteration given by (2) and ω is the relaxation parameter. When, in particular, $\omega \varepsilon (0,1)$ (7) represents the under-relaxed form of (2).

It should be pointed out that Theorem 3 guarantees the existence of a fixed point in A. This fixed point, however, need not be unique as illustrated by the mapping $F:[0,1]\to[0,1]$ defined by $F(x)=x$.

3.0 Time and Frequency Domain Mappings

Let h(n) be a sequence of length N which has an N-point Discrete Fourier Transform (DFT), H(k). When expressed in polar form, H(k) is given in terms of its magnitude and phase by

$$H(k) = |H(k)|\exp[j\Theta_h(k)] \tag{8}$$

* Convexity of A requires that if $x,y \varepsilon A$ and $\alpha \varepsilon [0,1]$, then $(1-\alpha)x+\alpha y \ \varepsilon \ A$.

In this section, we show that two mappings which incorporate partial information about h(n) in the time domain or about H(k) in the frequency domain are nonexpansive. In particular, we investigate a time domain mapping which substitutes known values of h(n) into an estimate x(n) of h(n) for certain indices n. Similarly, in the frequency domain, we examine mappings which either substitute $\Theta_h(k)$ or incorporate known values of H(k) into X(k). Another iterative procedure invokes the substitution of known values of $|H(k)|$ into X(k) [5,7,8], but these mappings do not have a nonexpansive property and, therefore, do not lie within the framework of this paper.

3.1 Notation and Framework

The mapping which incorporates known values of an N-point sequence h(n) into an arbitrary sequence x(n) will be denoted by T and is defined by

$$T[x(n)] = \begin{cases} x(n) & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{9}$$

where I_T is a subset of the interval [0,N-1] over which h(n) is known. Figure 1 illustrates the mapping in the case of timelimiting.

Likewise, in the frequency domain, the mapping which incorporates known values of H(k) into X(k) will be denoted by F_g. This mapping may be

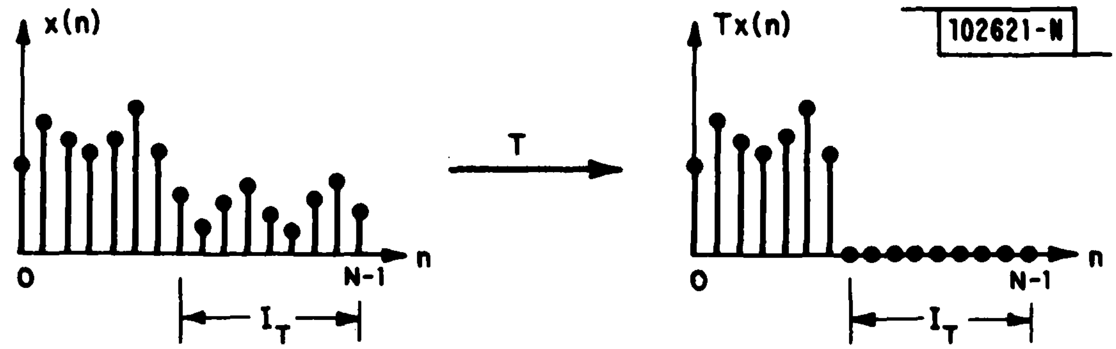

Fig. 1. Time-domain mapping T for the truncation case where $h(n)=0$ for $n \epsilon I_T$.

represented as $F_g = W^{-1}BW$ where W and W^{-1} denote the DFT and IDFT mappings respectively and

$$B[X(k)] = \begin{cases} X(k) & n \notin I_B \\ H(k) & n \in I_B \end{cases} \tag{10}$$

with I_B a subset of [0,N-1] over which H(k) is known. Figure 2 illustrates the mapping where constant magnitude replacement and linear phase replacement are made in a low-frequency region. Finally, the mapping which replaces the phase of X(k) with the known phase of H(k) is given by $F_p = W^{-1}\Phi W$ where

$$\Phi[X(k)] = |X(k)| \exp[j\Theta_h(k)] \tag{11}$$

3.2 Nonexpansive Properties

We now proceed to show that the mappings T, F_g and F_p defined in Section 3.1 are nonexpansive. To show that T in (9) is nonexpansive in the metric space R^N is straightforward. Specifically, for any $x,y \in R^N$,

$$d^2(x,y) = \sum_{n \in I_T} [x(n)-y(n)]^2 + \sum_{n \notin I_T} [x(n)-y(n)]^2 \tag{12}$$

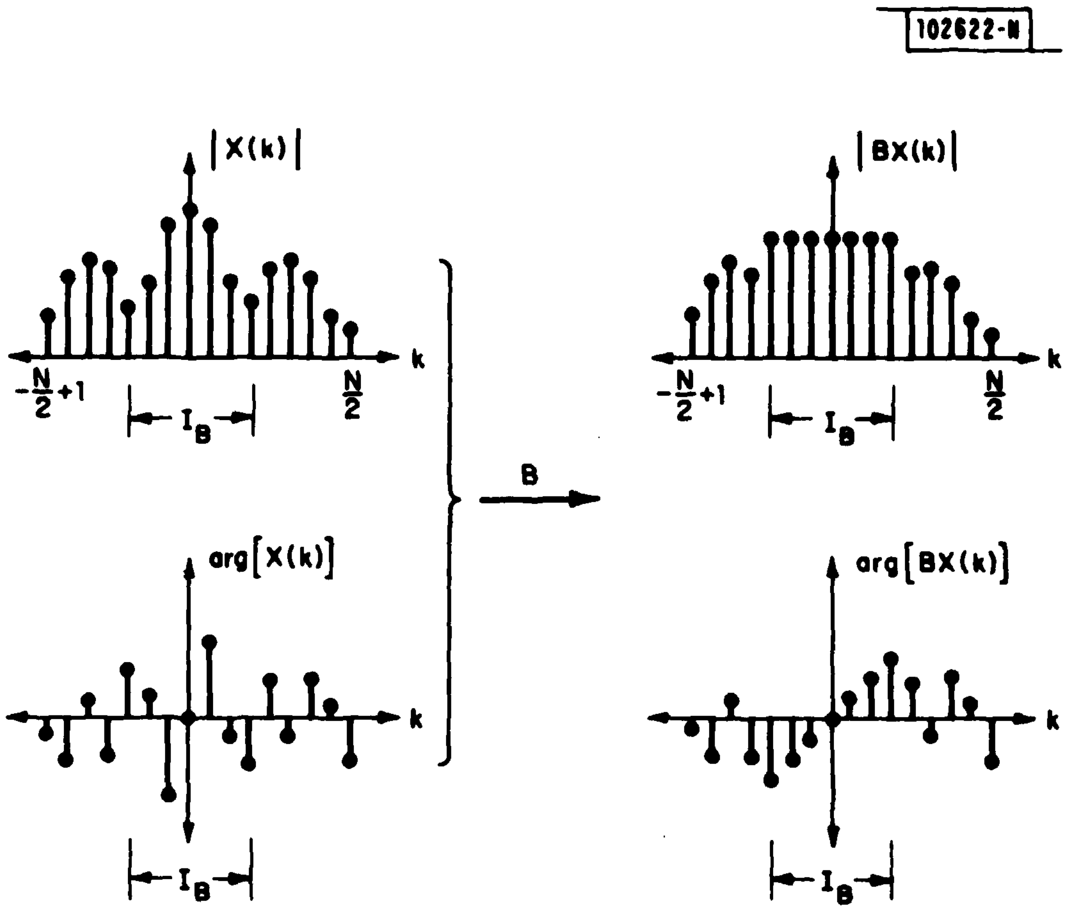

Fig. 2. Frequency-domain mapping B for constant magnitude and linear phase replacement in a low-frequency region I_B.

Thus,

$$d^2(x,y) \geq \sum_{n \notin I_T} [x(n)-y(n)]^2 \tag{13}$$

Since the right hand side of (13) equals $d^2(Tx,Ty)$, then

$$d(x,y) \geq d(Tx,Ty) \tag{14}$$

with equality if and only if $x(n)=y(n)$ for all $n \varepsilon I_T$.

The nonexpansiveness of the mapping B in (10) follows in a manner analogous to that for T. The discrete form of Parseval's Theorem may then be used to show that $F_g=W^{-1}BW$ is also nonexpansive. Therefore, since F_g is nonexpansive

$$d(x,y) \geq d(F_g x, F_g y) \tag{15}$$

with equality if and only if $X(k)=Y(k)$ for $k \varepsilon I_B$.

To show that the mapping F_p in (11) is nonexpansive, we proceed as follows:

$$d^2(X,Y) = \sum_{k=0}^{N-1} |X(k)-Y(k)|^2 \tag{16}$$

Using the triangle inequality for vector differences, (16) becomes

$$d^2(X,Y) \geq \sum_{k=0}^{N-1} \Big| |X(k)| - |Y(k)| \Big|^2 \tag{17}$$

Thus,

$$d^2(X,Y) \geq \sum_{k=0}^{N-1} \Big| |X(k)| e^{j\theta_h(k)} - |Y(k)| e^{j\theta_h(k)} \Big|^2 \tag{18}$$

Therefore,

$$d(X,Y) \geq d(\Phi X, \Phi Y) \tag{19}$$

with equality if and only if for each index k, $\Theta_x(k)=\Theta_y(k)$, or $|X(k)|=0$, or $|Y(k)|=0$. Thus, as was the case for F_g, since Φ is nonexpansive then so is F_p:

$$d(x,y) \geq d(F_p x, F_p y) \tag{20}$$

Finally, we note that the composition of two or more nonexpansive mappings is also nonexpansive. In particular, the compositions $G=TF_g$ and $P=TF_p$ are nonexpansive. A discussion of these mappings is the subject of the following section where we present sufficient conditions for the sequences $x_{k+1}=Gx_k$ and $x_{k+1}=Px_k$ to converge to a unique fixed point.

4.0 Iterative Solutions to Signal Reconstruction Problems

In this section, an iterative solution (2) for two specific signal reconstruction problems is examined. The first is the iteration defined by the composite mapping $G=TF_g$, i.e.,

$$x_{k+1} = Gx_k \tag{21}$$

where T and $F_g=W^{-1}BW$ are defined by (9) and (10), respectively. The resulting reconstruction algorithm is identical to the timelimited (bandlimited) extrapolation procedure described by Gerchberg [1] and Papoulis [2]. This algorithm seeks to determine the entire Fourier transform of a finite duration signal given knowledge of the spectrum only over a subset of the frequency domain.

The second iteration is defined by the composite mapping $P=TF_p$, i.e.,

$$x_{k+1} = Px_k \tag{22}$$

where T and $F_p = W^{-1}\Phi W$ are defined by (9) and (11), respectively. This iteration defines the phase-only algorithm for reconstructing a finite duration signal from the phase of its Fourier transform [5,13].

The objective of this section is to show that, under the appropriate set of conditions, both of these iterations converge to the desired sequence, h(n). First, we establish conditions which guarantee the uniqueness of the sequence which simultaneously satisfies the constraints imposed by the mappings of T and F_g or F_p. Under these conditions, we then show that both G and P are strictly nonexpansive and map a compact subset of R^N into itself. Finally, using Theorem 2 in Section 2.2, we show that the sequences (21) and (22) converge to the desired sequence, h(n).

4.1 Timelimited Extrapolation Iteration

There are applications such as diffraction-limited imaging for which it is desirable to determine the Fourier transform of a finite duration signal when only a segment of the transform is known. In order that this problem be well defined, however, we need to establish a set of conditions related to the uniqueness of the solution. Specifically, the following theorem can be deduced from the results in [12]:

THEOREM 4: Let h(n) be a sequence which is zero outside the interval [0,M-1]. Then h(n) is uniquely specified by M samples of its Fourier transform in the interval $[0,2\pi)$.

In particular, we assume frequency samples are derived from the DFT, and that only a segment of the corresponding Fourier transform is known in an interval I_B. To satisfy the sampling requirement of Theorem 4, the DFT then must be sufficiently long so that M samples of the DFT fall within this given interval. If, for example, the known segment is low-pass with cutoff frequency ω_c, there must exist M DFT samples in the region $[0,\omega_c] \cup [2\pi-\omega_c, 2\pi)$.

For any sequence which satisfies this finite duration constraint, there is a solution which expresses h(n) in terms of M samples of its Fourier transform. However, since this solution requires an MxM matrix inversion, an iterative solution may be preferable when M is large. One such iterative solution alternately imposes the time and frequency domain constraints at each step of the iteration. Mathematically, this iteration is expressed as

$$x_{k+1} = Gx_k = (TF_g)x_k \tag{23}$$

where F_g is defined by (10) with $I_B \subset [0,N-1]$ consisting of at least M points, and where T is the mapping (9) with $I_T = [M,N-1]$:

$$T[x(n)] = \begin{cases} x(n) & 0 \leq n < M \\ 0 & M \leq n < N \end{cases} \tag{24}$$

We now proceed to show that the sequence $x_k(n)$ in (23) converges to h(n) for any $x_0 \varepsilon R^N$.

Let S_ρ denote the closed sphere of radius ρ about h:

$$S_\rho = \{x \varepsilon R^N \mid d(x,h) \leq \rho\} \tag{25}$$

Since G is nonexpansive and Gh=h, then G maps S_ρ into itself. To see this, consider a point $x \varepsilon S_\rho$. Then

$$d(Gx,h) = d(Gx,Gh) \leq d(x,h) \leq \rho \tag{26}$$

so $Gx \varepsilon S_\rho$ and our claim follows. We note, in addition, that the continuity of G implies that $G(S_\rho)$ is compact.

Now consider the mapping $G: S_\rho \subset R^N \rightarrow R^N$ and recall from (14) and (15) that

$$d(Gx,Gy) = d(TF_g x, TF_g y) \leq d(F_g x, F_g y) \leq d(x,y) \tag{27}$$

We shall proceed to show that G is, in fact, strictly nonexpansive. Assume to the contrary that there exists $x,y \in S_\rho$ with $x \neq y$ such that $d(Gx,Gy) = d(x,y)$. It follows from (27) that

$$d(TF_g x, TF_g y) = d(F_g x, F_g y) = d(x,y) \tag{28}$$

The lefthand equality is true if and only if $F_g x = F_g y$ for $n \in I_T$. In addition, the DFTs of $F_g x$ and $F_g y$ are equal for $k \in I_B$.

Define now the sequence $z(n)$ by

$$z = F_g x - F_g y \tag{29}$$

The sequence $z(n)$ is of length M and its Fourier transform is zero over the set I_B which contains M points. Thus, Theorem 4 implies $z(n)=0$ for $0 \leq n \leq N-1$; and $d(F_g x, F_g y) = 0$. Consequently, from (28), $d(x,y) = 0$ which implies $x=y$, a contradiction. Therefore, G is strictly nonexpansive,

$$d(Gx,Gy) < d(x,y) \tag{30}$$

for all $x,y \in S_\rho$ when $x \neq y$.

Finally, we note that since G is strictly nonexpansive and maps a compact subset of R^N into itself, then from Theorem 2 the sequence (23) converges to h(n) for any $x_0 \varepsilon S_\rho$. Since the radius ρ of the sphere S_ρ was arbitrary, (23) converges for any $x_0 \varepsilon R^N$.

4.2 Phase-Only Iteration

Recently, several sets of conditions have been developed under which a sequence is uniquely specified, to within a scale factor, by the phase of its Fourier transform [6]. One such set of constraints is contained in the following

> THEOREM 5: Let h(n) be a real sequence which is zero outside the interval [0,M-1] with $h(0) \neq 0$. If the z-transform of h(n) has no zeroes in reciprocal pairs or on the unit circle, then h(n) is uniquely specified to within a scale factor by (M-1) samples of the phase (or tangent of the phase) of its Fourier transform in the open interval $(0,\pi)$.

A special case of this theorem arises when the phase samples of $H(\omega)$ are equally spaced around the unit circle and motivates an iterative technique for reconstructing h(n) from these phase samples. Specifically, we have the following Corollary [6] to Theorem 5:

Corollary: Let h(n) satisfy the constraints of Theorem 5 and denote its N-point DFT by

$$H(k) = |H(k)|\exp[j\theta_h(k)] \tag{31}$$

with $N \geq 2M$. If x(n) is any sequence which is zero outside the interval [0,M-1] with an N-point DFT of the form

$$X(k) = |X(k)|\exp[j\theta_h(k) + j\pi\alpha_k] \tag{32}$$

where $\alpha_k = 0, \pm 1$ for each k, then $x(n)=\beta h(n)$ for all n and some scalar β.

Motivated by this corollary, the following iterative technique has been proposed for reconstructing h(n) from $\theta_h(k)$:

$$x_{k+1} = Px_k = (TF_p)x_k \tag{33}$$

where $F_p = W^{-1}\Phi W$ is the mapping defined in (11) and T is the mapping (9) with $I_T = \{0\} \cup [M,N-1]$, i.e.,

$$T[x(n)] = \begin{cases} h(0) & n=0 \\ x(n) & 0 < n < M \\ 0 & M \leq n < N \end{cases} \tag{34}$$

We now show that the sequence (33) converges to h(n) for any $x_0 \varepsilon R^N$.

As in (25), let S_ρ denote the closed sphere of radius ρ about h. For reasons identical to those in (26), we note that since P is nonexpansive with Ph=h, then $P(S_\rho)$ is a compact subset of S_ρ. Now consider the mapping $P:S_\rho \subset R^N \to R^N$. From (14) and (19), we recall that

$$d(Px,Py) = d(TF_p x,TF_p y) \leq d(F_p x,F_p y) \leq d(x,y) \tag{35}$$

As before, we show that P is strictly nonexpansive by contradiction. Assume equality holds in (35) and that $x \neq y$. We then have

$$d(TF_p,TF_p y) = d(F_p x,F_p y) = d(x,y) \tag{36}$$

where the lefthand equality in (36) is true if and only if $F_p x = F_p y$ for $n \varepsilon I_T$. If we then define a sequence z(n) by

$$z = F_p x - F_p y \tag{37}$$

and note that z(n)=0 for $n \varepsilon I_T$ and that z(n) has an N-point DFT of the form (32), it then follows from the Corollary to Theorem 5 that $z(n)=\beta h(n)$ for all n and some scalar β. However, since z(0)=0 and $h(0)\neq 0$, $\beta=0$ and z(n)=0 for all n. Therefore, $d(F_p x,F_p y)=0$ and from (36), it follows that d(x,y)=0 and x=y, a contradiction. Consequently, P is strictly nonexpansive since

$$d(Px,Py) < d(x,y) \tag{38}$$

for all $x,y\epsilon S_\rho$ when $x\neq y$.

Again, Theorem 2 guarantees the convergence of the sequence (33) to $h(n)$ for any $x_0 \epsilon R^N$.

5.0 Extensions

In this section, we examine the issue of convergence when the iterative procedures (21) and (22) are modified to include a minimum or maximum value constraint in the time domain. We also briefly describe an adaptive relaxation technique which offers another convergent algorithm.

5.1 Minimum and Maximum Value Constraints

In some applications, the partial characterization of a sequence in the time domain includes a minimum or maximum value constraint. For example, a positivity (i.e., minimum value) constraint is frequently imposed in iterative procedures related to images [8] and spectra [9].

Within our framework, it is easy to show that the inclusion of a minimum or maximum value constraint into the nonexpansive mapping T in (9) does not alter the nonexpansive property of the mapping. Consider, for example, the case in which $h(n)$ is known to be positive for $n\not\epsilon I_T$, and let the mapping

T_0 be defined by

$$T_0[x(n)] = \begin{cases} |x(n)| & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{39}$$

In this case, we show that T_0 is nonexpansive as follows. Using the triangle inequality for vector differences, we have

$$d^2(x,y) \geq \sum_{n \notin I_T} [|x(n)|-|y(n)|]^2 = d^2(T_0x, T_0y) \tag{40}$$

It should be pointed out that the replacement of x(n) with $|x(n)|$ for $x \notin I_T$ in (39) is not the only one which preserves the nonexpansive property of T. For example, it is straightforward to show that the mapping

$$T^*_0[x(n)] = \begin{cases} \max[0,x(n)] & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{41}$$

is also nonexpansive.

A generalization of (39) and (41) to incorporate an arbitrary minimum value constraint in T also preserves the nonexpansive property. For example, suppose that h(n) is known to be greater than or equal to some number m for all $n \notin I_T$. Then the mappings

$$T_m[x(n)] = \begin{cases} m+|x(n)-m| & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{42}$$

and

$$T^*_m[x(n)] = \begin{cases} \max[m, x(n)] & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{43}$$

are both nonexpansive. This follows simply by noting that T_m in (42), for example, may be written as

$$T_m = S_m^{-1} T_0 S_m \tag{44}$$

where T_0 is the mapping (39) and where S_m and S_m^{-1} are defined by

$$S_m[x(n)] = x(n)-m \tag{45a}$$

$$S_m^{-1}[x(n)] = x(n)+m \tag{45b}$$

Since S_m and S_m^{-1} are nonexpansive mappings, (44) implies that T_m is also nonexpansive.

In a similar fashion, it follows that if a maximum value constraint is imposed in the mapping T, then the nonexpansive property of T is preserved. In addition, by recalling that the composition of nonexpansive mappings is also nonexpansive, we see that an arbitrary combination of minimum and maximum value constraints may be used. For example, if it is known that $M_1 \leq x(n) \leq M_2$ for $n \notin I_T$, then the mapping

$$T_{M_1,M_2}[x(n)] = \begin{cases} \min\{M_2, \max[M_1, x(n)]\} & n \notin I_T \\ h(n) & n \in I_T \end{cases} \tag{46}$$

is nonexpansive.

Finally, we note that if the minimum or maximum value constraints are consistent with the existence of a fixed point, the nonexpansive property of T is preserved and the iterations (20) and (21) still converge to the unique solution.

5.2 Fixed and Adaptive Relaxation

The motivation for examining convergence of the so-called under-relaxed form of the iteration given by (7) with $\omega \varepsilon (0,1)$ is two-fold. First, we would like to develop iterative procedures which converge in the absence of a unique solution. Secondly, this iterative formulation may possibly increase the rate of convergence of the iteration.

The iteration in (7) must converge if the conditions of Theorem 3 are met. Theorem 3 also implies that if a unique fixed point exists, the iteration will converge to that solution. The iterative procedures of section 4, in particular, satisfy the conditions of Theorem 3 and, in addition, have a unique fixed point. Thus, the under-relaxed forms of the iterations given by (21) and (22) yield a unique converging solution. In the case of multiple solutions, however, (7) will converge to any one of the possible solutions. Furthermore, the minimum or maximum value constraints of the previous section can be imposed with resulting convergence.

By modifying (7) slightly, an adaptive form of the relaxation technique has been realized. Specifically, the modified iteration becomes

$$x_{k+1} = (1-\omega_k)x_k + \omega_k F x_k \tag{47}$$

where ω_k is a relaxation scalar which is computed at each step of the iteration and which can be optimally chosen to minimize a residual error criterion.

For the case where a mean-square error is used, ω_k is relatively easy to derive, and the convergence characteristics of (47) are significantly improved [13]. Futhermore, if the optimum relaxation parameter is never allowed to take on values outside the interval (0,1), the convergence of (47) follows from a modified version of Theorem 3. A discussion of the more general case in which no restriction is placed on the range of values for ω_k may be found in [14].

6.0 Summary

In this report, we have demonstrated the convergence of two iterative procedures which involve repeated transformations between the time and frequency domains. The convergence proofs rely on the nonexpansive property of the mapping which defines the iteration as well as on the uniqueness of the desired signal. The generality of our approach is such that other iterative procedures and their under-relaxed forms which are characterized by nonexpansive mappings may similarly be shown to converge when the appropriate conditions are satisfied. For example, iterative deconvolution techniques studied by Mersereau and Schafer [9] and the iterative filtering procedure of Dudgeon [15] fall within our class of iterations.

We also considered the incorporation of a minimum or maximum value constraint into the iteration. With our approach, the convergence of this iteration was easily demonstrated. In a similar manner, other forms of constraints may be considered and, provided these constraints preserve the nonexpansive property of the mapping, convergence of the iteration follows.

When multiple solutions exist, the under-relaxed form of the iterations has been shown to converge. Finally, we also briefly examined the convergence of the under-relaxed formulation when the relaxation parameter is variable, and discussed the optimal choice of this parameter for increasing the rate of convergence.

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A convergence proof of a special class of iterative signal reconstruction problems is presented. The proof relies on the concept of nonexpansive mappings which impose partial knowledge of the unknown signal in both the time and frequency domains. Two examples studied in detail are timelimited extrapolation and phase-only reconstruction. The proof of convergence for the phase-only iteration is a new result. The generality of the approach allows the incorporation of nonlinear constraints such as positivity or minimum and maximum values. Finally, the under-relaxed form of these iterations is shown to converge when the solution is not necessarily unique.

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