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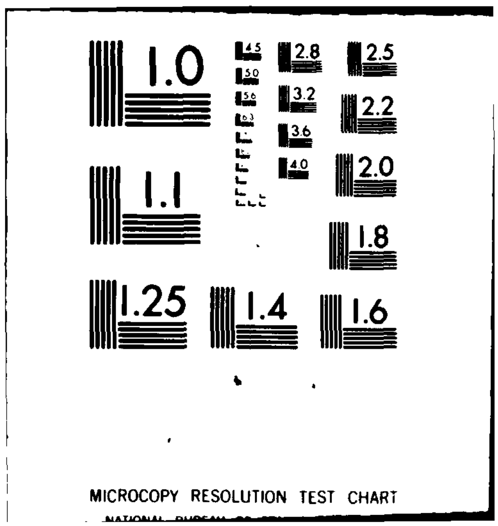
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CONVERGENCE OF NONLINEAR ELLIPTIC OPERATORS AND APPLICATION TO A QUASI VARIATIONAL INEQUALITY

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CONVERGENCE OF NONLINEAR ELLIPTIC OPERATORS AND APPLICATION TO A QUASI VARIATIONAL INEQUALITY

Maria Giovanna Garroni[1] and Jean-Pierre Gossez[2]

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ABSTRACT

This paper is composed of two parts. In the first part closedness and compactness results are given for a sequence of nonlinear elliptic operators of the form

$$Lu \equiv \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,u,\nabla u,\dots,\nabla^{m} u) ,$$

with monotone type assumptions ~~on the A_α's~~. These results are then used in the second part to derive existence theorems for a quasi variational inequality related to some questions from nonlinear heat flow. This quasi variational inequality involves a second order operator as above and an implicit obstacle of the Signorini type on the boundary.

AMS (MOS) Subject Classifications: 35J25, 35J40, 47H05, 47H17

Key Words: nonlinear elliptic boundary value problem, pseudo monotone operator, convergence of nonlinear operators, quasi variational inequality, conormal derivative, Signorini problem, nonlinear heat flow

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Quasi variational problems are characterized by the fact that the constraints are not given in advance. Typically, given a differential operator T acting on some function space V and a varying constraints set $Q(u) \subset V$, one asks for $u \in V$ satisfying

$$\begin{cases} u \in Q(u) , \\ (Tu, u - v) \leq 0 \quad \text{for all} \quad v \in Q(u) . \end{cases}$$

Variational inequalities correspond to $Q(u) \equiv Q$. Such quasi variational inequalities were introduced by Bensoussan and Lions for the study of some stochastic optimal control problems.

The quasi variational inequality considered in this paper is related to nonlinear heat flow. The constraints arise in the following way: the boundary temperature is required to remain at least equal to the exterior temperature, while the latter itself is influenced by the heat flux crossing the boundary. Existence theorems for stationary solutions are established under rather general nonlinear constitutive assumptions. They extend and sharpen previously known results relative to the linear case. One feature of this problem is the dependence of the constraints set on the derivatives of the temperature at the boundary. This precludes the use in the nonlinear case of the standard approach for solving quasi variational inequalities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

CONVERGENCE OF NONLINEAR ELLIPTIC OPERATORS AND APPPLICATION TO A QUASI VARIATIONAL INEQUALITY

Maria Giovanna Garroni[1] and Jean-Pierre Gossez[2]

0. INTRODUCTION

The purpose of this paper is to study the existence of solutions for a second order nonlinear elliptic equation with implicit Signorini type boundary conditions. The equation we consider is of the form

$$(0.1)\qquad Lu \equiv - \sum_{i=1}^{N} D^i A_i(x,u,\nabla u) + A_0(x,u,\nabla u) = f \quad \text{in} \quad \Omega ,$$

Ω a bounded open set of $\mathbb{R}^N$ with boundary Γ. The boundary conditions are the following:

$$(0.2)\qquad u \geq \Psi(u) \quad \text{on} \quad \Gamma ,$$

$$(0.3)\qquad \gamma_a u \geq 0 \quad \text{on} \quad \Gamma ,$$

$$(0.4)\qquad \gamma_a u \cdot (u - \Psi(u)) = 0 \quad \text{on} \quad \Gamma ,$$

where $\Psi(u)$, the obstacle on Γ, will be defined by means of an integro-differential operator Ψ on Γ. γ_a denotes the conormal derivative associated to L:

$$(0.5)\qquad \gamma_a u = \sum_{i=1}^{N} A_i(x,u,\nabla u)\nu_i ,$$

with ν_i the components of the unit exterior normal to Γ.

Equations and boundary conditions of this kind are related to some questions of nonlinear heat flow (see [21]). Consider a homogeneous rigid material Ω and let u

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denote the temperature inside Ω, s the heat production and q the heat flux. One wishes to keep u on Γ at least equal to some reference temperature h (e.g. the exterior temperature). For that purpose we assume that $q\cdot\nu$, the flux across Γ, vanishes whenever $u > h$ and is nonpositive whenever $u = h$. The first law of thermodynamics requires, for a stationary solution,

$$\operatorname{div} q = s \quad \text{in } \Omega .$$

Thus, for constitutive assumptions of the form

$$q_i = \sigma_i(u,\nabla u)$$

and $s = s_1(u,\nabla u) + s_2(x)$, we obtain a problem as (0.1)-(0.4) with $\Psi(u) \equiv h$ (the s_1 term here is partly for mathematical convenience; see section 2.5.e). Replacing now $h(x)$ above by an expression $\Psi(u)(x)$ which may depend on u or its derivatives means that one takes into account a possible variation of the reference temperature. This variation will be assumed to be proportional to the average flux $q\cdot\nu = -\gamma_a u$ across Γ:

(0.6) $$\Psi(u)(x) = h(x) - \int_\Gamma \gamma_a u(y)\varphi(y)d\Gamma_y$$

or more generally

(0.7) $$\Psi(u)(x) = h(x) - \int_\Gamma \gamma_a u(y)\varphi(x,y)d\Gamma_y ,$$

where h and φ are given on Γ. A dependence like (0.7) occurs for instance in the following situation: let Ω be surrounded by another material Ω_e and assume that Ω and Ω_e satisfy the Fourier law; then the exterior temperature on Γ is given by (0.7) with $\varphi(x,y)$ a Green's function associated to Ω_e. Similar problems may arise in fluid mechanics, when one deals with semi-permeable membranes (see [7,20]).

Existence results for problem (0.1)-(0.4), with Ψ of the form (0.6), L linear and f in $L^2(\Omega)$, were obtained by Joly-Mosco [12,20] when φ is ≥ 0, and by Boccardo-Dolcetta [2] when the norm of φ (in $H^{1/2}(\Gamma)$) is sufficiently small. Our purpose here is to study the nonlinear case, in particular that one corresponding in the above model to constitutive assumptions of the form

$$q_i = -K(|\nabla u|)\frac{\partial u}{\partial x_i}$$

with, for instance, $K(r) = r^{p-2}$, $1 < p < \infty$. The coefficients $A_i(x,u,\nabla u)$ and $A_0(x,u,\nabla u)$ of L will be assumed to verify either the usual (full) monotonicity conditions or conditions which are similar to but slightly stronger than the Leray-Lions conditions. These conditions involve among other things an exponent $1 < p < \infty$ (polynomial growth, coercivity, ...). We will prove the existence of solutions to problem (0.1)-(0.4), with Ψ of the form (0.6) and f in $L^{p'}(\Omega)$ (or more generally in a subspace $\Theta_{p'}(\Omega)$ of the order dual of $W_0^{1,p}(\Omega)$), when either $1 < p < 2$ or $p \geq 2$ and the negative part of φ has a sufficiently small norm (in $W^{1-1/p,p}(\Gamma)$). A similar result holds for an obstacle of the form (0.7).

Our general approach is classical in the theory of quasi variational inequalities in that the given problem is transformed into a fixed point equation via the resolution of an auxiliary variational inequality (the so-called variational selection). (For the theory and applications of quasi variational inequalities, see e.g. [1]). However a difficulty arises here due to the fact that $\Psi(u)$ explicitly contains the conormal derivative $\gamma_a u$ which, as is well-known, can only be defined via Green's formula under certain informations on Lu (see section 2.1). This difficulty is easily overcome in the linear case by working in the space

$$H_L^1(\Omega) = \{v \in H^1(\Omega);\ Lv \in L^2(\Omega)\}$$

(cf. [2,12,20]), but it is not clear how to adapt this method to the nonlinear case (for example, what are then the properties of the set corresponding to $H_L^1(\Omega)$?). To get around this difficulty, we consider the whole integral in $\Psi(u)$ as the parameter leading to the construction of the variational selection. (Another possibility is indicated in section 2.5.c, which is inspired by the variational formulation of the Neumann problems). The monotone case can then be treated rather simply. The problem is reduced to a fixed point equation in $\mathbf{R}$ when Ψ has the form (0.6), in $W^{1-1/p,p}(\Gamma)$ when Ψ has the form (0.7). In the nonmonotone case, in order to maintain a minimum of convexity, we are lead to replace in the auxiliary variational inequality the operator L by an operator L_w

obtained from L by freezing some of its terms (as e.g. in [2]). A second parameter w is thus introduced in the variational selection. To study then the dependence of the solutions of the auxiliary variational inequality with respect to w, we apply general closedness and compactness theorems relative to the convergence of a sequence of nonlinear elliptic operators of the form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,u,\nabla u,\dots,\nabla^{m} u) \;. \tag{0.8}$$

These theorems, which are proved in the first part of this paper and which seem to be of some interest in their own right, are somehow related to various recent results about stability, G-convergence, Γ-convergence, ... (see the references in [6]).

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The plan is as follows:

1. Closedness and compactness theorems
    - 1.1. Preliminaries
    - 1.2. Closedness theorems
    - 1.3. Compactness theorem
2. A quasi-variational inequality with obstacle on the boundary
    - 2.1. Conormal derivative
    - 2.2. Statement of the problem
    - 2.3. Nonmonotone case
    - 2.4. Monotone case
    - 2.5. Variations

1. CLOSEDNESS AND COMPACTNESS THEOREMS

1.1. PRELIMINARIES

Let V be a real reflexive Banach space. We denote by V' the dual of V, $\langle\ ,\ \rangle$ the pairing between V' and V, $\to$ (resp. $\rightharpoonup$) norm (resp. weak) convergence in V or V'.

DEFINITIONS 1.1. Let T_n, $n = 1,2,\dots$ and T be mappings from V to V'. We say that $T_n \xrightarrow{PM} T$ when (i) the T_n's are equibounded (i.e. $\bigcup_n T_n(B)$ is bounded in V' whenever B is bounded in V), (ii) for each sequence $k_n \to \infty$, $u_n \rightharpoonup u$ in V with $T_{k_n} u_n \rightharpoonup u'$ in V' and

$$\limsup \langle T_{k_n} u_n, u_n\rangle \leq \langle u', u\rangle \quad , \tag{1.1}$$

one has $Tu = u'$ and $\langle T_{k_n} u_n, u_n\rangle \to \langle u', u\rangle$. We say that $T_n \xrightarrow{S} T$ when (i) holds and for each sequence k_n, u_n as above, one has $Tu = u'$ and $u_n \to u$ in V.

These definitions are closely related to the notion of pseudo monotone homotopy which is used in the study of some strongly nonlinear problems (see [5,9,10]).

We recall that a sequence of sets $K_n \subset V$ is said to converge in the Mosco sense to a set $K \subset V$ (briefly $K_n \xrightarrow{M} K$) when

$$\text{s-}\liminf K_n = \text{w-}\limsup K_n = K \ ,$$

where

$$\text{s-}\liminf K_n = \{v \in V;\ \text{there exist } v_n \in K_n \text{ with } v_n \to v\} \ ,$$

$$\text{w-}\liminf K_n = \{v \in V;\ \text{there exist } k_n \to \infty \text{ and } v_n \in K_{k_n} \text{ with } v_n \rightharpoonup v\}$$

(see [19]).

One then has the following simple result concerning the convergence of solutions of variational inequalities (see e.g. [14]).

THEOREM 1.2. *Let* K_n *and* K *be closed convex sets in* V *with* $K_n \xrightarrow{M} K$. *Let* T_n *and* T *be mappings from* V *to* V' *with* $T_n \xrightarrow{PM} T$. *Let* $u'_n \to u'$ *in* V'. *Assume that* u_n *satisfies*

$$(1.2)\qquad \begin{cases} u_n \in K_n , \\ \langle T_n u_n, u_n - v\rangle \le \langle u_n', u_n - v\rangle \ \underline{\text{for all}} \ \ v \in K_n , \end{cases}$$

<u>and that</u> $u_n \rightharpoonup u$ <u>in</u> V. <u>Then</u> u <u>satisfies</u>

$$(1.3)\qquad \begin{cases} u \in K , \\ \langle Tu, u - v\rangle \le \langle u', u - v\rangle \ \underline{\text{for all}} \ \ v \in K , \end{cases}$$

<u>and</u> $\langle T_n u_n, u_n\rangle \to \langle Tu, u\rangle$. <u>If moreover</u> $T_n \xrightarrow{S} T$, <u>then</u> $u_n \to u$ <u>in</u> V.

PROOF. Passing to a subsequence, one can assume $T_n u_n \rightharpoonup u'$. Since $u \in K$, there exists $w_n \in K_n$ with $w_n \to u$; replacing in (1.2), we obtain

$$\lim\sup \langle T_n u_n, u_n\rangle \le \langle u', u\rangle ,$$

so that, by the convergence property of T_n, $Tu = u'$ and $\langle T_n u_n, u_n\rangle \to \langle Tu, u\rangle$. Let now $v \in K$. Taking $v_n \in K_n$ with $v_n \to v$ and replacing in (1.2), we get (1.3). Q.E.D.

1.2. <u>CLOSEDNESS THEOREMS</u>

We now give sufficient conditions for a sequence of mappings T_n associated with operators of the form (0.8) to converge in the above sense.

Consider, on a bounded open set Ω of $\mathbf{R}^N$ for which the Sobolev imbedding theorem holds, the operators

$$(1.4)\qquad L_n u \equiv \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha^n(x, u, \nabla u, \ldots, \nabla^m u), \qquad n = 1,2,\ldots$$

$$(1.5)\qquad Lu \equiv \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, \ldots, \nabla^m u) .$$

We will make the following assumptions, denoting as usual by $\zeta = (\zeta_\alpha)_{|\alpha|=m}$ (resp. $\eta = (\eta_\alpha)_{|\alpha|<m}$) the top (resp. lower) order part of a vector $\xi = (\xi_\alpha)_{|\alpha|\le m}$:

(1.6) each function $A_\alpha^n(x,\xi)$ and $A_\alpha(x,\xi)$ satisfies the Caratheodory conditions;

(1.7) there exist $1 < p < \infty$, $k_1(x) \in L^{p'}(\Omega)$ where $p' = p/(p-1)$ and a constant c_1 such that

$$|A_\alpha^n(x,\xi)| \le c_1|\xi|^{p-1} + k_1(x) ,$$

for a.e. x, all ξ, all α, n, and similarly for $A_\alpha(x,\xi)$, $|\alpha| \le m$;

(1.8) for a.e. x, all η, all n, one has

$$\sum_{|\alpha|=m} (A^n_\alpha(x,\eta,\zeta) - A^n_\alpha(x,\eta,\zeta'))(\zeta_\alpha - \zeta'_\alpha) > 0$$

for $\zeta \neq \zeta'$, and similarly for $A_\alpha(x,\eta,\zeta)$, $|\alpha| = m$;

(1.9) for a.e. x and all α, if $k_n \to \infty$ and $\xi_n \to \xi$, then $A^{k_n}_\alpha(x,\xi_n) \to A_\alpha(x,\xi)$.

In the case of a single operator, (1.6)-(1.8) are exactly the Leray-Lions conditions (see [16,17,4]) such as they were generalized recently by Landes [15]. Assumption (1.9) expresses the convergence of the coefficients of L_n to those of L.

Let V be a closed subspace of $W^{m,p}(\Omega)$ containing $W^{m,p}_0(\Omega)$, and define $T_n : V \to V'$ by the usual formula

$$\langle T_n u,v\rangle = a_n(u,v) \quad \text{for} \quad u,v \in V$$

where $a_n(u,v)$ is the Dirichlet form associated to L_n:

$$a_n(u,v) = \int_\Omega \sum_{|\alpha|\le m} A^n_\alpha(x,u,\nabla u,\ldots,\nabla^m u)D^\alpha v \ ;$$

one similarly has $T : V \to V'$ and $a(u,v)$ associated to L.

THEOREM 1.3. Assume (1.6)-(1.9). Then $T_n \xrightarrow{PM} T$.

The following additional condition yields a stronger conclusion:

(1.10) there exist $d_1 > 0$, $c_1 \ge 0$ and $\ell_1(x) \in L^1(\Omega)$ such that

$$\sum_{|\alpha|=m} A^n_\alpha(x,\eta,\zeta)\zeta_\alpha \ge d_1|\zeta|^p - c_1|\eta|^p - \ell_1(x)$$

for a.e. x, all η, ζ, all n, and similarly for $A_\alpha(x,\eta,\zeta)$, $|\alpha| = m$.

Note that this condition is implied by an inequality of the form

(1.11) $$\sum_{|\alpha|\le m} A^n_\alpha(x,\xi)\xi_\alpha \ge d_1|\xi|^p - \ell_1(x) \ .$$

THEOREM 1.4. Assume (1.6)-(1.10). Then $T_n \xrightarrow{S} T$.

Theorem 1.3 is, up to the use of [15], a particular case of theorem 5.1 of [9] which deals with a similar convergence problem (with a parameter $t \in [0,1]$ instead of $n = 1,2,\ldots$) in Orlicz-Sobolev spaces. In order to allow certain references and also to avoid to the reader the technicalities inherent to the situation considered in [9] (unbounded and non everywhere defined mappings, in nonreflexive spaces, ...), we will give below the main points of the proof. The result of theorem 1.4 is related to the notion of mapping of type (S_+) which was considered by F. E. Browder in some of his works (see e.g. [4]).

PROOF OF THEOREM 1.3. Let $u_n \rightharpoonup u$ in V, $k_n \to \infty$, $T_{k_n} u_n \rightharpoonup f$ in V' with

$$\limsup \langle T_{k_n} u_n, u_n \rangle \leq \langle f, u \rangle . \tag{1.12}$$

We must show that $Tu = f$ and $\langle T_{k_n} u_n, u_n \rangle \to \langle f, u \rangle$. For brevity, we will write T_n instead of T_{k_n}.

As $A^n_\alpha(\xi(u_n))$ remains bounded in $L^{p'}(\Omega)$, we can assume, passing to a subsequence, that $A^n_\alpha(\xi(u_n)) \rightharpoonup h_\alpha$ in $L^{p'}(\Omega)$; thus

$$\langle f, v \rangle = \int_\Omega \sum_{|\alpha| \leq m} h_\alpha(x) D^\alpha v \tag{1.13}$$

for all $v \in V$. We can also assume, passing to a further subsequence, that for $|\alpha| < m$, $D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$ and a.e. in Ω. We will show that this a.e. convergence also holds for $|\alpha| = m$. It then follows that $A^n_\alpha(\xi(u_n)) \to A_\alpha(\xi(u))$ a.e. for all α, so that, by lemma 1.5 below, $A_\alpha(\xi(u)) = h_\alpha$, and consequently, by (1.13), $Tu = f$.

We first note that

$$\limsup \int_\Omega \sum_{|\alpha| = m} (A^n_\alpha(\eta(u_n), \zeta(u_n)) - A^n_\alpha(\eta(u_n), \zeta(u)))(D^\alpha u_n - D^\alpha u) \leq 0 . \tag{1.14}$$

Indeed the integral in (1.14) is equal to

$$\langle T_n u_n, u_n \rangle - \int_\Omega \sum_{|\alpha| < m} A^n_\alpha(\xi(u_n)) D^\alpha u_n - \int_\Omega \sum_{|\alpha| = m} A^n_\alpha(\xi(u_n)) D^\alpha u$$
$$- \int_\Omega \sum_{|\alpha| = m} A^n_\alpha(\eta(u_n), \zeta(u))(D^\alpha u_n - D^\alpha u) ,$$

and since the last integral above converges to zero, (1.14) can be deduced from (1.12) and (1.13). As the integrand in (1.14) is ≥ 0 a.e. by (1.8), it converges to zero in $L^1(\Omega)$, and so, by passing to a subsequence, a.e. in Ω:

(1.15) $$\sum_{|\alpha|=m} (A^n_\alpha(\eta(u_n),\zeta(u_n)) - A^n_\alpha(\eta(u_n),\zeta(u)))(D^\alpha u_n - D^\alpha u) \to 0 \quad \text{a.e. in } \Omega .$$

Fix $x_0 \in \Omega$ (a.e.) and let us show that $\zeta(u_n)(x_0)$ remains bounded. Suppose the contrary. Then, writing $\xi_n = \xi(u_n)(x_0)$ and $\xi = \xi(u)(x_0)$, we get, for a subsequence, $|\zeta_n - \zeta| \geq 1$ and $(\zeta_n - \zeta)/|\zeta_n - \zeta| \to \zeta^* \neq 0$; but it follows from (1.8) that

$$\sum_{|\alpha|=m} (A^n_\alpha(\eta_n,\zeta + \zeta_n - \zeta) - A^n_\alpha(\eta_n,\zeta))(\zeta_{n\alpha} - \zeta_\alpha)$$

$$\geq \sum_{|\alpha|=m} (A^n_\alpha(\eta_n,\zeta + (\zeta_n - \zeta)/|\zeta_n - \zeta|) - A^n_\alpha(\eta_n,\zeta))(\zeta_{n\alpha} - \zeta_\alpha) \geq 0 ,$$

and so we deduce from (1.15), after dividing by $|\zeta_n - \zeta|$, that

$$\sum_{|\alpha|=m} (A_\alpha(\eta,\zeta + \zeta^*) - A_\alpha(\eta,\zeta))\zeta^*_\alpha = 0 ;$$

consequently, by (1.8), $\zeta^* = 0$, a contradiction. We can thus assume, passing to a subsequence (depending a priori on x_0), that $\zeta(u_n)(x_0) \to \zeta_0$; it then follows from (1.15) that at x_0,

$$\sum_{|\alpha|=m} (A_\alpha(\eta(u),\zeta_0) - A_\alpha(\eta(u),\zeta(u)))(\zeta_{0\alpha} - D^\alpha u) = 0 ,$$

and consequently, by (1.8), $\zeta_{0\alpha} = D^\alpha u(x_0)$ for $|\alpha| = m$. So $\zeta(u_n)(x_0)$ converges for the original sequence to $\zeta(u)(x_0)$. We have thus proved that for $|\alpha| = m$, $D^\alpha u_n \to D^\alpha u$ a.e.

It remains to see that $\langle T_n u_n, u_n\rangle \to \langle Tu,u\rangle$. As

$$\int_\Omega \sum_{|\alpha|<m} A^n_\alpha(\xi(u_n))D^\alpha u_n \to \int_\Omega \sum_{|\alpha|<m} A_\alpha(\xi(u))D^\alpha u ,$$

it suffices by (1.12) to show that

(1.16) $$\liminf \int_\Omega \sum_{|\alpha|=m} A^n_\alpha(\xi(u_n))D^\alpha u_n \geq \int_\Omega \sum_{|\alpha|=m} A_\alpha(\xi(u))D^\alpha u .$$

But (1.8) implies

$$\int_\Omega \sum_{|\alpha|=m} (A^n_\alpha(\eta(u_n),\zeta(u_n)) - A^n_\alpha(\eta(u_n),\zeta(u)))(D^\alpha u_n - D^\alpha u) \geq 0 ,$$

and (1.16) follows by passing to the limit. Q.E.D.

LEMMA 1.5 (cf. [16]). Let $r_n(x)$ be a bounded sequence in $L^p(\Omega)$, $1 < p < \infty$, with $r_n(x) \to r(x)$ a.e. in Ω. Then $r(x) \in L^p(\Omega)$ and for each $s(x) \in L^{p'}(\Omega)$, $r_n s \to rs$ in $L^1(\Omega)$.

PROOF OF THEOREM 1.4. We must show, using the notations of the above proof, that if (1.10) holds, then $u_n \to u$ in V. It clearly suffices to see that $D^\alpha u_n \to D^\alpha u$ in $L^p(\Omega)$ for $|\alpha| = m$, and since we already have a.e. convergence, it is enough to prove, by Vitali theorem, that the $|D^\alpha u_n|^p$ are equi absolutely integrable. Let $E \subset \Omega$. By (1.10),

$$\int_E \sum_{|\alpha|=m} |D^\alpha u_n|^p \leq c \int_E \sum_{|\alpha|=m} A_\alpha^n(\xi(u_n)) D^\alpha u_n + c \int_E \sum_{|\alpha|<m} |D^\alpha u_n|^p + c \int_E \ell_1(x)$$

where c denotes a constant independent of n and E. Given $\varepsilon > 0$, one deduces from (1.14) that there exists n_ε (independent of E) such that for $n \geq n_\varepsilon$,

$$\int_E \sum_{|\alpha|=m} A_\alpha^n(\xi(u_n)) D^\alpha u_n \leq \varepsilon + \int_E \sum_{|\alpha|=m} A_\alpha^n(\xi(u_n)) D^\alpha u$$
$$+ \int_E \sum_{|\alpha|=m} A_\alpha^n(\eta(u_n), \zeta(u))(D^\alpha u_n - D^\alpha u) \ ;$$

but each integrand on the right hand side converges in $L^1(\Omega)$, the first by lemma 1.5 and the second by a preceding argument. The conclusion follows. Q.E.D.

REMARK 1.6. The conclusion of theorems 1.3 and 1.4 still holds if the growth assumption (1.7) is weakened in the following way: it suffices that (i) the inequalities in (1.7) be verified with a constant c_1 and a function $k_1(x)$ possibly depending on n, (ii) if u remains bounded in $W^{m,p}(\Omega)$ and $n = 1,2,\ldots$, then $A_\alpha^n(\xi(u))$ remains bounded in $L^{p'}(\Omega)$, (iii) for $|\alpha| = m$, if u remains bounded in $W^{m,p}(\Omega)$, v is fixed in $W^{m,p}(\Omega)$ and $n = 1,2,\ldots$, then $A_\alpha^n(\eta(u), \zeta(v))$ varies in a compact set of $L^{p'}(\Omega)$. A similar remark applies to theorem 1.9 below.

REMARK 1.7. Similar results, with simpler proofs, can be given when L_n and L are monotone. Assume (1.6), (1.7), (1.9), and

$$\sum_{|\alpha|\leq m} (A_\alpha^n(x,\xi) - A_\alpha^n(x,\xi'))(\xi_\alpha - \xi'_\alpha) \geq 0$$

for a.e. x, all ξ, ξ' and all n, and similarly for $A_\alpha(x,\xi)$, $|\alpha| \leq m$. Then $T_n \xrightarrow{PM} T$. Moreover if the condition

$$\int_\Omega \sum_{|\alpha|\leq m} (A_\alpha^n(\xi(u_n)) - A_\alpha(\xi(u)))(D^\alpha u_n - D^\alpha u) \to 0$$

implies $u_n \to u$ in V (this will be the case if the coefficients A_α^n verify a strong monotonicity condition which is uniform with respect to n), then $T_n \xrightarrow{S} T$.

EXAMPLE 1.8. Consider

$$L_n u \equiv - \sum_{i=1}^{N} D^i(a_i^n(x,u)|\nabla u|^{p-2} D^i u) + a_0^n(x,u,\nabla u) ,$$

$$Lu \equiv - \sum_{i=1}^{N} D^i(a_i(x,u)|\nabla u|^{p-2} D^i u) + a_0(x,u,\nabla u) ,$$

where $1 < p < \infty$ and the functions a_i^n, a_0^n, a_i, a_0 satisfy the Caratheodory conditions together with:

(1.17) there are constants Λ and c_1, $k_1(x) \in L^{p'}(\Omega)$ such that

$$|a_i^n(x,\eta)| \text{ and } |a_i(x,\eta)| \leq \Lambda ,$$
$$|a_0^n(x,\eta,\zeta)| \text{ and } |a_0(x,\eta,\zeta)| \leq c_1|\eta|^{p-1} + c_1|\zeta|^{p-1} + k_1(x) ,$$

for a.e. x, all η, ζ, all i, n;

(1.18) for a.e. x and all i, if $k_n \to \infty$ and $(\eta_n,\zeta_n) \to (\eta,\zeta)$, then $a_i^{k_n}(x,\eta_n) \to a_i(x,\eta)$ and $a_0^{k_n}(x,\eta_n,\zeta_n) \to a_0(x,\eta,\zeta)$.

Let $T_n : V \to V'$ and $T : V \to V'$ be the corresponding mappings ($m = 1$ here). If for a.e. x, all η, all i, n,

$$a_i^n(x,\eta) \text{ and } a_i(x,\eta) > 0 ,$$

then $T_n \xrightarrow{PM} T$. And if

$$a_i^n(x,\eta) \text{ and } a_i(x,\eta) \geq \lambda > 0$$

(λ a constant) for a.e. x, all η, all i, n, then $T_n \xrightarrow{S} T$. Related results in the latter case have been obtained for $p = 2$ by Boccardo-Dolcetta [3].

1.3. COMPACTNESS THEOREM

In some applications (see section 2.3), one has a sequence of operators L_n as above for which the lower order coefficients A_α^n, $|\alpha| < m$, do not necessarily converge in the sense of (1.9). The following theorem yields a compactness result in this situation.

Let L_n be as in (1.4), $n = 1,2,\ldots$, and let $A_\alpha(x,\xi)$, $|\alpha| = m$, be functions. We will assume among other things:

(1.19) each function $A_\alpha^n(x,\xi)$, $|\alpha| < m$, $A_\alpha(x,\xi)$, $|\alpha| = m$ satisfies the Caratheodory conditions;

(1.20) each function $A_\alpha^n(x,\xi)$, $|\alpha| < m$, $A_\alpha(x,\xi)$, $|\alpha| = m$ satisfies a growth condition such as (1.7), with a constant c_1 and a function $k_1(x)$ independent of n;

(1.21) for a.e. x and all $|\alpha| = m$, if $k_n \to \infty$ and $\xi_n \to \xi$, then $A_\alpha^{k_n}(x,\xi_n) \to A_\alpha(x,\xi)$.

Let $T_n : V \to V'$ be the mapping associated to L_n.

THEOREM 1.9. _Assume_ (1.19), (1.20), (1.8), (1.10) _and_ (1.21). _If_ $u_n \rightharpoonup u$ _in_ V _and_ $T_n u_n \rightharpoonup f$ _in_ V' _with_

$$\limsup \langle T_n u_n, u_n \rangle \leq \langle f,u \rangle ,$$

then $u_n \to u$ _in_ V.

PROOF. The arguments are essentially the same as those in the proof of theorems 1.3 and 1.4 and we will not repeat them. Q.E.D.

2. A QUASI VARIATIONAL INEQUALITY WITH OBSTACLE ON THE BOUNDARY

2.1. CONORMAL DERIVATIVE

In this section we make precise the notion of conormal derivative for an operator of the form

$$Lu \equiv -\sum_{i=1}^{N} D^i A_i(x,u,\nabla u) + A_0(x,u,\nabla u) \tag{2.1}$$

on a bounded open set Ω of R^N with locally Lipschitzian boundary Γ.

We assume:

(2.2) the functions $A_i(x,\xi)$ and $A_0(x,\xi)$ satisfy the Caratheodory conditions;

(2.3) there exist $1 < p < \infty$, $k_1(x) \in L^{p'}(\Omega)$ and c_1 such that

$$|A_i(x,\xi)| \text{ and } |A_0(x,\xi)| \leq c_1|\xi|^{p-1} + k_1(x)$$

for a.e. x, all ξ, all i.

L is considered as a mapping from $W^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$; so

$$\langle Lu,v\rangle = a(u,v) \quad \text{for } u \in W^{1,p}(\Omega),\ v \in W_0^{1,p}(\Omega),$$

where $a(u,v)$ is the Dirichlet form associated to L and $\langle\ ,\ \rangle$ denotes the pairing in the distribution sense. $T : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is defined by

$$\langle\langle Tu,v\rangle\rangle = a(u,v) \quad \text{for } u \text{ and } v \in W^{1,p}(\Omega),$$

where $\langle\langle\ ,\ \rangle\rangle$ denotes the pairing between $(W^{1,p}(\Omega))'$ and $W^{1,p}(\Omega)$. We will also denote by $\langle\ ,\ \rangle$ the pairing between $W^{-(1-1/p),p'}(\Gamma)$ and $W^{1-1/p,p}(\Gamma)$.

PROPOSITION 2.1. *Let* X *be a subspace of* $W^{-1,p'}(\Omega)$ *and* $\pi : X \to (W^{1,p}(\Omega))'$ *be a linear mapping such that for* $f \in X$,

$$\langle\langle \pi f,v\rangle\rangle = \langle f,v\rangle \quad \text{for } v \in W_0^{1,p}(\Omega) \tag{2.4}$$

(i.e. πf *is an extension of the linear form* f *to* $W^{1,p}(\Omega)$*). Take* $u \in W^{1,p}(\Omega)$ *with* $Lu \in X$. *Then there exists an unique element in* $W^{-(1-1/p),p'}(\Gamma)$, *denoted by* $\gamma_a u$, *such that*

$$a(u,v) = \langle\langle \pi Lu,v\rangle\rangle + \langle\gamma_a u,\gamma_0 v\rangle \quad \text{for } v \in W^{1,p}(\Omega), \tag{2.5}$$

where γ_0 *denotes the usual trace on* Γ. *Moreover* (2.5) *is the unique decomposition of the form*

$$a(u,v) = ((F,v)) + (g,\gamma_0 v) \quad \underline{\text{for}} \quad v \in W^{1,p}(\Omega) ,$$

<u>with</u> F <u>in the range of</u> π <u>and</u> g <u>in</u> $W^{-(1-1/p),p'}(\Gamma)$.

PROOF. The expression

$$a(u,v) - ((\pi L u,v)) \quad \underline{\text{for}} \quad v \in W^{1,p}(\Omega)$$

depends only, and continuously, on the trace $\gamma_0 v$ (use a right inverse of γ_0). This implies the existence of $\gamma_a u$ and its uniqueness. The last part of the proposition follows easily from (2.4). Q.E.D.

If for f smooth in X, one has

$$((\pi f,v)) = \int_\Omega fv \quad \text{for} \quad v \in W^{1,p}(\Omega) , \tag{2.6}$$

then it is rather natural to call (2.5) the Green's formula associated to the extension mapping π.

EXAMPLE 2.2. Take $X = L^{p'}(\Omega)$ and define π by formula (2.6). We then write (f,v) instead of $((\pi f,v))$. Formula (2.5) becomes, for $u \in W^{1,p}(\Omega)$ with $Lu \in L^{p'}(\Omega)$,

$$a(u,v) = (Lu,v) + (\gamma_a u,\gamma_0 v) \quad \text{for} \quad v \in W^{1,p}(\Omega) . \tag{2.7}$$

EXAMPLE 2.3. Denote by $\Theta^+_{p'}(\Omega) \subset W^{-1,p'}(\Omega)$ the set of all restrictions to $W_0^{1,p}(\Omega)$ of the positive continuous linear forms on $W^{1,p}(\Omega)$, and write $\Theta_{p'}(\Omega) = \Theta^+_{p'}(\Omega) - \Theta^+_{p'}(\Omega)$. This space has been introduced and studied for $p = 2$ by Hanouzet-Joly [11] in relation with the interpretation of solutions of some variational inequalities. Some of their results extend easily to the case $p \neq 2$, as remarked in [8]. In particular one can define an extension mapping $\pi : \Theta_{p'}(\Omega) \to (W^{1,p}(\Omega))'$ which verifies (2.6) by writing, for $f \in \Theta^+_{p'}(\Omega)$ and $v \in W^{1,p}(\Omega)$, $v \geq 0$ a.e.,

$$((\pi f,v)) = \sup\{\langle f,w\rangle;\ w \in W_0^{1,p}(\Omega) \text{ and } 0 \leq w \leq v \text{ a.e.}\} . \tag{2.8}$$

For L linear with smooth coefficients, Hanouzet-Joly proved that γ_a defined by (2.5) by using this π is the continuous extension of the usual conormal derivative operator (0.5) on $C^\infty(\bar{\Omega})$. Denoting by E^* the order dual of E (i.e. the set of differences of positive continuous linear forms), one has the following strict inclusions:

$\Theta_{p'}(\Omega) \subset (W_0^{1,p}(\Omega))^* \subset (W_0^{1,p}(\Omega))'$ and $\pi\,\Theta_{p'}(\Omega) \subset (W^{1,p}(\Omega))^* \subset (W^{1,p}(\Omega))'$; moreover $L^{p'}(\Omega) \subset \Theta_{p'}(\Omega)$ strictly, and (2.6) holds for $f \in L^{p'}(\Omega)$. See [11,8].

In the following we will use the extension mapping of example 2.2. The more general result obtained by considering the extension mapping of example 2.3 will be mentioned in section 2.5.d.

2.2. STATEMENT OF THE PROBLEM

We now start the study of problem (0.1)-(0.4) itself, with Ψ of the form (0.6). The case of the obstacle (0.7) will be treated in section 2.5.a.

Let L be given by (2.1), with coefficients satisfying (2.2) and (2.3). The functions h and φ are given in $W^{1-1/p,p}(\Gamma)$ and we consider, for $w \in W^{1,p}(\Omega)$ with $Lw \in L^{p'}(\Omega)$, the obstacle

$$\Psi(w) = h - \langle \gamma_a w, \varphi \rangle \tag{2.9}$$

where $\gamma_a w$ is defined by (2.7). Let

$$Q(w) = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq \Psi(w) \text{ a.e. on } \Gamma\} \tag{2.10}$$

be the corresponding closed convex set. We are also given f in $L^{p'}(\Omega)$.

For $u \in W^{1,p}(\Omega)$, equation (0.1) is interpretated in the distribution sense in Ω, condition (0.2) as $\gamma_0 u \geq \Psi(u)$ a.e. on Γ, condition (0.3) in the sense of the dual of $W^{1-1/p,p}(\Gamma)$, and condition (0.4) as $\langle \gamma_a u, \gamma_0(u - \Psi(u))\rangle = 0$. Then one easily verifies that stated in this way, the problem of finding $u \in W^{1,p}(\Omega)$ verifying (0.1)-(0.4) is equivalent to solving the quasi variational inequality

$$\begin{cases} u \in W^{1,p}(\Omega) \text{ with } Lu \in L^{p'}(\Omega)\ , \\ u \in Q(u)\ , \\ \langle\langle Tu, u - v\rangle\rangle \leq \langle f, u - v\rangle \text{ for all } v \in Q(u)\ . \end{cases} \tag{2.11}$$

Examples can easily be constructed (for $N = 1$ and $Lu \equiv -u'' + u$) which show that this problem may have no, one, two or infinitely many solutions.

2.3. NONMONOTONE CASE

It will be useful (see example 2.5 below) to distinguish in the coefficient $A_0(x,u,\nabla u)$ a dependence on u which yields monotonicity and coercivity from one of perturbation type. We write for this purpose $A_0(x,u,u,\nabla u)$ instead of $A_0(x,u,\nabla u)$, so that the operator L becomes

$$Lu \equiv -\sum_{i=1}^{N} D^i A_i(x,u,\nabla u) + A_0(x,u,u,\nabla u) \; .$$

We will make the following assumptions (compare with the standard Leray-Lions conditions):

(2.12) each function $A_i(x,\xi)$ and $A_0(x,\eta_1,\eta_2,\zeta)$ satisfies the Caratheodory conditions;

(2.13) there exists $1 < p < \infty$, $k_2(x) \in L^{p'}(\Omega)$ and a constant c_2 such that

$$|A_i(x,\eta,\zeta)| \leq c_2|\zeta|^{p-1} + k_2(x) \; ,$$
$$|A_0(x,\eta_1,\eta_2,\zeta)| \leq c_2|\eta_1|^{p-1} + k_2(x) \; ,$$

for a.e. x, all η, η_1, η_2, ζ, all i;

(2.14) for a.e. x, all η,

$$\sum_{i=1}^{N} ((A_i(x,\eta,\zeta) - A_i(x,\eta,\zeta'))(\zeta_i - \zeta_i') > 0$$

if $\zeta \neq \zeta'$; for a.e. x, all η_1, η_1', η_2, ζ,

$$(A_0(x,\eta_1,\eta_2,\zeta) - A_0(x,\eta_1',\eta_2,\zeta))(\eta_1 - \eta_1') \geq 0 \; ;$$

(2.15) there exist $d_2 > 0$ and $\ell_2(x) \in L^1(\Omega)$ such that

$$\sum_{i=1}^{N} A_i(x,\eta,\zeta)\zeta_i \geq d_2|\zeta|^p - \ell_2(x) \; ,$$

$$A_0(x,\eta_1,\eta_2,\zeta)\eta_1 \geq d_2|\eta_1|^p - \ell_2(x) \; ,$$

for a.e. x, all η, η_1, η_2, ζ.

THEOREM 2.4. Let the conditions (2.12)-(2.15) be satisfied, and let h and φ be given in $W^{1-1/p,p}(\Gamma)$, f in $L^{p'}(\Omega)$. Then problem (2.11) has a solution when either $1 < p < 2$ or $p \geq 2$ and $\|\varphi^-\|_{W^{1-1/p,p}(\Gamma)}$ is sufficiently small (depending on Ω, h, f and the various constants and functions in (2.13) and (2.15)).

PROOF. Let us write, for $\lambda \in \mathbb{R}$,

$$Q_\lambda = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq h - \lambda \text{ a.e. on } \Gamma\},$$

and for $w \in W^{1,p}(\Omega)$,

$$L_w(u) \equiv -\sum_{i=1}^{N} D^i A_i(x,w,\nabla u) + A_0(x,u,w,\nabla w),$$

and let $T_w : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ be the mapping corresponding to L_w. By (2.12)-(2.15), T_w is monotone, continuous and coercive, so that the variational inequality

$$(2.16)\qquad \begin{cases} u \in Q_\lambda, \\ \langle\langle T_w u, u - v\rangle\rangle \leq \langle f, u - v\rangle \text{ for all } v \in Q_\lambda \end{cases}$$

has solutions. Defining

$$\theta(\lambda,w) = \{(\langle\gamma_{a_w} u, \varphi\rangle, u);\ u \text{ solution of (2.16)}\} \subset \mathbb{R} \times W^{1,p}(\Omega)$$

where γ_{a_w} denotes the conormal derivative associated to L_w, we are reduced to finding a fixed point of the multivalued mapping $(\lambda,w) \to \theta(\lambda,w)$ in $\mathbb{R} \times W^{1,p}(\Omega)$.

$\theta(\lambda,w)$ is closed and convex. Indeed the set of solutions of (2.16) is closed, convex, and if u_1 and u_2 are two such solutions, then (2.14) implies that $\nabla u_1 = \nabla u_2$, so that, using (2.7) for L_w, we see that $\gamma_{a_w}(u_1) = \gamma_{a_w}(u_2)$.

A priori estimate. Denote by $v \to \tilde{v}$, from $W^{1-1/p,p}(\Gamma)$ into $W^{1,p}(\Omega)$, a right inverse of the trace mapping γ_0. Let $\bar{\lambda} \geq 0$ and let u be a solution of (2.16) with $\lambda \geq -\bar{\lambda}$ and $w \in W^{1,p}(\Omega)$. Then, if we put $v = \tilde{h} - (-\bar{\lambda})$ in (2.16), we deduce from (2.13) and (2.15) that

$$d_2\|u\|_\Omega^p \leq c + c\|u\|_\Omega^{p-1} + c\bar{\lambda}\|u\|_\Omega^{p-1} + c\|u\|_\Omega + c\bar{\lambda},$$

where c denotes various constants independent of u, λ, $\bar{\lambda}$ and w, and $\|\ \|_\Omega$ denotes the norm in $W^{1,p}(\Omega)$. Consequently

$$(2.17)\qquad \|u\|_\Omega \leq c\bar{\lambda} + c,$$

so that, using (2.7) for L_w and (2.13),

$$\|\gamma_{a_w} u\|_{\Gamma'} \leq c\bar{\lambda}^{p-1} + c \,, \tag{2.18}$$

where $\| \ \|_{\Gamma'}$ denotes the norm in $W^{-(1-1/p),p'}(\Gamma)$. Thus θ transforms $[-\bar{\lambda}, +\infty[\times W^{1,p}(\Omega)$ into a bounded set. Moreover, since $\gamma_{a_w} u$ is a positive element of the dual of $W^{1-1/p,p}(\Gamma)$ (this follows from (2.16) by taking $v = u + \tilde{z}$ with $z \in W^{1-1/p,p}(\Gamma)$, $z \geq 0$ a.e. on Γ, and using (2.7) for L_w), we deduce from (2.18) that

$$\langle \gamma_{a_w} u, \varphi \rangle \geq -(c_1\bar{\lambda}^{p-1} + c_2)\|\varphi^-\|_\Gamma \,,$$

where we have written $\varphi = \varphi^+ - \varphi^-$ and denoted by $\| \ \|_\Gamma$ the norm in $W^{1-1/p,p}(\Gamma)$. Consequently, if $1 < p < 2$, then, φ being given, there exists $\bar{\lambda}$ such that $\langle \gamma_{a_w} u, \varphi \rangle \geq -\bar{\lambda}$. Such a $\bar{\lambda}$ also exists when $p \geq 2$ provided $\|\varphi^-\|_\Gamma$ is sufficiently small:

$$\|\varphi^-\|_\Gamma \leq \max_{\lambda \geq 0} \lambda/(c_1\lambda^{p-1} + c_2) \,.$$

In any case we have found $\bar{\lambda} > 0$ and $R > 0$ such that θ transforms $[-\bar{\lambda}, R] \times B_R$ into itself, where B_R denotes the closed ball centered at zero of radius R in $W^{1,p}(\Omega)$.

θ transforms a bounded set into a relatively compact set. Indeed, let $\lambda_n \to \lambda$ and $w_n \rightharpoonup w$ in $W^{1,p}(\Omega)$, and let u_n be a corresponding solution of (2.16):

$$\left\{ \begin{array}{l} u_n \in \mathcal{Q}_{\lambda_n} \,, \\ \langle\langle T_{w_n}(u_n), u_n - v \rangle\rangle \leq \langle f, u_n - v \rangle \quad \text{for all} \quad v \in \mathcal{Q}_{\lambda_n} \,, \end{array} \right. \tag{2.19}$$

where T_{w_n} is associated to the operator

$$L_{w_n}(u) \equiv \sum_{i=1}^{N} D^i A_i(x, w_n(x), u) + A_0(x, u, w_n(x), \nabla w_n(x)) \,.$$

One immediately has $\mathcal{Q}_{\lambda_n} \xrightarrow{M} \mathcal{Q}_\lambda$. Moreover u_n remains bounded in $W^{1,p}(\Omega)$, as seen before, so that, passing to a subsequence, we can assume $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and, using (2.13), $T_{w_n}(u_n) \rightharpoonup g$ in $(W^{1,p}(\Omega))'$. We first deduce $u \in \mathcal{Q}_\lambda$, and then the existence of $v_n \in Q_{\lambda_n}$ with $v_n \to u$; replacing in (2.19) and going to the limit, we obtain

$$\limsup \langle\langle T_{w_n}(u_n), u_n\rangle\rangle \leq \langle\langle g, u\rangle\rangle .$$

We can now apply theorem 1.9 (after passing to a further subsequence to have $w_n \to w$ a.e.) and conclude that $u_n \to u$ in $W^{1,p}(\Omega)$.

Let us write $C = \mathrm{cl\ conv}\ \theta([-\bar{\lambda}, R] \times B_R)$. C is convex, compact, and $\theta(C) \subset C$. In order to apply Kakutani's theorem and thus complete the proof, we must verify that the graph of θ is closed. Let $\lambda_n \to \lambda$, $w_n \to w$ in $W^{1,p}(\Omega)$, and let u_n be a solution of (2.19) with $u_n \to u$ in $W^{1,p}(\Omega)$ and $r_n = \langle \gamma_{a_{w_n}}(u_n), \varphi\rangle \to r$. As above, $Q_{\lambda_n} \xrightarrow{M} Q_\lambda$. Moreover, passing to a subsequence so that $w_n \to w$ and $\nabla w_n \to \nabla w$ a.e., we deduce from theorem 1.4 that $T_{w_n} \xrightarrow{S} T_w$. It then follows from theorem 1.2 that u satisfies (2.16). Finally (2.7) implies that $\gamma_{a_{w_n}}(u_n) \to \gamma_{a_w}(u)$ in $W^{-(1-1/p),p'}(\Gamma)$, and consequently $r = \langle \gamma_{a_w}(u), \varphi\rangle$. Q.E.D.

EXAMPLE 2.5. The assumptions of theorem 2.4 are satisfied by the operator

$$Lu \equiv - \sum_{i=1}^{N} D^i(a_i(x,u)|\nabla u|^{p-2}D^i u) + a_0(x,u,\nabla u)|u|^{p-2}u$$

if the functions a_i and a_0 verify the Caratheodory conditions together with

$$0 < \lambda \leq a_i(x,\eta) \leq \Lambda ,$$
$$0 < \lambda \leq a_0(x,\eta,\zeta) \leq \Lambda ,$$

for some constants λ and Λ, a.e. x, all η, ζ, all i.

2.4. MONOTONE CASE

We suppose now that L, given by (2.1), satisfies (2.2), (2.3), and

(2.20) for a.e. x, all ξ, ξ',

$$\sum_{i=1}^{N} (A_i(x,\xi) - A_i(x,\xi'))(\zeta_i - \zeta_i') + (A_0(x,\xi) - A_0(x,\xi'))(\eta - \eta') \geq 0 ;$$

(2.21) there exist $d_2 > 0$ and $\ell_2(x) \in L^1(\Omega)$ such that

$$\sum_{i=1}^{N} A_i(x,\xi)\zeta_i + A_0(x,\xi)\eta \geq d_2|\xi|^p - \ell_2(x)$$

for a.e. x, all ξ.

THEOREM 2.6. Assume (2.2), (2.3), (2.20) and (2.21), let $h, \varphi \in W^{1-1/p,p}(\Gamma)$, $f \in L^{p'}(\Omega)$. Then the conclusion of theorem 2.4 holds.

PROOF. Define Q_λ as before and consider the variational inequality

$$(2.22) \qquad \begin{cases} u \in Q_\lambda \,, \\ \langle\langle Tu, u - v\rangle\rangle \leq \langle f, u - v\rangle \text{ for all } v \in Q_\lambda \,. \end{cases}$$

Writing

$$\theta(\lambda) = \{\langle\gamma_a u, \varphi\rangle;\ u \text{ solution of (2.22)}\} \subset \mathbb{R}\,,$$

we are reduced to finding a fixed point of the multivalued mapping $\lambda \to \theta(\lambda)$ in $\mathbb{R}$. The arguments are rather similar to those in the proof of theorem 2.4, but simpler, and we will not describe them any furhter. Let us just mention that the convexity of $\theta(\lambda)$ follows from the fact that since γ_a is continuous on the (convex) set of solutions of (2.22),

$$\{\gamma_a u;\ u \text{ solution of (2.22)}\} \subset W^{-(1-1/p),p'}(\Gamma)$$

is connected. Q.E.D.

REMARK 2.7. Assume (2.2), (2.3), (2.20) or (2.23), and (2.21), where:

(2.23) for a.e. x, all η,

$$\sum_{i=1}^{N} (A_i(x,\eta,\zeta) - A_i(x,\eta,\zeta'))(\zeta_i - \zeta_i') > 0$$

if $\zeta \neq \zeta'$.

Let u_λ be a solution of (2.22). Then $\gamma_a u_\lambda \longrightarrow 0$ in $W^{-(1-1/p),p'}(\Gamma)$ as $\lambda \to +\infty$ (and consequently $\theta(\lambda) \to 0$ as $\lambda \to +\infty$). Indeed, for a subsequence, $u_\lambda \longrightarrow u$ in $W^{1,p}(\Omega)$ and $Tu_\lambda \longrightarrow g$ in $(W^{1,p}(\Omega))'$. Taking $v_\lambda \in Q_\lambda$ with $v_\lambda \to u$ (e.g.

$v_\lambda = \sup(u, \tilde{h} - \lambda))$ and replacing in (2.22), we obtain

$$\limsup \langle\langle Tu_\lambda, u_\lambda \rangle\rangle \leq \langle\langle g, u \rangle\rangle ,$$

and consequently, by the pseudo-monotonicity of T, $g = Tu$ and $\langle\langle Tu_\lambda, u_\lambda \rangle\rangle \to \langle\langle Tu, u \rangle\rangle$. Take now any $v \in W^{1,p}(\Omega)$, and $v_\lambda \in Q_\lambda$ with $v_\lambda \to v$; replacing again in (2.22), we deduce

$$\langle\langle Tu, v \rangle\rangle = \langle f, v \rangle ,$$

so that $\gamma_a u = 0$. But by (2.7), $\gamma_a u_\lambda \longrightarrow \gamma_a u$ in $W^{-(1-1/p),p'}(\Gamma)$. It thus follows that, without passing to any subsequence, $\gamma_a u_\lambda \longrightarrow 0$.

EXAMPLE 2.8. The assumptions of theorem 2.6 (as well as those of theorem 2.4) are verified in the linear case

$$Lu \equiv - \sum_{i,j=1}^{N} D^i (a_{ij}(x) D^j u) + a_0(x) u ,$$

where a_{ij} and a_0 are in $L^\infty(\Omega)$ and satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$$

together with

$$a_0(x) \geq \lambda ,$$

λ a strictly positive constant. Theorem 2.6 thus includes the results of [12,20,2] referred to in the introduction.

2.5. VARIATIONS

a. Consider the obstacle (0.7), or more generally an obstacle of the form

$$\Psi(u) = h(x) - [\phi^+(\gamma_a u) - \phi^-(\gamma_a u)] \tag{2.24}$$

where $h \in W^{1-1/p,p}(\Gamma)$ and $\phi^\pm$ are mappings from $W^{-(1-1/p),p'}(\Gamma)$ into $W^{1-1/p,p}(\Gamma)$. We assume $\phi^\pm$ continuous, compact, positive (i.e. $\phi^\pm(\sigma) \geq 0$ a.e. on Γ when σ is a positive element of the dual of $W^{1-1/p,p}(\Gamma)$), with an estimate of the form

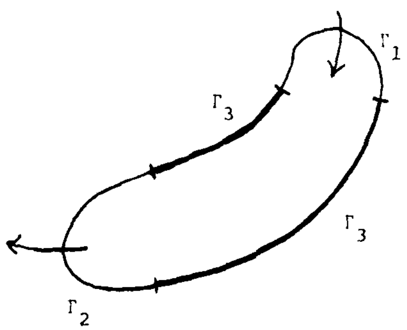
$$\|\phi^-(\sigma)\|_{L^\infty(\Gamma)} \leq a_1 \|\sigma\| + a_2 .$$

We look for a solution u of (2.11), where $f \in L^{p'}(\Omega)$ and $Q(w)$ is defined by (2.10), Ψ being now given by (2.24). Then, under the assumptions (2.12)-(2.15), this problem has a solution when either $\sigma(p-1) < 1$ or $\sigma(p-1) > 1$ and a_1 is sufficiently small. The proof of theorem 2.4 can be adapted to this situation. One replaces Q_λ by Q_ℓ, where Q_ℓ is defined for $\ell \in W^{1-1/p,p}(\Gamma)$ by

$$Q_\ell = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq h - \ell \text{ a.e. on } \Gamma\} ;$$

the mapping θ now operates in $W^{1-1/p,p}(\Gamma) \times W^{1,p}(\Omega)$. One also has an analogous result in the monotone case, i.e. under the assumptions (2.2), (2.3), (2.20) and (2.21). However here we are led to impose the strict monotonicity in (2.20) in order to guarantee that θ is convex valued.

b. The method of sections 2.3 and 2.4 can also be applied to the situation where Γ is composed of two parts Γ_1 and Γ_2 separated by a third part Γ_3 and one requires (0.2)-(0.4) on Γ_1, (0.2)-(0.4) with reverse inequality signs on Γ_2, and the Neumann boundary condition on Γ_3. In the language of fluid mechanics, one has a pipe with a semi-permeable membrane at each extremity:



Signorini problems of this type, with obstacles which do not depend on the solution, were considered recently by Kawohl [13].

c. As remarked in the introduction, the obstacle (2.9) is not defined for an arbitrary $w \in W^{1,p}(\Omega)$. One way of avoiding talking about $\Psi(w)$ unless $Lw \in L^{p'}(\Omega)$ is described in sections 2.3 and 2.4. Here is another possibility. Write, for $w \in W^{1,p}(\Omega)$,

$$\bar{\Psi}(w) = h - \langle\langle Tw, \tilde{\varphi} \rangle\rangle + \langle f, \tilde{\varphi} \rangle ,$$

$$P(w) = \{v \in W^{1,p}(\Omega);\ \gamma_0 v \geq \bar{\Psi}(w) \text{ a.e. on } \Gamma\} ,$$

and consider the problem of finding u solution of the quasi variational inequality

$$(2.25) \qquad \begin{cases} u \in W^{1,p}(\Omega) , \\ u \in P(u) , \\ \langle\langle Tu, u - v \rangle\rangle \leq \langle f, u - v \rangle \text{ for all } v \in P(u) . \end{cases}$$

Problems (2.11) and (2.25) are equivalent because, by (2.7), $P(w)$ and $Q(w)$ coincide when $w \in W^{1,p}(\Omega)$ verifies $Lw = f$. Formulation (2.25) allows a more traditional approach, by defining (in, say, the nonmonotone case) the variational selection $\theta(w)$, $w \in W^{1,p}(\Omega)$, as the set of all solutions u of the variational inequality

$$\begin{cases} u \in P(w), \\ \langle\langle T_w u, u - v \rangle\rangle \leq \langle f, u - v \rangle \text{ for all } v \in P(w) . \end{cases}$$

The results for (2.11) that we have obtained along these lines are however weaker than those in sections 2.3 and 2.4. But the above approach has proved useful in other similar problems.

d. By using in (2.9) the conormal derivative corresponding to the extension mapping π of example 2.3, one can get the conclusion of theorems 2.4 and 2.6 for a right hand side f in $\Theta_{p'}(\Omega)$. More precisely, for h and φ in $W^{1-1/p,p}(\Gamma)$, f in $\Theta_{p'}(\Omega)$, the problem of finding u verifying

$$(2.26) \qquad \begin{cases} u \in W^{1,p}(\Omega) \text{ with } Lu \in \Theta_{p'}(\Omega) , \\ u \in Q(u) , \\ \langle\langle Tu, u - v \rangle\rangle \leq \langle\langle \pi f, u - v \rangle\rangle \text{ for all } v \in Q(u) \end{cases}$$

has a solution when either $1 < p < 2$ or $p > 2$ and $\|\bar{\varphi}\|_\Gamma$ is sufficiently small. Note that (2.26) can still be shown to be in this more general situation equivalent to (0.1)-(0.4), see [8].

e. Coming back to the heat flow problem described in the introduction, we see that the $s_1(u,\nabla u)$ term represents in our results some cooling effect inside Ω (e.g. $s_1(u,\nabla u) = -|u|^{p-2}u$). The need for such a term is physically understandable since no restriction has been imposed on the forcing term $s_2(x)$. The case $s_1 \equiv 0$ will be studied elsewhere.

f. We conclude with a regularity result in the case where L is of the form

$$Lu \equiv -\sum_{i=1}^{N} D^i(a_i(x)|\nabla u|^{p-2}D^i u) + a_0(x)|u|^{p-2}u \; .$$

Assume Γ of class C^3, $a_i \in W^{1,\infty}(\Omega)$, $a_0 \in L^\infty(\Omega)$, $a_i(x)$ and $a_0(x) \geq \lambda > 0$, $h \in W^{1+1/p,p}(\Gamma)$, $\varphi \in W^{1-1/p,p}(\Gamma)$ and $f \in W^{1-1/p,p'}(\Omega)$. It then follows from proposition 3 in [8] that any solution u of (2.11) satisfies

$$u \in B^{1+1/p(p-1),p}(\Omega) \quad \text{if} \quad 2 < p < \bar{p} \; ,$$
$$u \in B^{1+(p-1)/p,p}(\Omega) \quad \text{if} \quad \underline{p} < p < 2 \; ,$$

where $\bar{p}$ and $\underline{p}$ are given by

$$(\bar{p} - 1)^3 - \bar{p} = 0 = (\underline{p} - 1)^2\underline{p} - 1$$

and where $B_q^{\sigma,p}(\Omega)$, $\sigma > 0$ different from an integer, $1 < p < \infty$, $1 \leq q \leq \infty$, is the Besov space defined by interpolation:

$$B_q^{\sigma,q}(\Omega) = [W^{1+[\sigma],p}(\Omega), W^{[\sigma],p}(\Omega)]_{1+[\sigma]-\sigma,q}$$

(see [18]).

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This paper is composed of two parts. In the first part closedness and compactness results are given for a sequence of nonlinear elliptic operators of the form

$$Lu = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,u,\nabla u,\ldots,\nabla^m u),$$

with monotone type assumptions on the A_α's. These results are then used in the second part to derive existence theorems for a quasi variational inequality

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20. ABSTRACT - Cont'd.

related to some questions from nonlinear heat flow. This quasi variational inequality involves a second order operator as above and an implicit obstacle of the Signorini type on the boundary.