On the First Passage Time Distribution
For a Class of Markov Chains,

by

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Consider a stochastically monotone Markov chain with monotone paths on a partial-ly ordered countable set S. Let C be an increasing subset of S with finite com-plement. Then the first passage time from i∈S to C is shown to be IFRA (in-creasing failure rate on the average). Several applications are presented in-cluding coherent systems, shockmodels, and convolutions of IFRA distributions.

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ABSTRACT

Consider a stochastically monotone Markov chain with monotone paths on a partially ordered countable set S. Let C be an increasing subset of S with finite complement. Then the first passage time from $i \in S$ to C is shown to be IFRA (increasing failure rate on the average). Several applications are presented including coherent systems, shock models, and convolutions of IFRA distributions.

1. __Introduction__ Let S be a countable set with a partial ordering denoted by $\leq$. Consider a discrete time Markov chain $\{X_n, n \geq 0\}$ with state space S, and transition matrix P. Define the Markov chain to have monotone paths if $Pr(X_{n+1} \geq X_n) = 1$. Define $C \subset S$ to be an increasing set if $i \in C$ and $j \geq i$ implies $j \in C$. Define the Markov chain to be stochastically monotone if $i \leq j$ implies $P(i, C) \leq P(j, C)$ for all increasing sets C.

For a state i and set C define $T(i,C)$ to be the first passage time from i to C, with $T(i,C) = 0$ if $i \in C$, and $T(i,C) = \infty$ if C is never reached. Our main result (theorem 1) is that for a stochastically monotone Markov chain with monotone paths on a partially ordered countable set, $T(i,C)$ is IFRA for all states i and all increasing sets C with finite complement. (See section 2 for a definition of IFRA).

Conversely, every discrete IFRA distribution is either a first passage time distribution of the above described type or the limit of a sequence of such distributions (corollary 1).

Several applications of theorem 1 are presented in section 5. These include coherent systems, shock models, convolutions of IFRA distributions, multinomial distributions, and sampling. In these applications S is a subset of R^n and the partial ordering is defined by $\underline{x} \leq \underline{y}$ if and only if $x_i \leq y_i$, $i = 1, \ldots n$. The possibility exists for applications to a wider class of partially ordered sets, for example those studied in combinatorial theory (Rota [14]).

The question arises as to whether the conclusion of theorem 1 can be strengthened from IFRA to IFR. An example is given to show that even for a totally ordered set the first passage time need not be IFR. However for a Markov chain on the positive integers with TP_2 transition matrix the first passage time from 1 to $\{i: i>n\}$, $n = 1, 2, \ldots$ is IFR. This is proved in section 6.2.

Examples are given which demonstrate that even for a totally ordered set the IFRA conclusion does not follow for a stochastically monotone chain without monotone paths nor for a non-stochastically monotone chain with monotone paths.

By a uniformization and total positivity argument the result (theorem 1) extends to continuous time Markov chains with countable state space. An analogue of the theorem 1 undoubtedly holds for continuous state space. The restriction that the complement of C be finite appears to be a limitation of our methodology rather than an essential condition.

2. **Main result.** A random variable Y taking values in $\{0, 1, ..., \infty\}$ is defined to be IFRA if either $Pr(Y = 0) = 1$, or $Pr(Y = 0) = 0$ and $[Pr(Y > k)]^{1/k}$ is decreasing in $k = 1, 2, ...$. Included as IFRA is the case $Pr(Y = \infty) = 1$. Note that if Y is IFRA and $Pr(Y < \infty) > 0$ then $Pr(Y < \infty) = 1$.

Thus if T(i, C) is IFRA then either $i \in C$ (in which case $T(i, C) \equiv 0$), or starting in i it's impossible to reach C (in which case $T(i, C) \equiv \infty$), or $Pr(T(i, C) = 0) = Pr(T(i, C) = \infty) = 0$ and $[Pr(T(i, C) > k)]^{1/k}$ is decreasing.

Theorem 1   Let $\{X_n, n \geq 0\}$ be a stochastically monotone Markov chain with monotone paths on the partially ordered countable set S. Let C be an increasing set with $\tilde{C}$ (the complement of C in S) finite. Then T(i, C) the first passage time from state i to set C, is IFRA.

Proof   Define $\bar{F}_k(i, C) = Pr(T(i,C) > k)$. Our goal is to prove that $[\bar{F}_k(i, C)]^{1/k}$ is decreasing, equivalently that:

(1) $[\bar{F}_k(i, C)]^{k+1} \geq [\bar{F}_{k+1}(i, C)]^k$, $k = 1, 2, ...$

The proof is by induction. We first show that (1) holds for k = 1. It is trivially true for $i \in C$. For $i \notin C$ by the monotone path assumption:

$$(2) \qquad \bar{F}_2(i, C) = \sum_{b \geq i, b \in \tilde{C}} P(i, b)P(b, \tilde{C})$$

By stochastic monotonicity $b \geq i$ implies $P(i, \tilde{C}) \geq P(b, \tilde{C})$, thus from (2):

$$(3) \qquad \bar{F}_2(i, C) \leq P(i, \tilde{C}) \sum_{b \geq i, b \in \tilde{C}} P(i, b) = (\bar{F}_1(i, C))^2$$

Assume now that (1) holds for k = 1, ..., ℓ, for <u>all</u> increasing sets with finite complement. Define P_k to be the k step transition matrix for the Markov chain. Now:

$$(4) \qquad (\bar{F}_{\ell+1}(i,C))^{\ell+2} = \bar{F}_{\ell+1}(i, C)[\sum_{b \in \tilde{C}} P_\ell(i, b)P(b, \tilde{C})]^{\ell+1}$$

By stochastic monotonicity $P(b, \tilde{C})$ is decreasing in b. We can therefore label the finitely many points in $\tilde{C}$ by $b_1, \ldots, b_m$ in such a way that $P(b_1, \tilde{C}) \leq P(b_2, \tilde{C}) \ldots \leq P(b_m, \tilde{C})$ and for $i < j$, $b_i \nleq b_j$ (b_i is not smaller than b_j under the partial ordering). It follows that the sets $D_0 = \tilde{C}$, and $D_j = \tilde{C} - \{b_1, \ldots b_j\}$, j = 1, ... m - 1, are finite sets which are complements of increasing sets. The induction hypothesis is thus applicable to these sets. Define $\alpha_1 = P(b_1, \tilde{C})$ and $\alpha_j = P(b_j, \tilde{C}) - P(b_{j-1}, \tilde{C})$, j = 2, ... m, and note that the b's have been labeled so that:

$$(5) \qquad \alpha_j \geq 0, \; j = 1, \ldots m$$

Next:

$$(6) \qquad \bar{F}_{\ell+1}(i, C) = \sum_{r=1}^{m} P_\ell(i, b_r) \sum_{j=1}^{r} \alpha_j = \sum_{j=1}^{m} \alpha_j P_\ell(i, D_{j-1})$$

By the induction hypothesis:

$$(7) \qquad P_\ell(i, D_{j-1}) \geq [P_{\ell+1}(i, D_{j-1})]^{\frac{\ell}{\ell+1}}, \; j = 1, \ldots m$$

Note that in (7) if $i \notin D_{j-1}$, then both sides equal zero, and (7) is valid.

By (4), (5), (6) and (7):

$$(8) \quad [F_{\ell+1}(i, C)]^{\ell+2} \geq [(F_{\ell+1}(i, C))^{\frac{1}{\ell+1}} \sum_{j=1}^{m} \alpha_j (P_{\ell+1}(i, D_{j-1}))^{\frac{\ell}{\ell+1}}]^{\ell+1}$$

But, $\tilde{C} \supset D_{j-1}$; by this fact and the monotone path assumption:

$$(9) \quad F_{\ell+1}(i, C) = P_{\ell+1}(i, \tilde{C}) \geq P_{\ell+1}(i, D_{j-1}), \quad j = 1, \ldots m.$$

The result now follows from (6) (applied to $\ell + 1$), (8) and (9) . □

3. <u>Continuous time</u>   Consider a stochastically monotone Markov chain with monotone paths and countable state space. Suppose that changes of state occur according to a Poisson process with rate λ. Define $T^*(i, C)$ to be the first passage time (in continuous time) from state i to C, an increasing set with finite complement. Now:

$$Pr(T^*(i, C) > t) = \sum_{k=1}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} Pr(T(i, C) > k)$$

By theorem 1 and theorem 3.6, page 93 of Barlow and Proschan [1], $T^*(i, C)$ is IFRA.

Next, consider a continuous time Markov chain with infinitesimal matrix A taking values in a partially ordered countable set. Assume that the Markov chain has monotone paths, equivalently that $A(i, j) \neq 0$ implies $i \leq j$. Assume that the process is uniformizable, i. e. that $\sup(-A(i,i)) < \infty$.

By Feller [8] p. 312, the Markov process is representable as a Markov chain with transition matrix $P = I + \frac{A}{\lambda}$ for which transitions occur according to a Poisson process of rate λ. The Markov chain with transition matrix P inherits monotone paths. In view of monotone paths, the Markov chain is stochastically monotone if and only if $A(i, C) = \sum_{\ell \in C} A(i, \ell) \leq A(j, C)$ for

all increasing sets C with finite complement and $i \leq j$ with both i and j
in $\tilde{C}$. It follows from theorem 1 and the remarks at the beginning of this
section that the first passage time to C for the continuous time process
is IFRA. To summarize:

Corollary 1 (i) Consider a Markov chain of the type described in theorem 1.
Suppose that transitions occur according to a Poisson process. Then the
first passage time in continuous time from $i \notin C$ to C, an increasing set
with finite complement, is IFRA.

(ii) Consider a Markov chain in continuous time with countable partially
ordered state space and infinitesimal matrix A. Assume that:

(i) $\sup(-A(i,i)) < \infty$

(ii) $A(i, j) \neq 0$ implies $i \leq j$

(iii) $A(i, C) \leq A(j, C)$ for all increasing
sets C with finite complement, and pairs
of states (i, j) with $i \leq j$ and i, j both in $\tilde{C}$.

Then the first passage time from $i \notin C$ to C, an increasing set with
finite complement, is IFRA.

4. **Converse**   We will reinterpret a result of Birnbaum, Esary, and Marshall [2]
(described in Barlow and Proschan [1], lemma 2.13, page 88) in terms of
Markov chains. The analogue of their result in the discrete case provides
a converse to theorem 1.

The result is that every IFRA distribution in continuous time is either
the system life for a coherent system of independent exponential components,
or is the limit distribution of a sequence of such system lives. But a
coherent system of n independent exponential components is a continuous
time Markov process on the set of the 2^n n-tuples of 0's and 1's with partial

ordering $\underline{x} \leq \underline{y}$ if and only if $x_i \leq y_i$, $i = 1, \ldots n$. Moreover the process
is stochastically monotone with monotone paths, and the system lifetime
is the first passage time to an increasing set. Thus every IFRA distribution
in continuous time is either a first passage time distribution of the above
described type or the limit of a sequence of such first passage time distri-
butions.

Replacing continuous time by discrete time and exponential distributions
by geometric distributions, we can imitate their proof and obtain the following
result.

Lemma 1 Every IFRA distribution in discrete time is either the first passage
time distribution to an increasing set for a stochastically monotone Markov
chain with monotone paths on a partially ordered finite set, or else is
the limit of a sequence of such distributions.

5. Applications

(5.1) Coherent Systems Consider a coherent system of n independent components
(Barlow and Proschan [1], chapter 1, section 2). Assume that the survival function
for component i is given by $(\bar{F}(t))^{\lambda_i}$, $i = 1, \ldots, n$ where F is continuous.
This is known as the proportional hazard assumption. Each component starts
off in state 0, remains in 0 until failure, at which time it switches to
state 1 and stays there forever. The system fails as soon as the state
vector visits B, an increasing set. In this case S is the set of 2^n n-tuples
of 0's and 1's with partial order $\underline{x} \leq \underline{y} \Leftrightarrow x_i \leq y_i$, $i = 1, \ldots n$.

Consider the embedded discrete time process, starting in state $(0, \ldots, 0)$,
and changing state each time a component fails. Under the proportional
hazard assumption this process is a Markov chain. If the set of working

components at a given instant is A, then given that a change of state occurs, the probability that component $i \in A$ failed is given by $\lambda_i / \sum_A \lambda_j$. The Markov chain is easily seen to be stochastically monotone with monotone paths. Therefore by theorem 1 the number of component failures until system failure is IFRA. This generalizes a result of Ross, Shashahani, and Weiss [13], who proved the IFRA property in the case of i.i.d. components with continuous distribution $(\lambda_i \equiv \lambda)$.

By corollary 1, if the component lifetimes are exponential then the time to first failure for the continuous time process is IFRA. This is a special case of the IFRA closure theorem (Barlow and Proschan [1] p. 85).

__5.2__ __Shock models__ Assume that Y_1, Y_2, ... are i.i.d. random vectors. The set S consists of vectors of the form $(\lambda_1 w_1 \ldots \lambda_k w_k)$ where $w = (w_1, \ldots w_k)$ is a fixed vector of positive numbers and the ℓ's are non-negative integer valued. Define $S_n = \sum_1^n Y_i$, $n = 1, 2, \ldots$. Then $\{S_n, n \geq 1\}$ is a stochastically monotone Markov chain with monotone paths, and theorem 1 is applicable. Let g be a function, $R_+^k \to R_+ = [0, \infty)$, which is increasing in each argument and which goes to ∞ as $x_i \to \infty$ with $(x_1 \ldots x_{i-1}, x_{i+1}, \ldots x_k)$ held fixed, for $i = 1, \ldots k$. Then the first n such that $g(S_n)$ exceeds a given constant γ is IFRA. This is true because the set $\{\underline{x}: g(\underline{x}) > \gamma\}$ is an increasing set with finite complement.

Interpreting Y_i as the joint damage to k components due to the ith shock, g as the rule for converting component damage into system damage, and γ as the damage threshold beyond which the system fails, it follows that the number of shocks required for system failure is IFRA.

More generally the damage caused by a shock may depend on the previously accumulated damage. As long as damage is non-negative and the Markov chain remains stochastically monotone, the IFRA conclusion will hold.

If shocks occur in time according to a Poisson process then by lemma 1 the waiting time (in continuous time) until system failure is IFRA.

Finally if the damages are not of arithmetic type, i.e. concentrated on a set S as described above, but are still componentwise non-negative, then by passage to the limit through approximating vectors of arithmetic type it follows that the time to system failure is IFRA. This is true for the discrete time process and for the continuous time process when shocks occur according to a Poisson process.

Thus the classic shock model results of Esary, Marshall, and Proschan [7] are derivable from theorem 1 and corollary 1, as well as the IFRA character of the number of shocks until system failure. Moreover the current results are more general in that they allow for vector valued component damages and general system damage function g.

5.3 Multinomial distributions

Consider i.i.d. observations from a multinomial distribution. Define $S_n = (S_{n1} \ldots S_{nK})$ where S_{ni} is the number of times category i appears in the first n observations. Then $\{S_n, n \geq 1\}$ is a Markov chain of the type considered in example (5.1), i.e. the partial sums of i.i.d. random vectors with arithmetic distribution.

The following random variables are thus IFRA:

The number of observations required until each category appears at least once; until at least r categories each appear at least m times; until $g(S_n)$ exceeds γ where g is described in section (5.2).

(5.4) Sampling with or without replacement

Sampling with replacement is covered by (5.2) above. For sampling without replacement suppose that an urn contains m_i balls of color i, $i = 1, \ldots k$. Then the following random

variables are IFRA. The number of balls sampled until all balls of at least
one color are depleted; until at least one ball of each color is sampled;
until at least r_i balls of type i, i = 1, ... k are sampled; until at least
ℓ of the events A_1 ... A_k occur, where A_i = {at least r_i balls of type i
are sampled}. The last case includes the first three.

(5.5) <u>Convolutions</u> Consider two Markov chains of the type described in theorem 1,
with finite state spaces S_1 and S_2, and transition matrices Q_1 and Q_2. Let
C_1 be an increasing set in S_1. Define a new Markov chain on $S_1 \times S_2$, with
product partial ordering, by

$$(10) \qquad P((i, j), (i', j')) = \begin{cases} Q_1(i, i'), & i \in \tilde{C}_1 \\ Q_1(i, i')Q_2(j, j'), & i \in C_1 \end{cases}$$

It follows that the Markov chain is stochastically monotone with monotone
paths and finite state space. Theorem 1 is thus applicable.

Consider the set $C_1 \times C_2$ where C_2 is an increasing set in S_2. Since
both S_1 and S_2 are finite so is $C_1 \times C_2$. Starting in state (i, j) with
$i \in \tilde{C}_1$, $j \in \tilde{C}_2$, the first passage time from (i, j) to $C_1 \times C_2$ is the sum
of $T(i, C_1)$ and $T(j, C_2)$ where the two are independent. Therefore the convo-
lution of two IFRA distributions, each of which is a first passage time
distribution to an increasing set in a stochastically monotone Markov chain
with monotone paths on a finite state space, is IFRA. But by lemma 1 every
IFRA distribution is either of this form or the limit of a sequence of IFRA
distributions of this form. It follows that the class of discrete IFRA
distributions is closed under convolutions.

The closure of IFRA distributions under convolutions in the continuous
case was proved by Block and Savits [3]. In [4] Block and Savits prove
that closure in the continuous case follows from closure in the discrete case.

6. Comments and Additions.

(6.1)   The following example shows that under the conditions of theorem
1 the conclusion cannot be strengthened from IFRA to IFR even when the state
space is totally ordered. Let $\{Y_i, i \geq 1\}$ be i.i.d. with $Pr(Y = 9) = Pr(Y = 2) = \frac{1}{2}$.
Define $S_n = \sum_1^n Y_i$, $n \geq 1$; S_n is a Markov process on the integers. It is sto-
chastically monotone and has monotone paths. Consider, T, the first passage
time to $\{10, 11, 12, \ldots\}$. Now $Pr(T = 2|T \geq 2) = \frac{3}{4}$ while $Pr(T = 3|T \geq 3) = \frac{1}{2}$;
therefore the failure rate at 3 is less than the failure rate at 2 and
the distribution is not IFR.

(6.2)   Consider a Markov chain on $[1, 2, \ldots]$. Assume that the transi-
tion matrix is TP_2, i.e. that $P(i, j)P(i', j') \geq P(i, j')P(i', j)$ for all
$i < i'$, $j < j'$. Lemma 2 below shows that the first passage time from state
1 to $C_n = \{i: i > n\}$ is IFR (increasing failure rate) for $n = 1, 2, \ldots$.
A random variable T is defined to be IFR if $\lambda_k = Pr(T = k)/Pr(T \geq k)$ is
increasing.

<u>Lemma 2</u>   In a Markov chain with state space $\{1, 2, \ldots\}$, and TP_2 transition
matrix P, the first passage time $T(1, C_n)$, from state 1 to $C_n = \{i: i > n\}$
is IFR for $n = 1, 2, \ldots$.

<u>Proof</u>   Define $P_k(1, i) = Pr(T(1, C_n) > k, X_k = i | X_0 = 1)$, $i = 1, \ldots n$.
Then λ_k, the failure rate for $T(1, C_n)$ at k is given by:

(11)
$$\lambda_k = \frac{\sum_1^n P_{k-1}(1,i)P(i,C_n)}{\sum_1^n P_{k-1}(1,i)}, \quad k = 1, 2, \ldots$$

Now, from (11):

(12)
$$\lambda_{k+1} - \lambda_k = \frac{\sum_{1 \leq j < i \leq n} [P(i,C_n)-P(j,C_n)][P_{k-1}(1,j)P_k(1,i)-P_{k-1}(1,i)P_k(1,j)]}{P_{k+1}(1, \tilde{C}_n)P_k(1, \tilde{C}_n)}$$

It follows from theorem 2.2 of Karlin [10] that:

(13) $$P_{k-1}(1, j)P_k(1, i) - P_{k-1}(1, i)P_k(1, j) \geq 0$$

And by stochastic monotonicity (which is implied by TP_2):

(14) $$P(i, C_n) \geq P(j, C_n) \text{ for } i \geq j$$

Thus from (12), (13) and (14), λ_K is increasing and $T(1, C_n)$ is thus IFR. $\sqcap$

The above proof also shows that in a TP_2 Markov chain if for some i_0 $P(i, j) = 0$ for all $i \geq i_0$, $j < i$, then $T(i_0, C_n)$ is IFR for $n \geq i_0$. In particular if the Markov chain has monotone paths then the above holds for all i_0.

It follows from lemma 2 that for a ± 1 random walk with reflecting barrier at 0, starting at 0, that the number of trials required to reach m for the first time is IFR. Using the argument of corollary 1 which extends the discrete time result to continuous time, it follows that for a birth and death process the time required to go from 0 to the first visit to m is IFR for all m. This result was obtained using differents methods by Keilson [11], and Derman, Ross and Schechner [6].

(6.3)   If the Markov chain is stochastically monotone but without monotone paths then the IFRA conclsion need not hold. For example in the following chain the first passage time from state 2 to state 3 is DFRA (decreasing failure rate on the average).

$$P = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

One might attribute the above example to the fact that starting in state 2 it's possible to go to state 1 which has smaller failure rate than

state 2 (i.e. $P(1, 3) < P(2, 3)$). This might be eliminated if we required the chain to start in its smallest state. In the following example however the first passage time from state 1 to state 3 is not IFRA.

$$P = \begin{pmatrix} .18 & .09 & .73 \\ .17 & 0 & .83 \\ 0 & 0 & 1 \end{pmatrix}$$

In this example the average failure rate $\gamma_k = \frac{1}{k}\sum_1^k \Pr(T = j | T \geq j)$ equals .755763 for $k = 18$, and .755696 for $k = 19$. Since γ_k is not increasing it follows from [12], lemma 1, that T is not IFRA. It is true however that T is NBU (see section 6.5).

(6.4)   In the following example the Markov chain has monotone paths but is not stochastically monotone. The first passage time from state 1 to state 3 is DFRA.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

(6.5)   For a stochastically monotone Markov chain on the real numbers, $T(x, C)$ is stochastically decreasing in x for all increasing sets C. This is true because by Lehmann [12] p. 73, for $x < y$, we can construct a bivariate version of the Markov chain, $\{\binom{X_n}{Y_n}, n \geq 0\}$ with $X_0 = x$, $Y_0 = y$, and $X_n \leq Y_n$ for all n; in this construction $\{X_n, n \geq 0\}(\{Y_n, n \geq 0\})$ is distributed as the given Markov chain starting in $x(y)$.

It immediately follows that if the state space is a subset of $[a, \infty)$ which includes a then $T(a, C)$ is NBU for all increasing sets C. A random

variable on {0, 1, 2, ...} is defined to be NBU if $Pr(T \geq r + s) \leq Pr(T \geq r)Pr(T \geq s)$, for $r, s = 1, 2, \ldots$. The NBU property is weaker than IFRA.

Define a partially ordered set S to be of type L if $P_1(C) \leq P_2(C)$ for all increasing sets C implies that there exist random elements $\binom{X}{Y}$ with $X \sim P_1$, $Y \sim P_2$ and $Pr(X \leq Y) = 1$. By Lehmann [12] p. 73, the real line with the usual stochastic ordering is of type L. Kamae, Krengel, and O'Brien [9] show that a Polish space endowed with a closed partial ordering is of type L.

It follows that for a stochastically monotone Markov chain on a type L partially ordered set S that $T(x, C)$ is stochastically decreasing in x for all increasing sets C. Moreover, if $a \leq x$ for all $x \in S$, then $T(a, C)$ is NBU.

(6.6)   The definition of IFRA for discrete distributions followed here (section 2) is the same as in Barlow-Proschan [1], p. 94, and Block and Savits [4]. Ross, Shashahani and Weiss [13] use the term SSLSF (star-shaped log survival function) for what we call IFRA, and reserve the term IFRA for the weaker property, $\frac{1}{k} \sum_{i=1}^{k} Pr(T = i | T \geq i)$ increasing.

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