A ZERO-ONE DICHOTOMY THEOREM FOR
r-SEMI-STABLE LAWS ON INFINITE
DIMENSIONAL LINEAR SPACES

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A ZERO-ONE DICHOTOMY THEOREM FOR r-SEMI-STABLE LAWS
ON INFINITE DIMENSIONAL LINEAR SPACES

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ABSTRACT

Let \( \mu \) be an \( r \)-semistable probability measure on a real linear space \( E \). It is shown that the \( \mu \)-measure of any translate of an arbitrary measurable linear subspace over certain countable subfield of reals is 0 or 1. This result yields immediately the 0-1 laws for stable measures of Dudley-Kanter (Proc. Amer. Math. Soc., 45(1974), 245 - 252) and also a more recent 0-1 law of Fernique for quasi-stable measures which is included in his ISI lectures of September, 1978. It is also shown that \( r \)-semi-stable measures - like stable ones - are continuous, i.e., they assign zero mass to singletons.

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I. INTRODUCTION

Let \((E, \mathcal{F})\) be a measurable vector space in the sense of [2], and \(\mu\) a stable probability measure (p.m.) on \(E\). Recently, Dudley-Kanter [2] have shown that the \(\mu\)-measure of certain measurable subspaces of \(E\) is 0 or 1. More recently Fernique exhibited a similar 0-1 law for what he calls quasi-stable p.m. measures. A natural and non-trivial generalization of stable p.m. measures is the class of \(r\)-semi-stable p.m. measures, which was first introduced and studied on the real line \(\mathbb{R}\) by P. Lévy [6]. Later Kruglov [3] obtained a quite explicit form of the characteristic function of \(r\) semi-stable p.m. measures on \(\mathbb{R}\) and showed that this class have many properties similar to those exhibited by stable probability measures. (This in Hilbert space setting is also shown in Kruglov [4] and Kumar [5]). Partly motivated from these papers we raised and completely answered the question whether \(r\)-semi-stable p.m. measures share with stable measures the 0-1 dichotomy results obtained in [2]. Explicitly we prove that if \((E, \mathcal{F})\) is a measurable vector space over \(\mathbb{R}\), \(\mu\) a \(r\)-semi-stable p.m. (see §2) on \((E, \mathcal{F})\) and \(G\) a measurable subspace over the field \(Q(c)\), the smallest subfield containing \(Q\), the rationals, and \(c = c(r)\), then 

\[
\mu(G - z) = 0 \text{ or } 1, \text{ for every } z \in E \text{ (Theorem 3.1).}
\]

This result includes and, in fact, extends the 0-1 theorems for stable p.m. measures obtained in [2] (Corollary 3.2); also the method of proof of the result includes a recent 0-1 dichotomy theorem.
of Fernique (ISI Calcutta, Lectures '78) for quasi-stable p.
measures (Corollary 3.3). Further, we also show that, like stable
p. measures, non-degenerate r-semistable p. measures are continuous;
that is, they assign zero mass to singletons (Corollary 3.4). Our
proof of the 0 - 1 dichotomy theorem seems new as well as simpler
than those in [2] (we use only the definition of canvolution and
Fubini's theorem); in particular, we do not require any number theory
results which was not the case in the proofs of [2].

2. PRELIMINARIES

Let \((E, \mathcal{F})\) be a measurable vector space and \(\mu\) be a p.m. on
\(\mathcal{F}\). Let \(r \in (0, 1)\); then \(\mu\) is called r-semistable if there is
a constant \(c(r) = c\) with \(0 < c \neq 1\) and a semigroup \(\{\mu^s; s > 0\}\)
of p. measures on \(\mathcal{F}\) and a sequence \(\{x_m\}\) in \(E\) such that the
following hold

\[
\mu^1 = \mu, \tag{2.1}
\]

\[
\mu^m = T_{c^m} \mu \ast \delta_{x_m}, \tag{2.2}
\]

for each \(m = 1, 2, \ldots\), where for \(a > 0\), \(T_a \mu\) denotes the measure
\(T_a \mu(B) = \mu(a^{-1} B)\), for every \(B \in \mathcal{F}\) and \(\ast\) denotes the usual
convolution.

The above definition is motivated from a characterization of a
class of measures also called r-semistable on locally convex
topological vector spaces (LCTVS) obtained in [1]. It follows from [1] that our results are applicable for r-semistable (and hence stable and Gaussian) measures studied in [1].

3. 0 - 1 DICHOTOMY THEOREM FOR r-SEMI-STABLE MEASURES

The main result we propose to prove is the following:

**Theorem 3.1.** Let \( \mu \) be a r-semistable p.m. on a measurable vector space \((E, \mathcal{F})\) over \( \mathbb{R}\) and let \( G \) be a subspace over the subfield \( \mathbb{Q}(c) \) such that \( G \in \mathcal{F} \) (\( c \) is the constant appearing in (2.2)). Then \( \mu(G - z) = 0 \) or 1, for all \( z \in E \).

**Proof.** Let \( z \in E \) and assume that \( \mu(G-z_1) > 0 \). We will show that \( \mu(G-z_1) = 1 \). Choose an integer \( n_r \) so that \( 0 < 1/n_r < 1-r \). Let

\[
\mathcal{F} = \{ G - x \mid \mu(G-x) > 0 \text{ or } \mu^{1/n_r}(G-x) > 0 \} \subseteq E/G.
\]

\( \mathcal{F} \) is linear span of \( \mathcal{F} \) in \( E/G \) over the field \( \mathbb{Q}[c] \), and \( G_0 \) is inverse image of \( \mathcal{F} \) under natural projection \( \cup \langle \mathcal{F} \rangle \).

Then \( G_0 \) is a vector subspace of \( E \) over \( \mathbb{Q}(c) \) and clearly, \( G_0 \in \mathcal{F} \), since \( G_0 \) is a countable union of sets in \( \mathcal{F} \).

For the sake of clarity, the remainder of the proof will be divided into seven parts.

(1) \( \mu^{1-r} \ast \delta_x(1)(G_0) = 1 \).

**Proof of (1).** Observe that \( \mu(G_0 - r^m / n) = 0 \), for all \( y \in G_0 \), and that
\[
\mu = \mu^{r} \ast \mu^{1-r} = T_c \mu \ast \mu^{1-r} \ast \delta_x(1).
\]

Thus
\[
0 < \mu(G_0) = \int_E T_c \mu \cdot G_0 - y ) \mu^{1-r} \ast \delta_x(1) (dy)
= \int_{G_0} \mu(G_0 - c^{-1} y) \mu^{1-r} \ast \delta_x(1) (dy)
= \mu(G_0) \mu^{1-r} \ast \delta_x(1) (G_0).
\]

Consequently, \( \mu^{1-r} \ast \delta_x(1) (G_0) = 1 \).
(ii) \( \mu^{1/nr}(G_0) = 1 \).

**Proof of (ii).** Since \( \mu = \mu^{1/nr} \ast (\mu^{1/nr})^{(nr-1)} \), we have
\[
0 < \mu(G-z_1) = \int_E \mu^{1/nr}(G-z_1-y) (\mu^{1/nr})^{(nr-1)}(dy).
\]
Thus there exists \( y \in E \) so that \( \mu^{1/nr}(G-z_1-y) > 0 \), and hence \( \mu^{1/nr}(G_0) > 0 \).

Now \( \mu^{1-r} \ast \delta_x(1) = \mu^{1/nr} \ast \mu^{1-r-1/nr} \ast \delta_x(1) \), and so, from (i),
\[
1 = \mu^{1-r} \ast \delta_x(1)(G_0) = \int_E \mu^{1-r-1/nr} \ast \delta_x(1)(G_0-y) \mu^{1/nr}(dy),
\]
which implies that \( \mu^{1-r-1/nr} \ast \delta_x(1)(G_0-y) = 1 \) a.s. \([\mu^{1/nr}] \). Since \( \mu^{1/nr}(G_0) > 0 \), it follows that \( \mu^{1-r-1/nr} \ast \delta_x(1)(G_0) = 1 \).

Consequently,
\[
1 = \mu^{1-r} \ast \delta_x(1)(G_0) = \int_{G_0} \mu^{1/nr}(G_0-y) \mu^{1-r-1/nr} \ast \delta_x(1)(y)
= \mu^{1/nr}(G_0) \mu^{1-r-1/nr} \ast \delta_x(1)(G_0)
= \mu^{1/nr}(G_0).
\]

(iii) \( \mu(G_0) = 1 \).

**Proof of (iii).** It follows from (ii) that
\[
\mu(G_0) = \int_{G_0} (\mu^{1/nr})^{(nr-1)}(G_0-y) \mu^{1/nr}(dy)
= (\mu^{1/nr})^{(nr-1)}(G_0) \mu^{1/nr}(G_0)
= (\mu^{1/nr})^{(nr-1)}(G_0)
= (\mu^{1/nr})(G_0)^{nr-1}
= 1.
\]

We will use the fact that \( \mu(G_0) = 1 \) to conclude that \( \mu(G-z_1) = 1 \) (see (vii)).

To this end, we proceed.

Recall that \( G_0 \) is a countable (possibly finite) union of disjoint cosets of \( G \). Let \( \{x_1, x_2, \ldots\} \) be a sequence of distinct points in \( E \) so that
$G_0 = \bigcup_k G-x_k$ (disjoint union). Clearly, we may assume, without loss of generality, that $u(G-x_1) \geq u(G-x_2) \geq \cdots$. Let $N_1$ be the largest integer so that $u(G-x_1) = u(G-x_{N_1})$. For the sake of simplicity of notation, let $t = t(m) = c^m$, $m = 1, 2, \ldots$, and let $v_t = u^1 - r^m \delta_x(m)$. Then

$u = T_u + v_t$, for any $t$.

(iv) For each $t$, $v_t \left( \bigcup_{k=1}^{N_1} G-x_n + tx_k \right) = 1$, for $1 \leq n \leq N_1$.

Proof of (iv). Observe that if $y \in G-x_k$, then $G-x_n - ty = G-x_n + tx_k$, for all $n$ and $k$. Thus,

$$\begin{align*}
u(G-x_n) &= \int_{G_0} v_t(G-x_n - ty) u(dy) \\
&= \sum_{k} v_t(G-x_n + tx_k) u(G-x_k), \hspace{1cm} (3.1)
\end{align*}$$

for $n = 1, 2, \ldots$. Now, for $1 < n < N_1$, we have

$$\begin{align*}
u(G-x_n) &= \sum_{k} v_t(G-x_n + tx_k) u(G-x_k) \\
&\leq \nu(G-x_n) \sum_{k} v_t(G-x_n + tx_k) \\
&= \nu(G-x_n) \sum_{k} v_t(G-x_n + tx_k) \\
&\leq \nu(G-x_n).
\end{align*}$$

Thus

$$\begin{align*}
u(G-x_n) v_t(G-x_n + tx_k) &= \nu(G-x_n) v_t(G-x_n + tx_k),
\end{align*}$$

for $1 \leq n \leq N_1$ and any $k$, which implies that $v_t(G-x_n + tx_k) = 0$ for $1 \leq n \leq N_1$ and $k > N_1$.

Thus

$$\begin{align*}
u(G-x_n) &= \sum_{k=1}^{N_1} v_t(G-x_n + tx_k) u(G-x_k), \\
&= \nu(G-x_n) \sum_{k=1}^{N_1} v_t(G-x_n + tx_k) \\
&= \nu(G-x_n) v_t(\bigcup_{k=1}^{N_1} G-x_n + tx_k),
\end{align*}$$

for $1 \leq n \leq N_1$. 
Hence
\[ N_1 \]
\[ 1 = \nu_t(\bigcup_{k=1}^{n} G-x_n+tx_k), \]
for \( 1 \leq n \leq N_1 \), since \( \nu(G-x_n) = \nu(G-x_1) > 0 \) for \( 1 \leq n \leq N_1 \).

(v) \( N_1 = 1 \) or, equivalently, \( (G-x_1) > (G-x_k) \),
for all \( k > 1 \).

Proof of (v). Suppose \( N_1 \geq 2 \) and consider the \( 2 \times N_1 \) array \( M_1 \):
\[
\begin{array}{cccc}
G-x_1+tx_1 & G-x_1+tx_2 & \ldots & G-x_1+tx_{N_1} \\
G-x_2+tx_1 & G-x_2+tx_2 & \ldots & G-x_2+tx_{N_1}
\end{array}
\]
By (iv), the \( \nu_t \)-measure of row 1 of \( M_1 \) is 1. Thus there is an integer \( k_1 \),
\( 1 \leq k_1 \leq N_1 \), so that \( \nu_t(G-x_1+tx_{k_1}) > 0 \), for infinitely many values of \( t \).

Now, the \( \nu_t \)-measure of row 2 is also 1 (by (iv) again), which implies that
\( G-x_1+tx_{k_1} \) intersects row 2, for infinitely many values of \( t \). Thus there is
an integer \( k_2 \), \( 1 \leq k_2 \leq N_1 \), so that \( G-x_1+tx_{k_1} = G-x_2+tx_{k_2} \), for infinitely
many values of \( t \). Consequently, there are integers \( k_1 \) and \( k_2 \), \( 1 \leq k_1 \leq N_1 \)
\( 1 \leq k_2 \leq N_1 \), so that
\[
G-x_1+x_2 = G-t(x_{k_2}-x_{k_1}), \quad (3.2)
\]
for infinitely many values of \( t \). In particular, these exist \( t_1 \) and \( t_2 \), \( t_1 \neq t_2 \),
so that \( G-t_1(x_{k_2}-x_{k_1}) = G - t_2(x_{k_2}-x_{k_1}) \) which implies that
\( G = G+(t_1-t_2)(x_{k_2}-x_{k_1}) \) and so, \( (t_1-t_2)(x_{k_2}-x_{k_1}) \in G \) from which it follows that
\( G-x_{k_1} = G-x_{k_2} \). Consequently, since \( G_o \) is a disjoint union, we have \( k_1 = k_2 \)
which implies, from (3.2), that \( G-x_1 = G-x_2 \). But \( G-x_1 \neq G-x_2 \).
Hence (v) follows.
(vi) For each \( t \), \( v_t(G-x_1+tx_1) = 1 \).

Proof of (vi). This is immediate from (iv) and (v).

(vii) \( u(G-z_1) = u(G_0) \).

Proof of (vii). Suppose \( u(G-z_1) < u(G_0) \). Then \( u(G-x_2) > 0 \). Let \( N_2 \) be the largest integer so that \( u(G-x_2) = u(G-x_{N_2}) \).

Observe that, by (vi), we have that for each \( t \), \( v_t(G-x_n+tx_1) = 0 \), for all \( n > 2 \); otherwise, we get \( G-x_n = G-x_1 \), for some \( n \geq 2 \).

Thus, by (3.1), for \( 2 \leq n \leq N_2 \),

\[
\begin{align*}
u(G-x_n) &= v_t(G-x_n+tx_1) + \sum_{k \geq 2} v_t(G-x_n+tx_k)u(G-x_k) \\
&= \sum_{k \geq 2} v_t(G-x_n+tx_k)u(G-x_k) \\
&\leq u(G-x_n) \sum_{k \geq 2} v_t(G-x_n+tx_k) \\
&= u(G-x_n)v_t(\bigcup_{k=2} G-x_n+tx_k) \\
&\leq u(G-x_n).
\end{align*}
\]

It follows that

\[
u(G-x_n)v_t(G-x_n+tx_k) = u(G-x_k)v_t(G-x_n+tx_k), \quad \text{for } 2 \leq n \leq N_2 \text{ and any } k \geq 2,
\]

which implies that \( v_t(G-x_n+tx_k) = 0 \), for \( 2 \leq n \leq N_2 \) and \( k > N_2 \).

Consequently,

\[
u(G-x_n) = \sum_{k=2}^{N_2} v_t(G-x_n+tx_k)u(G-x_k) \\
= u(G-x_n) \sum_{k=2}^{N_2} v_t(G-x_n+tx_k) \\
= u(G-x_n)v_t(\bigcup_{k=2} G-x_n+tx_k),
\]

for \( 2 \leq n \leq N_2 \).

Hence, for all \( t \), \( N_2 \)

\[
1 = v_t(\bigcup_{k=2} G-x_n+tx_k), \quad (3.3)
\]

for \( 2 \leq n \leq N_2 \), since \( u(G-x_2) > 0 \), for \( 2 \leq n \leq N_2 \).
Observe that, by (vi), $\nu_t(G-x_2+tx_2) = 0$; otherwise, $G-x_2+tx_2 = G-x_1+tx_1$ which implies that $G-x_1 = G-x_2$. Consequently, from (3.3), $N_2 \geq 3$, and so, $G_0$ contains at least three disjoint cosets of $G$. Now consider the $2x(N_2-1)$ array $M_2$:

$$
\begin{array}{cccc}
G-x_2+tx_2 & G-x_2+tx_3 & G-x_2+tx_4 & \ldots & G-x_2+tx_{N_2} \\
G-x_3+tx_2 & G-x_3+tx_3 & G-x_3+tx_4 & \ldots & G-x_3+tx_{N_2} \\
\end{array}
$$

Observe that the $\nu_t$-measure of each row of $M_2$ is equal to 1. Now proceed, as in (v), to show that there exist integers $k_1$ and $k_2$, $2 \leq k_1 \leq N_2$, $2 \leq k_2 \leq N_2$, $k_1 \neq k_2$, so that

$$G-x_2+x_3 = G-t(x_{k_2}-x_{k_1}), \quad (3.4)$$

for infinitely many values of $t$. It follows, from (3.4), like in (v), that $k_1 = k_2$. Consequently, by (3.4), $G-x_2 = G-x_3$. This is a contradiction! Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$. Hence our initial assumption must be false and it follows that $\mu(G-z_1) = \mu(G_0)$.

To complete the proof of the theorem, observe that, by (iii) and (vii), we have $\mu(G-z_1) = \mu(G_0) = 1$.

In view of the last sentence of the previous section, we have the analogue of Theorem 3.1 for stable and Gaussian measures if the measures are K-regular and are defined on the Borel σ-algebra of a complete LCTVS. In the following corollary, we show, however, that the same result can be recovered from Theorem 3.1 even if the stable measures $\mu$ is defined on a measurable vector space $(E, \mathcal{F})$ provided $\mu$ has the index; i.e., there exists an $\alpha > 0$ such that for every $a > 0$, $b > 0$, $T_a u * T_b u = T_{(a^\alpha + b^\alpha)1/\alpha} u * \delta_x$, for some $x \in E$. This corollary contains and extends various results of [2]; we do not, however, deal with $0 - 1$. 


Corollary 3.2: Let \((E, \mathcal{F})\) be a measurable vector space and let \(G\) be a rational subspace of \(E, G \in \mathcal{F}\). Then

(i) If \(\mu\) is a strictly stable p.m. of index \(\alpha\) on \((E, \mathcal{F})\), then for all \(z \in E\), \(\mu(G - z) = 0\) or \(1\).

(ii) If \(\mu\) is a stable p.m. of index \(\alpha\) on \((E, \mathcal{F})\), then \(\mu(G) = 0\) or \(1\).

Proof: (i) Assume \(\mu\) is strictly stable of index \(\alpha\) and set \(\mu^s = T_s^{1/\alpha} \mu\). Then \(\{\mu^s \mid s > 0\}\) is a semigroup with \(\mu^1 = \mu\) and (2.1), (2.2) are satisfied for all \(r > 0\), with \(\chi(m) = 0\), and \(c = s^{1/\alpha}\). Then, it is easy to see that \(\mu\) is a \(r\)-semistable p.m. for all \(0 < r < 1\). Choose \(r_0, 0 < r_0 < 1\), so that \(r_0^{-1/\alpha}\) is rational. Then \(Q(r_0^{-1/\alpha}) = Q\). Now apply Theorem 3.1 to obtain the desired result.

(ii) Let \(\mu\) be a stable p.m. of index \(\alpha\) and assume that \(\mu(G) > 0\). Let \(v = \mu * T_{-1} \mu\) be the symmetrization of \(\mu\). Then \(v\) is a strictly stable p.m. of index \(\alpha\). Observe that

\[
v(G) = \int_E \mu(G + y) \mu(dy)
\geq \int_G \mu(G + y) \mu(dy)
= (\mu(G))^2 > 0.
\]

Thus, by (i), \(v(G) = 1\), and so \(\mu(G + y) = 1\) a.s. \((\mu)\) which implies that \(\mu(G) = 1\).
The following corollary shows that the method of proof of Theorem 3.1 also yields the 0 - 1 dichotomy theorem for quasi-stable measures recently obtained by Fernique who uses a non-trivial inequality of Kantor for his proof. Our proof, as we noted earlier, uses only elementary facts about convolution. Now we recall the definition of quasi-stable as introduced by Fernique. Let \( \mu \) be a p. measure on a measurable vector space \((E, \mathcal{F})\), then \( \mu \) is said to be quasi-stable if \( \mu^2 = T_c \mu \), for some \( c > 0 \), \( c \neq 1 \).

**Corollary 3.3:** Let \((E, \mathcal{F})\) be a measurable vector space and \( \mu \) be quasi-stable on \( E \). Let \( G \) be \( Q(c) \) vector space which belongs to \( \mathcal{F} \). Then \( \mu(G - z) = 0 \) or \( 1 \), for every \( z \in E \).

**Proof:** Let \( \mu(G - z_1) > 0 \) and let \( \mathcal{K}' = \{ G - x : \mu(G - x) > 0 \} \) and define \( G_0 \) as in the beginning of the proof of Theorem 3.1 with \( \mathcal{K} \) replaced by \( \mathcal{K}' \). Since

\[
0 < \mu(G_0) = T_c \mu(G_0) = \mu^2(G_0) = \int_{G_0} \mu(G_0 - x) \mu(dx)
\]

(as \( x \in G_0^c \) implies \( \mu(G_0 - x) = 0 \)), we have \( \mu(G_0) = 1 \). Now the definition of quasi-stability implies \( \mu^2 = T_c \mu \); hence

\[
\mu = T_{(1/c)^m} \mu^2 = T_{(1/c)^m} \mu^2 - 1 \ast T_{(1/c)^m} \mu.
\]

Setting \( (1/c)^m = t(m) \) and \( T_{(1/c)^m} \mu^2 - 1 = v \), we see that \( \mu = v \ast T_t \mu \).

Now repeating the proof of (iv) to (vii) of Theorem 3.1 without any change at all, one shows \( \mu(G - z_1) = 1 \). Completing the proof.
The following corollary shows that nondegenerate \( r \)-semistable measures cannot have positive point mass.

**Corollary 3.4:** Let \( \mu \) be a nondegenerate \( r \)-semistable measure of index \( \alpha \) on a measurable vector space \((E, \mathcal{F})\). Assume that \( \{x\} \in \mathcal{F} \), for all \( x \in E \). Then \( \mu(x) = 0 \), for all \( x \in E \).

**Proof:** Let \( G = \{0\} \) and \( x \in E \). If \( \mu(G + x) = \mu(x) > 0 \), then, by Theorem 3.1, \( \mu\{x\} = 1 \). Hence \( \mu \) is degenerate, a contradiction.

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A 0-1 Dichotomy Theorem for r-Semistable Laws on Infinite Dimensional Linear Spaces

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Let \( \nu \) be an \( r \)-semistable probability measure on a real linear space \( E \). It is shown that the \( r \)-measure of any translate of an arbitrary measurable linear subspace over certain countable subfield of reals is 0 or 1. This result yields Dudley-Kantor 0-1 laws for stable laws and Fernique 0-1 law for quasistable laws. It is also shown that the \( r \)-semistable laws - like stable ones - are continuous: i.e. they assign zero mass to singletons.