ANOTHER GENERALIZATION OF CARATHÉODORY'S THEOREM (U)

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by

Victor Klee

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Another Generalization of Carathéodory's Theorem

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1. Abstract

When \( P \) is a \( d \)-dimensional convex polytope with vertex-set \( V \), we use the term \( V \)-simplex to denote a \( d \)-simplex whose vertices all belong to \( V \). A slight variant of Carathéodory's theorem asserts that for each \( v \in V \) there is a collection \( \mathcal{S} \) of \( V \)-simplices such that \( P = \cup \mathcal{S} \) and \( v \in \mathcal{S} \). In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection \( \mathcal{S} \) of \( V \)-simplices such that \( P = \operatorname{con}_n u \mathcal{S} \) and \( \dim \mathcal{S} = d \). For \( 0 \leq k < d \) the present note establishes a theorem concerning the generation of \( P \) by \( V \)-simplices in conjunction with the operation \( \operatorname{con}_{k+1} \), where \( \operatorname{con}_n X \) is the set of all convex combinations of \( n \) or fewer points of \( X \). When \( k = 0 \) the theorem is Carathéodory's and when \( k = d-1 \) it is a slight sharpening of Goodearl's conjecture.
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When $P$ is a $d$-dimensional convex polytope with vertex-set $V$, we use the term $V$-simplex to denote a $d$-simplex whose vertices all belong to $V$. A slight variant of Carathéodory's theorem [2] asserts that for each $v \in V$ there is a collection $\mathcal{S}$ of $V$-simplices such that $P = u\mathcal{S}$ and $v \in n\mathcal{S}$. In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection $\mathcal{S}$ of $V$-simplices such that $P = \text{con } u\mathcal{S}$ and $\dim n\mathcal{S} = d$. (This result is used in [4].) For $0 \leq k < d$ the present note establishes a theorem concerning the generation of $P$ by $V$-simplices in conjunction with the operation $\text{con}_{k+1}$, where $\text{con}_X$ is the set of all convex combinations of $n$ or fewer points of $X$. When $k = 0$ the theorem is Carathéodory's and when $k = d-1$ it is a slight sharpening of Goodearl's conjecture.

THEOREM Suppose that $P$ is a $d$-dimensional convex polytope with vertex-set $V$, $0 \leq k < d$, and $F$ is a $k$-face of $P$. Then there is a collection $\mathcal{S}$ of $V$-simplices such that

$$P = \text{con}_{k+1} u\mathcal{S} \quad \text{and} \quad \dim (P \cap (n\mathcal{S})) = k.$$

When $k = d-1$ the intersection $n\mathcal{S}$ is $d$-dimensional. If $V$ is in general position then $\text{con}_{k+1}$ may be replaced by $\text{con}_{[d/(d-k)]}$.

Proof. Observe first that if $H$ is a $(j-1)$-flat in a $j$-flat $G$, $Q$ is one of the two closed halfspaces into which $H$ divides $G$, and $\mathcal{B}$ is a finite collection of $j$-dimensional convex subsets of $Q$ such that the set $C = \cap_{B \in \mathcal{B}}$ is $(j-1)$-dimensional, then $n\mathcal{B}$ is $j$-dimensional. Indeed, choose points $c$ and $q$ in the relative interiors of $C$ and $Q$ respectively, and note that for each $B \in \mathcal{B}$ there exists $\lambda_B > 0$ such that $(1-\lambda_B)c + \lambda_Bq$. With $e = \min \{\lambda_B : B \in \mathcal{B}\} > 0$, $n\mathcal{B}$ contains the $j$-dimensional set $\text{con } (C \cup ((1-e)c + eq))$. 

Whenever $P$ is a $d$-polytope with vertex-set $V$, $0 \leq k \leq d$, and $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k$ is a sequence of faces of $P$ with $\dim F_i = i$ for each $i$, let $\mathcal{S}_P(F_0, \ldots, F_k)$ denote the collection of all sets of the form $\con \{v_0, \ldots, v_d\}$ such that

(i) for $0 \leq i \leq k$, $v_i \in F_i$

(ii) for $1 \leq i \leq d$, $v_i \in \con \{v_0, \ldots, v_{i-1}\}$.

Plainly each member of $\mathcal{S}_P(F_0, \ldots, F_k)$ is a $V$-simplex. A straightforward induction on $i$, based on the observation of the preceding paragraph, shows that for $0 \leq i \leq k$,

$$\dim \cap_{F_i} \mathcal{S}_P(F_0, \ldots, F_i) = i.$$ 

To construct the $\mathcal{S}$ whose existence is claimed by the theorem, simply set $\mathcal{S} = \mathcal{S}_P(F_0, \ldots, F_k)$ for an arbitrary sequence of faces $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k$ with $F_k = F$ and $\dim F_i = i$ for all $i$. Plainly $\dim (F \cap \mathcal{S}) = k$, for $\mathcal{S} \supseteq \mathcal{S}_{F_k}(F_0, \ldots, F_k)$.

And since

$$\mathcal{S}_P(F_0, \ldots, F_{d-1}) = \mathcal{S}_P(F_0, \ldots, F_{d-1}, P),$$

$\mathcal{S}$ is $d$-dimensional when $k = d-1$.

It remains to show that $P = \con \cup \mathcal{S}$ with $r = k+1$ in general and $r = \lceil d/(d-k) \rceil$ (the smallest integer $\geq d/(d-k)$) when $V$ is in general position. With $v_0 \in F_0$, consider an arbitrary point $p \in P \setminus \{v_0\}$ and let $q$ be the last point of the ray from $v_0$ through $p$ that belongs to $P$. If $q \in \con \cup \mathcal{S}$ then $p \in \con \cup \mathcal{S}$ because $p \in \{v_0, q\}$ and each member of $\mathcal{S}$ is a convex set that contains $v_0$.

Let $j$ denote the dimension of the smallest face $G$ of $P$ that contains $q$. By Carathéodory's theorem, $q \in \con X$ for an affinely independent set $X$ consisting of $j+1$ points of $V \cap G$. If $G \subseteq F_k$ then $j < k$ and for each $x \in X$ there is a member $S_x$ of $\mathcal{S}$ which contains $x$. Hence $q \in \con \cup \mathcal{S}$.

Suppose, on the other hand that $G \not\subseteq F_k$, and let $W$ be the vertex-set of an arbitrary member of $\mathcal{S}_{F_k}(F_0, \ldots, F_k)$. Let $\mathcal{W}$ denote the cardinality of $\mathcal{S}$.
maximal affinely independent subsets of \( W \cup X \). From the facts that \( W \cap G \) and \( X \cap F_k \) it follows that \( m > k \) and \( m < j \). Since \( W \) is affinely independent, there is a set \( Y < X \) such that the set \( W \cup Y \) is affinely independent and of cardinality \( m+1 \), whence \( |Y| = m-k \). Plainly \( W \cup Y \) lies in a member of \( \mathcal{S}_a \) as does each of the \((j+1)-(m-k) \) remaining points of \( X \). Hence \( p \in \text{con}_{r+1} W \) with \( r = (j+1)-(m-k) \leq k \).

Now suppose, finally, that the vertex-set \( V \) of \( P \) is in general position, meaning that each set of \( d+1 \) points of \( V \) is affinely independent. Then all proper faces of \( P \) are simplices, and \( \mathcal{S}_a \) consists merely of all \( V \)-simplices that contain \( F_k \). Consider \( v_0, p, q, G, X, W \) as described earlier. Then \( W \cup Y \) is affinely independent for each set \( Y < X \) with \( |Y| \leq d-k \). Hence \( X \cup W \) can be covered by \( \lceil (j+1)/(d-k) \rceil \) members of \( \mathcal{S}_a \), and since \( j < d \) it follows that \( q \) (and hence \( p \)) belongs to \( \text{con}_{d/(d-k)} W \). That completes the proof.

To see that the theorem cannot be improved by reducing the subscripts \( k+1 \) and \( d/(d-k) \), consider a \( d \)-polytope \( P = \text{con} V \) where \( V \) is the union of the vertex-set \( W \cup F_k \)-simplices \( F \) and the vertex-set \( X \) of a \((d-l)\)-simplex. Let \( \mathcal{S}_a \) be the collection of all \( V \)-simplices \( S \) such that 
\[
\text{dim}(P \cap S) = k
\]
for each \( S \in \mathcal{S}_a \), whence the centroid of \( \text{con} X \) does not belong to \( \text{con}_{d/(d-k)} W \). If a translate \( W' \) of \( W \) is contained in \( X \) then \( |W' \cap S| = 1 \) for each \( S \in \mathcal{S}_a \), whence the centroid of \( \text{con} W' \) does not belong to \( \text{con}_{k} W \).
y other generalizations of Carathéodory's theorem appear in the
re. Some of them can be found in the references below.

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