MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS

UM 016294-F

AD A088662

A Study of Hybrid Computing Techniques for Transonic Flow Fields

T. C. ADAMSON JR.

A. F. MESSITER

February, 1980

A

THE UNIVERSITY OF MICHIGAN 1817

80 9 2 059

Conditions of Reproduction

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
AFOSR-TR-80-0693	AD-A088 662	

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
A STUDY OF HYBRID COMPUTING TECHNIQUES FOR TRANSONIC FLOW FIELDS.	FINAL rept. 1 Aug 78 — 31 Dec 79,
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
T. C. ADAMSON, JR. A. F. MESSITER	AFOSR-78-3635

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
DEPARTMENT OF AEROSPACE ENGINEERING THE UNIVERSITY OF MICHIGAN ANN ARBOR, MICHIGAN 48109	2307/A4 61102F

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA BUILDING 410 BOLLING AFB, DC 20332	February 1980
	13. NUMBER OF PAGES
	72

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

FAR FIELD SOLUTIONS
TRANSONIC FLOW
SUPERSONIC FLOW
COMPRESSIBLE CHANNEL FLOW

SHOCK WAVES
VISCOUS TRANSONIC FLOW

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two problems in which analytical solutions might be used to improve the accuracy of or reduce the computation time of numerical computations are considered. In the first, axisymmetric flow at large distances from a slender body is studied for supersonic and transonic speeds, and the length scales for which viscous effects become important are derived in terms of the slenderness parameter and the Reynolds number. An analytical solution is obtained for a case which appears most likely to be important at ⟶ over

DD FORM 1473 1 JAN 73

402605

supersonic speeds. In the second, solutions are derived for transonic flow in a two-dimensional channel with a shock wave, at low Reynolds number. A relationship is derived from which the effects of Reynolds number on the position of the sonic line within the shock wave can be found.

A STUDY OF HYBRID COMPUTING TECHNIQUES
FOR TRANSONIC FLOW FIELDS

by

T. C. Adamson, Jr.
A. F. Messiter

Department of Aerospace Engineering
The University of Michigan

1. INTRODUCTION

This final technical report is concerned with work done under Grant AFOSR 78-3635, A Study of Hybrid Computing Techniques for Transonic Flow Fields. The grant was started on August 1, 1978 and was to run for a period of 14 months. A three month no-cost extension was granted so that the final date for technical work was December 31, 1980.

The purpose of this work is the investigation of the use of asymptotic techniques to derive analytical solutions for use in conjunction with numerical methods, to improve the accuracy of solutions or to reduce computing times or both. The two problems chosen for study are

(1) The use of asymptotic far field solutions in viscous flow field computations

(2) The accurate determination of the location of a shock wave in a flow field, using composite asymptotic solutions in the neighborhood of the shock wave.

The first problem is concerned with the use of analytical far field solutions rather than boundary conditions at infinity to set conditions which the numerical computations must approach as the solution proceeds away from the body in question. For inviscid flow fields, such outer solutions have been used in the past with considerable success; as points farther from the body are considered, the numerical solution is compared with the far field solution and computation may be stopped when the difference between the two is within the allowable error. In the present work, covered in Section 2, far field solutions for viscous flow fields are considered for axisymmetric, supersonic and transonic flows. For a given set of parameters, the results allow determination of the length scales at which viscous effects are important and therefore show the general nature of the solutions valid at the various length scales. Then, depending upon the accuracy desired, a decision can be made as to the type of outer solution to be used. Analytical solutions are obtained for the case judged to be probably the most important.

1

In the second problem, covered in Section 3, the flow problem chosen as an example problem is that of steady transonic flow through a two-dimensional channel in which a shock wave is located. Because several regions of flow must be considered, adding to the general complexity of the solution, the solutions in this section are presented in detail.

Because two independent problems were studied, the sections in which each is considered, Sections 2 and 3, are independent and self-contained.

The work described herein represents only the initial phases of a study of hybrid computing techniques in that only analytical solutions are considered for only two problems.

2. FAR-FIELD DIFFUSION EFFECTS
IN AXISYMMETRIC SUPERSONIC AND TRANSONIC FLOW

A. F. Messiter

Introduction

The propagation of a weak pressure disturbance in a gas which initially is at rest and has uniform properties is described by the linear wave equation, provided that the distance through which the wave travels is not too large. Nonlinear and diffusion effects, however, may eventually become important; the details differ for plane and cylindrically symmetric flows. The present work is concerned with axisymmetric supersonic and transonic flow past a slender body of revolution in the case when nonlinearity is an essential feature at distances smaller than the distance at which a fully viscous region appears.

The inviscid far field, nonlinear because shock waves and characteristics are slightly inclined from their linearized positions, was studied by Whitham[1]; a summary appears in his book.[2] A detailed account of viscosity effects on sound waves was given by Lighthill.[3] In particular, he, and later Ryzhov[4] and others, noted that for cylindrically symmetric motions a fully viscous region finally develops, which is described by an equation similar to the Burgers equation but modified because of the geometry. The velocity field beyond this fully viscous region was studied by

[1]Whitham, G. B., The Flow Pattern of a Supersonic Projectile, Comm. Pure Appl. Math., 5 (1952), 301-348.

[2]Whitham, G. B., Linear and Nonlinear Waves, John Wiley and Sons, 1974.

[3]Lighthill, M. J., Viscosity Effects in Sound Waves of Finite Amplitude, in Surveys in Mechanics, eds. G. K. Batchelor and R. M. Davies, Cambridge University Press, 1956, pp. 250-351.

[4]Ryzhov, O. S., Influence of Viscosity and Thermal Conductivity on Propagation of Sound Impulses, P.M.M., 30 (1966), 296-302. Engl. transl. J. Appl. Math. Mech., 30 (1966), 362-369.

Salathe[5], Chong and Sirovich,[6] and Sanchez-Palencia-Hubert.[7] Chong and Sirovich[8] also considered the combined effects of diffusion and non-linearity. For a body moving at sonic speed, these effects have been studied by Szaniawski[9] and Ryzhov and Shefter.[10] Review articles by Sichel[11] and by Ryzhov[12] include discussions of these flows and list additional references.

The present study has two main purposes. First, for small but nonzero viscosity and thermal conductivity, the far-field solution for inviscid super-sonic flow can be modified to include the details of the velocity variation through the front and rear shock waves. This requires calculation of second-order solutions for the velocity in the region between the shock waves and for the velocity distribution within the shock waves, so that the uncertainty in shock-wave location is smaller than the shock-wave thickness. These

5 Salathe, E. P., The Fundamental Matrix in Three-Dimensional Dissipative Gasdynamics, J. Fluid Mech., 39 (1969), 209-225.

6 Chong, T. H. and Sirovich, L., Structure of Three-Dimensional Supersonic and Hypersonic Flow, Phys. Fluids, 13 (1970), 1990-1999.

7 Sanchez-Palencia-Hubert, J., Asymptotic Study at Infinity of Supersonic Flows of a Dissipative Fluid, Int. J. Eng. Sci., 14 (1976), 567-584.

8 Chong, T. H. and Sirovich, L., Nonlinear Effects in Steady Supersonic Dissipative Gasdynamics. Part 2. Three-Dimensional Axisymmetric Flow, J. Fluid Mech., 58 (1973), 53-63.

9 Szaniawski, A., The Asymptotic Structure of Weak Shock Waves in Flows over Symmetrical Bodies at Mach Number Unity, Acta Mech., 5 (1968), 189-203.

10 Ryzhov, O. S. and Shefter, G. M., On the Effect of Viscosity and Thermal Conductivity on the Structure of Compressible Flows, P.M.M., 28 (1964), 996-1007. Engl. transl. J. Appl. Math. Mech., 28 (1964), 1206-1218.

11 Sichel, M., Two-Dimensional Shock Structure in Transonic and Hypersonic Flow, in Advances in Applied Mechanics, 11, ed. C. S. Yih, Academic Press, 1971, pp. 131-207.

12 Ryzhov, O. S., Viscous Transonic Flows, in Ann. Rev. Fluid Mech., 10 (1978), 65-92.

solutions are derived, the second-order shock-wave position is obtained, and a simple multiplicative composition then gives a solution which is uniformly valid in the streamwise coordinate. Second, the flow properties depend on the Mach number, the Reynolds number, and the body slenderness ratio, and different approximations are needed for different parameter and coordinate ranges. In particular, at distances where nonlinear effects first become essential the shock waves are thin and one-dimensional; somewhat closer to the body the shock-wave structure becomes two-dimensional and somewhat farther away the shock waves can no longer be considered thin. In the existing literature, for both supersonic and transonic flows, the identification of different approximations and their domains of validity is incomplete. Several extensions and corrections are given here, so that the dominant physical effects can be identified as functions of position and of the nondimensional parameters.

Basic Equations

Dimensional quantities are indicated by an overbar; the subscript ∞ refers to values in the undisturbed uniform flow ahead of the body. Thus, e.g., $\bar{u}_\infty$, $\bar{a}_\infty$, $\bar{\rho}_\infty$, $\bar{\mu}_\infty$, and $\bar{\lambda}_\infty$ are, respectively, the undisturbed values of the flow speed, sound speed, fluid density, and first and second viscosity coefficients. The nondimensional flow speed q, sound speed a, enthalpy h, density ρ, pressure p, and stress tensor τ are defined by

$$q = \frac{\bar{q}}{\bar{u}_\infty} \qquad a = \frac{\bar{a}}{\bar{u}_\infty} \qquad h = \frac{\bar{h}}{\bar{u}_\infty^2}$$

$$\rho = \frac{\bar{\rho}}{\bar{\rho}_\infty} \qquad p = \frac{\bar{p}}{\bar{\rho}_\infty \bar{u}_\infty^2} \qquad \tau = Re \frac{\bar{\tau}}{\bar{\rho}_\infty \bar{u}_\infty^2}$$

(2.1)

The Reynolds number Re is based on the body length $\bar{L}$ and a reference viscosity coefficient specified later to be $2\bar{\mu}_\infty + \bar{\lambda}_\infty$; the same reference viscosity is used for the Prandtl number Pr and the nondimensional first and

second viscosity coefficients μ and λ. In terms of these nondimensional variables, the differential equations describing the fluid motion are

$$\operatorname{div} \rho \vec{q} = 0 \tag{2.2}$$

$$\rho \vec{q} \cdot \nabla \vec{q} + \operatorname{grad} p = \operatorname{Re}^{-1} \operatorname{div} \tau \tag{2.3}$$

$$\rho \vec{q} \cdot \nabla h_o = \operatorname{Re}^{-1} \operatorname{div}(\operatorname{Pr}^{-1} \operatorname{grad} h) + \operatorname{Re}^{-1} \operatorname{div}(\tau \vec{q}) \tag{2.4}$$

where h_o is the nondimensional total enthalpy. Also

$$\tau = \mu(\operatorname{def} \vec{q}) + \lambda(\operatorname{div} \vec{q})I \tag{2.5}$$

where $\operatorname{def} \vec{q}$ and I are the deformation and identity tensors. A useful combination of the differential equations is

$$a^2 \operatorname{div} \vec{q} - \vec{q} \cdot \nabla \frac{q^2}{2} = (\gamma - 1)\vec{q} \cdot \nabla h_o - \operatorname{Re}^{-1} \frac{\gamma}{\rho} \vec{q} \cdot \operatorname{div} \tau \tag{2.6}$$

A perfect gas with constant specific heats is assumed; γ is the ratio of specific heats.

Cylindrical coordinates $x = \bar{x}/\bar{L}$, $r = \bar{r}/\bar{L}$, and θ are introduced, with the origin at the nose of the body and x measured in the direction of the undisturbed stream. For axisymmetric flow the velocity vector is

$$\vec{q} = (1+u)\vec{e}_x + v\vec{e}_r \tag{2.7}$$

where $\vec{e}_x$ and $\vec{e}_r$ are unit vectors; if the body is slender, $u^2 + v^2 \ll 1$. Again for axisymmetric flow

$$\operatorname{div} \tau = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{r} \frac{\partial(r\tau_{xr})}{\partial r} \right)\vec{e}_x + \left(\frac{\partial \tau_{rx}}{\partial x} + \frac{1}{r} \frac{\partial(r\tau_{rr})}{\partial r} - \frac{\tau_{\theta\theta}}{r} \right)\vec{e}_r \tag{2.8}$$

where, with $\Theta = \operatorname{div} \vec{q}$,

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \Theta \qquad\qquad \tau_{xr} = \tau_{rx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \right)$$

$$\tag{2.9}$$

$$\tau_{rr} = 2\mu \frac{\partial v}{\partial r} + \lambda \Theta \qquad\qquad \tau_{\theta\theta} = 2\mu \frac{v}{r} + \lambda \Theta$$

6

Supersonic Far Field with Small Diffusion

For r = O(1) the flow past a slender body is described in a first approx-
imation by a velocity potential which satisfies the wave equation in the coor-
dinates x and r. For a body of revolution at zero incidence the flow is axi-
symmetric, and the solution is expressed in terms of a distribution of sour-
ces along the axis, with strength equal to the rate of change $\epsilon^2 S'(x)$ of the
body cross-section area, where $\epsilon \ll 1$. As $r \to \infty$, this solution gives

$$u \sim - \frac{\epsilon^2}{2\pi(2\,Br)^{1/2}} \int_0^{x-Br} \frac{S''(x^*)dx^*}{(x-Br-x^*)^{1/2}} \tag{2.10}$$

for $x > Br$, where $B = (M_\infty^2 - 1)^{1/2}$ and $M_\infty = \bar{u}_\infty / \bar{a}_\infty$; the body has been
taken to have a pointed nose so that $S'(0) = 0$. The result, however, is not
uniformly valid for large r because nonlinear and viscous effects have not
been taken into account.

In the case considered here, the first error encountered as r becomes
large arises because the characteristics have slopes slightly different from
the undisturbed value $dx/dr = B$. The weak shock waves from the nose and
rear of the body likewise have slopes slightly different from B. At a still
larger distance, viscous forces would also have to be included in the first
approximation. If the displacement of the characteristics from their linear-
ized locations is O(1) when $1/r = O(\delta)$, where $\delta = \delta(\epsilon)$, it is convenient to
introduce a coordinate $\eta = \delta(\epsilon)r$. Since $x - Br = O(1)$ in the disturbed
portion of the flow, a second suitable coordinate is $\xi = x - Br$. The equa-
tions are then studied in a limit as $\epsilon \to 0$ with the coordinates $\xi = x - Br$
and $\eta = \delta(\epsilon)r$ held fixed. As $\eta \to 0$, the solution found in terms of ξ and η
must match with the solution (2.10) which is correct for $1 \ll r \ll \delta^{-1}$.
Since the solution (2.10) gives $u = O(\epsilon^2 r^{-1/2})$, it follows from the matching
that $u = O(\epsilon^2 \delta^{1/2})$ when $r = O(\delta^{-1})$. The requirement that a nonlinear
term appear in the differential equation for the potential then gives $\delta = O(\epsilon^4)$;
the simplest choice is $\delta(\epsilon) = \epsilon^4$. It is also seen from this equation that
the next terms in u are $O(Re^{-1})$ and $O(\epsilon^8)$. In the case considered here,

7

$\epsilon^8 \ll Re^{-1} \ll \epsilon^4$; that is, a limit is taken with $\epsilon \to 0$ and $Re \to \infty$ such that $\epsilon^4 Re \to \infty$ and $\epsilon^8 Re \to 0$. The coordinates and the expansion for the velocity are then given by

$$\xi = x - Br, \qquad\qquad \eta = \epsilon^4 r \qquad\qquad (2.11)$$

$$u = \epsilon^4 u_1(\xi, \eta) + Re^{-1} u_2(\xi, \eta) + \ldots \qquad\qquad (2.12)$$

The shock wave from the nose of the body is located at $\xi = \xi_s(\eta)$, where

$$\xi_s = \xi_{s1}(\eta) + (\epsilon^4 Re)^{-1} \xi_{s2}(\eta) + \ldots \qquad\qquad (2.13)$$

The rear shock wave is represented in a similar way. Some general features of the flow for $\epsilon^4 r = O(1)$, and in the more distant fully viscous region discussed below, are shown in Fig. 1; coordinates used here and later are shown in Fig. 2.

The differential equation for u_1, found from Eq. (2.6) in the manner described above, is

$$2u_{1\eta} + \frac{u_1}{\eta} + \frac{(\gamma+1)M_\infty^4}{B} u_1 u_{1\xi} = 0 \qquad\qquad (2.14)$$

After multiplication by η, the differential equation states that $\eta^{1/2} u_1$ is constant along characteristics $d\xi / d\eta^{1/2} = (\gamma+1)M_\infty^4 B^{-1} \eta^{1/2} u_1$. The solution which satisfies the matching condition for $\eta \to 0$ is given by

$$u_1 = -(2B\eta)^{-1/2} F(X), \qquad\qquad X = \xi + k\eta^{1/2} F(X) \qquad\qquad (2.15, 16)$$

$$F(X) = \frac{1}{2\pi} \int_0^X \frac{S''(x^*)dx^*}{(X-x^*)^{1/2}} \qquad\qquad (2.17)$$

where $k = (\gamma+1)M_\infty^4/(2B^3)^{1/2}$. This is exactly Whitham's solution.[1,2] The shape $X = X_s(\eta)$ of the front shock wave is most easily found by use of the known results that a weak shock wave approximately bisects the angle between upstream and downstream characteristics; from Eqs. (2.13) and (2.16), it follows that

$$\xi'_{s1}(\eta) = -\frac{1}{4} k \eta^{-1/2} F(X_s) \qquad\qquad (2.18)$$

8

The result found for $X_s(\eta)$ is given implicitly by

$$\frac{1}{2} k \, \eta^{1/2} \, F^2(X_s) = \int_0^{X_s} F(X^*) dX^*$$

(2.19)

Evaluation of u_1 near the shock wave requires an expansion in the form

$$u_1(\xi, \eta) = u_1(\xi_{s1}, \eta) + \frac{\xi_{s2}}{\epsilon^4 Re} u_{1\xi}(\xi_{s1}, \eta) + \ldots + (\xi - \xi_s) u_{1\xi}(\xi_{s1}, \eta) + \ldots$$

(2.20)

where $u_{1\xi}(\xi_{s1}, \eta)$ is found by differentiating the solution (2.15) for u_1 and the definition (2.16) of X, and combining with the derivative of Eq. (2.18) for the shock-wave shape; the result is

$$u_{1\xi}(\xi_{s1}, \eta) = -\frac{4B}{(\gamma+1)M_\infty^4} \frac{d}{d\eta} \ln F(X_s)$$

(2.21)

For the calculation of u_2, the first approximation to the diffusion terms is obtained using $v \sim -Bu$ and $\partial u/\partial r \sim -B\partial u/\partial x$. From Eq. (2.4) it is found that changes in h_o along a streamline are $O(\epsilon^4 Re^{-1})$; entropy changes are also of this order. It follows from the vorticity equation, found by taking the curl of Eq. (2.3), that the vorticity is $O(\epsilon^8/Re)$, and so the flow remains irrotational to the order required here. The differential equation for u_2 is

$$2u_{2\eta} + \frac{u_2}{\eta} + \frac{(\gamma+1)M_\infty^4}{B}(u_1 u_2)_\xi = \frac{1}{B}\left(1 + \frac{\gamma-1}{Pr}\right)M_\infty^4 u_{1\xi\xi}$$

(2.22)

In the right-hand side, the viscosity coefficients appear in the combination $2\mu + \lambda \sim 1$, the reference value having been taken to be $2\bar{\mu}_\infty + \bar{\lambda}_\infty$. Equation (2.22) can be rewritten as an equation for $\eta^{1/2} u_2$ in terms of coordinates $\eta^{1/2}$ and X, where X is constant along a characteristic. The solution can then be found as

$$u_2 = -\left(1 + \frac{\gamma-1}{Pr}\right) \frac{M_\infty^4 F''}{B(2B\eta)^{1/2}(1-k\eta^{1/2}F')k^2 F'^2} \left\{ \ln(1-k\eta^{1/2}F') + \frac{k\eta^{1/2}F'}{1-k\eta^{1/2}F'} \right\}$$

(2.23)

9

where $F' = F'(X)$; an arbitrary function of X has been chosen as that u_2 remains bounded as $\eta \to 0$.

Shock-Wave Structure and Location

In order to find the second term $\xi_{s2}(\eta)$ in the shock-wave shape, it is necessary to obtain both first- and second-order solutions for the velocity distribution within the shock wave. Coordinates $\tilde{x}$ and y are defined to be of order one at points within the shock wave when $r = O(\epsilon^{-4})$ and are measured, respectively, normal to and along a curve which remains within the shock wave as $\epsilon \to 0$ and $Re \to \infty$ (Fig. 2). This curve is chosen such that the normal velocity component is sonic at every point, and its angle of inclination $\beta(\eta)$ from the radial direction is related to the expansion (2.13) of $\xi_s(\eta)$ by

$$\operatorname{ctn} \beta(\eta) = B + \epsilon^4 \xi'_{s1}(\eta) + Re^{-1} \xi'_{s2}(\eta) + \ldots \qquad (2.24)$$

The coordinate transformation from x, r to $\tilde{x}, y$ can be carried out in three steps. First, x and r are replaced by $\xi - \xi_s(\eta)$ and η; next, at each point the coordinate directions are rotated through the angle $\beta(\eta)$; and finally, the coordinate measured normal to the shock wave is stretched by a factor $\Delta_s(y) \ll 1$, where $\Delta_s(y)$ is a suitably defined shock-wave thickness, as obtained below. The transformation of derivatives can be summarized in the form

$$\frac{\partial}{\partial x} = \frac{1}{\Delta_s} \sin \beta \frac{\partial}{\partial \tilde{x}} + \epsilon^4 \cos \beta \left(\frac{\partial}{\partial y} - \frac{\tilde{x}}{\Delta_s} \frac{d\Delta_s}{dy} \frac{\partial}{\partial \tilde{x}} \right) \qquad (2.25)$$

$$\frac{\partial}{\partial r} = -\frac{1}{\Delta_s} \cos \beta \frac{\partial}{\partial \tilde{x}} + \epsilon^4 \sin \beta \left(\frac{\partial}{\partial y} - \frac{\tilde{x}}{\Delta_s} \frac{d\Delta_s}{dy} \frac{\partial}{\partial \tilde{x}} \right) \qquad (2.26)$$

The velocity perturbations U and V, in the $\tilde{x}$ and y directions respectively, are

$$U = u \sin \beta - v \cos \beta \qquad (2.27)$$

$$V = u \cos \beta + v \sin \beta \qquad (2.28)$$

Changes in entropy and in total enthalpy within the shock wave are $O(\epsilon^8)$. The vorticity is extremely small, of order ϵ^{16}, since the shock-wave curvature is very small, and it follows that $\Delta_s^{-1} V_{\tilde{x}} \sim \epsilon^4 U_y$. The differential equation for U is found to be

$$\frac{1}{M_\infty^3} (\text{ctn}^2\beta - B^2) \frac{1}{\Delta_s} U_{\tilde{x}} - (\gamma+1) \frac{1}{\Delta_s} U U_{\tilde{x}}$$

$$- \frac{\epsilon^4 B}{M_\infty^2} \left\{ \frac{U}{\eta} + 2 M_\infty U_y - M_\infty \frac{\tilde{x}}{\Delta_s} \frac{d\Delta_s}{dy} U_{\tilde{x}} \right\} = - \left\{ 1 + \frac{\gamma-1}{Pr} \right\} \frac{1}{\Delta_s^2 Re} U_{\tilde{x}\tilde{x}} + \dots$$

$$(2.29)$$

where $d\eta/dy = M_\infty^{-1}$. Equation (2.29) is to be studied in the limit as $\epsilon \rightarrow 0$ and $Re \rightarrow \infty$ with coordinates $\tilde{x}$ and y held fixed.

The velocity U is represented in the form

$$\frac{U}{\Delta_u} = -1 + U_1(\tilde{x}) - \frac{\xi'_{s2}(\eta)}{\epsilon^4 Re \xi'_{s1}(\eta)} + \frac{1}{\epsilon^4 Re} U_2(\tilde{x}; y) + \dots \qquad (2.30)$$

where $2\Delta_u$ is the first approximation to the jump in U across the shock wave, and $\xi'_{s1}(\eta)$ is given by Eq. (2.18). The shock-wave thickness Δ_s used in the definition of $\tilde{x}$ is found by equating the coefficients of the nonlinear and diffusion terms in Eq. (2.29), with a factor of two included for later convenience. Thus Δ_u and Δ_s are given by

$$\Delta_u = \epsilon^4 \frac{M_\infty F(X_s)}{2(2B\eta)^{1/2}} \qquad\qquad \Delta_s = \frac{2}{(\gamma+1)\Delta_u} \left\{ 1 + \frac{\gamma-1}{Pr} \right\} \frac{1}{Re} \qquad (2.31)$$

As for a one-dimensional shock wave of the same strength, $\epsilon^4 \Delta_u$ is the first approximation to the difference between the normal velocity component ahead of the shock wave and the critical sound speed based on that velocity.

11

Since $\tilde{x}$ has been defined such that the normal velocity component is sonic at $\tilde{x} = 0$, it follows that $U_1(0) = 0$; similarly, for the second approximation it is seen that $U_2(0) = 0$. As $\tilde{x} \to -\infty$, $U_1 \to 1$ and $U_2 \to \xi'_{s2}/\xi'_{s1}$.

The differential equations for U_1 and U_2 are

$$U_{1\tilde{x}\tilde{x}} - 2U_1 U_{1\tilde{x}} = 0 \tag{2.32}$$

$$U_{2\tilde{x}\tilde{x}} - 2(U_1 U_2)_{\tilde{x}} = A\left\{\left(\frac{1}{\eta} + \frac{2}{\Delta_u}\frac{d\Delta_u}{d\eta}\right)(-1 + U_1) + \frac{1}{\Delta_u}\frac{d\Delta_u}{d\eta}\tilde{x}U_{1\tilde{x}}\right\} \tag{2.33}$$

where $A = 2B(\gamma+1)^{-1}M_\infty^{-2}\epsilon^8 Re\Delta_s\Delta_u^{-1} = O(1)$. The solutions which satisfy the required boundary conditions are given by

$$U_1(\tilde{x}) = -\tanh\tilde{x} \tag{2.34}$$

$$\frac{1}{A}(4\cosh^2\tilde{x})U_2(\tilde{x};\eta) = (e^{2\tilde{x}} - 2\tilde{x} - 1)\frac{d}{d\eta}\ln F(X_s)$$

$$- (e^{2\tilde{x}} + \tilde{x} - 1)\tilde{x}\frac{d}{d\eta}\ln[\eta^{-1/2}F(X_s)]$$

$$- \left\{(\sinh 2\tilde{x})\ln(1 + e^{2\tilde{x}}) + \int_1^{e^{2\tilde{x}}}t^{-1}\ln(1 + t)dt\right\}\frac{d}{d\eta}\ln[\eta^{1/2}F(X_s)]$$

$$- \frac{4\xi'_{s2}(\eta)}{A\xi'_{s1}(\eta)}(\tilde{x} + \sinh\tilde{x}\cosh\tilde{x}) \tag{2.35}$$

The result for $U_1(\tilde{x})$, with the definition of Δ_s given above, is the familiar solution for a weak one-dimensional shock wave. As $\tilde{x} \to \infty$, the solution for $U_2(\tilde{x})$ has the form

$$U_2 \sim -A(2\tilde{x} - 1)\frac{d}{d\eta}\ln F(X_s) - \frac{\xi'_{s2}(\eta)}{\xi'_{s1}(\eta)} \tag{2.36}$$

After the coordinate transformation is taken into account, the first term in U_2 is found to match correctly with the result of Eq. (2.21). Matching the constant terms of order Re^{-1} in the expansions (2.12) and (2.30) of u, evaluated for $\xi \to \xi_s$ and for $\tilde{x} \to \infty$ respectively, involves terms which appear

in Eqs. (2.20), (2.23), (2.30), and (2.36). The condition obtained gives a first-order differential equation for the second term $\xi_{s2}(\eta)$ appearing in the shock-wave shape (2.13):

$$\frac{d}{d\eta}(F\xi_{s2}) = -(1 + \frac{\gamma-1}{Pr}) \frac{M_\infty^4 F}{2B(1 - k\eta^{1/2}F')^2} \left\{ \frac{F''}{2F'} + \frac{F'}{F}(1 - k\eta^{1/2}F') \right.$$

$$\left. + \frac{(1 - k\eta^{1/2}F')F''}{2k\eta^{1/2}F'^2} \ln(1 - k\eta^{1/2}F') \right\} \tag{2.37}$$

where F, F', and F'' are evaluated at the shock wave $X = X_s(\eta)$. As $\eta \to 0$, $F(X_s) = O(X_s^{1/2})$ and $X_s = O(\eta)$, and the right-hand side of Eq. (2.37) is $O(\eta^{-1/2})$. Integration of Eq. (2.37) should be carried out from $\eta = 0$, so that $\xi_{s2}(\eta)$ is bounded as $\eta \to 0$.

A solution for the velocity u which is uniformly valid for all ξ can be constructed as the product of $\epsilon^4 u_1$ with suitable factors which account for the front and rear shock waves. For the rear shock wave, a solution can be derived in a form analogous to Eq. (2.30), in terms of a coordinate $\tilde{x}_r$ analogous to $\tilde{x}$. The composite solution then can be written as

$$u \sim \frac{1}{4}\epsilon^4(1 + \tanh \tilde{x})(1 - \tanh \tilde{x}_r)u_1(\xi, \eta) \tag{2.38}$$

To the degree of approximation needed here, the coordinate transformation given by Eqs. (2.25) and (2.26) reduces to

$$M_\infty \Delta_s(y)\tilde{x} = \xi - \xi_{s1}(\eta) - \frac{1}{\epsilon^4 Re} \xi_{s2}(\eta), \qquad y = M_\infty \eta \tag{2.39}$$

where, as shown in Fig. 2, $\tilde{x}$ and y are measured, respectively, normal to and along the curve $\xi = \xi_{s1}(\eta) + (\epsilon^4 Re)^{-1}\xi_{s2}(\eta)$; a corresponding definition can be given for $\tilde{x}_r$. Since ξ_{s2} has been calculated, the error in the location of the curve $\xi = \xi_s(\eta)$, and therefore in the origin for $\tilde{x}$, is of higher order than the shock-wave thickness $\Delta_s = O(\epsilon^{-4}Re^{-1})$.

<u>Diffusion Effects Elsewhere</u>

Small values of X identify characteristics which intersect the body near the nose, which has been assumed pointed so that $S''(x)$ remains finite as $x \to 0$. Thus for $X \ll 1$ the flow differs only slightly from the flow past a slender circular cone having surface $r = \epsilon \lambda x$. The definition (2.17) gives $F(X) \sim 2\lambda^2 X^{1/2}$ as $X \to 0$, and the first approximation (2.18) to the shock-wave shape shows that $X_s(\eta) \sim \frac{9}{4} k^2 \lambda^4 \eta$ as $\eta \to 0$. The expansion (2.13) for the shock-wave shape becomes

$$\xi_s(\eta) = -\frac{3}{4} k^2 \lambda^4 \eta - \Delta(1 + \frac{\gamma-1}{Pr}) \frac{2B^2}{3(\gamma+1)^2 M_\infty^4} (-1 + \frac{3}{2} \ln 3) + \dots \qquad (2.40)$$

where $\Delta = (\epsilon^4 Re)^{-1}$. The second term remains small in comparison with the first term provided that $\eta \gg \Delta$. As $\eta \to 0$ the largest terms in the differential equation (2.29) become

$$(\frac{U}{\Delta_u})_{\widetilde{x}\widetilde{x}} - 2(1 + \frac{U}{\Delta_u})(\frac{U}{\Delta_u})_{\widetilde{x}} \sim \frac{2B\epsilon^4}{(\gamma+1)M_\infty^2 \Delta_u} \left\{ \frac{U/\Delta_u}{\eta/\Delta_s} + 2M_\infty \frac{\partial(U/\Delta_u)}{\partial(y/\Delta_s)} \right\} \qquad (2.41)$$

Since both Δ_u and Δ_s approach constant values as $\eta \to 0$, the right-hand side is small, and the first approximation leads to the one-dimensional velocity distribution $U \sim \Delta_u(-1 + U_1)$, only if $\eta \gg \Delta$. If $\eta = O(\Delta)$, and so $r = O(\epsilon^{-8} Re^{-1})$, all the terms shown in Eq. (2.41) must be retained in the first approximation. The differential equation obtained is related to Burgers' equation, but contains an additional term because the flow is axisymmetric.

As $\eta \to \infty$, $X_s(\eta) \to X_o = $ constant, where $X = X_o$ defines the characteristic at which $u_1 = 0$. Then Eq. (2.19) gives

$$F^2(X_s) \sim \frac{2}{k\eta^{1/2}} \int_0^{X_o} F(X^*)dX^* \qquad (2.42)$$

so that $F(X_s) = O(\eta^{-1/4})$. Also $\xi_{s1} = O(\eta^{1/4})$, $\Delta_u = O(\epsilon^4 \eta^{-3/4})$, and $\Delta_s = O(\Delta \eta^{3/4})$, where again $\Delta = (\epsilon^4 Re)^{-1}$, proportional to the ratio of second-order to first-order terms in the preceding results. It follows (Fig. 1) that

14

the shock-wave thickness is no longer small in comparison with the distance between the front and rear shock waves when $\eta = O(\Delta^{-2})$. Then also $\xi = O(\Delta^{-1/2})$ and $u = O(\epsilon^4 \Delta^{3/2})$ in this fully viscous region. The largest terms in the differential equation (2.6), all of the same small order of magnitude, give an equation

$$\epsilon^4 (2u_\eta + \frac{u}{\eta}) + \frac{(\gamma+1)M_\infty^4}{B} uu_\xi \sim \frac{1}{Re} (1 + \frac{\gamma-1}{Pr}) \frac{M_\infty^4}{B} u_{\xi\xi} \qquad (2.43)$$

which, like Eq. (2.41), is similar to Burgers' equation.

At still larger distances, for $\eta \gg \Delta^{-2}$, the linear terms are all of the same order of magnitude if $\xi = O(\Delta^{1/2}\eta^{1/2})$, and the nonlinear terms are smaller provided that $\epsilon^{-4}\Delta^{-3/2}u \ll 1$. The result is the one-dimensional diffusion equation for $(\Delta^2\eta)^{1/2}(\epsilon^4\Delta^{3/2})^{-1}u$. Self-similar solutions have the form

$$(\epsilon^4\Delta^{3/2})^{-1}u = (\Delta^2\eta)^{-\alpha - \frac{1}{2}} g(t) \qquad (2.44)$$

where $t = \Delta^{-1/2}\xi/\eta^{1/2}$ and g satisfies the ordinary differential equation

$$\frac{1}{2} (1 + \frac{\gamma-1}{Pr}) \frac{M_\infty^4}{B} g'' + \frac{1}{2} t g' + \alpha g = 0 \qquad (2.45)$$

The nonlinear terms are of higher order if $\alpha > -1/2$. If at large distances the body appears as a point source, the velocity potential according to linear inviscid-flow theory is proportional to $\epsilon^2(x^2 - B^2r^2)^{-1/2}$ for $x > Br$. Therefore $u = O(\epsilon^4\xi^{-3/2}\eta^{-1/2})$ as $x(Br)^{-1} - 1 \to 0$. A solution of the form (2.44), which includes the effects of viscosity, can match as $t \to \infty$ with this potential-flow solution if $\alpha = 3/4$; this is equivalent to the results of Refs. 5 and 6. The solution for g(t) can be written in terms of confluent hypergeometric functions, whose properties are known[13]. The result is exponentially

---

[13] Abramowitz, M. and Stegun, I. A. (eds.), Handbook of Mathematical Functions, National Bureau of Standards, 1964.

15

small as $t \to -\infty$ (upstream) and is of order $t^{-3/2}$ as $t \to +\infty$ (downstream).
The integral of $f(t)$ over the range $-\infty < t < \infty$ is found, by direct integra-
tion of the differential equation (2.45), to be zero. The second term in u is
the simple dipole solution for $\alpha = 1$, which is exponentially small both up-
stream and downstream; the nonlinear term is still of higher order. If
there were no net mass outflow of order ϵ^2 from the neighborhood of the
body, the solution for $\alpha = 1$ would be the leading term in u.

Transonic Flows

The definition (2.11) of η and the expansion (2.12) for u could have been
modified so as to show explicitly the dependence on the parameter B. The
orders of magnitude for r and u would again be found by specifying that a
nonlinear term must appear in the differential equation (2.14) and that the
solution for u must match asymptotically with the solution (2.10). If M_∞
is then allowed to decrease toward one, so that $B \to 0$, the nonlinearity
appears for $r = O(\epsilon^{-4} B^3)$, with $u = O(\epsilon^4 B^{-2})$, whereas the linearized flow
description, in terms of a source distribution along the axis, gives the solu-
tion for $r = O(B^{-1})$. These distances are no longer distinct when $B = O(\epsilon)$,
corresponding to the usual transonic small-disturbance theory derived with
the similarity parameter $K = (M_\infty^2 - 1)/\epsilon^2$ held fixed as $\epsilon \to 0$ and $M_\infty \to 1$.
For $B = O(\epsilon)$, then, $u = O(\epsilon^2)$ and the first approximation satisfies the non-
linear transonic small-disturbance equation in the variables x and $\tilde{r} = \epsilon r$.
The ratio of second-order to first-order terms is $O(\Delta_t)$, where $\Delta_t = (\epsilon^2 Re)^{-1}$
for the transonic limit; the shock-wave thickness is also of this order.
At small values of $\tilde{r}$ such that $\tilde{r} = O(\Delta_t)$, the shock wave is no longer one-
dimensional but is described instead by the "viscous-transonic" equation,
which is the transonic small-disturbance equation augmented by the diffusion
term proportional to u_{xx}. At large values of $\tilde{r}$, again as $B \to 0$, the previous
solutions for shock-wave location and thickness give $\xi_{s1} = O(\epsilon B^{-3/4} r^{1/4})$
and $\Delta_s = O(\epsilon^{-1} B^{-1/4} Re^{-1} r^{3/4})$. These two lengths are no longer distinct if
$r = O(\epsilon^4 B^{-1} Re^2)$ and $\xi = O(\epsilon^2 B^{-1} Re^{1/2})$. In the fully viscous region defined
in this way, the velocity is $u = O(\epsilon^{-2} B Re^{-3/2})$. For $B = O(\epsilon)$, the solution

for $\widetilde{r} = O(1)$ no longer has the form shown by Eqs. (2.15) through (2.17), but for $\widetilde{r} \gg 1$ the quantities ξ_{s1} and Δ_s have the same power-law behavior as for $B \gg \epsilon$, now with unknown numerical factors. The fully viscous region for $B = O(\epsilon)$ is then defined by $\widetilde{r} = O(\Delta_t^{-2})$ and $\xi = O(\Delta_t^{-1/2})$, and can again be described by the modified form of Burgers' equation required for axi-symmetric flow.

If the free-stream speed is exactly sonic, the inviscid-flow solution as $\widetilde{r} \to \infty$ has the self-similar form $u = \epsilon^2 \widetilde{r}^{-6/7} f(x/\widetilde{r}^{4/7})$. Then $u u_x = O(\epsilon^4 \widetilde{r}^{-16/7})$ and $Re^{-1} u_{xx} = O(Re^{-1} \epsilon^2 \widetilde{r}^{-2})$ as $\widetilde{r} \to \infty$. There now is a single shock wave having thickness $\Delta_s = O(Re^{-1} \epsilon^{-2} \widetilde{r}^{6/7})$, whereas the inviscid-flow solution implies $x = O(\widetilde{r}^{4/7})$. By either comparison it is seen that the fully viscous region, described by the viscous-transonic equation, appears for $\widetilde{r} = O(\Delta_t^{-7/2})$. The self-similar solution then is correct for $1 \ll \widetilde{r} \ll \Delta_t^{-7/2}$. If now B is increased from zero, a linear term $B^2 u_x$ will first contribute to the description of the fully viscous region when B^2 is no longer small in comparison with u; for this condition $B/\epsilon = O(\Delta_t^{3/2})$. The previous result $r = O(\epsilon^4 B^{-1} Re^2)$ for the distance to the fully viscous region is still correct for B/ϵ as small as $O(\Delta_t^{3/2})$, but is replaced by $\widetilde{r} = O(\Delta_t^{-7/2})$ for $B/\epsilon \ll \Delta_t^{3/2}$. The orders of magnitude for the locations of regions in which different approximations are required are shown schematically in terms of Mach number in Fig. 3.

For subsonic flow with M_∞ slightly smaller than one, the similarity parameter is $K = (1-M_\infty^2)/\epsilon^2$, and the transonic small-disturbance equation is obtained in coordinates x and $\widetilde{r}$ if $(1-M_\infty^2)/\epsilon^2 = O(1)$. For $\widetilde{r} \gg 1$, the flow is purely subsonic, and the body (if closed) appears as a doublet with $\epsilon^{-2} \widetilde{r}^3 u$ equal to a function of $x/\widetilde{r}$. The linear term $(1-M_\infty^2) u_x$ is then of order $(1-M_\infty^2)\epsilon^2 \widetilde{r}^{-4}$, whereas the diffusion term is proportional to $Re^{-1} u_{xx} = O(Re^{-1}\epsilon^2 \widetilde{r}^{-5})$. That is, the diffusion term decays faster and never appears in the first approximation as $\widetilde{r} \to \infty$. If, instead, $0 < 1 - M_\infty^2 \ll \epsilon^2$, the first approximation for $\widetilde{r} = O(1)$ is

the same as for $M_\infty = 1$, and $(1-M_\infty^2)u_x = O((1-M_\infty^2)\epsilon^2\tilde{r}^{-10/7})$ as $\tilde{r} \to \infty$. This term is no longer small in comparison with the nonlinear term when u and $(1-M_\infty^2)$ are of the same order, for $\tilde{r} = O(K^{-7/6})$. The shock wave then extends to a distance $\tilde{r} = O(K^{-7/6})$ and its thickness remains small in comparison with $\Delta x = O(\tilde{r}^{4/7})$ provided that $K^{-7/6} \ll \epsilon^7 Re^{7/2}$, i.e., $(1-M_\infty^2)/\epsilon^2 \gg \Delta_t^{-3}$. In other words, the doublet describes the far field for $(1-M_\infty^2)/\epsilon^2 \gg \Delta_t^{-3}$ and a fully viscous region appears for large values of $\tilde{r}$ only in the very narrow Mach number range $(1-M_\infty^2) = O(\epsilon^{-4}Re^{-3})$.

Sample numerical values indicate that viscous effects are very small, as expected, and in some respects become less important as the Mach number decreases toward one. A convenient pair of values is $\epsilon = 0.1$, $Re = 10^6$. At supersonic speeds, nonlinear effects become essential for $\eta = \epsilon^4 r = O(1)$, i.e., for $r = O(10^4)$. At this distance the shock-wave thickness is $O(\Delta)$, where $\Delta = (\epsilon^4 Re)^{-1} = 0.01$; the fully viscous region appears for $\eta = O(\Delta^{-2})$, i.e., for $r = O(10^8)$. At $M_\infty = 1$, the shock-wave thickness for $\tilde{r} = \epsilon r = O(1)$ is $O(\Delta_t)$, where $\Delta_t = (\epsilon^2 Re)^{-1} = 10^{-4}$, and the fully viscous region appears for $\tilde{r} = O(\Delta_t^{7/2})$, i.e., for $r = O(10^{15})$. The solutions for $M_\infty = 1$ are correct in the extremely narrow Mach-number range $|B|^2/\epsilon^2 = O(\Delta_t^3)$, i.e., $|M_\infty^2 - 1| = O(10^{14})$. Some of these features are apparent in Fig. 3.

References

1. Whitham, G. B., The Flow Pattern of a Supersonic Projectile, Comm. Pure Appl. Math., 5 (1952), 301-348.

2. Whitham, G. B., Linear and Nonlinear Waves, John Wiley and Sons, 1974.

3. Lighthill, M. J., Viscosity Effects in Sound Waves of Finite Amplitude, in Surveys in Mechanics, eds. G. K. Batchelor and R. M. Davies, Cambridge Univ. Press, 1956, pp. 250-351.

4. Ryzhov, O. S., Influence of Viscosity and Thermal Conductivity on Propagation of Sound Impulses, P. M. M. 30 (1966), 296-302. Engl. transl. J. Appl. Math. Mech., 30 (1966), 362-369.

5. Salathe, E. P., The Fundamental Matrix in Three-Dimensional Dissipative Gasdynamics, J. Fluid Mech. 39 (1969), 209-225.

6. Chong, T. H. and Sirovich, L., Structure of Three-Dimensional Supersonic and Hypersonic Flow, Phys. Fluids, 13 (1970), 1990-1999.

7. Sanchez-Palencia-Hubert, J., Asymptotic Study at Infinity of Supersonic Flows of a Dissipative Fluid, Int. J. Eng. Sci., 14 (1976), 567-584.

8. Chong, T. H. and Sirovich, L., Nonlinear Effects in Steady Supersonic Dissipative Gasdynamics. Part 2. Three-Dimensional Axisymmetric Flow, J. Fluid Mech., 58 (1973), 53-63.

9. Szaniawski, A., The Asymptotic Structure of Weak Shock Waves in Flows over Symmetrical Bodies at Mach Number Unity, Acta Mech., 5 (1968), 189-203.

10. Ryzhov, O. S. and Shefter, G. M., On the Effect of Viscosity and Thermal Conductivity on the Structure of Compressible Flows, P. M. M. 28 (1964), 996-1007. Engl. transl. J. Appl. Math. Mech., 28 (1964), 1206-1218.

11. Sichel, M., Two-Dimensional Shock Structure in Transonic and Hypersonic Flow, in *Advances in Applied Mechanics*, 11, ed. C. S. Yih, Academic Press, 1971, pp. 131-207.

12. Ryzhov, O. S., Viscous Transonic Flows, in *Ann. Rev. Fluid Mech.*, 10 (1978), 65-92.

13. Abramowitz, M. and Stegun, I. A. (eds.), *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.
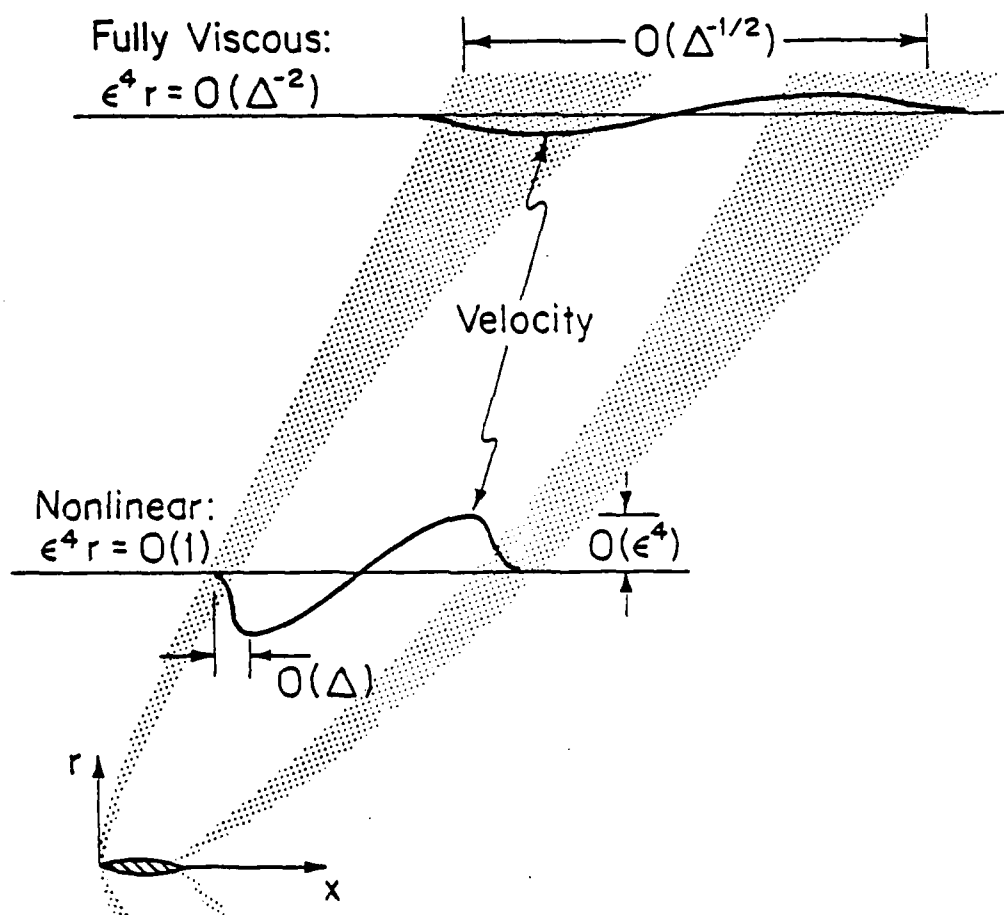
Figure 1. Flow regions for $\Delta = (\epsilon^4 \, Re)^{-1} \ll 1$.

Figure 2. Coordinate systems.

Fully Viscous:
$r = O(\epsilon^4 B^{-1} Re^2)$

Transonic:
$r = O(\epsilon^{4/3} B^{-7/3})$

Nonlinear:
$r = O(\epsilon^{-4} B^3 M_\infty^{-8})$

Linear: $r = O(B^{-1})$

$B^2 = M_\infty^2 - 1$

$B = O(\epsilon^{-2} Re^{-3/2})$

$B = O(\epsilon)$

r

Fully Viscous:
$r = O(\epsilon^6 Re^{7/2})$

Transonic:
$r = O(\epsilon^{4/3} |B^2|^{-7/6})$

Self-Similar
Solution
(for $M_\infty = 1$)

Transonic ($M_\infty = 1$)
$r = O(\epsilon^{-1})$

$B^2 = O(\epsilon^2)$

Linear:
$r = O(|B^2|^{-1/2})$

Figure 3. Mach-number ranges for various approximations.

3. LOCATION OF A SHOCK WAVE IN LOW REYNOLDS NUMBER FLOW

T. C. Adamson, Jr.

Introduction

It is well known that in numerical solutions of supersonic flow fields the location of a shock wave is known only to within three or four mesh points when the so-called shock capturing technique is used. That is, the shock wave appears not as a discontinuity, but 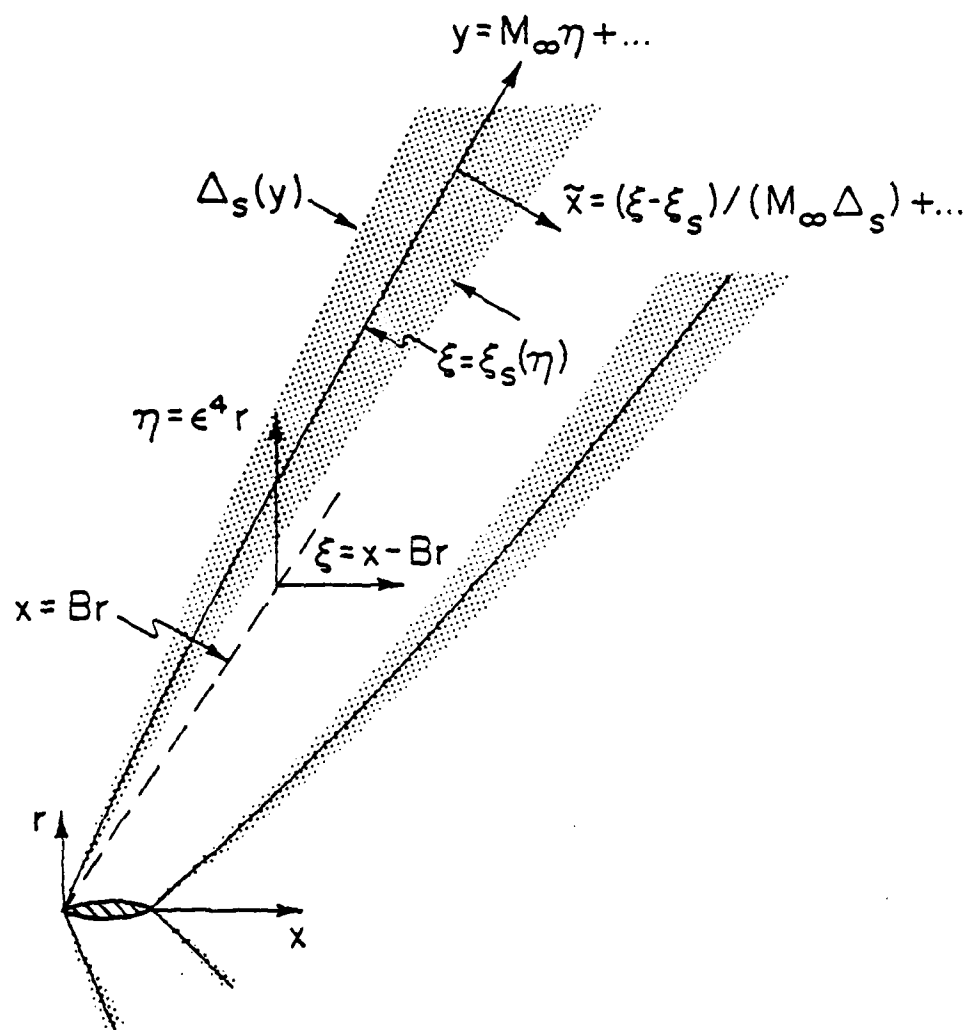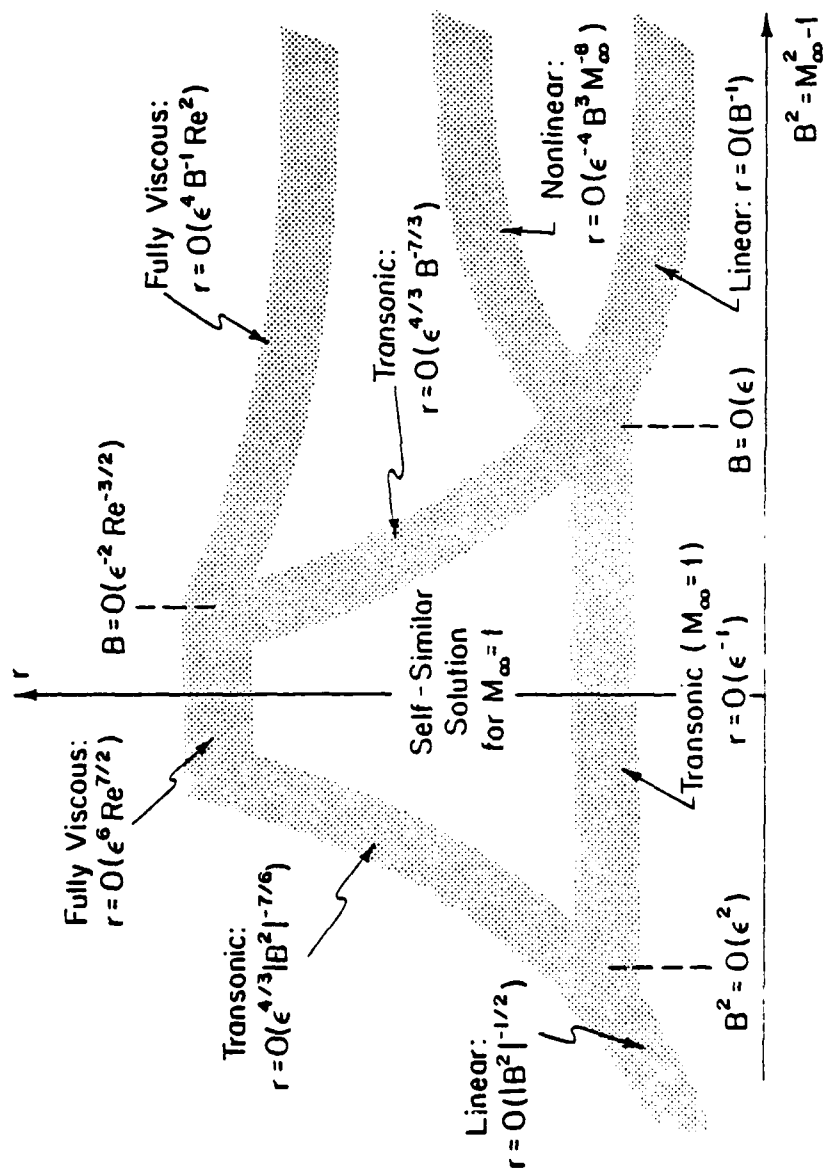as a thick region in which flow properties vary relatively rapidly. Often this is not a serious drawback because only a single shock wave occurs and it is not necessary to locate the shock wave in the flow field more accurately in order to calculate forces on the body to the desired accuracy. The fact that the pressure distribution at the intersections of the shock wave and the wall is spread out over a region large compared to the thickness of the shock wave is not in itself incorrect in view of the fact that this is what occurs in the interaction region caused by the intersection of a shock wave and a boundary layer; if the extent of this expanded pressure distribution is not too much different from that of the interaction region, the errors incurred may not be prohibitive. However, before accurate results using known solutions in the interaction region can be calculated, it is necessary to locate the shock wave accurately. Moreover, in flows over complex bodies and in internal flows, where complicated shock wave patterns and reflections are important, such a lack of preciseness in location of the shock wave does not suffice.

One of the methods for calculating the location of the shock wave accurately is that of shock fitting (e. g., see reference 1). Here, at each iteration of the solution, the shock wave is positioned such that the proper

---

[1] Hafez, M. M. and Cheng, H. K., Shock Fitting in Transonic Flow Computation, Transonic Flow Problems in Turbomachinery, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 328-335.

jump conditions are met and the solution on either side of the shock wave merges with the known numerical solution. Unfortunately, the logic involved in this calculation is complicated for even a single shock wave and, at least in the present form, too unwieldy to consider for complex shock wave patterns. It is clear that a simple method for locating the shock wave would be of considerable value.

This section covers the initial investigation of a method different from those considered before, for locating a shock wave. It is applicable in those cases where mass, momentum, and energy are conserved in the "thick" shock wave given by the numerical solutions. The essential idea is that the thick shock corresponds to a shock wave in a relatively low Reynolds number flow. An asymptotic solution for, say, the distribution of velocity through a shock wave is to be compared with the numerical solution to give the values of the necessary parameters. Then, the analytical solution is to be used to locate the shock wave in the actual relatively high Reynolds number flow by simply using the solution valid in the limit as Reynolds number tends to infinity. In actuality, because the location of the sonic line within the shock wave corresponds to the location of the shock wave in the real flow case, where the shock is very thin, the calculation reduces to finding the dependence of the location of the sonic line on the Reynolds number. As it turns out, in the flow problem considered there is a range of parameters for which the sonic line (within the shock wave) is independent of the Reynolds number to the accuracy desired. In this event, the shock wave is very easily located by locating the sonic line. For smaller Reynolds numbers, the equation derived for the first order correction to the location of the shock wave indicates an apparent dependence upon Mach number. However, this equation contains a term which has not been evaluated; there is a possibility that the final equation will be independent of Reynolds number.

In order to test the utility of the proposed method, it is necessary to choose a flow problem for which solutions in the form of numerical computations can be found relatively easily and for which analytical solutions are known. The test then consists of comparing the shock locations found using

25

on the one hand the numerical solution in conjunction with the shock wave locating method and on the other hand the analytical solution. The test problem chosen is that of steady, transonic flow through a 2-dimensional channel with arbitrary wall shape. Solutions for inviscid flow upstream and downstream of the shock wave are known in the form of asymptotic expansions uniformly valid to the second order, [2, 3, 4] both for steady and unsteady flow. It is necessary to modify these solutions somewhat and to add a higher order term and solutions for the structure of a shock wave to complete the relations needed here. Since only the flow downstream of a sonic nozzle is really needed for testing purposes, the numerical computations should not be very difficult.

In this report, then, we first present the analytical solutions for the inviscid flow, to be used as a basis for comparison. In these solutions, the shock wave appears, of course, as a discontinuity. Next, we consider the same channel flow, but now for the case where the Reynolds number is relatively low, but still large compared to unity. In this case, the shock is relatively thick, and it is this solution which is to be compared with the numerical solution to find the equivalent Reynolds number, and other necessary parameters. It is also this solution, then, which is to be used to find the change in location of the shock wave as Reynolds number becomes very large; the change is used in correcting the numerical solution.

It should be noted that the method of computation used has much to do with the form of "thick shock wave" found in numerical computations. That is, with a straightforward finite difference scheme, one might find a monotonic

[2] Messiter, A. F. and Adamson, T. C., Jr., On the Flow near a Weak Shock Wave Downstream of a Nozzle Throat, J. Fluid Mech., 69 (1975), 97-108.

[3] Richey, G. K. and Adamson, T. C., Jr., Analysis of Unsteady Transonic Channel Flow with Shock Waves, AIAA J., 14 (1976), 1054-1061.

[4] Adamson, T. C., Jr., Messiter, A. F. and Liou, M. S., Large Amplitude Shock-Wave Motion in Two-Dimensional, Transonic Channel Flows, AIAA J., 16 (1978), 1240-1247.

decrease in velocity, for example, which resembles the distribution found in a shock wave; with other formulations, an oscillation may be found in the velocity in the region of the shock. In the latter case, the oscillations can be removed with the addition of an artificial viscosity and it is supposed that this would be done if the present method for locating the shock wave were to be used.

Problem Descriptions - Solutions for Inviscid Flow

We consider a steady transonic flow in a two-dimensional channel. A sketch illustrating the coordinate system and notation used is shown in figure (1). The specific case chosen here is that where the flow accelerates through a sonic throat to a supersonic velocity; at some point downstream of the throat, the channel walls become parallel to each other and terminate in a plenum chamber, at which a back pressure is applied such that a shock wave forms in the channel.

The gas is assumed to follow the perfect gas law and to have constant specific heats. The flow upstream of the shock wave is isentropic, and the shock itself is weak enough that a velocity potential may be used to the order desired. Coordinates x and y are made dimensionless with respect to the throat half width, $\bar{L}$, and velocity components with respect to the undisturbed critical sound speed, $\bar{a}*$. (Overbars denote dimensional quantities.) The pressure, p, density, ρ, and temperature, T, are made dimensionless with respect to their critical values, and the enthalpy, H, is referred to $\bar{a}*^2$.

The wall shape is written as follows, for symmetric channels, where $\epsilon \ll 1$:

$$y_w = \pm (1 + \epsilon^2 f(x)) \qquad (3.1)$$

where $f(0) = f'(0) = 0$. Thus, ϵ^2 may be related to the radius of curvature of the walls at the throat, $x = 0$.

Composite solutions, uniformly valid to order ϵ^2, may be written for the flow under consideration as follows:[2, 3, 4]

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \tag{3.2a}$$

$$v = \epsilon^2 v_2 + \epsilon^{5/2} \zeta_y^* + \ldots \tag{3.2b}$$

$$p = 1 - \epsilon \gamma u_1 - \epsilon^2 \gamma u_2 + \ldots \tag{3.2c}$$

$$\rho = 1 - \epsilon u_1 - \epsilon^2 (u_2 + \frac{(\gamma-1)}{2} u_1^2) + \ldots \tag{3.2d}$$

$$T = 1 - \epsilon(\gamma-1) - \epsilon^2 (\gamma-1)(u_2 + \frac{u_1^2}{2}) + \ldots \tag{3.2e}$$

where $\gamma = c_p / c_v$ is the ratio of specific heats and where

$$u_1 = \pm \sqrt{\frac{2}{(\gamma+1)} f(x) + C_w} \tag{3.3a}$$

$$u_2 = f'' \frac{y^2}{2} + h_x + \zeta_{x^*}^* , \qquad v_2 = f' y \tag{3.3b, c}$$

$$h_x = -\frac{1}{6}[f'' + u_1^2 (2\gamma - 3)] + \frac{C_2}{u_1} \tag{3.3d}$$

$$\zeta^* = 4 f_0'' [(\gamma+1)u_{10}]^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(\pi n)^3} \cos(n \pi y) \cdot$$

$$\exp\{-n\pi x^* /[(\gamma+1)u_{10}]^{1/2}\} \qquad x > x_s \tag{3.3e}$$

$$\zeta^* = 0 \qquad\qquad x < x_s \tag{3.3f}$$

$$x^* = (x - x_{so}) \epsilon^{-1/2} \tag{3.3g}$$

In eqn. (3.3a), C_w is an arbitrary constant determined by the value of the velocity at the throat; i.e. if, there, the flow is subsonic or supersonic, $C_w = 0$. The constant C_2 is arbitrary and may have different values upstream and downstream of the shock wave. In the equations above, $f' = df/dx$, etc. Also, f_0' and u_{10} are f' and the positive value of u_1,

28

respectively, evaluated at x_{so}, the zero-order approximation to the location of the shock wave, x_s, where x_s is expanded as follows:

$$x_s = x_{so} + \epsilon x_{s1} + \epsilon^{3/2} x_{s3/2}(y) + \epsilon^2 x_{s2}(y) + \ldots \qquad (3.4)$$

That is, $dx_s/dy = [v]/[u] = O(\epsilon^{5/2}/\epsilon) = O(\epsilon^{3/2})$, where the square brackets indicate the jump in the quantity enclosed across the shock wave at the point y and $x_s(y)$, but it can be shown[2] that there is a term of $O(\epsilon)$ in the expansion. Hence, in eqn. (3.4), x_{so} and x_{s1} are constants for steady flow. It should be noted that in this inviscid flow solution, the shock wave is a discontinuity.

The quantity ζ^* which appears in eqns. (3.2b) and (3.3b) and which is defined in eqns. (3.3e) and (3.3f) is an additional potential function needed in the neighborhood of the shock wave. That is, it can be shown[2] that the channel flow solutions satisfy the jump conditions across the shock wave to first order but not in second and higher order terms. Hence, it is necessary to consider a region of order $\epsilon^{1/2}$ in thickness in the neighborhood of the shock wave; in this region the flow must adjust from the conditions required by the shock wave to those associated with the outer channel flow. In general, because the flow is supersonic upstream of the shock wave, $\zeta^* = 0$ there and an adjustment region is needed only downstream of the shock wave. There, x^* (defined in eqn. (3.3g) is O(1) and as shown in eqn. (3.3e), ζ^* goes to zero exponentially as x^* increases. As mentioned previously, the solutions shown in equations (3.3) are composite solutions; the solutions in the inner region have been included and the solutions, as written, are uniformly valid to order ϵ^2 throughout the channel.

The location of the shock wave may be found in either of two ways. On the one hand, the composite solutions may be used in conjunction with the jump conditions across the shock wave. On the other hand, conservation of mass, enforced by equating $\int_0^{y_w} \rho u\, dy$ evaluated upstream and downstream of the shock wave, may be used. The former method involves the evaluation of higher order solutions in the inner region if higher order approximations to the location of the shock wave are desired, and so the latter method is employed

29

here. As pointed out in reference 2, it is necessary to calculate the jump in entropy across the shock wave and ρu to order ϵ^4 if one wishes to calculate x_{s1}. In addition, u_3, the third order outer solution for u must be known; again, if the mass flow is evaluated far enough upstream and downstream of the shock wave, then inner solutions are unnecessary. The solution for u_3 shown below has been found using the methods described in reference (2). The required equations for ρu and the jump in entropy across the shock wave, also shown below, are essentially those given in reference 2; a few corrections have been incorporated:

$$u_3 = (\gamma+1)[u_1 f''(\frac{y^4}{24} - \frac{y^2}{12}) + \frac{u_1^3}{6} y^2]_{xx} + g_x \tag{3.5a}$$

$$g_x = (\gamma+1)[\frac{7}{360} u_1 f'' - \frac{u_1^3}{18}]_{xx} + [\frac{1}{4} - \frac{\gamma(12-\gamma)}{36}]u_1^3$$

$$- \frac{(2\gamma-3)}{3} C_2 - \frac{1}{u_1}[\frac{(f'')^2}{90} + \frac{C_2^2}{2u_1^2} + \frac{(f')^2}{6(\gamma+1)} + \frac{f^2}{2(\gamma+1)} - C_3] \tag{3.5b}$$

$$\frac{\Delta S_{sh}}{(\gamma+1)} = \epsilon^3 \frac{2\gamma}{3} u_{10}^3 + \epsilon^4 \gamma u_{10}^2 \{f_0''(y^2 - \frac{1}{3}) - \frac{2}{3}\gamma u_{10}^2$$

$$+ 2x_{s1} u_{1x0} + 2C_{2u}/u_{10}\} + \ldots \tag{3.5c}$$

$$\rho u = 1 - \epsilon^2 \frac{(\gamma+1)}{2} u_1^2 - \epsilon^3 (\gamma+1)[u_1 u_2 + \frac{(2\gamma-3)}{6} u_1^3]$$

$$- \epsilon^4 (\gamma+1)[u_1 u_3 + \frac{(2\gamma-3)}{2} u_1^2 u_2 + \frac{u_2^2}{2} + \frac{v_2^2}{2(\gamma+1)}$$

$$+ \frac{(2-\gamma)(1-2\gamma)}{8} u_1^4] + \ldots - \Delta S_{sh} \tag{3.5d}$$

30

where the entropy, S, is made dimensionless with respect to the gas constant, $\bar{R}$.

If eqn. (3.5d) is integrated at $x = 0$ and at $x = X$, say, where $X > x_s$, and the results are equated, the terms of order ϵ^2 reduce to eqn (3.3a), and the terms of order ϵ^3 and ϵ^4 give, respectively,

$$C_{2d} = C_{2u} - \frac{2\gamma}{3} u_{10}^3 \qquad (3.6a)$$

$$C_{3d} = C_{3u} + \frac{2\gamma^2}{3} u_{10}^4 - 2\gamma x_{s1} u_{10}^2 u_{1x0} - 2\gamma u_{10} C_{2u} \qquad (3.6b)$$

where u_{1x0} refers to the value of $u_{1x}(x_{so})$ immediately upstream of the wave. Thus, setting the constants C_{2d} and C_{2u}, equivalent to setting the pressure to second order at points upstream and downstream of the shock wave, allows one to calculate u_{10}, from eqn. (3.6a); therefore, from eqn. (3.3a), x_{so} can be calculated. Next, setting C_{3d} and C_{3u}, equivalent to setting the pressure at the same points to third order, one may calculate x_{s1} from eqn. (3.6b). Thus, with the inviscid flow solutions shown here, the position of the shock wave can be calculated to first order accuracy.

Solutions Including Structure of the Shock Wave

The solutions in the previous section are found under the assumption that the shock wave is a discontinuity, and are valid as long as the thickness of the shock wave is small compared to the order of the accuracy to which the shock is located. That is, if the order of the dimensionless (with respect to the throat half width), thickness of the shock wave is given by L_s, then, as indicated in the following sketch, x_s is known to order ϵ as long as $L_s \ll \epsilon$.

31
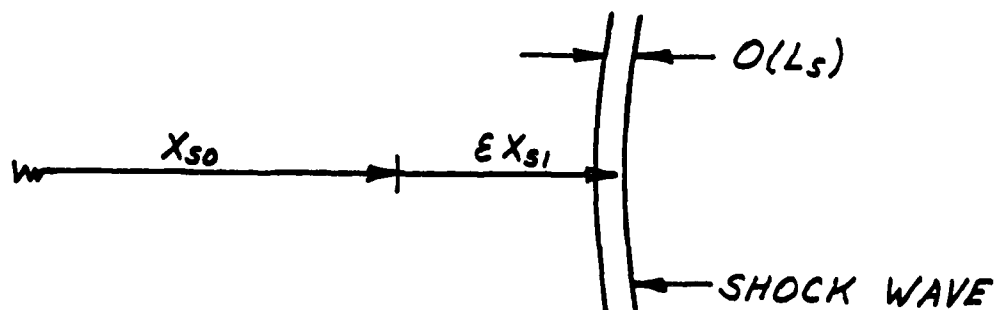
In order to relate the orders of L_s and ϵ to be used in the present application, it is necessary to estimate the thickness of a shock wave "captured" in a typical numerical computation relative to the characteristic length in the problem under consideration. For example, in typical computations of flows over airfoils, the shock thickness can vary from roughly three percent (Reference 5) to eight percent (Reference 6) of the chord; in computations of the flow through a stator blade, with artificial viscosity added, this thickness can be as high as 12 percent of the chord.[7] If the effects of adding artificial viscosity are neglected for the present, it appears that a good estimate of the thickness of the captured shock wave is five percent of the characteristic length in the flow direction. In the present case, the throat width is the characteristic length, i.e., $\overline{L}_s = 0.05(2\overline{L})$, or

$$L_s = \frac{\overline{L}_s}{\overline{L}} = \frac{1}{20} \left(\frac{2\overline{L}}{\overline{L}} \right) = \frac{1}{10} \tag{3.7}$$

If the Reynolds number, $\tilde{R}e$, (based on $\overline{L}$ and critical flow conditions) and the Prandtl number, $\tilde{P}_r$, are written in terms of the longitudinal viscosity, then following Illingworth[8]

[5] Yoshihara, H., Finite Difference Procedure for Unsteady Transonic Flows: A Review, Transonic Flow Problems in Turbomachinery, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 139-162.

[6] Caughey, D. A. and Jameson, A., Calculation of Transonic Potential Flowfields About Complex Three-Dimensional Configurations, Transonic Flow Problems in Turbomachinery, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 274-293.

[7] Farn, C. L. S. and Whirlow, D. K., Application of Time-Dependent Finite Volume Method to Transonic Flow in Large Turbines, Transonic Flow Problems in Turbomachinery, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 208-227.

[8] Illingworth, C. R., Shock Waves, Modern Developments in Fluid Mechanics, High Speed Flow, Vol. I, ed. L. Howarth, Oxford Press, 1953, 105-137.

$$L_s = \frac{4}{\widetilde{Re}\,\epsilon\,(\gamma+1)}\left(1 + \frac{\gamma-1}{\widetilde{Pr}}\right) \qquad (3.8a)$$

$$\cong 4k_s \qquad (3.8b)$$

where equation (3.8b) defines the parameter k_s, to be used hereafter as being of the order of the thickness of the shock wave. It is seen, from eqns. (3.7) and (3.8b), that the value of k_s corresponding to the estimated thickness of the shock wave found in numerical studies is $k_s \cong 1/40$ and further, from eqn. (3.8a), that for $\widetilde{P}_r = 1$, $\overline{Re}\,\epsilon \cong 40\,\gamma/(\gamma+1)$. Finally, a typical value for ϵ is $1/10$.

It appears from the above estimates, that solutions from either of two limit processes might be useful in applications to numerical problems. The two possibilities are

(a) $k_s = O(\epsilon^2);\quad \widetilde{Re} = O(\epsilon^{-3})$

(b) $k_s = O(\epsilon);\quad \widetilde{Re} = O(\epsilon^{-2})$.

In case (a), setting $k_s = m\epsilon^2$ with m being a constant of order unity, a numerical value of 2.5 for m would result in the desired relative values of k_s and ϵ. In case (b), with $k_s = m\epsilon$, the corresponding value of m is $1/4$.

In the following section, solutions for case (b), $k_s = m\epsilon$, are considered. This case was chosen for the initial study because solutions are valid for case (a) also, in the present application. It will be shown that it is possible to obtain the desired solutions in terms of an integral which can be evaluated using known functions.

(a) Channel Flow Solutions; $k_s = m\epsilon$

The solutions considered here are for a flow in which the longitudinal viscosity is large enough that its effects are found in the second order outer channel flow solutions. In addition, it will be seen that the shock wave is

thick enough that the effects of streamline curvature are found in second order solutions for the shock structure; as a result, the Rankine-Hugoniot jump conditions do not hold across the shock wave. These two effects cause formidable difficulties in obtaining higher order solutions.

The governing equations are the Navier-Stokes, continuity, and energy equations for steady flow. Thus:

$$uu_x + vu_y = \frac{1}{\rho \tilde{R}e} R(u) \tag{3.9a}$$

$$uv_x + vv_y = \frac{1}{\rho \tilde{R}e} R(v) \tag{3.9b}$$

$$uH_x + vH_y = \frac{1}{\rho \gamma}(up_x + vp_y) + \frac{1}{\rho \tilde{R}e} R(H) \tag{3.9c}$$

where

$$R(u) = [\tilde{\mu} u_x - (\frac{2}{3}\mu - \mu')v_y]_x + [\mu(u_y + v_x)]_y \tag{3.10a}$$

$$R(v) = [\mu(v_x + u_y)]_x + [\tilde{\mu} v_y - (\frac{2}{3}\mu - \mu')u_x]_y \tag{3.10b}$$

$$R(H) = [\frac{\tilde{\mu}}{\tilde{P}r} H_x]_x + [\frac{\tilde{\mu}}{\tilde{P}r} H_y]_y + \tilde{\mu}[u_x^2 + v_y^2]$$

$$- (\frac{4}{3}\mu - 2\mu')u_x v_y + (u_y + v_x)^2 \tag{3.10c}$$

The dimensionless longitudinal coefficient of viscosity, Reynolds number, and Prandtl number are defined as follows where $\bar{\mu}'$ is the bulk viscosity and $\bar{\mu}$ the usual viscosity coefficient;

$$\bar{\tilde{\mu}} = \frac{4}{3}\bar{\mu} + \bar{\mu}' \tag{3.11a}$$

$$\tilde{\mu} = \bar{\tilde{\mu}}/(\bar{\tilde{\mu}})* \qquad \mu = \bar{\mu}/(\bar{\tilde{\mu}})* \tag{3.11b}$$

$$\widetilde{Re} = \frac{\bar{\rho}* \bar{a}* \bar{L}}{(\bar{\mu})^*} \qquad (3.11c)$$

$$\widetilde{Pr} = \bar{C}_p \, \bar{\mu}/\bar{\lambda} \qquad (3.11d)$$

The equations of state and relationship between enthalpy and temperature are, respectively, in dimensionless terms,

$$P = \rho T \qquad (3.12a)$$

$$H = \frac{T}{\gamma-1} \qquad (3.12b)$$

Finally, three equations which are of importance in the following analysis are the gasdynamic equations and those governing the vorticity and entropy:

$$(u^2 - a^2)u_x + uv(u_y + v_x) + (v^2 - a^2)v_y$$

$$= \frac{1}{\rho \widetilde{Re}} [uR(u) + vR(v) - (\gamma-1)R(H)] \qquad (3.13a)$$

$$u\Omega_x + v\Omega_y = -\Omega(u_x + v_y) + \frac{1}{\gamma}(T_x S_y - T_y S_x)$$

$$+ \frac{1}{\widetilde{Re}} [(\frac{R(v)}{\rho})_x - (\frac{R(u)}{\rho})_y] \qquad (3.13b)$$

$$\frac{\rho T}{\gamma}(u S_x + v S_y) = \frac{1}{\widetilde{Re}} R(H) \qquad (3.13c)$$

where $a^2 = T$ and $\Omega = v_x - u_y$ are the dimensionless speed of sound with respect to $\bar{a}*$ and z component of the vorticity vector respectively.

Following the same methods employed in obtaining the inviscid flow solutions,[2] one can find the outer channel flow solutions valid now for the case $k_s = m\epsilon$. In order to simplify the calculations, $\widetilde{Pr} = 1$ and $\mu' = 0$ are taken to be the case; more general values will add complexity but not change the fundamental conclusions. First, if the stagnation enthalpy, H_t,

is expanded as follows,

$$H_t = H + \frac{u^2 + v^2}{2} = \frac{T}{\gamma-1} + \frac{u^2 + v^2}{2} = \frac{\gamma-1}{2(\gamma-1)} + \beta_1 H_{t_1} + \ldots \qquad (3.14)$$

where $\beta_1 = \beta_1(\epsilon) \ll 1$, then from the governing equations for H_t,

$$u(H_t)_x + v(H_t)_y = \frac{1}{\rho \tilde{R}e} [u R(u) + v R(v) + R(H)] \qquad (3.15)$$

one can show that $\beta_1 = \epsilon^{9/2}$, so that $T = a^2$ can be calculated from the equations:

$$T + \frac{(\gamma-1)}{2}(u^2 + v^2) = \frac{\gamma+1}{2} + O(\epsilon^{9/2}) \qquad (3.16)$$

Also, from eqn. (3.13c), one can show that $\Delta s = O(\epsilon^3)$ and from eqn. (3.13b) that $v_x - u_y = O(k_s \epsilon^3) = O(\epsilon^4)$ so that to order ϵ^3 a velocity potential exists. Finally, the boundary conditions are given by the relation

$$v(x, \pm y_w) = \pm y_w' \, u(x, \pm y_w) \qquad (3.17)$$

where, now, y_w includes the displacement thickness of the boundary layer for the low Reynolds number under consideration. Now, because an interaction occurs at the intersection of a shock wave and a boundary layer (assumed to be laminar here), it is important to ascertain the effects of the interaction on the displacement thickness. In order to estimate these effects, the solutions for the flow within the boundary layer in the interaction region, given by Brilliant and Adamson[9], may be used with the orders of the velocity perturbations valid outside the boundary layer found in the present calculations. It is not difficult to show that at least in lowest order,

---

[9] Brilliant, H. M. and Adamson, Jr., T. C., Shock Wave Boundary-Layer Interactions in Laminar Transonic Flow, AIAA J., 12 (1974), 323-329.

the increase in v due to the effects of an interaction between the shock wave and the boundary layer is small compared to the increase caused by the change in displacement thickness over the extent of the interaction region. Moreover, when the solutions valid within the shock wave itself are considered, it will be seen that through $O(\epsilon^{5/2})$, the expression for v is simply a continuation of the outer solution for v and thus satisfies the undisturbed flow boundary conditions at the wall to the same order. This is not the case for the term $O(\epsilon^3)$; the term $O(\epsilon^3)$ in the solution for v within the shock wave does not satisfy the boundary condition associated with undisturbed flow at the wall. It is possible that this could lead to corrections of $O(\epsilon^3)$ in the displacement thickness downstream of the shock wave and thus to corrections in second and higher order terms in the solutions for u downstream of the shock wave; these corrections are not considered here.

If eqn. (3.16) is used for a^2 in eqn. (3.13a), eqns. (3.8) with $\widetilde{Pr} = 1$ are used for $\widetilde{Re}$, and the boundary conditions to each order of approximation are obtained from eqn. (3.17), the following results are found for u and v written, as in the inviscid flow case, in terms of asymptotic expansions:

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots \tag{3.18a}$$

$$v = \epsilon^2 v_2 + \epsilon^3 v_3 + \dots \tag{3.18b}$$

where u_1 is given by eqn. (3.3a) and

$$u_2 = \frac{f'' y^2}{2} + h_x \qquad\qquad v_2 = f' y \tag{3.19a, b}$$

$$h_x = -\frac{1}{6}\left[f'' + u_1^2(2\gamma - 3)\right] + \frac{C_2}{u_1} + \frac{mu_{1x}}{u_1} \tag{3.19c}$$

$$u_3 = (\gamma + 1)\left[u_1 f''(\frac{y^4}{24} - \frac{y^2}{12}) + \frac{y^2 u_1^3}{6}\right]_{xx} + g_x \tag{3.19d}$$

$$g_x = (\gamma + 1) \left[\frac{7}{360} u_1 f'' - \frac{u_1^3}{18} \right]_{xx} + \left[\frac{1}{4} - \frac{\gamma(12-\gamma)}{36} \right] u_1^3$$

$$- \left(\frac{2\gamma-3}{3} \right) C_2 - m \left(\frac{2\gamma}{3} + k(\gamma-1) \right) u_{1x}$$

$$- \frac{1}{u_1} \left[\frac{(f'')^2}{90} + \frac{(C_2 + m u_{1x})^2}{2u_1^2} + \frac{(f')^2}{6(\gamma+1)} + \frac{f^2}{2(\gamma+1)} \right.$$

$$- m \left\{ \left(f + (1 - \frac{\gamma}{3}) u_1^2 + \frac{(C_2 + m u_{1x})}{u_1} \right)_x + \gamma u_1 u_{1x} \right.$$

$$\left. \left. - \gamma \int_{x_{so}}^{x} (u_{1x})^2 dx \right\} - C_3 \right] \tag{3.19e}$$

$$v_3 = (\gamma + 1) \left[\frac{u_1 f''}{6} (y^3 - y) + \frac{\gamma u_1^3}{6} \right]_x \tag{3.19f}$$

Thus, the solutions are similar to the corresponding solutions in the inviscid flow case, the difference being the terms depending upon m. The same notation is used in the two cases, for h and g; there should be no confusion since either inviscid flow or viscous higher order flow solutions are employed exclusively in any given case. As in the inviscid flow case, constants C_2 and C_3 are arbitrary and may have different values upstream and downstream of the shock wave, denoted by subscripts u and d, respectively. In the integral in eqn. (3.19e), the lower limit may, of course, be chosen arbitrarily, the choice being reflected in the value of C_3 as boundary conditions are applied. Here, the choice is x_s, the position of the shock wave; upstream of the shock wave, the integral has a negative value and downstream of the wave a positive value. Since only the lowest order value of the integral is required here, x_s is replaced with x_{so}. Again, in eqn. (3.19e) k refers to the exponent of the temperature in the expression for the viscosity, taken to be

$$\tilde{\mu} = T^k \tag{3.20}$$

38

It may be noted that if $k_s = O(\epsilon^2)$, then u_2 would be independent of m and only linear terms in m would occur in u_3.

In order to consider the flow in the region of order $\epsilon^{1/2}$ in thickness downstream of the shock wave, it is necessary to know the flow variables just downstream of the wave. In the inviscid flow case, the jump conditions across the shock wave supplied these conditions. In the present case, it is not clear that the jump conditions hold; their validity must be checked. In this regard, the continuity equation $(\rho u)_x + (\rho v)_y = 0$ provides a simple test. Thus, if this equation is integrated across the shock wave, then the jump in ρu, indicated by $[\rho u]$, is

$$[\rho u] = O(k_s (\rho v)_y)$$

since $\rho = O(1)$ and $v = O(\epsilon^2)$, $[\rho u] = O(k_s \epsilon^2) = O(\epsilon^3)$. Thus, the divergence of the streamlines within the shock wave begins to affect $[\rho u]$ in third order terms. However, from eqn. (3.5d), it is seen that the third order terms in ρu involve second order terms in u. Hence, for $k_s = O(\epsilon)$, the jump conditions break down in second order terms; the shock wave is a Hugoniot shock to first order only. As a result, solutions valid within the shock wave itself must be used to continue the solutions across the shock.

(b) Solutions in Inner Structure Region

and Calculations of x_{l_0} and x_{l_1}

The following sketch shows the various regions under consideration,
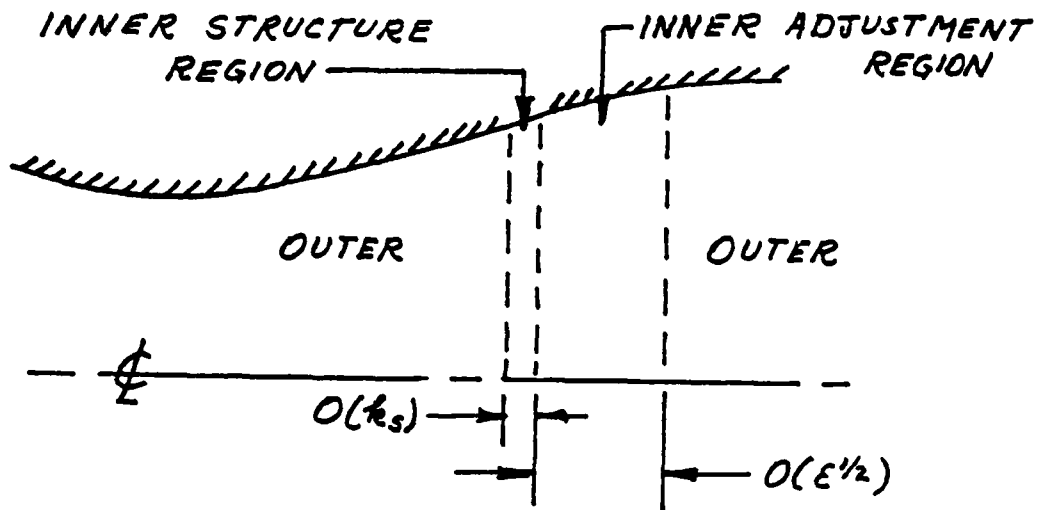
and the orders of thickness of the inner regions. Thus the outer regions have lengths of $O(1)$. The inner adjustment region, employed also in the inviscid flow calculations mentioned previously, has a length scale of order $\epsilon^{1/2}$; the inner structure region has a thickness of order k_s in general, so that for the present calculation, its thickness is $O(\epsilon)$.

Because the length scale in the inner structure region is the order of the thickness of the shock wave, the independent variables in this inner region are taken to be x^+ and y^+, where

$$x^+ = \frac{x - x_l(y)}{k_s} \qquad\qquad y^+ = y \qquad\qquad (3.21a, b)$$

40

and where $x_\ell(y)$ is the location of the sonic line within the shock wave. That is, in general, the sonic line is chosen to represent the position of a shock wave with finite thickness, and the distributions of velocity, pressure, etc. are written with respect to the sonic line. In the limit as $Re \to \infty$ and $k_s \to 0$, it is clear that $x_\ell = x_s$, the location of the shock wave in the inviscid flow solution. Hence it is seen that a similar expansion may be used for x_ℓ; i.e.,

$$x_\ell = x_{\ell\,o} + \epsilon x_{\ell\,1} + \epsilon^{3/2} x_{\ell\,3/2}(y) + \epsilon^2 x_{\ell\,2}(y) + \dots \tag{3.22}$$

The goal, then, is to find $x_{\ell\,o}$ and $x_{\ell\,1}$ in terms of m; as $m \to 0$ $(k_s/\epsilon \to 0)$ $x_{\ell\,o} \to x_{so}$ and $x_{\ell\,1} \to x_{s1}$.

If the outer solutions upstream and downstream of the shock wave are expanded about x_ℓ and written in terms of inner variables, the following results are found:

$$u = 1 + \epsilon u_1(x_{\ell\,o}) + \epsilon^2 \{(x_{\ell\,1} + mx^+)u_{1x}(x_{\ell\,o}) + u_2(x_{\ell\,o}, y) + \zeta^*_{x*}(0, y)\}$$

$$+ \epsilon^{5/2}\{x_{\ell\,3/2}u_{1x}(x_{\ell\,o}) + (x_{\ell\,1} + mx^+)\zeta^*_{x*x*}(0, y)\}$$

$$+ \epsilon^3\{x_{\ell\,2}u_{1x}(x_{\ell\,o}) + \frac{(x_{\ell\,1} + mx^+)^2}{2} u_{1xx}(x_{\ell\,o}) + (x_{\ell\,1} + mx^+)u_{2x}(x_{\ell\,o}, y)$$

$$+ x_{\ell\,3/2}\,\zeta^*_{x*x*}(0, y) + \frac{(x_{\ell\,1} + mx^+)^2}{2} \zeta^*_{x*x*x*}(0, y)$$

$$+ u_3(x_{\ell\,o}, y) + \eta^*_{x*}(0, y)\} + \dots \tag{3.23a}$$

$$v = \epsilon^2 f'(x_{\ell\,o})y + \epsilon^{5/2} \zeta^*_y(0, y)$$

$$+ \epsilon^3\{(x_{\ell\,1} + mx^+)(f''(x_{\ell\,o}) + \zeta^*_{yx*}(0, y)) + v_3(x_{\ell\,o}, y)\} + \dots \tag{3.23b}$$

where the expansion for x_ℓ (eqn. 3.22) has been used and where it has been anticipated that a composite solution similar to that used in the inviscid flow

problem will be necessary. Thus, ζ^* and η^* are the second and third order potential functions, respectively, which are needed in the inner adjustment region downstream of the shock wave; they are both zero upstream of the wave. Anticipating, further, that the lowest order terms in the relations for the locations of the sonic line and the inviscid flow shock wave are the same, $x_{so} = x_{\ell o}$, it may be noted that the notation introduced previously for functions of x_{so} will be used in evaluating equations (3.23) in what follows; i.e., $f(x_{\ell o}) = f_o$, $u_1(x_{\ell o}) = u_{1o}$ upstream of the wave, etc.

In view of eqns. (3.23) and (3.21), it is seen that in the inner structure region, the expansions for u and v and the derivative transformations are

$$u = 1 + \epsilon u_1^+ + \epsilon^2 u_2^+ + \epsilon^{5/2} u_{5/2}^* + \epsilon^3 u_3^+ + \dots \tag{3.24a}$$

$$v = \epsilon^2 v_2^+ + \epsilon^{5/2} v_{5/2}^+ + \epsilon^3 v_3^+ + \dots \tag{3.24b}$$

$$\frac{\partial}{\partial x} = \frac{1}{k_s} \frac{\partial}{\partial x^+} \qquad\qquad \frac{\partial}{\partial y} = -\frac{x_\ell'}{k_s} \frac{\partial}{\partial x^+} + \frac{\partial}{\partial y^+} \tag{3.24c, d}$$

where

$$x_\ell' = \epsilon^{3/2} x_{\ell 3/2}' + \epsilon^2 x_{\ell 2}' + \dots \tag{3.25}$$

If eqns. (3.24) and an expansion for H_t are substituted into eqn. (3.15), with $\widetilde{\mathrm{Pr}} = 1$ and $\bar{\mu}' = 0$, it can be shown that within the shock wave

$$H_t = \frac{T}{\gamma - 1} + \frac{1}{2}(u^2 + v^2) = \frac{(\gamma + 1)}{2(\gamma - 1)} + O(\epsilon^{7/2}) \tag{3.26}$$

so H_t = constant to the order retained; thus, expansions can be obtained for $T = a^2$ in terms of the velocity components. In the same manner, eqns. (3.13c) and then (3.13b) can be used to show that within the shock wave, for the case considered ($k_s = O(\epsilon)$, $\widetilde{\mathrm{Pr}} = 1$, $\bar{\mu}' = 0$)

$$\Delta S = O(\epsilon^2) \tag{3.27a}$$

$$\Omega = v_x - u_y = O(\epsilon^{5/2}) \tag{3.27b}$$

42

It may be noted that Ω is at most $O(\epsilon^{5/2})$; i.e. this is an order estimate of Ω and not a solution so that the term of order $\epsilon^{5/2}$ could be zero in which case Ω would be of even higher order. If eqns. (3.24) are substituted into eqn. (3.27b), it is found that

$$v^+_{2x^+} = m u^+_{1y^+} \tag{3.28a}$$

$$v^+_{5/2x^+} = -x'_{\ell\,3/2} u^+_{1x^+} \tag{3.28b}$$

$$v^+_{3x^+} = m u^+_{2y^+} - x'_{\ell\,2} u^+_{1x^+}$$

If, as will be shown later, $u^+_1 = u^+_1(x^+)$, then $v^+_2 = v^+_2(y^+)$; also, the equation for $v^+_{5/2}$ is easily integrated. If these two results are written for $x^+ \to \infty$ and $x^+ \to -\infty$ and matched with the corresponding outer solutions calculated from eqn. (3.23b), it is easily shown that

$$v^+_2 = f'_o y^+ \tag{3.29a}$$

$$v^+_{5/2} = x'_{\ell\,3/2}(u_{10} - u^+_1) \tag{3.29b}$$

$$x'_{\ell\,3/2} = \frac{1}{2u_{10}} \zeta^*_y(0, y) \tag{3.29c}$$

It is seen that eqn. (3.29c) states that to lowest order, $dx_\ell/dy = [v]/[u]$ where the brackets indicate the jump in the quantity enclosed <u>across</u> the shock wave at the point in question. Of course, the shape of a discontinuous shock wave is given by the same equation. Thus, even for this thick shock wave case, the differential equation (lowest order) for the location of the sonic line within the shock wave is the same as that for the location of a discontinuous shock wave (i.e., shock in the limit as $Re \to \infty$), with the same conditions upstream and downstream of the wave. This is indicated by the fact that $x'_{\ell\,3/2}$, in eqn. (3.29c), is independent of m. Of course, it still

43

remains to calculate the value of x_ℓ at $y = 0$, i.e. $x_{\ell 0}, x_{\ell 1}$, and $x_{\ell 3/2}(0)$ before one can say that the location of the sonic line gives the location of the shock wave independent of Re.

If the expansions for u and v, eqns. (3.24a, b) and (3.29a, b) and the inner variable transformations, eqns. (3.24c, d) are substituted first into eqn. (3.26) to obtain a^2, and then into eqn. (3.13a), the following governing equations are found for the u_i^+.

$$u_1^+ u_{1x^+}^+ = u_{1x^+x^+}^+ \tag{3.30a}$$

$$\left(u_1^+ u_2^+ + \frac{u_1^{+3}}{6}\right)_{x^+} - \frac{mf_o'}{(\gamma+1)} = u_{2x^+x^+}^+ + \left(\tilde{\mu}_1 u_{1x^+}^+\right)_{x^+} + 2u_1^+ u_{1x^+x^+}^+ \tag{3.30b}$$

$$\left(u_1^+ u_{5/2}^+\right)_{x^+} - \frac{m}{(\gamma+1)} x_{\ell 3/2}'' (u_{10} - u_1^+) = u_{5/2x^+x^+}^+ \tag{3.30c}$$

$$\left(u_1^+ u_3^+ + \frac{(u_2^+)^2}{2} + \frac{(u_1^+)^2}{2} u_2^+\right)_{x^+} - \frac{m}{(\gamma+1)}\left(v_{3y^+}^+\right.$$

$$+ \frac{(x_\ell')^2}{m} u_{1x^+}^+ - (\gamma-1)u_1^+ f_o'\Bigg) = u_{3x^+x^+}^+ + \left(\tilde{\mu}_1 u_{2x^+}^+\right)_{x^+}$$

$$+ \left(\tilde{\mu}_2 u_{1x^+}^+\right)_{x^+} + 2u_1^+ u_{2x^+x^+}^+ + 2u_2^+ u_{1x^+x^+}^+$$

$$+ \frac{(\gamma+1)}{2} u_1^{+2} u_{1x^+x^+}^+ - S_2^+ u_{1x^+x^+}^+ + 2u_1^+\left(\tilde{\mu}_1 u_{1x^+}^+\right)_{x^+}$$

$$+ u_1^{+2} u_{1x^+x^+}^+ \tag{3.30d}$$

where S_2^+ is the lowest order term in the expansion for the entropy and arises from the expansion for ρ in eqn. (3.13a). Thus,

$$\rho = T^{\frac{1}{\gamma - 1}} e^{-\Delta S}$$

and if eqn. (3.26) is used to find an expansion for T in terms of the velocity components, and if $S = S_o + \epsilon^2 S_2^+ + \dots$ then it is found that

$$\rho_1 = - u_1^+ \tag{3.31a}$$

$$\rho_2^+ = - u_2^+ - \frac{(\gamma - 1)}{2} (u_1^+)^2 + S_2^+ \tag{3.31b}$$

Finally, from eqn. (3.13c), it can be shown that the differential equation for S_2^+ is easily integrated to give

$$S_2^+ = - \left(\gamma + 1\right) u_{1x^+}^+ \tag{3.32}$$

where the function of integration is found to be zero from matching considerations.

The solution to eqn. (3.30a) is the well known Taylor solution,

$$u_1^+ = - u_{10} \tanh r^+ \tag{3.33a}$$

$$r^+ = \frac{u_{10}}{2} x^+ \tag{3.33b}$$

where the constant u_{10} in eqn. (3.33a) results from matching with the first order term in eqn. (3.23a) in the limits $x^+ \to - \infty$, $x \to x_{\ell o}^{(-)}$ and $x^+ \to + \infty$, $x \to x_{\ell o}^{(+)}$. In general, eqn. (3.33b) should include a function of integration,

$$r^+ = \frac{u_{10}}{2} x^+ + G(y).$$

However, if a composite solution is formed from the first order inner and outer solutions and the condition is enforced that at the sonic line $u^2 + v^2 = a^2$, which reduces to $u = 1$ for terms up to first order, it is easily shown that $G(y) = 0$.

The solution to eqn. (3.30b) may be written as follows:

$$u_2^+ = -\left(\frac{2}{u_{10}}\right)^2 \frac{m\,f'_o}{(\gamma+1)} \left\{ \frac{r^+}{2}\,\tanh r^+ - \frac{(1+\tanh^2 r^+)}{8} \right.$$

$$\left. + \left(\frac{r^+}{2\cosh r^+}\right)^2 \right\} + k(\gamma-1)u_{10}^2\,\frac{\ln\cosh r^+}{\cosh^2 r^+}$$

$$+ u_{10}^2 \left\{ \frac{\tanh^2 r^+}{2} - \frac{\ln\cosh r^+}{\cosh^2 r^+} \right\}$$

$$- \frac{g_4(y)}{u_{10}} \left\{ \tan r^+ + \frac{r^+}{\cosh^2 r^+} \right\} + \frac{g_5(y)}{\cosh^2 r^+} \qquad (3.34)$$

Then eqn. (3.28c) may be employed to find v_3^+. Thus,

$$v_3^+ = -\frac{2mg'_4}{u_{10}^2}\,r^+\,\tanh r^+ + \frac{2m}{u_{10}}\,g'_5\,\tanh r^+$$

$$- x'_{\ell 2}u_1^+ + g_7(y) \qquad (3.35)$$

where in eqns. (3.34) and (3.35), the $g_i(y)$ are functions of integration. Equation (3.35) may, then, be used to calculate v_{3y}^+ which is needed in order to solve eqn. (3.30d) for u_3^+.

At this point, it is convenient to use the solutions found so far to illustrate the manner in which the conditions downstream of the shock wave can be found and to calculate $x_{\ell o}$. That is, as indicated previously, the jump conditions across a discontinuous shock wave hold for the present "thick" wave only to first order. That they do hold in first order is shown by comparing the jump condition with the first order inner structure solution, eqn. (3.33a); the inner solution indicates that as r^+ goes from $-\infty$ to $+\infty$, $u_{1d}^+ = -u_{1u}^+$ in agreement with the jump conditions. Now, the outer solutions evaluated at the shock wave position satisfy this first order condition

because u_1 may have either a positive or a negative value. As mentioned previously, in the case of inviscid flow the second order jump conditions are not satisfied by the outer solution; as a result it became necessary to consider the flow in an inner adjustment region. In the present case, although the Hugoniot jump conditions no longer hold across the shock wave to this order, the second order inner structure solutions are used to obtain similar conditions, which again are not satisfied by the outer solutions. Hence, an inner adjustment region is still necessary. In summary, then, the essential difference between the present analysis and that employed in the inviscid flow case is that here, the higher order flow conditions downstream of the shock wave, used as the upstream boundary conditions for the flow in the adjustment region, are found from solutions in the inner structure region.

As $r^+ \to \pm\infty$, then, u_2^+ becomes:

$$u_2^+ \sim \frac{2}{u_{10}^2} \frac{mf'_o}{(\gamma+1)} \left(r^+ + \frac{1}{2}\right) + \frac{u_{10}^2}{2} + \frac{g_4(y)}{u_{10}} \qquad (r^+ \to -\infty) \qquad (3.36a)$$

$$u_2^+ \sim -\frac{2}{u_{10}^2} \frac{mf'_o}{(\gamma+1)} \left(r^+ - \frac{1}{2}\right) + \frac{u_{10}^2}{2} - \frac{g_4(y)}{u_{10}} \qquad (r^+ \to +\infty) \qquad (3.36b)$$

If these equations are matched with the corresponding outer solutions, calculated using eqn. (3.23a), one finds the following relations:

$$\frac{x_{\ell 1} f'_o}{u_{10}(\gamma+1)} - \frac{g_4(y)}{u_{10}} = -f''_o\left(\frac{y^2}{2} - \frac{1}{6}\right) + \frac{\gamma u_{10}^2}{3} - \frac{C_{2u}}{u_{10}} \qquad (3.37a)$$

$$\frac{-x_{\ell 1} f'_o}{u_{10}(\gamma+1)} + \frac{g_4(y)}{u_{10}} = -f''_o\left(\frac{y^2}{2} - \frac{1}{6}\right) + \frac{\gamma u_{10}^2}{3} + \frac{C_{2d}}{u_{10}} - \zeta^*_{x^*}(0,y) \qquad (3.37b)$$

Hence,

$$\zeta^{*}_{x^{*}}(0, y) = -f''_{o}(y^2 - \frac{1}{3}) + \frac{2\gamma}{3} u^2_{10} + \frac{C_{2d} - C_{2u}}{u_{10}} \tag{3.38}$$

which is the same result found in reference 2 for the inviscid flow problem.
Here, just as in reference 2, the outer velocity components downstream of
the shock wave are expanded about the lowest order position of the shock
wave, $x_{\ell o}$, and thus in terms of x^{*}; then $\zeta^{*}_{x^{*}}$ and ζ^{*}_{y} are added to these ex-
pansions. The resulting expressions are used for the velocity components in
the inner adjustment regions in which the flow variables change from their
values immediately downstream of the shock wave to forms given by the outer
channel flow solutions; the additional potential function $\zeta^{*}(x^{*}, y)$ provides
these variations. It is not difficult to show that in the present problem,
because the expansions for the outer velocity components differ from their
counterparts in the inviscid flow problem only by constants, the governing
equation, boundary conditions, and hence solution for $\zeta^{*}(x^{*}, y)$ (eqns.
(3.3e) and (3.3f)) are the same as in reference 2. Thus, since ζ^{*} satisfies
a Laplace equation and the normal derivatives on the remaining boundaries
are zero, the condition

$$\int_{0}^{1} \zeta^{*}_{x^{*}}(0, y)dy = 0$$

must be met. When this condition is applied to eqn. (3.38) the results are

$$C_{2d} = C_{2u} - \frac{2\gamma}{3} u^3_{10} \tag{3.39}$$

and

$$\zeta^{*}_{x^{*}}(0, y) = -f''_{o}(y^2 - \frac{1}{3}) \tag{3.40}$$

which are the results found in reference 2. It is worthwhile to emphasize
that the results found in eqns. (3.39) and (3.40) are not found if second order
shock wave conditions are used to evaluate conditions downstream of the wave;
again, the Hugoniot jump conditions simply do not hold here.

48

Now, eqn. (3.39) involves $u_{10} = u_1(x_{\ell\,0})$ and thus may be used to find $x_{\ell\,0}$. That is, by choosing the arbitrary constants C_{2u} and C_{2d}, one can calculate u_{10} and from eqn. (3.3a), then, $x_{\ell\,0}$, for a given wall shape and c_w. Because eqn. (3.39) is independent of m and in fact precisely the same as the equation found for the infinitesimally thin shock wave[2], it is clear that

$$x_{\ell\,0} = x_{so} \qquad (3.41)$$

i.e., to lowest order the location of the sonic line within the shock wave and the location of the shock wave under the same flow conditions except that Re $\to \infty$, are the same.

It should be noted that a composite solution involving ζ^* and the outer velocity components may be formed, just as in reference (2); in fact, such a composite solution written to third order in u and thus involving a third order potential function, $\eta^*(x^*, y)$, has been used to write eqn. (3.23). In this regard, eqns. (3.13c) and (3.13b) can be used to show that $\Omega = O(\epsilon^{7/2})$ in the inner adjustment region, to confirm that a potential function does exist at least to $O(\epsilon^3)$.

In order to find $x_{\ell\,1}$, it is necessary to follow the same procedure as was used in calculating $x_{\ell\,0}$, except that now third order rather than second order terms in u are employed. That is, the solution for $u_{5/2}^+$, is not involved in the solution for $x_{\ell\,1}$; it is necessary, however, to find u_3^+.

The governing equation for u_3^+ is found by calculating v_{3y}^+, eqn. (3.35), and substituting this relation into eqn. (3.30d). The resulting equation may be integrated once without too much difficulty, to give,

$$u^+_{3r^+r^+} + 2(\tanh r^+)u^+_3 = \frac{1}{u_{10}}((u_2^+)^2 - 3(u_1^+)^2 u_2^+)$$

$$+ \frac{m}{(\gamma+1)}\left[\frac{8mg_4''}{u_{10}^4} \int_0^{r^+} t \tanh t\, dt \right.$$

$$- \frac{4}{u_{10}}\left(\frac{2m}{u_{10}^2} g_5'' + x_{\ell 2}''\right)\ln\cosh r^+ - \frac{4}{u_{10}^2} g_7' \, r^+$$

$$+ \frac{2}{m}(x_{\ell 3/2}')^2 \tanh r^+ - \frac{4}{u_{10}}(\gamma-1)f_o' \ln\cosh r^+ \Bigg]$$

$$- (\tilde{\mu}_1 u^+_{2r^+} + \tilde{\mu}_2 u^+_{1r^+}) - \frac{(\gamma-3)}{4u_{10}}(u_1^+)^4$$

$$- \frac{8}{u_{10}} \frac{mf_o'}{(\gamma+1)} \ln\cosh r^+ - \frac{(\gamma+1)}{4} u_{10}(u^+_{1r^+})^2 + g_8$$

$$(3.42)$$

where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are found from eqn. (3.20), with the terms in the expansion for T calculated using eqn. (3.26). Another integration of eqn. (3.42), to find u_3^+, would be very difficult if not impossible. Fortunately, the desired information can be obtained by using eqn. (3.42)[†]. First, the expressions to which u_3^+ must match as $r^+ \to \pm\infty$ are found from eqn. (3.23a). Thus, $u_3^+(r^+ \to -\infty, y)$ and $u_3^+(r^+ \to +\infty, y)$ are known. If they are used to evaluate the left hand side of eqn. (3.42), and the known inner solutions (u_1^+ and u_2^+) are used to evaluate the right hand side, in the limits as $r^+ \to \pm\infty$, the resulting equations can be used to obtain a relation for $x_{\ell 1}$. That is, from

---

[†] The author is grateful to Professor Messiter for having suggested the procedure used.

eqn. (3.23a) it is seen that

$$(u_3^+)_{u,d} = A_{u,d}(x^+)^2 + B_{u,d}x^+ + C_{u,d} \qquad (3.43)$$

where the subscripts u and d refer to the limits $r^+ = -\infty$ and $r^+ = +\infty$ respectively, and where

$$A_u = \frac{m^2}{2} u_{1xxo} \qquad (3.44a)$$

$$B_u = m\, x_{\ell 1} u_{1xxo} + m(u_{2x})_u \qquad (3.44b)$$

$$C_u = x_{\ell 2} u_{1xo} + \frac{x_{\ell 1}^2}{2} u_{1xxo} + x_{\ell 1}(u_{2x})_u + u_{3u} \qquad (3.44c)$$

$$u_{1xxo} = \frac{f''_o}{(\gamma+1)u_{10}} - \frac{(f'_o)^2}{(\gamma+1)^2 u_{10}^3} \qquad (3.44d)$$

In addition,

$$A_d = \frac{m^2}{2} [-u_{1xxo} + \zeta^*_{x^*x^*x^*}(0,y)] \qquad (3.45a)$$

$$B_d = mx_{\ell 1}[-u_{1xxo} + \zeta^*_{x^*x^*x^*}(0,y)] + m(u_{2x})_d \qquad (3.45b)$$

$$C_d = -x_{\ell 2}u_{1xo} + \frac{x_{\ell 1}^2}{2}[-u_{1xxo} + \zeta^*_{x^*x^*x^*}(0,y)]$$

$$+ x_{\ell 1}(u_{2x})_d + x_{\ell 3/2}\zeta^*_{x^*x^*}(0,y) + u_{3d}$$

$$+ \eta^*_{x^*}(0,y) \qquad (3.45c)$$

51

In eqns. (3.44) and (3.45), the subscripts u and d used with outer solutions refer to values immediately upstream and downstream of the shock wave respectively.

If eqn. (3.42) is evaluated at $r^+ \to -\infty$ and $r^+ \to \infty$, as described above, and coefficients of like powers of x^+ (i.e. r^+) on either side of the equation are equated, one obtains six equations, three for each limit.

The coefficients of $(x^+)^2$ give equations which are identities, and the coefficients of x^+ give equations which may be used to evaluate $g'_7(y)$. It is the constant terms which involve $x_{\ell 1}$; they are found to be:

$$\frac{2B_i}{u_{10}} \mp 2C_i = \frac{1}{u_{10}} \left[\frac{mf'_o}{(\gamma + 1)u_{10}^2} \pm \frac{g_4}{u_{10}} + \frac{u_{10}^2}{2} \right]^2 - 3u_{10} \left[\frac{mf'_o}{(\gamma + 1)u_{10}^2} \right.$$

$$\left. \pm \frac{g_4}{u_{10}} + \frac{u_{10}^2}{2} \right] + \frac{4m \ln 2}{(\gamma + 1)u_{10}} \left[\frac{2mg''_5}{u_{10}^2} + x'_{\ell 2} + (\gamma - 3)f'_o \right]$$

$$\mp \frac{2(x'_{\ell 3/2})^2}{(\gamma + 1)} - \frac{(\gamma - 3)u_{10}^2}{4} + g_8 + \frac{2h(\gamma - 1)mf'_o}{(\gamma + 1)u_{10}} \tag{3.46}$$

where the subscript i = u when the upper signs are used and i = d when the lower signs are used. When one equation is subtracted from the other, then, one obtains

$$\frac{1}{u_{10}}(B_u - B_d) - (C_u + C_d) = 2 \left[\frac{mf'_o}{(\gamma + 1)u_{10}^4} - 1 \right] g_4 - \frac{2(x'_{\ell 3/2})^2}{(\gamma + 1)} \tag{3.47}$$

The desired relation is found by substituting for B_u, B_d, C_u, and C_d from eqns. (3.44) and (3.45), using the known outer solutions (eqns. (3.19)) to evaluate $(u_{2x})_u$, u_{3u}, etc. After considerable algebraic manipulation, it is found that

$$\eta^*_{x*}(0,y) + x_{\ell\,3/2}\,\zeta^*_{x^*x^*}(0,y) + \frac{x^2_{\ell 1}f''_o}{(\gamma+1)u_{10}} + x_{\ell 1}\left[\,f'''_o\left(y^2 - \frac{1}{3}\right)\right.$$

$$\left. - \frac{2}{3}(2\gamma - 3)\frac{f'_o}{(\gamma+1)} + (C_{2d} - C_{2u})\frac{f'_o}{(\gamma+1)u^3_{10}} + \frac{2m}{(\gamma+1)u^2_{10}}\left(f''_o - \frac{(f'_o)^2}{(\gamma+1)u^2_{10}}\right)\right]$$

$$+ (\gamma + 1)\left[\left(\left(u_1 f''\right)_{xx}\right)_u + \left(\left(u_1 f''\right)_{xx}\right)_d\right]\left(\frac{y^4}{24} - \frac{y^2}{12}\right)$$

$$+ (\gamma + 1)\left[\left(\left(\frac{u_1^3}{6}\right)_{xx}\right)_u + \left(\frac{u_1^3}{6}\right)_{xx}\right)_d\right]\frac{y^2}{2} + (g_x)_u + (g_x)_d$$

$$+ m\frac{(C_{2u} + C_{2d})f'_o}{(\gamma+1)u^4_{10}} + 2\left(\frac{mf'_o}{(\gamma+1)u^4_{10}} - 1\right)g_4 - \frac{2(x'_{\ell\,3/2})^2}{(\gamma+1)} = 0 \qquad (3.48)$$

where g_x is given in eqn. (3.19e), g_4 in (3.37a), and ζ^* in eqns. (3.3e) and (3.3f).

At this point, in the second order calculation, the integral condition on $\zeta^*_{x*}(0,y)$ was used to derive an equation from which $x_{\ell\,o}$ could be found. In this third order calculation, the same procedure is followed, the integral conditions being represented by I_1, where

$$I_1 \equiv \int_0^1 \eta^*_{x*}(0,y)dy \qquad (3.49)$$

In general, the governing equation for η^* will be a Poisson equation, with any forcing function depending upon lower order, known functions. Hence, although the solution for I_1 is not given here, it is clear that it can be calculated without the necessity of performing any more calculations in the outer or inner shock structure regions. If eqn. (3.48) is integrated, then,

53

from $y = 0$ to $y = 1$, and in the resulting relation, eqn. (3.39) is used to substitute for C_{2d}, one finds finally the following equation, from which $x_{\ell 1}$ may be calculated.

$$C_{3d} = C_{3u} + 2\gamma \left(\frac{\gamma u_{10}^4}{3} - C_{2u} u_{10} \right)$$

$$- \frac{2x_{\ell 1} u_{10}}{(\gamma + 1)} \left(\gamma f'_o - \frac{mf''_o}{u_{10}^2} \right) + \frac{x_{\ell 1}^2 f''_o}{(\gamma + 1)} + u_{10} I_1 \qquad (3.50)$$

If eqn. (3.50) is compared with equation (3.6b), it is seen that all of the terms in the inviscid flow solution are contained in eqn. (3.50). In addition, there is a term depending upon m, one involving $x_{\ell 1}^2$, and one containing I_1. The following conclusions can be reached.

(i) As $m \rightarrow 0$, and thus for $Re \gg \epsilon^{-2}$ ($k_s \ll \epsilon$) the solution for $x_{\ell 1}$ must reduce to that for x_{s1}; the location of the sonic line is independent of Reynolds number through terms $O(\epsilon)$, and thus is the same as the location of the shock wave as $Re \rightarrow \infty$, to this order.

(ii) Although the Hugoniot jump conditions across a shock wave do not hold to the order needed, for $Re = O(\epsilon^{-2})$ and therefore $k_s = O(\epsilon)$, it is possible to derive a solution for $x_{\ell 1}$ for this thick shock case.

(iii) Although it appears that $x_{\ell 1}$ depends upon m and thus upon Reynolds number for $k_s = O(\epsilon)$, this is not necessarily the case. That is, in all probability, I_1 depends upon m also; this dependence could be such that $x_{\ell 1}$ is independent of m. Support for this possibility is given by the fact that to lowest order, dx_ℓ/dy is independent of m and thus $dx_\ell/dy = dx_s/dy$. to this order. If, indeed, x_ℓ is independent of m through terms $O(\epsilon)$, the location of the sonic line in the "thick" shock waves found in numerical solutions in channel flows could be used to

54

locate the shock wave in most practical cases even with the inclusion of artificial viscosity. In addition, if this result is indeed found, extension to more general flows and high flow Mach numbers should be considered.

Finally, it should be noted that composite solutions can be formed including now the solutions valid in the inner structure region. In doing so, it would be necessary to write the solutions in two parts, one for $x < x_\ell$ and one for $x > x_\ell$.

Shock Wave in the Neighborhood of the Throat

In the work described in the previous sections, the shock wave was located in the channel a distance $O(1)$ downstream of the nozzle throat. The question arises as to the validity of these solutions as a sonic nozzle throat is approached and the flow Mach number tends to unity. Because the wall shape at the throat is such that, in general, $f(0) = f'(0) = 0$, then in the neighborhood of the throat

$$f(x) = \frac{x^2}{2} \ f''(0) + \ldots$$

i.e., the wall shape is parabolic. Hence, in investigating the throat region, there is no loss of generality in considering a parabolic wall such that the channel flow solutions are,

$$f(x) = bx^2 \tag{3.51a}$$

$$u_1 = \pm \sqrt{\frac{2}{(\gamma + 1)} \ f(x)} = \pm Bx; \qquad B = \sqrt{\frac{2b}{\gamma + 1}} \tag{3.51b, c}$$

$$v_2 = 2bxy \tag{3.51d}$$

where u_1, with $c_w = 0$ because the flow is sonic at the throat, is given by eqn. (3.3a) and v_2 by eqn. (3.19b). With the above relations for u_1 and v_2

$$u = 1 \pm \epsilon Bx + \ldots \qquad\qquad v = \epsilon^2 2bxy + \ldots \tag{3.52a, b}$$

and so if $x = O(\alpha)$, say, in the neighborhood of the throat, then, expansions

for u and v would be such that $u = 1 + O(\epsilon\alpha)$ and $v = O(\epsilon^2\alpha)$. Because, from equation (16), it is seen that

$$a^2 = 1 - \frac{(\gamma - 1)}{2} (u-1) + \ldots \qquad (3.53)$$

so that $u^2 - a^2 = (\gamma + 1)(u-1)/2$, the important terms in the gasdynamic equation (3.13a) are found to have orders as shown in the following, where the order of each term is shown beneath it:

$$(u^2 - a^2)u_x - a^2 v_y = \frac{1}{\rho \tilde{Re}} [uR(u) - (\gamma - 1) R(H)]$$

$$\epsilon\alpha \cdot \frac{\epsilon\alpha}{\alpha} \qquad \epsilon^2\alpha \qquad \frac{1}{\tilde{Re}} \frac{\epsilon\alpha}{\alpha^2}$$

Thus, the terms on the left hand side of the equation, those which are associated with channel flow solutions, are $O(\epsilon^2\alpha)$. The viscous term on the right hand side, generally important in lowest order solutions only within a shock structure region, is $O(\epsilon/\tilde{Re}\alpha)$. As long as $\alpha \gg (\tilde{Re}\ \epsilon)^{-1/2}$, the viscous term is negligible, and the channel flow solutions hold in lower order; the viscous term could only become important in a thin shock structure region imbedded in the region in question, so the solution essentially remains that illustrated in the previous sections. However, when $\alpha = O[(\tilde{Re}\ \epsilon)^{-1/2}]$, then all three terms are of the same order. The solution is not that given previously, and it is this case we wish to investigate.

The region under study in the neighborhood of the throat is thus of order $\alpha = O[(\tilde{Re}\ \epsilon)^{-1/2}]$ in thickness and so we define an inner variable and velocity expansions as follows:

$$\hat{x} = \frac{x}{\alpha} \qquad (3.54a)$$

$$u = 1 + \epsilon\alpha\ \hat{u}_1(\hat{x}, y) + \ldots \qquad v = \epsilon^2\alpha\ \hat{v}_1(\hat{x}, y) + \ldots \qquad (3.54b, c)$$

Because $M^2 - 1 = O(u^2 - 1)$, and α can be written as $\alpha = O[(\tilde{Re}\ \epsilon\alpha)^{-1}]$, it is clear that

56

$$\alpha = O\left[\frac{1}{\widetilde{Re}(M^2-1)}\right]$$

and thus that α is a measure of the thickness of the shock, which fills the whole region here, just as k_s measures the shock thickness in the previous analysis. Hence, we define α in the same general form as k_s;

$$\alpha = \frac{1}{\widetilde{Re}\,\epsilon\alpha\,(\gamma+1)}\left(1+\frac{\gamma-1}{\widetilde{P}_r}\right) \tag{3.55}$$

If eqns. (3.54) and (3.55) are substituted in eqns. (3.10) and then in eqns. (3.9c), (3.13c), (3.13b) and (3.13a), it is found that

$$R(u) = O(\epsilon/\alpha) = R(H) \qquad\qquad R(v) = O(\epsilon) \tag{3.56a, b, c}$$

$$H_t = \frac{\gamma+1}{2(\gamma-1)} + O(\epsilon^2\alpha^2) \tag{3.56d}$$

$$\Delta s = O(\epsilon^2\alpha^2) \tag{3.56e}$$

$$\Omega = O(\epsilon^2\alpha^2) \tag{3.56f}$$

and finally that the governing equation for $\hat{u}_1$ is,

$$\hat{u}_1\hat{u}_{1\hat{x}} - \frac{1}{(\gamma+1)}\,\hat{v}_{1y} = \hat{u}_{1\hat{x}\hat{x}} \tag{3.57}$$

Thus, the lowest order governing equation is the viscous transonic equation.[10]

If $\Omega = v_x - u_y$ is evaluated, using eqns. (3.56f) and (3.54), the result is, to lowest order in each term,

$$\epsilon^2\frac{\partial\hat{u}_1}{\partial\hat{x}} + \ldots - \epsilon\alpha\,\frac{\partial\hat{u}_1}{\partial y} + \ldots = O(\epsilon^2\alpha^2) \tag{3.58}$$

---

[10] Cole, J., Problems in Transonic Flow, Ph.D. Thesis, California Institute of Technology, 1949.

Thus, from eqn. (3.55), the orders of the two terms are ϵ^2 and $\sqrt{\epsilon/\widetilde{Re}}$
If the Reynolds number is large enough, that $\epsilon^{3/2} \gg \widetilde{Re}^{-1/2}$, then
$\Diamond_{1\hat{x}} = 0$ or $\Diamond_1 = \Diamond_1(y)$, and such a solution cannot satisfy the matching
conditions given by the outer solutions for the v velocity component, eqn.
(3.51d). Moreover, for this case, $\alpha = (\widetilde{Re}\,\epsilon)^{-1/2} \ll \epsilon^2$. Now, if the
outer solution for u is expanded for $x \ll 1$ to second order, again for
wall shape and u_1 as given in eqns. (3.51) and with u_2 as given in eqns.
(3.19a) and (3.19c), we obtain

$$u \sim 1 \pm \epsilon\,bx + \ldots + \epsilon^2 \left(\frac{f''_0}{2} (y^2 - \frac{1}{3}) + \frac{\pm C_2 + m)}{x} \right) + \ldots$$

Thus, as $x \to 0$, the second order term becomes the same order as the
first order term for $x = O(\epsilon^{1/2})$ and $x = O(\epsilon)$, indicating the possi-
bility that solutions in two more regions should be considered. However,
it was shown in reference 2 that it is possible to write the solution for u
in a form such that the singular terms in u_2 do not arise, being contained
in u_1 through the use of an expansion for C_w. Hence, solutions in the
region $O(\epsilon^{1/2})$ in thickness do not differ significantly from the channel
flow solutions already being used. However, in the thinner region
$(x = O(\epsilon))$, it is clear that the flow is two dimensional even in lowest
order and thus quite different from the outer channel flow described by
eqns. (3.19a) and (3.19c); the governing equation for inviscid flow is the
nonlinear transonic small disturbance equation. Hence, for $\alpha \ll \epsilon$,
the throat region under consideration would be contained within the region
of order ϵ in thickness and the "outer" solutions to be used for match-
ing would be the solutions in the latter region, not the outer channel
solutions given in the previous section.

If $\alpha = O(\epsilon)$, then $\hat{v}_{1\hat{x}} = \hat{u}_{1y}$ and a velocity potential may be intro-
duced into the viscous transonic equation (3.57); the lowest order solu-
tions for u are two dimensional and the outer solutions to which the inner
solutions must match are those given in eqns. (3.52a, b).

Finally, if $\alpha \gg \epsilon$, then from equation (3.58), $\hat{u}_{1y} = 0$, so $\hat{u}_1 = \hat{u}_1(\hat{x})$. In this case, for which $\widetilde{Re} \ll \epsilon^{-3}$ is the condition, the velocity components must match with the outer solutions, eqns. (3.52a) and (3.52b) as $|x| \to 0$ and $|\hat{x}| \to \infty$. The Reynolds number under consideration, $\widetilde{Re} = O(\epsilon^{-2})$ so that $\alpha = O(\epsilon^{1/2})$, fulfills this condition and so it is this case which will be analyzed here. It will be shown that it is possible to derive an exact solution to the viscous transonic equation for an arbitrary (parabolic) wall shape.

It may be noted that because the whole idea of these calculations is to derive solutions comparable to the thick shock wave results found in numerical solutions, it is necessary to consider a Reynolds number small compared to those associated with typical channel flows. Nevertheless, the solutions to be shown are of more general interest than the limited Reynolds number range would indicate. That is, they may be used to illustrate the manner in which a shock wave forms at the throat and then evolves into a thin wave as it moves downstream in response to a pressure condition imposed on the flow downstream of the shock wave. Thus, the solutions will match not only with the outer channel flow solutions when the shock wave is within this inner throat region, but also with the solutions including the shock structure when the shock wave has moved downstream of the throat into the overlap region.

For the case considered, then, $\widetilde{Re} = O(\epsilon^{-2})$ and from eqn. (3.55), if the same constant of proportionality between $\widetilde{Re}$ and ϵ^{-2} is to be used as in the previous section (eqns. (3.8) with $k_s = m\epsilon$),

$$\alpha = (m\epsilon)^{1/2} \tag{3.59}$$

Now, from eqn. (3.58), $\hat{u}_1 = \hat{u}_1(\hat{x})$ and so from eqn. (3.57), $\hat{v}_{1y} = F_1(\hat{x})$, so

$$\hat{v}_1 = F_1(\hat{x})y + F_2(\hat{x}) \tag{3.60}$$

Also, as $|\hat{x}| \to \infty$, $v = \epsilon^2 \alpha \hat{v}_1 + \ldots$ (with $\hat{v}_1$ as in eqn. (3.60)) must match with eqn. (3.52b), written in terms of inner avariables. Hence, $F_2(\hat{x}) = 0$ and $F_1(\hat{x}) = 2b\hat{x}$ so that apparently

$$\hat{v}_1 = 2b\hat{x}y \qquad (3.61)$$

throughout this inner region, subject to the boundary conditions at the wall being satisfied. These conditions are given by eqn. (3.17) with $y_w = 1 + \epsilon^2 f(x) = 1 + \epsilon^2 bx^2$, where eqn. (3.51a) has been used for $f(x)$. Thus, the boundary conditions at the wall reduce to

$$\hat{v}_1(\hat{x}, \pm 1) = \pm 2b\hat{x} \qquad (3.62)$$

and are indeed satisfied by eqn. (3.61).

As noted previously, y_w is taken to include the displacement thickness of the boundary layer. Again, the effects of the interaction between the shock wave and the boundary layer may be estimated using the solutions valid within the boundary layer in an interaction region[9] and the orders of the velocity perturbations ($\epsilon\alpha$ for the u component and $\epsilon^2\alpha$ for the v component) in the flow outside the boundary layer; again, it is easily shown that these effects are small compared to the change in displacement thickness of the undisturbed boundary layer over the interaction region. Since $\epsilon^2\alpha\hat{v}_1$, the lowest order solution for v within this thick shock region, satisfies the boundary condition associated with the undisturbed boundary layer displacement thickness, it is only in higher order terms that corrections might be needed (e.g., terms $O(\epsilon^2\alpha^2) = O(\epsilon^3)$), because these boundary conditions were not satisfied. Such higher order terms are not considered here.

With $\hat{v}_1$ as given in eqn. (3.61), the governing equations for $\hat{u}_1$, eqn. (3.57), can be written as follows:

$$\hat{u}_1 \hat{u}_{1\hat{x}} - \frac{2b\hat{x}}{(\gamma + 1)} = \hat{u}_{1\hat{x}\hat{x}}$$

which can be integrated once to give

$$\hat{u}_{1\hat{x}} = \frac{\hat{u}_1^2}{2} - \frac{B^2\hat{x}^2}{2} - 2BA \qquad (3.63)$$

where the constant of integration has been written as $-2BA$ for later

convenience. Equation (3.63) can be transformed[11] into a standard form of Weber's equation, the solution of which can be written in terms of parabolic cylinder functions[12]. In terms of the standard solutions, $U(A, X)$ and $V(A, X)$ given in reference 12, where

$$X = \sqrt{B}\, \hat{x}, \tag{3.64}$$

the solution for $\hat{u}_1$ may be written in two parts. For $\hat{x} > 0$,

$$\frac{\hat{u}_1}{\sqrt{B}} = \frac{X[U(A, X) - C_1 V(A, X)] + (2A+1)U(A+1, X) - C_1(2A-1)V(A-1, X)}{U(A, X) + C_1 V(A, X)} \tag{3.65}$$

and for $\hat{x} < 0$

$$\frac{\hat{u}_1}{\sqrt{B}} = X - (2A+1)\, \frac{U(A+1, -X)}{U(A, -X)} \tag{3.66}$$

where in order that the solutions agree at $X = 0$,

$$C_1 = \frac{2\Gamma(\frac{1}{2} - A)\tan\left[\pi\left(\frac{1}{4} + \frac{A}{2}\right)\right]}{1 - \tan^2\left[\pi\left(\frac{1}{4} + \frac{A}{2}\right)\right]} \tag{3.67}$$

In eqn. (3.67), $\Gamma(z)$ is the Gamma Function[12].

Equation (3.63) was obtained and solved first by Kopystynski and Szaniawski[13] in their study of flow in a nozzle throat. They wrote the solution in terms of confluent hypergeometric functions; the solution here, written in terms of parabolic cylinder functions is equivalent to theirs and somewhat more convenient to use. It is interesting to note that Kopystynski and Szaniawski showed velocity distributions which appeared to be those for

---

[11] Murphy, G. M., Ordinary Differential Equations and Their Solutions, Van Nostrand Company, Inc., 1960.

[12] Handbook of Mathematical Functions, Eds. M. Abramowitz and I. A. Stegun, N.B.S. Applied Mathematics Series .55, 1964.

[13] Kopystynski, J. and Szaniawski, A., Structure of Flow in a Nozzle Throat, Archiwum Mechaniki Stosowanej, 3, 17 (1965), 453-466.

very thick shock waves but did not evaluate their solution in the limit as this relatively rapid variation in velocity moves downstream of the throat. Thus, they did not show that this solution evolves into that for a thin shock wave across which Hugoniot jump conditions hold. It is this point which is the subject of the present section.

It is possible, then to write an exact solution for $\hat{u}$, in terms of known functions. Now, eqn. (3.63) is a relatively simple first order nonlinear differential equation which it is quite easy to integrate numerically using only a programmable hand calculator. Nevertheless, the analytical solution is useful in that it may be used to demonstrate that it becomes exactly that given in the previous section as the shock wave moves downstream of the throat.

The expansions necessary for matching with the outer, channel flow solutions, are found by writing U and V for $|X| \gg 1$. From reference 12, one finds that for X large and A moderate, for $X \gg |A|$,

$$U(A, X) \sim e^{-\frac{X^2}{4}} \; X^{-A - \frac{1}{2}} \left\{ 1 - \frac{(A + \frac{1}{2})(A + \frac{3}{2})}{2X^2} + \ldots \right\} \tag{3.68a}$$

$$V(A, X) \sim \sqrt{\frac{2}{\pi}} \; e^{\frac{X^2}{4}} \; X^{A - \frac{1}{2}} \left\{ 1 + \frac{(A - \frac{1}{2})(A - \frac{3}{2})}{2X^2} + \ldots \right\} \tag{3.68b}$$

Hence, from eqns. (3.64), (3.65), and (3.66), as $\hat{x} \to \infty$

$$\hat{u}_1 = -B\hat{x} - \frac{(2A - 1)}{\hat{x}} + \ldots \tag{3.69}$$

and as $\hat{x} \to -\infty$

$$\hat{u}_1 = B\hat{x} + \frac{(2A + 1)}{\hat{x}} + \ldots \tag{3.70}$$

Equations (3.69) and (3.70) can be checked by substitution into eqn. (3.63) and, indeed, could have been derived from it. Now, the present solutions hold for the case where the shock wave is in the neighborhood of the throat.

Although the flow upstream of the throat could, in general, be either subsonic or supersonic, we consider here the case where it is subsonic. Hence, as $\hat{x} \to -\infty$ and $\hat{x} \to +\infty$, the inner solutions should match with outer channel flow solutions which are valid for subsonic flow upstream of the throat and subsonic flow downstream of the shock wave, respectively, both written in the limit as $x \to 0$. For the solutions valid downstream of the shock wave, this means also that $x_{so} \to 0$ and as a result, $u_{10} \to 0$. The required outer solutions are found from eqns. (3.18a), (3.19a), (3.54a) and (3.59), using eqns. (3.51) for $f(x)$, and u_1. Thus, for $x \to 0$ from upstream of the throat

$$u = 1 + \epsilon\alpha \left\{ B\hat{x} + \frac{C_{2u} + mB}{mB\hat{x}} \right\} + \ldots \tag{3.71}$$

and for $x \to 0$ from downstream of the throat,

$$u = 1 - \epsilon\alpha \left\{ B\hat{x} + \frac{C_{2d} - mB}{mB\hat{x}} \right\} + \ldots \tag{3.72}$$

where both equations are written in terms of the inner variable $\hat{x}$. Matching the inner and outer solutions, one finds that,

$$C_{2u} = 2mBA = C_{2d} \tag{3.73}$$

If this result is compared with that given in eqn. (3.39), it is seen that for $u_{10} = Bx_{so} = \alpha\, B\hat{x}_{so} = (m\epsilon)^{1/2} B\hat{x}_{so}$, the two results agree, in lowest order. Thus, in the throat region, choosing a value for $C_{2d} = C_{2u}$ is equivalent to choosing a value for A, and as will be seen, this means setting the location of the shock wave just as in the outer region.

For $A = -1/2$, $C_1 = 0$ (eqn. 3.67) and the solutions for $\hat{u}_1$ (eqns. (3.65) and (3.66)) reduce to

$$\hat{u}_1 = \sqrt{B}\, X = B\hat{x} \tag{3.74}$$

This is the solution for a flow accelerating from subsonic to supersonic velocities with no shock wave; that is, any shock wave which might occur is positioned downstream of the throat region under consideration. On the other hand, for values of A greater than $-1/2$, solutions for $\hat{u}_1$ show shocklike

behavior, as illustrated in figure 2. The solutions in figure 2 were found by integrating eqn. (3.63) numerically, with an H.P. 29C programmable calculator, using eqn. (3.70) to obtain initial values for $\hat{u}_1/\sqrt{B}$. They could have been obtained also by using eqn. (3.65) and (3.66) with numerical values for $U(A, X)$ and $V(A, X)$ obtained from tables in reference 12. The solutions show that there is a certain value for A, (somewhat larger than -0.30) greater than which there is no shock wave since no supersonic flow exists. The flow is simply a viscous channel flow. As A decreases below this value and tends toward $-1/2$, a shock wave forms and moves downstream. The more closely A approaches $-1/2$, the closer is the flow upstream of the jump in velocity to the supersonic inviscid channel flow solution $\hat{u}_1 = \sqrt{B}X = B\hat{x}$ and the thinner is the region associated with the jump in velocity. Downstream of the jump, the solution approaches the subsonic inviscid channel flow solution, $\hat{u}_1 = -\sqrt{B}X = -B\hat{x}$. The fact that the Hugoniot shock wave jump conditions are not satisfied in the immediate region of the jump is clear; i.e., these jump conditions are $\hat{u}_{1d} = -\hat{u}_{1u}$ and in figure 2 it can be seen that the solutions are not symmetric about $\hat{u}_1 = 0$.

The solutions shown in figure 2 apparently show the evolution of a shock wave as it forms at the throat and moves downstream in response to a pressure condition impressed upon the flow downstream of the shock wave. In order to prove this contention, it must be shown that as the shock wave moves downstream, this inner solution matches with the solutions valid for the case where the shock wave is located at a distance of O(1) downstream of the throat, but written now in the limit as the shock wave moves toward the throat. From the solutions shown in figure 2, it is clear that the inner limit process involves $A \to -1/2$ as $\epsilon \to 0$.

The outer solutions to which the inner solution must match is a composite solution, valid to first order, formed from the solutions found in the previous section. Thus, if the flows upstream of the shock wave and in the inner structure regions are considered, the composite solution is the sum of the channel flow and inner structure solutions minus the common (found

from matching) terms.

$$u = 1 + \epsilon u_1(x) + \epsilon u_1^+(x^+) - \epsilon u_{10} + \ldots \qquad (3.75)$$

Equation (3.75) is valid for $x < x_\ell$, $(x^+ < 0)$, because the supersonic value of $u_1(x)$ was used in determining the common term. The corresponding equation valid for $x > x_\ell$ would be $u = 1 + \epsilon u_1(x) + \epsilon u_1^+(x^+) + \epsilon u_{10} + \ldots$. Now, because $x^+ = (x - x_\ell)/k_s$ and $x_\ell = x_{\ell o} + \epsilon x_{\ell 1} + \epsilon^{3/2} x_{\ell 1} + \ldots$, with $k_s = O(\epsilon)$, it is seen that for eqn. (3.75) to be uniformly valid to $O(\epsilon)$, it is necessary only to include the first two terms in the expansion for x_ℓ, i.e., $x_\ell = x_{\ell o} + \epsilon x_{\ell 1}$, and this is understood to be the case in what follows. Finally, because when the shock wave is within the throat region, matching between the inner throat region solution and the outer channel flow solutions has already been demonstrated, it only remains to demonstrate that the solutions match in the immediate vicinity of the shock wave as it moves toward the throat. Therefore, the outer solution, eqn. (3.75) is written in the limit as $x \to x_\ell$, for $x_\ell \ll 1$ such that $x^+ = O(1)$. Then, to lowest order, eqn. (3.75), and u_{10} become

$$u = 1 - \epsilon u_{10} \tanh\left(\frac{u_{10}}{2} x^+\right) + \ldots \qquad (3.76a)$$

$$u_{10} = B x_{\ell o} \qquad (3.76b)$$

where for $x \ll 1$, $u_1 = Bx$, and where the solutions for x_1^+, eqn. (3.33a) has been employed.

The corresponding limit to be used in evaluating the inner solution is

$$X = X_\ell + X - X_\ell \qquad (3.77a)$$

$$\frac{X - X_\ell}{X_\ell} \ll 1 \ll X_\ell \qquad (3.77b)$$

$$A = -\frac{1}{2} - \frac{\omega}{2} \qquad (3.77c)$$

$$\omega = \omega(\epsilon) \ll 1 \qquad (3.77d)$$

65

When C_1, eqn. (3.67), is evaluated using eqn. (3.77c), for A, it is found that

$$C_1 = \omega \pi + O(\omega^2) \tag{3.78}$$

Next, if eqn. (3.78) and the asymptotic expressions for U(A, X) and V(A, X), eqns. (3.68a) and (3.68b) respectively, are substituted into the equation for $\hat{u}_1$ valid for $\hat{x} > 0$, eqn. (3.65), the following result is obtained for $X \gg 1$, and thus $\hat{x} \gg 1$.

$$\frac{\hat{u}_1}{\sqrt{B}} = -X \frac{\left[1 - (\omega\sqrt{2\pi})^{-1} X e^{-\frac{X^2}{2}} - X^{-2} + \cdots \right]}{\left[1 + (\omega\sqrt{2\pi})^{-1} X e^{-\frac{X^2}{2}} + X^{-2} + \cdots \right]} \tag{3.79}$$

Now, starting with eqn. (3.77a) and using the definitions of X, $\hat{x}$ and x^+, eqns. (3.64), (3.54a) and (3.21a) respectively, one can write X as

$$X = \frac{u_{10}}{\sqrt{\epsilon m B}} + \sqrt{\frac{\epsilon B}{m}} \, x_{\ell 1} + \sqrt{\epsilon m B} \tag{3.80}$$

where x_ℓ has been expanded using only the first two terms, as mentioned previously. In addition, eqn. (3.76b) has been used for $x_{\ell o}$ and eqn. (3.59) for α. When eqn. (3.80) is substituted into eqn. (3.79), it is seen that if to lowest order

$$\omega(\epsilon) = \frac{u_{10}}{\sqrt{2mB\pi\epsilon}} \exp \left\{ -\left(\frac{u_{10}^2}{2mB\epsilon} + \frac{u_{10} x_{\ell 1}}{m} \right) \right\} \tag{3.81}$$

then, again to lowest order,

$$\hat{u}_1 = -u_{10} \frac{\left[1 - e^{-u_{10} x^+} \right]}{\left[1 + e^{-u_{10} x^+} \right]} = -u_{10} \tanh r^+ \tag{3.82}$$

where, again, $r^+ = u_{10} x^+/2$. Therefore, this solution from the inner throat region matches the corresponding outer solution in the immediate vicinity of the shock wave, eqn. (3.76a). Evidently, the solutions found in this section do represent the evolution of a shock wave as it moves downstream of the throat.

66

changing from a thick shock across which Hugoniot conditions are not satisfied to a thin shock wave across which they are satisfied.

Finally, it is interesting to note that the solutions shown in figure 2 are quite similar to those given previously for steady channel flow by Sichel[14] (two-dimensional) and Sichel and Yin[15] (axisymmetric), and for unsteady two-dimensional flow by Adamson and Richey[16]. In all of these analyses, a similarity transformation was used to reduce the viscous transonic equation to an ordinary differential equation. Then, in references 14 and 15, numerical solutions were obtained; in reference 16, numerical results were obtained for thick shock waves and analytical solutions for thin shock waves. In all cases, as the solution for the perturbation in u went through the jump caused by the shock wave, it overshot the solution associated with decelerating subsonic flow downstream of the shock and then approached this solution asymptotically from beneath. As seen in figure 2, the present solutions do not show this feature; instead, $\hat{u}_1$ varies monotonically from its peak value to the solution for subsonic decelerating flow ($\hat{u}_1 = - B\hat{x}$). This difference occurs because the similarity solutions for simple supersonic accelerating flow and subsonic decelerating flow are not symmetric about $\hat{u}_1 = 0$. That is, at any given location the solution for subsonic decelerating flow is not the negative of the solution for accelerating supersonic flow. Hence, these solutions cannot, in themselves, satisfy the jump condition across a shock wave, $\hat{u}_{1d} = - \hat{u}_{1u}$; an overshoot in the solution results when a shock wave occurs.

---

[14] Sichel, M., The Effect of Longitudinal Viscosity on the Flow at a Nozzle Throat, J. Fluid Mech., 25 (1966), 769-786.

[15] Sichel, M. and Yin, Y. K., An Axisymmetric Similarity Solution for Viscous Transonic Nozzle Flow, J. Fluid Mech., 28 (1967), 512-522.

[16] Adamson, T. C., Jr. and Richey, G. K., Unsteady Transonic Flows with Shock Waves in Two-Dimensional Channels, J. Fluid Mech., 60 (1973), 363-382.

Perhaps the most important difference between the similarity solutions and those presented here is that the former cannot be employed for arbitrary wall shapes. Thus, a wall must be associated with a streamline calculated from the similarity solution; no arbitrariness is allowed. Not only does this limit the usefulness of similarity solutions in practical applications, it leads to wall shapes with bends or angles at the point where the streamline associated with the wall passes through the shock wave, depending upon the thickness of the shock wave. The present solutions allow arbitrary wall shapes to be considered; in those cases where the streamline variation through the shock wave is different from that of the wall (including the effects of the displacement of the boundary layer), i. e., when interaction effects are important, corrections can be added.

References

1. Hafez, M. M. and Cheng, H. K., Shock Fitting in Transonic Flow Computation, <u>Transonic Flow Problems in Turbomachinery,</u> eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 328-335.

2. Messiter, A. F. and Adamson, T. C., Jr., On the Flow near a Weak Shock Wave Downstream of a Nozzle Throat, J. Fluid Mech., <u>69</u> (1975), 97-108.

3. Richey, G. K. and Adamson, T. C., Jr., Analysis of Unsteady Transonic Channel Flow with Shock Waves, AIAA J., <u>14</u> (1976), 1054-1061.

4. Adamson, T. C., Jr., Messiter, A. F. and Liou, M. S., Large Amplitude Shock-Wave Motion in Two-Dimensional, Transonic Channel Flows, AIAA J., <u>16</u> (1978), 1240-1247.

5. Yoshihara, H., Finite Difference Procedure for Unsteady Transonic Flows: A Review, <u>Transonic Flow Problems in Turbomachinery</u>, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 139-162.

6. Caughey, D. A. and Jameson, A., Calculation of Transonic Potential Flowfields About Complex Three-Dimensional Configurations, <u>Transonic Flow Problems in Turbomachinery</u>, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 274-293.

7. Farn, C. L. S. and Whirlow, D. K., Application of Time-Dependent Finite Volume Method to Transonic Flow in Large Turbines, <u>Transonic Flow Problems in Turbomachinery</u>, eds. T. C. Adamson, Jr. and M. F. Platzer, Hemisphere Publishing Corp., 1977, 208-227.

8. Illingworth, C. R., Shock Waves, <u>Modern Developments in Fluid Mechanics, High Speed Flow, Vol. I,</u> ed. L. Howarth, Oxford Press, 1953, 105-137.

9. Brilliant, H. M. and Adamson, Jr., T. C., Shock Wave Boundary-Layer Interactions in Laminar Transonic Flow, AIAA J., $\underline{12}$ (1974), 323-329.

10. Cole, J., Problems in Transonic Flow, Ph. D. Thesis, California Institute of Technology, 1949.

11. Murphy, G. M., <u>Ordinary Differential Equations and Their Solutions,</u> Van Nostrand Company, Inc., 1960.

12. <u>Handbook of Mathematical Functions,</u> Eds. M. Abramowitz and I. A. Stegun, N. B.S. Applied Mathematics Series. 55, 1964.

13. Kopystynski, J. and Szaniawski, A., Structure of Flow in a Nozzle Throat, Archiwum Mechaniki Stosowanej, $\underline{3}$, 17 (1965), 453-466.

14. Sichel, M., The Effect of Longitudinal Viscosity on the Flow at a Nozzle Throat, J. Fluid Mech., $\underline{25}$ (1966), 769-786.

15. Sichel, M. and Yin, Y. K., An Axisymmetric Similarity Solution for Viscous Transonic Nozzle Flow, J. Fluid Mech., $\underline{28}$ (1967), 512-522.

16. Adamson, T. C., Jr. and Richey, G. K., Unsteady Transonic Flows with Shock Waves in Two-Dimensional Channels, J. Fluid Mech., $\underline{60}$ (1973), 363-382.
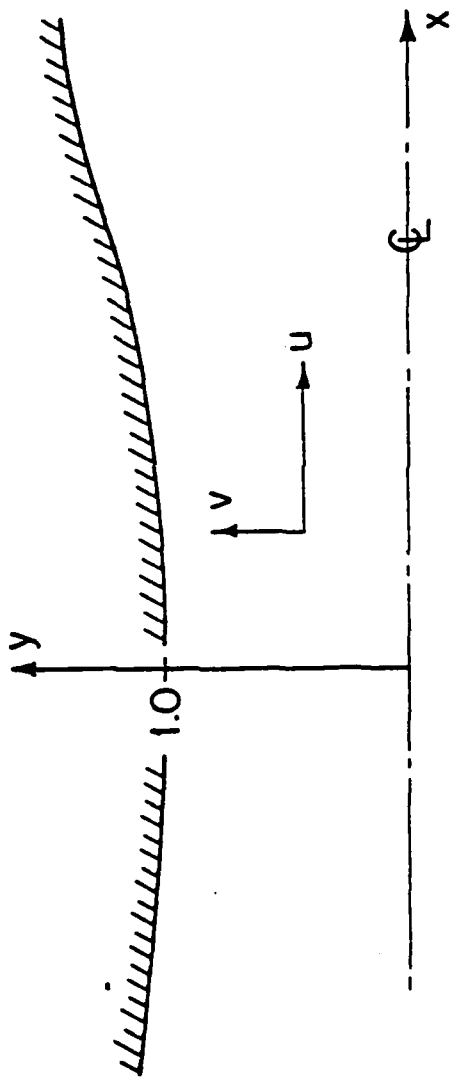
Figure 1. Sketch of symmetric channel flow showing coordinate
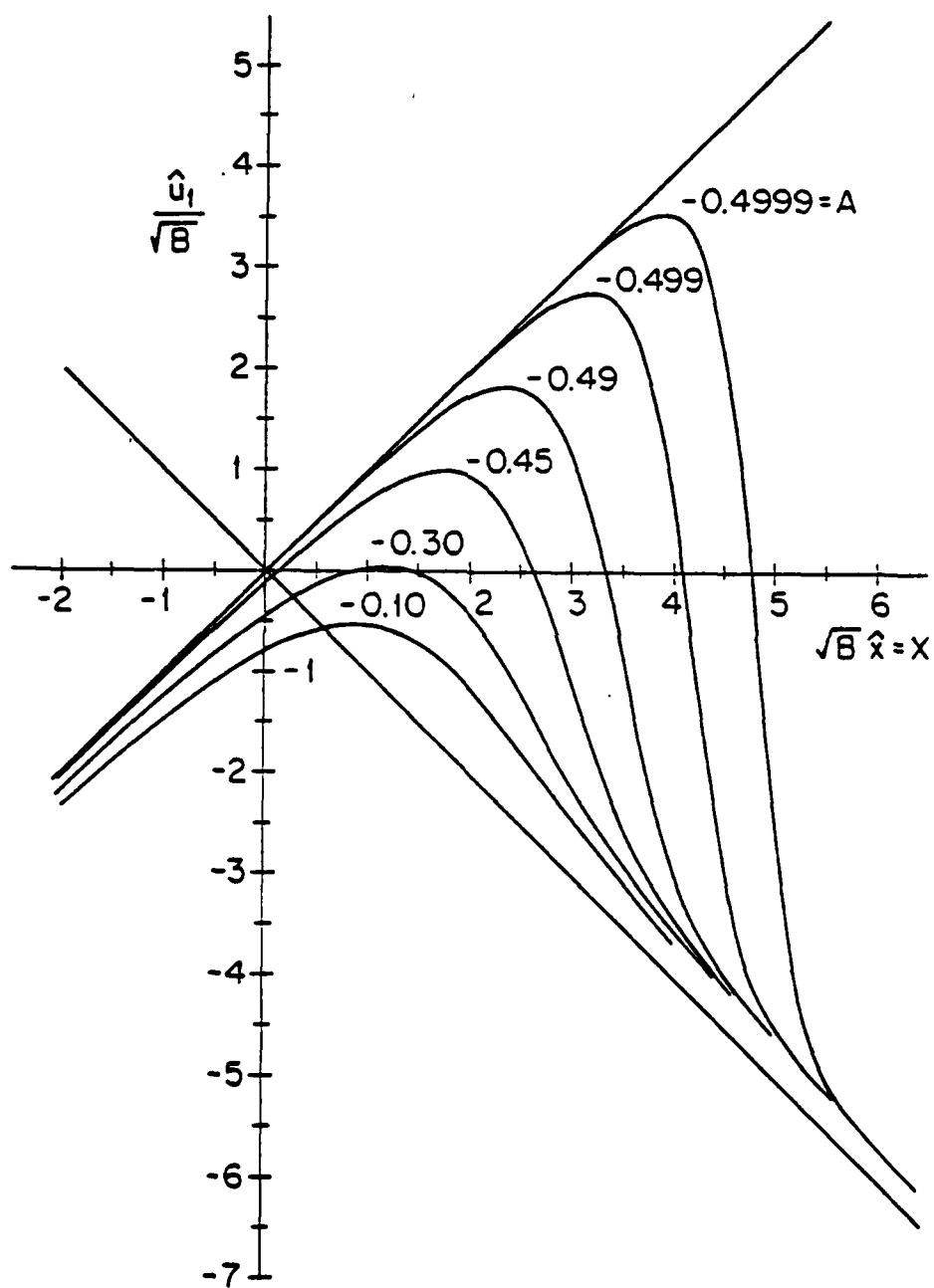system and notation used.

Figure 2. Distribution of perturbation velocity, $\hat{u}_1$, for various values of A.

ND

DATE

ILMED