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RADAR EMITTER RECOGNITION USING PULSE REPETITION INTERVAL

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S U M M A R Y

A method for estimating radar emitter pulse repetition interval parameters from intercepted emissions is presented. Adaptive parameter estimates based on certain measurements are derived. The measurements are obtained by a readily implementable observation procedure. The approach can deal with emitters with variable pulse repetition interval in a dense environment. Computer simulation results for the techniques are included.

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1. INTRODUCTION

In this paper a method for determining radar emitter pulse repetition interval (PRI) in a dense environment is presented. The PRI estimates can be used for emitter identification. The method is based on statistical techniques.

This work is done under task DST 76/004.

The aim of an emitter classification system is to identify possible threats by analysing intercepted signals. The importance of emitter identification in an EW system can be judged from the fact that it is a necessary first step in implementing countermeasures against threat emitters.

To deal with the complex signal environments likely to be encountered one emitter parameter alone cannot be relied on to achieve positive identification. Thus, in this paper it is assumed that some analysis of the environment using such parameters as radio frequency and direction of arrival will be carried out in conjunction with PRI analysis.

The electromagnetic environment can be divided into two classes of emitters, namely, those emitters whose intercepted pulse amplitude is heavily modulated (such as search radars) and those emitters whose intercepted pulse amplitude has a modest amount of or no modulation (such as tracking radars). The range of PRI to be encountered will tend in practice to be from about 10^2 μs to 10^4 μs. Search radars tend to have longer PRI while tracking radars usually have relatively short PRI.

Standard methods for emitter recognition using pulse repetition interval include the pulse train sorter and the time of arrival method. Experience has revealed certain limitations for these methods. The main limitations are in handling a variable PRI and in extending their use to a dense environment.

It is important to be able to recognize emitters with variable pulse repetition interval. Variable PRI is used in radar emitters to overcome the blind speed problem with moving target indication (MTI) and as an antijamming technique. In the MTI application the PRI may vary typically ± 10% whilst in the antijamming application ± 50% has been considered.

The recognition methods in this paper can cope with variable PRI and offer a simple, systematic approach to analysing the dense environment problem.

A shortened version of this paper appeared in reference 1.

The method proposed in this paper will be sketched briefly now. In order to monitor the emitter environment, data are collected over a series of consecutive observation intervals. The time between changes to the environment of emitters is assumed to be long compared with an observation interval. Each observation interval is divided up into gate intervals and, at the end of each observation interval, the relative frequency of the various possible number of pulses which can occur during a gate interval is recorded. It turns out that the possible number of pulses which can occur is of the order of the number of emitters present. Using the recorded relative frequencies it is possible to deduce the types of PRI (whether fixed or M-position stagger, say) and estimate the PRI parameters (for a 2-position stagger the two PRI used) when more than one emitter is present. The data processing to accomplish this involves only a few simple rational operations. The method adopted is adaptive and uses a certain algorithm to select the length of the gate interval. Changes which occur to the environment are analysed using the relative frequencies recorded at the end of each observation interval. In this way the PRI parameters of the emitter(s) which started or stopped being intercepted are determined. This adaptive approach yields more accurate parameter estimates than is possible with a non-adaptive approach.

An example of the achievable accuracy is as follows. With up to five non-scanning emitters present and one-second observation intervals, the r.m.s. percentage error in the estimates of fixed PRIs is less than 0.4%. The main advantage of the method is that only a small amount of data has to be manipulated.

The reader who is not interested in the mathematics should go directly to Sections 9 and 10 for a discussion of possible applications and further developments.

2. OBSERVATION PROCEDURE AND DEFINITIONS

In this section a readily implementable observation procedure is introduced for the determination of radar emitter PRI from intercepted emissions. Suppose that the IF output of a typical electronic surveillance receiver is examined during the time interval from s_0 to s_T. This interval, which will be called the observation interval, is partitioned as $s_0, s_1, \ldots, s_{T-1}, s_T$ where $s_t - s_{t-1} = \tau$ for all t. The quantity τ will be called the gate time and the subintervals of length τ will be called the gate intervals. Note that $T\tau = s_T - s_0$. The observation procedure is to determine the number of pulses which occur during each gate interval and set n_t equal to the number of pulses which occur during gate interval t. Thus, a sequence of numbers $n_1, n_2, \ldots, n_T$ is obtained. Suppose that during the observation interval the maximum and minimum number of pulses detected in any gate interval is N_{max} and N_{min} respectively. A set of quantities, x_t, is derived from the observations, n_t, as defined by

$$x_t = n_t - N_{min} \tag{1}$$

The PRI of an emitter is defined as the time interval between successive pulses. The sequence of PRI, $y_1, y_2, \ldots, y_N$ is assumed to be a realization of the stochastic process, $\{Y_n, n = 1, 2, \ldots, N\}$. Four types of emitter PRI will be considered in this paper. Each type can be characterized by its PRI probability density function, $p(y_n)$, as follows:

(a) Fixed PRI emitters with

$$p(y_n) = \delta(y_n - z) \tag{2}$$

so that $y_n = z$ for all n.

(b) M-position staggered PRI emitters with

$$p(y_n) = M^{-1} \sum_{m=1}^{M} \delta(y_n - z_m) \tag{3}$$

The PRI sequence is cyclic. For example, if M = 2 then $y_{2n} = z_1$ and $y_{2n+1} = z_2$ for all n.

(c) Uniform PRI emitters with

$$p(y_n) = \begin{cases} (b - a)^{-1}, & \text{if } a \leqslant y_n \leqslant b \\ 0 & \text{, otherwise} \end{cases} \tag{4}$$

The PRI sequence is independently and identically distributed.

(d) Sinusoidal PRI emitters with

$$p(y_n) = \begin{cases} \dfrac{2}{(b - a)\pi\sqrt{1 - \left(\dfrac{2y_n - a - b}{b - a}\right)^2}}, & \text{if } a \leqslant y_n \leqslant b \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

The PRI sequence is independently and identically distributed.

In the above $\delta(.)$ denotes the Dirac delta function. When referring to a dense environment of emitters, to denote the k^{th} emitter a superscript k will be added to algebraic symbols. For the sake of simpler notation it will be omitted where no ambiguity arises.

Suppose that Q + 1 possible states for x_t can occur. The relative frequencies of occurrence of the states of the process, P_q, q = 0, 1, ..., Q, during the observation interval can be calculated via

$$P_q = N_q/T \tag{6}$$

$$N_q = \sum_{t=1}^{T} N_q^t \tag{7}$$

$$N_q^t = \begin{cases} 1, & \text{if } x_t = q \\ 0, & \text{if otherwise} \end{cases} \tag{8}$$

3. ASSUMPTIONS

The assumptions made throughout this paper about the radar emitters and the observation procedure are stated in this section.

A mathematical model will be developed for the observed data. The proposed observation procedure generates pseudo random sequences of numbers. The equations derived for the mathematical model are based on certain assumptions concerning these sequences. The validity of these assumptions should be verified either by theoretical analysis or by empirical tests. (See references 2 and 3 for a discussion of standard random number generators). Theoretical tests would give much more understanding of the method than the results of empirical tests. In this paper conditions concerning the PRI of the emitters and the observation procedure will be given which are believed to be sufficient for the mathematical model to adequately represent the observed data. The results of simulations of the proposed PRI estimation technique which have been done justify this belief. Theoretical proofs of the results will be given in Appendix I.

The basic assumptions made are as follows:

(a) The pulse trains of the emitters are unsynchronized.

(b) The pulse trains of the emitters are assumed to be unsynchronized with the gate interval partition of the observation procedure.

(c) If overlapping pulses occur it is possible to detect how many pulses occurred.

(d) The environment does not change during the observation interval.

(e) All pulses transmitted are detected - including those of small amplitude in antenna side lobes.

These assumptions are not overly restrictive. For emitters with fixed PRI assumptions 3(a) and 3(b) are tantamount to assuming that the PRIs of the radar emitters are incommensurable (i.e., not divisible without remainder by the same quantity) and that the PRIs of the radar emitters and τ are incommensurable. In practice, radar emitters with commensurable PRIs will be quite rare and can possibly be treated as a single emitter with variable PRI if present together. Assumption 3(b) can always be satisfied by suitable choice of τ. It is believed that assumptions 3(c), (d) and (e) are realistic in that for practical application the techniques can be modified to deal with problems which will occur. This will be discussed further in Sections 7 and 9.

4. EMITTERS WITH FIXED PRI

4.1 Single emitter

If a single emitter with PRI z is present and the gate time is selected so that

$$Wz < \tau < (W + 1)z \tag{9}$$

where W is a nonnegative integer then during any gate interval either W or W + 1 pulses will be observed. Thus, the possible states for x_t are 0 and 1 and occur with probabilities denoted by p_0 and $p_1 = 1 - p_0$ respectively.

An expression for the probability, p_0, in terms of z, W and τ is now derived. Let θ_t be the time measured from the start of gate interval t to where the next pulse occurs. Note that $0 < \theta_t \leqslant z$. Furthermore, if assumption 3(b) is invoked then θ_t may be considered to be a sample value of a random variable uniformly distributed on the interval 0 to z. The proof will be given in Appendix I. In figure 1 the number of pulses occurring during gate interval t is shown as θ_t varies from 0 to z for two cases W = 0 and 3. In general,

$$\begin{aligned} x_t &= 1, \text{ if } 0 < \theta_t \leqslant \tau - Wz \\ x_t &= 0, \text{ if } \tau - Wz < \theta_t \leqslant z \end{aligned} \tag{10}$$

Since θ_t is uniformly distributed on 0 to z the expression for p_0 is

$$p_0 = \{(W + 1)z - \tau\}/z \tag{11}$$

Simple algebraic manipulation yields

$$z = \tau/\{(W + 1) - p_0\} \tag{12}$$

These results are summarized in Table 1.

4.2 Dense environment

Suppose that K fixed PRI emitters are present. The observation x_t is related to the observations for the individual emitters comprising the environment, x_t^k, via

$$x_t = \sum_{k=1}^{K} x_t^k \tag{13}$$

The possible values of x_t are the set $\{0, 1, \ldots, K\}$ and $K = N_{max} - N_{min}$. The probability that state j occurs in a dense environment of K emitters is denoted by $P_{j,K}$. In order to derive an expression for $P_{j,K}$ in terms of the quantities, p_0^k, the sequences of states x_t^k are assumed to be mutually statistically independent. It is believed that assumption 3(a) guarantees mutual statistical independence. The proof together with simulation results supporting this statement is given in Appendix I. In Appendix II the quantities $P_{j,K}$ are expressed in terms of the quantities p_0^k and it is proved that, given the $P_{j,K}$, the probabilities p_0^k are zeroes of the polynomial, $f_K(\lambda)$, of degree K in λ given by

$$f_K(\lambda) = \lambda^K + a_{K-1,K}\lambda^{K-1} + \ldots + a_{1,K}\lambda + a_{0,K} \tag{14}$$

where the coefficients can be expressed as

$$a_{r,K} = (-1)^{K-r} \sum_{j=0}^{r} P_{j,K} \binom{K-j}{r-j} \tag{15}$$

and (.) denotes the combinatorial symbol.

The following example illustrates this result.

Example 1 (K = 3) Referring to Appendix II

$$P_{0,3} = p_0^1\, p_0^2\, p_0^3$$

$$P_{1,3} = (1 - p_0^1)\, p_0^2\, p_0^3 + (1 - p_0^2)\, p_0^1\, p_0^3 + (1 - p_0^3)\, p_0^1\, p_0^2$$

$$P_{2,3} = (1 - p_0^1)(1 - p_0^2)\, p_0^3 + (1 - p_0^1)(1 - p_0^3)\, p_0^2 + (1 - p_0^2)(1 - p_0^3)\, p_0^1$$

Simple algebraic manipulation yields

$$p_0^1\, p_0^2\, p_0^3 = P_{0,3}$$

$$p_0^1\, p_0^2 + p_0^1\, p_0^3 + p_0^2\, p_0^3 = P_{1,3} + 3P_{0,3}$$

$$p_0^1 + p_0^2 + p_0^3 = P_{2,3} + 2P_{1,3} + 3P_{0,3}$$

Recall that the coefficients of a polynomial with zeroes p_0^1, p_0^2 and p_0^3 can be computed by multiplying out the expression

$$(\lambda - p_0^1)\ (\lambda - p_0^2)\ (\lambda - p_0^3)$$

$$= \lambda^3 - (p_0^1 + p_0^2 + p_0^3)\ \lambda^2 + (p_0^1\ p_0^2 + p_0^1\ p_0^3 + p_0^2\ p_0^3)\ \lambda - p_0^1\ p_0^2\ p_0^3$$

This verifies the dense environment theorem of Appendix II for K = 3, viz.,

$$a_{0,3} = -P_{0,3}$$

$$a_{1,3} = P_{1,3} + 3P_{0,3}$$

$$a_{2,3} = -(P_{2,3} + 2P_{1,3} + 3P_{0,3})$$

Each emitter has a unique z^k and, once τ is specified, a unique p_0^k from equation (11). The significance of the above result is that in a dense environment each possible combination of emitters which can be present will have a unique set of probabilities $P_{j,K}$, $j = 0,1,\ldots,K$. The dense environment of emitters is said to be identifiable by observing the sequence x_t.

From reference 4, p392, Ex 12.23 the maximum likelihood estimate of $P_{j,K}$, denoted $\tilde{P}_{j,K}$, is given by

$$\tilde{P}_{j,K} = P_j \tag{16}$$

where P_j is given in equations (6) to (8). Using the invariance property of maximum likelihood estimates (reference 5, Ch VII, Section 2.2) the maximum likelihood estimates of the p_0^k, denoted $\tilde{p}_0^k$, are zeroes of the polynomial $f_K(\lambda)$ of equation (14) with coefficients

$$a_{r,K} = (-1)^{K-r} \sum_{j=0}^{r} \tilde{P}_{j,K} \binom{K-j}{r-j} \tag{17}$$

Again using the invariance property, the maximum likelihood estimate of z^k, denoted $\tilde{z}^k$, is given from equation (12) by

$$\tilde{z}^k = \tau / \{(W^k + 1) - \tilde{p}_0^k\} \tag{18}$$

For discussion of the properties of maximum likelihood estimates see reference 6, Ch 4. In equation (18) W^k is assumed to be known. This can usually be arranged if a rough idea of the PRI range to be encountered is known. For example, if $\tau < z^k$ then $W^k = 0$.

5. EMITTERS WITH VARIABLE PRI

5.1 Single emitters

With variable PRI emitters the possible number of states for x_t can be greater than two. The number of states which can occur depends on the type of emitter and the gate time, τ. The probability that state i occurs will be denoted p_i.

In Tables 2 and 3 the statistical properties of the observed sequence, x_t, are given for two position stagger and three position stagger emitters respectively. The derivation is similar to that for fixed PRI emitters. The formulae in Table 3 apply to both the 123 and 132 permutations of PRI order.

The results obtained for fixed and staggered PRI emitters can be applied to deriving the statistical properties of the observed sequence, x_t, for emitters with PRIs, y_n, which are independently and identically distributed samples of a random variable taking values between a and b ($a < b$) with probability density function $p(y_n)$.

The range of gate times is restricted to $\tau < 3a$. Also, the width of the distribution of PRIs is limited by assuming that $3a > 2b$. This implies $2a > b$ which means that the number of states which can occur will be three or two depending on τ. More general results can be derived but the required analysis becomes tedious and is omitted here.

The possible number of pulses which can occur during the gate interval is shown in Table 4 for the range of τ of interest together with the formulae for the state probabilities, p_i. These probabilities can be determined using the results of Table 1 for $\tau < 2a$ and the results of Table 2 for $2a < \tau < 3a$ and averaging over the PRI range a to b. The results of Table 2 must be used for the higher gate time range because pairs of pulse repetition intervals must be considered for this case. To form the averages the probability density function $y\ p\ (y)\ \tau_1^{-1}$ for $\tau < 2a$ and $(y_1 + y_2)\ p(y_1)\ p(y_2)\tau_2^{-1}$ for $2a < \tau < 3a$ must be used, where

$$\tau_1 = \int_a^b p(y)y\ dy \tag{19}$$

$$\tau_2 = \int_a^b \int_a^b p(y_1)p(y_2)(y_1 + y_2)dy_1\ dy_2 \tag{20}$$

$$\tau_2 = 2\tau_1 \tag{21}$$

Equation (21) follows from the fact that

$$\int_a^b p(y_i)dy_i = 1 \tag{22}$$

The use of these probability density functions to form the averages takes into account the fact that a shorter PRI occupies fewer gate intervals than a longer PRI on the average. Note that when two states only can occur equivalent results are obtained if it is assumed that a fixed PRI emitter is present with PRI equal to the average PRI, τ_1, of the actual emitter.

The results of Table 4 are now specialized to two commonly occurring types of PRI variation. Upon substituting the appropriate function p(y) the p_i for uniform PRI emitters given in Table 5 are obtained. The corresponding result for sinusoidal PRI emitters and $a < \tau < b$ is

$$p_0 = \{\pi^{-1} A_1 + (\tau_1 - \tau)\ (0.5 - \pi^{-1} \text{ arsin } A_2)\} \tau_1^{-1} \tag{23}$$

$$p_1 = \{-2\pi^{-1} A_1 + (2\tau_1 - \tau)(0.5 + \pi^{-1} \text{ arsin } A_2) + \tau(0.5 - \pi^{-1} \arcsin A_2)\} \tau_1^{-1} \tag{24}$$

$$p_2 = \{\pi^{-1} A_1 + (\tau - \tau_1)(0.5 + \pi^{-1} \arcsin A_2)\} \tau_1^{-1} \tag{25}$$

where

$$A_1 = \sqrt{\{V^2 - (\tau - \tau_1)^2\}} \tag{26}$$

$$A_2 = (\tau - \tau_1)/V \tag{27}$$

$$\tau_1 = (a + b)/2 \tag{28}$$

$$V = (b - a)/2 \tag{29}$$

In Table 6 the state probabilities p_0 and p_2 are given for four types of emitter PRI at $\tau = \tau_1$ and $\tau = 2\tau_1$ and for $\tau < a$. These probabilities form the basis of the algorithm used to decide the type of emitter PRI variation and estimate the parameters τ_1, z, etc from the relative frequencies of occurrence of the states of the observed data.

An algorithm for PRI recognition and parameter estimation is shown in figure 2. The class of emitter is assumed to be either fixed, two position stagger or uniform PRI. The algorithm first decides the type of PRI present and then estimates the appropriate PRI parameters. This is achieved by observing the relative frequencies of occurrence of the states at selected values of τ. Parallel processing could possibly be used to obtain the required data at the different gate times. Provided that the correct PRI type is recognized the parameter estimates obtained can be shown to be maximum likelihood estimates.

The algorithm can be generalized for other PRI variations. Some difficulty may be experienced in distinguishing between uniform and sinusoidal PRIs. For M-position staggers $p_0 = p_2 = 0$ when $\tau = M\tilde{\tau}_1$. This fact can be used to distinguish between complex PRI staggers.

5.2 Dense environment

In a dense environment of K emitters the observation x_t is again related to the observations for the individual emitters comprising the environment x_t^k via

$$x_t = \sum_{k=1}^{K} x_t^k \tag{30}$$

The set of states which can occur now depends on τ. The probability that the dense environment process is in state j, $P_{j,K}$, can be related to the quantities p_i^k. A polynomial $f(\lambda)$ defined by

$$f_L(\lambda) = \lambda^L + a_{L-1,L}\,\lambda^{L-1} + \ldots + a_{1,L}\,\lambda + a_{0,L} \tag{31}$$

can be used to represent a given environment of emitters where the coefficients are given by

$$a_{r,L} = (-1)^{L-r} \sum_{j=0}^{r} P_{j,K} \binom{L-j}{r-j} \tag{32}$$

The quantities p_i^k are available upon finding the factors of polynomial $f_L(\lambda)$. If τ is less than the minimum a^k then L = K and the analysis of Section 4.1 is applicable with τ_1^k replacing z^k.

It is possible to generalize the analysis for other values of τ. If for emitter k the value of τ gives rise to three states the linear factor $\lambda - p_0^k$ in the polynomial of equations (31) and (32) is replaced by the quadratic factor $\lambda^2 - \lambda(2p_0^k + p_1^k) + p_0^k$ and L increases by 1. By observing the relative frequencies of occurrence of the states it is possible to estimate the polynomial coefficients. If the polynomial is factored estimates of p_0^k and p_1^k can be obtained which allow the emitter type to be determined and the parameters τ_1^k, z_m^k, a^k and b^k estimated.

6. ADAPTIVE IDENTIFICATION

Suppose that no emitters are present. It is required to identify a number of emitters which will then start transmitting. It is assumed that when the environment of emitters changes only one emitter will start or stop transmitting and the time between changes to the environment is long compared with a specified observation interval. Operating under these conditions the computational requirements for emitter identification using PRI can be greatly reduced and improved accuracy achieved. The reason for this is that the identification method of this paper requires a given polynomial to be factored. Normally this entails a considerable amount of computational effort. However, if only changes to the environment need be determined the problem simplifies because, in splitting a polynomial into two factors, the second factor is easily found once the first factor is specified. This approach will be called adaptive identification.

Suppose that a number of emitters are present and the polynomial associated with this environment is given by

$$f_L^1(\lambda) = \lambda^L + a_{L-1}\,\lambda^{L-1} + \ldots + a_1\,\lambda + a_0 \tag{33}$$

(In this section $a_{r,L}$ is replaced by a_r to simplify the notation).
If another emitter starts transmitting prior to the next observation interval and for this emitter two states are observed at the value of τ selected, then the new environment can be represented by the polynomial

$$f^2_{L+1}(\lambda) = \lambda^{L+1} + b_L \lambda^L + \ldots + b_1 \lambda + b_0 \tag{34}$$

It is easy to show that the state probability, p_0, for this emitter is given by

$$p_0 = a_{L-1} - b_L = -b_0/a_0 = (a_{i-1} - b_i)/a_i \qquad i = 1,2,\ldots, L-1 \tag{35}$$

Thus, it is possible to estimate the new emitter parameter with simple rational operations if the prior knowledge about the environment is used. The coefficients a_i are assumed known and the coefficients b_i are estimated from observations of the relative frequencies of occurrence of the states.

If the new emitter has three states at the value of τ selected the new environment is represented by the polynomial

$$f^2_{L+2}(\lambda) = \lambda^{L+2} + b_{L+1} \lambda^{L+1} + \ldots + b_1\lambda + b_0 \tag{36}$$

The state probabilities p_0 and p_1 for this emitter are related to the polynomial coefficients via

$$c_1 = -(2p_0 + p_1) \tag{37}$$

$$c_0 = p_0 \tag{38}$$

$$b_0 = c_0 a_0 \tag{39}$$

$$b_1 = c_1 a_0 + c_0 a_1 \tag{40}$$

$$b_i = a_{i-2} + c_1 a_{i-1} + c_0 a_i, \; i = 2, \ldots, L - 1 \tag{41}$$

$$b_L = a_{L-2} + c_1 a_{L-1} + c_0 \tag{42}$$

$$b_{L+1} = a_{L-1} + c_1 \tag{43}$$

A theoretical framework for deriving optimal adaptive parameter estimates is not known at this stage. Further discussion of adaptive identification is postponed until Section 8.

7. PULSE DETECTION

In order to identify radar emitters using PRI the number of pulses occurring within the gate interval must be determined. If the pulses do not overlap it is relatively easy to determine the number of pulses which have occurred. In practice, the probability that pulses will overlap is small and it may be possible to assume that overlap never occurs. Nevertheless, in this section the possibility of detecting overlapping pulses will be investigated.

Suppose that the received RF signal, $r_{RF}(s)$, from K emitters is represented at the time s by

$$r_{RF}(s) = \sum_{k=1}^{K} A^k(s) \cos(2\pi f^k s + \varphi^k) \tag{44}$$

where the pulse amplitude modulation waveform is denoted by $A^k(s)$. The radio frequency and phase of emitter k are denoted by f^k and φ^k respectively.

Let the centre frequency of the RF filter be f_0. The IF signal in the receiver can be expressed as

$$r_{IF}(s) = A_{IF}(s) \cos\{2\pi f_{IF} s + \alpha_{IF}(s)\} \tag{45}$$

where the amplitude, $A_{IF}(s)$, and phase, $\alpha_{IF}(s)$, are given by

$$A_{IF}(s) = \left[\sum_{k=1}^{K} \{A^k(s)\}^2 + \sum_{\substack{k=1 \\ k \neq r}}^{K} \sum_{r=1}^{K} A^k(s) A^r(s) \cos\{2\pi (f^k - f^r)\, s + \varphi^k - \varphi^r\} \right]^{\frac{1}{2}} \tag{46}$$

$$\alpha_{IF}(s) = \tan^{-1} \left[\frac{\sum_{k=1}^{K} A^k(s) \sin\{2\pi (f^k - f_0) s + \varphi^k\}}{\sum_{k=1}^{K} A^k(s) \cos\{2\pi (f^k - f_0) s + \varphi^k\}} \right] \tag{47}$$

The derivation of equations (45) to (47) is discussed in more detail in reference 7. In what follows the radar pulses received from each emitter are assumed to be approximately rectangular in shape.

If $f^k \neq f^r$ for $k \neq r$ then the time average of $A_{IF}(s)$ will increase sharply whenever a new pulse occurs no matter if other pulses are already present. If $f^k = f^r$ for some $k \neq r$ then except under freak conditions the arrival of a new pulse will be accompanied by a sharp change in the average phase or amplitude of the IF. Similarly, when a particular pulse finishes the IF phase and amplitude will change sharply.

From the preceding analysis it would appear that it is possible in principle to detect overlapping pulses. The practical implementation of a suitable detector could present problems. The use of Bragg cell technology can avoid all these problems with detecting overlapping pulses. At present these devices are relatively expensive and limited in dynamic range. Two alternate approaches are considered next.

It may be possible to develop a PRI measurement system where, after a pulse has been detected, all further signal activity is ignored for a specified dead time. The dead time would be comparable to the longest expected pulse width, say 10 μs. The statistical analysis of this paper would need to be modified to take this dead time into account. The equations would be a little more complicated but the problem of detecting overlapping pulses could then be ignored.

Another approach is to use bandpass filters and logic gates. Consider the following simple illustrative example. Suppose that signal activity at two different RF values is detected. Two bandpass filters with pulse detectors connected to an exclusive OR gate as shown in figure 3 could be used to facilitate PRI analysis. Provided that exactly coincident pulses with identical pulse width do not occur (a low probability event) overlapping pulses will enter the pulse counter as two well defined pulses. The PRI parameter measurement method proposed in this paper could then be applied using the data obtained with this simple device.

If missing pulses occur (or pulses in a train are not detected) then the PRI measurements will of course be degraded. Precautions would need to be taken to avoid errors caused by missing pulses. Detailed discussions of this topic are left for later work.

8. EXPERIMENTAL RESULTS

In this section computer simulations and laboratory work which have been performed to test the emitter PRI measurement method of this paper are described.

8.1 Fixed PRI emitters

The results presented in figures 4 and 5 demonstrate the accuracy which can be achieved and show the importance of the choice of τ. If the maximum likelihood estimates are accurate then the zeroes of the polynomial of equation (14) will be real. For short observation intervals the estimates, $\tilde{P}_{j,K}$, may be inaccurate resulting in complex zeroes for the polynomial. In this case the PRI estimation method can not be used as equation (18) is no longer valid. To investigate this problem simulations were carried out with K = 2, z^1 = 103 μs and z^2 = 223 μs (nominally). In figure 4 the percentage of runs for which complex zeroes occur, C_R, is plotted versus τ for three observation intervals of duration 0.01, 0.1 and 1.0 s. The percentage of complex zeroes decreases for longer observation times and is considerably reduced for certain values of τ. In figure 5 a performance index P_I to be defined next is plotted versus τ for the same three observation intervals. The performance index is given by

$$P_I = 2(1 - C_R/100)\ (E^1 + E^2)^{-1} \tag{48}$$

where E^k is the r.m.s. (root mean square) percentage error in $\tilde{z}^k$ over a number of runs with θ_0^k sampled from a uniform distribution on 0 to z^k. The number of runs is 10^4, 10^3 and 10^2 for the 0.01, 0.1 and 1.0 s observation intervals respectively. The greater the value of this index the nearer the PRI estimates are expected to be to the true PRIs. The estimates may be deemed to be acceptable if $P_I > 1$. Notice that certain values of τ yield more approximate PRI estimates.

In figure 6 the discriminant of the quadratic function $f_2(\lambda)$, $(a_{1,2}^2 - 4\,a_{0,2})$, is plotted versus τ. Referring to the results of figures 4 and 5 it can be seen that improved performance is achieved for values of τ for which the discriminant of the quadratic function is greater.

It may be possible to select τ to obtain optimal discrimination between emitters if the scenario to be encountered is known. However, in many cases a fixed value of τ, say 100 μs, will be necessary due to practical limitations. This case is examined next.

The results obtained in the figure 7 show the accuracy which can be obtained with a fixed value of τ. The graphs in figures 7(a) and 7(b) are of the percentage of runs for which complex zeroes occur (K = 2) and r.m.s. percentage error (K = 1 and K = 2) respectively versus observation time. The value of τ is 100 μs. The number of runs is 1000 and for each run the PRIs were selected randomly from the range 200 to 5000 μs. When K = 2 the zeroes of a quadratic are determined to obtain PRI estimates for the two emitters present. As can be seen the results are not very good. A large error results if only two emitters are present and complex zeroes occur quite frequently, which renders the method almost useless. These problems could be overcome by altering τ from its 100 μs value. Perhaps a better way is to use the adaptive approach of Section 6.

Recall that with the adaptive approach all the factors of the polynomial representing the environment need not be found as the change to the environment can be deduced simply by comparing polynomial coefficients. Prior knowledge of the environment allows the coefficients, a_i, to be precisely calculated. The coefficients, b_i, of the polynomial associated with the new environment are estimated using the relative frequencies of occurrence of states of the observed data. Two estimates of PRI given by equations (50) and (51) are considered

$$\tilde{z} = \tau/\{(W+1) - \tilde{p}_0\} \tag{49}$$

$$\tilde{p}_0 = (L+1)^{-1}\left\{a_{L-1} - \tilde{b}_L - \tilde{b}_0/a_0 + \sum_{i=1}^{L-1}(a_{i-1} - \tilde{b}_i)/a_i\right\} \tag{50}$$

$$\tilde{p}_0 = a_{L-1} - \tilde{b}_L \tag{51}$$

where L is the number of fixed PRI emitters present prior to the transmission of the new emitter.

The results for the estimates of equations (50) and (51) are shown in figures 8 and 9 respectively. The number of runs, the value of τ and the PRI ranges considered are given on the figures. A curve is labelled L emitters present if (L - 1) emitters with precisely known PRI comprise the environment prior to the transmission of the L^{th} emitter, whose PRI is to be estimated. The r.m.s. % error in the resulting estimate is plotted. It is seen that the results with the estimate of equation (51) are superior and for up to five emitters present and one second observation times the r.m.s. percentage error in the estimates obtained is less than 0.4%.

8.2 Variable PRI emitters

The generalisation of the adaptive identification approach to variable PRI emitters is discussed now.

If τ is such that for the new variable PRI emitter only two states are observed then the approach of Section 8.1 can be used to estimate the average PRI, τ_1, with $\tilde{\tau}_1$ replacing $\tilde{z}$ in equation (49). If τ is such that for the new variable PRI emitter three states are observed p_0 and p_1 are estimated by $\tilde{p}_0$ and $\tilde{p}_1$ given as

$$\tilde{p}_0 = \tilde{c}_0 \tag{52}$$

$$\tilde{p}_1 = -\tilde{c}_1 - 2\tilde{c}_0 \tag{53}$$

$$\tilde{c}_1 = \tilde{b}_{L+1} - a_{L-1} \tag{54}$$

$$\tilde{c}_0 = \tilde{b}_L - a_{L-2} - \tilde{c}_1 \, a_{L-1} \tag{55}$$

where the degree of the polynomial representing the initial environment is L. These estimates are selected because it appears that estimates are best calculated using the highest order coefficients in the polynomial. The estimates $\tilde{p}_i$ and $\tilde{\tau}_1$ can be used in the algorithm of Section 5.1 to decide the type of variable PRI for the new emitter and estimate the parameters τ_1, z_m, a and b of the new emitter.

Simulation results for adaptive identification applied to variable PRI emitters are given in figures 10 and 11. Background environments of 0, 1, 2 and 3 emitters are considered. The emitters for the background environment are selected from a 614 μs fixed, a 401/491 μs 2 position stagger and a 950 to 1050 μs uniform. The new emitter can be either a 761 μs fixed, a 721/887 μs 2 position stagger or a 813 to 997 μs uniform. For a given background environment the type of PRI of the new emitter is determined and then the parameters τ_1, z_m a and b are estimated. In figure 10 the probability of error in determining the emitter type is plotted versus observation time for each of the three new emitters considered. The algorithm of figure 2 is used with the decision thresholds ϵ_1 and ϵ_2 shown on figure 10. The r.m.s. percentage error in the estimates of τ_1, z_m a and b are plotted versus observation time in figure 11 assuming that the new emitter is correctly classified in each case. These results show that the type of emitter PRI can be recognized and parameter estimates subsequently extracted using the techniques proposed in this paper.

The implementation of these concepts using both analogue and digital techniques is being investigated. The digital techniques show the greatest promise. The system under development consists of a pulse detector, a 4-bit counter and an Intel 8080 microprocessor. The environment of emitters for testing the identification system is provided by a microwave laboratory simulator. The 4-bit counter is used to count the number of pulses which occur during the gate interval. The gate time is controlled by the micro-processor. More details will be given in a later report.

9. DISCUSSION

9.1 Introduction

Radar emitter recognition in a rapidly changing dense environment of emitters with parameter agility is a challenging problem.

"One signal processing area that needs immediate improvement is the handling of complex time domain modulations. IFMs and other wide open receivers can handle RF agility, but time domain agility remains a serious problem for all receiver types. It is a very difficult task to correctly process a signal having multiposition stagger or PRI shifts in the presence of other emitters without generating false emitters or incorrect IDs. Angle of arrival is a powerful tool in the solution of this problem(ref.8)".

This quotation testifies to the importance of the signal processing problem addressed in this paper. The suggestion of directional antennas may not always provide a solution. It may not always be possible to thin the environment sufficiently using direction of arrival. The limitations of cost, space, time and higher priority tasks such as direction finding and directive countermeasures have to be taken into account also.

Several recognition methods using PRI measurement are adopted in practice and this paper puts forward yet another approach. Each method relies on the validity in the real world of the particular assumptions involved. Other methods of radar emitter PRI sorting are described in this section and then some features of the method proposed here are discussed.

9.2 Other methods

Current methods of PRI analysis are mainly applicable to emitters with fixed PRI. The pulse train sorter passes the received pulse train and a delayed version through a coincidence circuit. If the delay is equal to the emitter PRI an output is obtained. Another method uses the times of arrival of the incoming pulses. The time interval between the next two unsorted pulses is measured and then if pulses are found at 2, 3, 4 ..., r times this interval from the first pulse an emitter with this PRI is deemed to be present. Depending on the search method employed and the frequency with which missing pulses occur both these methods are prone to respond to subharmonic pulse repetition frequencies.

Calculating the autocorrelation function of the incoming pulse train is also suggested as a possible means of PRI sorting. For a single emitter with fixed PRI the autocorrelation function bears a striking resemblance to the original pulse train. Calculating the autocorrelation function digitally would involve a considerable amount of computational effort, but optical signal processing techniques may allow the autocorrelation function to be calculated almost instantaneously. Unfortunately, a systematic method of analysing the autocorrelation function for a dense environment of variable PRI emitters does not appear to be available and handling the data rate involved would not be a trivial task.

A common problem with these standard methods is that, if implemented on a microprocessor, the processor rapidly becomes overloaded in a dense environment. Even if elaborate buffering is provided it may be necessary to ignore some received data.

9.3 Blanking

Blanking is a technique for thinning the environment. Time domain blanking can be used with all methods of emitter identification. After an emitter is identified further pulses which are deemed to have come from this emitter are blanked or ignored in the analysis of future received data. This approach is quite successful with emitters with fixed PRI in a dense environment where the emitters appear one at a time and at a rate which does not saturate the system.

Time domain blanking could well be applied to boost the number of emitters which can be successfully identified in a dense environment using the PRI measurement method proposed in this paper.

9.4 Scanning antennas

A requirement of the PRI measurement method proposed in this paper is that the environment does not change rapidly. Sufficient data must be collected while the environment is static to obtain accurate PRI estimates. If emitters with scanning antennas are present and only main lobe pulses can be detected this requirement may not be met. Possible solutions to this problem are discussed in what follows.

To increase the number of observed pulses in dealing with search radars the threshold of the pulse detection circuit could be lowered to detect pulses in the antenna side lobes. The drawback with this is that it may preclude emitter sorting using other parameters such as direction of arrival and radio frequency. Also the scan information for the emitter concerned will be lost and in a noisy environment the resulting false alarm rate could be too high. To achieve the required receiver sensitivity a narrow bandwidth would be necessary. This would no doubt reduce the probability of intercept.

A better approach is suggested now. Suppose that data are collected using a series of consecutive observation intervals each of which is shorter than the shortest main lobe pulse burst to be encountered. Accurate PRI estimates could not be obtained using data from any individual observation interval. However, it should be possible to determine when the main lobe of a particular emitter occurs by correlating the data from all the consecutive observation intervals. By averaging the data recorded during successive main lobe bursts it is conjectured that accurate PRI parameter measurements can be obtained for emitters with scanning antennas. The data from observation intervals in between main lobe pulse bursts will be useful too if nonscanning radar emitters are present.

9.5 Very short PRI

In practice the gate time can not be made arbitrarily short because a certain amount of processing must be done during this time. PRI of the order of a few microseconds can occur for some emitters. This problem can be dealt with by treating such emitters as a special case. They will produce a recognizable factor in the polynomial representing the environment for any given value of τ.

9.6 Recognition of known emitters

Often it is necessary to be able to quickly recognize that a particular emitter has started transmitting. This is the case with certain threat emitters. The polynomial factor associated with such an amitter could be stored in a library and the adaptive method of this paper used to detect the presence of this emitter. This approach would be advantageous for threats with PRI agility.

9.7 Sorting using pulse amplitude

The adaptive technique for PRI measurement presented in this paper examines changes in the environment. It is possible to artificially obtain the effect of an emitter starting or stopping transmitting using pulse amplitude selection. Sorting using pulse amplitude in conjunction with the adaptive PRI measurement technique would appear to be a promising method for emitter identification.

9.8 Summary

The PRI measurement method proposed here can be viewed as a generalization of the familiar pulse counter to deal with variable PRI in a dense environment. The method is adaptive in that it analyses changes to the environment to obtain parameter measurements. One advantage of the method is its simplicity which makes it ideal for implementation on a microprocessor.

It does not involve time consuming parameter searches even to deal with variable PRI emitters. It allows emitters to be identified with a high probability of intercept. Further work is in progress to implement the concept using microprocessor technology.

10. CONCLUSION

The technique described in this paper should have applications to surveillance problems in electronic warfare. Although alternate techniques exist, the method of this paper allows parameter agile radar emitters to be identified in a dense environment comparatively cheaply and simply.

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NOTATION

For the following symbols if a superscript, k, is added this denotes that the symbol is associated with emitter k. Also, if ~ is placed over a symbol this indicates an estimate of the quantity denoted by the symbol.

a	parameter of a variable PRI emitter specifying PRI distribution
$a_{r,j}$	coefficient r in $f_j(\lambda)$
a_r	coefficient r in $f^1_j(\lambda)$
b	parameter of variable PRI emitter specifying PRI distribution
b_r	coefficient r in $f^2_j(\lambda)$
c_0	coefficient of quadratic factor defined in equation (37)
c_1	coefficient of quadratic factor defined in equation (38)
$f_j(\lambda)$	polynomial $\lambda^j + a_{j-1,j}\lambda^{j-1} + \ldots + a_{1,j}\lambda + a_{0,j}$
$f^1_j(\lambda)$	polynomial of degree j representing initial environment
$f^2_j(\lambda)$	polynomial of degree j representing environment after new emitter starts transmitting
f^k	radio frequency of k^{th} emitter
f_{IF}	IF frequency
k	denotes emitter number
n_t	number of pulses observed in gate interval t
p_i	probability that $x_t = i$
$p(y_n)$	probability density function of PRI
$r_{IF}(s)$	IF signal at time s
$r_{RF}(s)$	RF signal at time s
s	time variable
s_t	time point t
t	subscript denoting discrete time points
x_t	observation derived from n_t for gate interval t
x^k_t	observation at time point t when emitter k only is present
y_n	PRI sample
z	PRI of fixed PRI emitter
z_m	staggered PRI parameter

A_1	defined in equation (26)
A_2	defined in equation (27)
$A^k(s)$	amplitude of component of RF signal due to emitter k at time s
$A_{IF}(s)$	amplitude of IF signal at time s
C_R	percentage of simulations for which complex zeroes occur
E^k	root mean square percentage error in $\tilde{z}^k$
K	number of emitters present
M	number of positions for PRI stagger
N_q	defined in equation (7)
N_q^t	defined in equation (8)
N_{min}	minimum number of pulses detected in any gate interval during observation interval
N_{max}	maximum number of pulses detected in any gate interval during observation interval
P_I	performance index of equation (48)
$P_{j,K}$	probability of state j occurring in dense environment of K emitters
P_q	relative frequency of occurrence of state q
Q	number of observed states is Q + 1
T	number of discrete time points (or gate intervals)
V	half width of PRI distribution
W	non-negative integer specifying gate time limits
$\alpha_{IF}(s)$	phase of IF signal
$\delta(.)$	Dirac delta function
λ	polynomial variable
θ_t	time measured from start of gate interval to where a specified pulse occurs
τ	gate time
τ_1	average PRI
τ_2	twice average PRI
φ^k	phase of emitter k
$\binom{r}{j}$	combinatorial symbol $r!/\{(r-j)!j!\}$
$\{Y_n, n=1,2,\ldots,N\}$	stochastic process representing PRI

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APPENDIX I

VERIFICATION OF STATISTICAL MODEL

I.1 (Section 4.1) θ_t may be considered to be a sample value of a random variable uniformly distributed from 0 to z

Assume that the gate interval partition and the radar pulse train are synchronized to a clock of period, γ, which can be made arbitrarily short. Thus, $\tau = n\gamma$ and $z = m\gamma$ where n and m are positive integers. Assumption 3(b) then becomes equivalent to assuming that m and n are relative prime. Also, $\theta_t = j_t\gamma$ where $1 \leq j_t \leq m$. It is required to prove that j_t is uniformly distributed on the set 1,2,3,...,m . Note that

$$j_{q+r} \equiv j_q + r\{(W + 1)m - n\} \quad (\mathrm{mod}\ m)$$

where W is defined in Table 1. The sequence, $\{j_t\}$, is periodic with period m. There are mn clock periods between the pulses associated with j_q and j_{q+m} and each value of j_t occurs once during a period. If this were not true then for some q and i, $1 < i < m$, $j_q = j_{q+i}$. This would contradict the assumption that m and n are relatively prime.

This result can be generalized further to the case where j_t can be real, that is $j_t \in (0,m]$. This corresponds to the case where the radar pulse train is not exactly synchronized with the clock. Suppose that for some t

$$\theta_t = (i_t - \Delta)\gamma$$

for some $\Delta \in [0,1)$ and $i_t \in \{1,2,\ldots,m\}$. Then θ_t can be expressed in this form for all t. The sequence, $\{i_t\}$, is periodic as before and j_t is uniformly distributed on $\{1 - \Delta, 2 - \Delta, \ldots, m - \Delta\}$.

For practical purposes the gate interval partition and the radar pulse train are unsynchronized if, for some choice of Δ and γ, relatively prime m and n can be found (with m sufficiently large) which allow an accurate model to be obtained. If these conditions are met then the distribution of θ_t will closely approximate a uniform distribution on 0 to z, independent of the initial phase of the radar pulse train relative to the gate interval partition.

I.2 (Section 4.2) The sequence of states x_t^k associated with the individual emitters are mutually statistically independent

Assume that the gate interval partition and the radar pulse trains are synchronized to a clock of period, γ, which can be made arbitrarily short. Thus $\tau = n\gamma$ and $z^k = m^k\gamma$ where n and m^k are positive integers. Assumptions 3(a) and (b) then become equivalent to assuming that integers n, $m^1, \ldots, m^K$, are relatively prime. Note that $\theta_t^k = j_t^k\gamma$, where $1 \leq j_t^k \leq m^k$. Using a similar argument to that in (I.1) above the vector sequence, $J_t^K = \{j_t^1,\ldots,j_t^K\}$, is periodic with period, $m^1 \ldots m^K$, and each possible value of J_t^K occurs once for this number of successive gate intervals. Thus, the sequence,

$\theta_t^K = \{\theta_t^1, \ldots, \theta_t^K\}$, can be regarded as uniformly distributed on $(0, z^1] \times \ldots \times (0, z^K]$. Since the value of θ_t^k determines the value of x_t^k this is tantamount to having mutually statistically independent sequences of states, x_t^k.

I.3 Model validation via simulations for fixed PRI emitters

The results in figures 12 and 13 are submitted as evidence that the model introduced faithfully represents the observed data. The radar emitter pulse trains and the observation procedure are simulated in double precision FORTRAN on a 32 bit machine so that a number of runs with varying z^k, τ and θ_0^k can be performed. For each run z^k and τ are samples from uniform distributions with specified limits and θ_0^k is a sample from a uniform distribution on the interval 0 to z^k. Consequently, the probability that assumptions 3(a) and 3(b) are invalid is vanishingly small. The error for run R, E_R, is recorded as

$$E_R = (K+1)^{-1} \sum_{j=0}^{K} \{(\tilde{P}_{j,K} - P_{j,K})/P_{j,K}\}^2$$

and then the root mean square percentage error, E, computed via

$$E = 100 \left\{ N_R^{-1} \sum_{R=1}^{N_R} E_R \right\}^{\frac{1}{2}}$$

For the simulations carried out $N_R = 1000$. In figures 12 and 13 E is plotted versus the duration of the observation interval. In both figures τ lies between 90 and 100 μs. In figure 12 z^k lies between 103 and 303 μs and curves are plotted for K = 2, 3 and 4. In figure 13 K = 2 and z^k lies between 103 and 303 μs for curve C and z^k lies between 103 and 1103 μs for curve D. The fact that the curves in figures 12 and 13 all decrease to fairly small values indicates that the model introduced is valid. Notice that convergence is slower when K is larger and when the range of allowable PRIs is greater.

I.4 Model validation via simulations for two position stagger PRI emitters

The simulations described next justify the use of the proposed models for two positions stagger PRI emitters. The radar emitter pulse trains and the observation procedure are simulated in double precision FORTRAN on a 32 bit machine so that a number of runs with varying z_m^k, τ and θ_0^k can be performed. For each run z_m^k and τ are samples from uniform distributions with specified limits and θ_0^k is a sample from a uniform distribution on either 0 to z^k or 0 to $z_1^k + z_2^k$ as appropriate. Consequently the probability that assumptions 3(a) and (b) are invalid is vanishingly small. The root mean square percentage error, E, defined above is computed for $N_R = 1000$. The curves shown in figure 14 are for one fixed and one staggered PRI emitter present and for one fixed and two staggered PRI emitters present. In figure 14 E is plotted versus the length of the observation interval. For the emitter with fixed PRI z^1 lies between 103 and 303 μs. For the two position stagger PRI emitters $z_1^k + z_2^k$ lies between 203 and 603 μs and z_1^k and τ are selected with the following constraints

$$50 < z_1^k < \{z_1^k + z_2^k\}_{min}/2$$

$$\{z_1^k\}_{max} < \tau < \{z_2^k\}_{min}$$

$$\tau < z^1$$

The fact that the curves both decrease to small values indicates that the models introduced are valid. Convergence is slower when more emitters are present.

APPENDIX II

DENSE ENVIRONMENT THEOREM

This appendix contains the proof of the result of Section 4.2 that, given the $P_{j,K}$, the probabilities p_0^k are zeroes of the polynomial, $f_K(\lambda)$, of degree K in λ given by

$$f_K(\lambda) = \lambda^K + a_{K-1,K}\lambda^{K-1} + \ldots + a_{1,K}\lambda + a_{0,K}$$

where the coefficients can be expressed as

$$a_{r,K} = (-1)^{K-r}\sum_{j=0}^{r} P_{j,K}\binom{K-j}{r-j}$$

and (.) denotes the combinatorial symbol.

To simplify notation in this appendix p_0^k is replaced by p_k.

Using the statistical independence of the individual processes x_t^k discussed in Appendix I the expression for $P_{j,K}$ is

$$P_{0,K} = p_1\, p_2 \ldots p_K$$

$$P_{j,K} = \sum_{\substack{i_1, i_2, \ldots, i_j = 1 \\ i_1 < i_2 < \ldots < i_j}}^{K} (1 - p_{i_1})(1 - p_{i_2})\ldots(1 - p_{i_j})\, p_{i_j + 1}\, p_{i_j + 2} \cdots p_{i_K}$$

$$\{i_j + 1, i_j + 2, \ldots, i_K\} = \{1, 2, \ldots, K\} - \{i_1, i_2, \ldots, i_j\}$$

$$, \quad j = 1, 2, \ldots, K.$$

Another useful relation is for $1 < j < K$

$$P_{j,K+1} = P_{j,K}\, p_{K+1} + P_{j-1,K}\,(1 - p_{K+1})$$

The theorem is proved by means of the following three lemmas.

Lemma 1: Let $p_1, p_2, \ldots, p_K$ be zeroes of the polynomial of degree K in λ

$$f_K(\lambda) = \lambda^K + a_{K-1,K}\lambda^{K-1} + \ldots + a_{1,K}\lambda + a_{0,K}. \tag{II.1}$$

That is, for r = 1, 2, ..., K-1

$$a_{r,K} = (-1)^{K-r} \sum_{\substack{i_1, i_2, \ldots, i_{K-r}=1 \\ i_1 < i_2 < \ldots < i_{K-r}}}^{K} p_{i_1} p_{i_2} \ldots p_{i_{K-r}} \tag{II.2}$$

and

$$a_{0,K} = (-1)^K p_1 p_2 \ldots p_K \tag{II.3}$$

$$a_{K,K} = 1 \tag{II.4}$$

Then,

$$a_{0,K+1} = -p_{K+1}\, a_{0,K} \tag{II.5}$$

$$a_{r,K+1} = -p_{K+1}\, a_{r,K} + a_{r-1,K}, \quad 0 < r \leqslant K \tag{II.6}$$

$$a_{K+1,K+1} = 1 \tag{II.7}$$

Proof: Equations (II.5) and (II.7) are trivial to verify. Equation (II.6) is obtained by noting that, apart from the sign, $a_{r,K+1}$ is a sum of terms $p_{i_1} p_{i_2} \ldots p_{i_{K+1-r}}$ where the $i_1, i_2, \ldots, i_{K+1-r}$ are all possible combinations of K+1 - r subscripts chosen from the set $\{1, 2, \ldots, K+1\}$. This is equivalent to summing all terms $p_{i_1} p_{i_2} \ldots p_{i_{K+1-r}}$ where $i_1, i_2, \ldots, i_{K+1-r}$ are all possible combinations of K+1 - r subscripts from the set $\{1, 2, \ldots, K\}$ and then adding this to the sum of all terms $p_{i_1} p_{i_2} \ldots p_{i_{K-r}}$ where $i_1, i_2, \ldots, i_{K-r}$ are all possible combinations of K-r subscripts from the set $\{1, 2, \ldots, K\}$ multiplied by p_{K+1}. Apart from the sign these two sums are none other than $a_{r-1,K}$ and $a_{r,K}$ multiplied by p_{K+1} respectively. Since $a_{r,K}$ has a sign term, $(-1)^{K-r}$, a negative sign must be included in equation (II.6).

Lemma 2: Suppose that $P_{j,K}$, $j = 0, 1, \ldots, K$ are defined in terms of the $p_1, p_2, \ldots, p_K$ as above. Let $p_1, p_2, \ldots, p_K$ be zeroes of the polynomial of degree K in λ, see equation (II.1) below

$$f_K(\lambda) = \lambda^K + a_{K-1,K}\,\lambda^{K-1} + \ldots + a_{1,K}\,\lambda + a_{0,K}$$

Then the quantities $P_{j,K}$ can be expressed in terms of the polynomial coefficients as

$$P_{j,K} = (-1)^{K-j} \sum_{r=0}^{j} a_{r,K} \binom{K-r}{j-r}, \quad j = 0, 1, 2, \ldots, K \qquad \text{(II.8)}$$

Proof: The proof is by induction. For K = 2 the $P_{j,K}$ are defined by

$$P_{0,2} = p_1\, p_2$$

$$P_{1,2} = p_1 + p_2 - 2\,p_1\,p_2$$

$$P_{2,2} = 1 - p_1 - p_2 + p_1\,p_2$$

and the polynomial coefficients $a_{r,K}$ are defined by

$$a_{0,2} = p_1\, p_2$$

$$a_{1,2} = -(p_1 + p_2)$$

$$a_{2,2} = 1$$

Simple algebraic rearrangement yields

$$P_{0,2} = a_{0,2}$$

$$P_{1,2} = -(2a_{0,2} + a_{1,2})$$

$$P_{2,2} = a_{0,2} + a_{1,2} + a_{2,2}\,.$$

which are none other than equations (II.8) for K = 2.

To complete the inductive proof assume that equations (II.8) are true for K = H and show they are true for K = H+1. Now

$$P_{j,H+1} = P_{j,H}\,p_{H+1} + P_{j-1,H}\,(1 - p_{H+1})$$

$$= p_{H+1}\,(P_{j,H} - P_{j-1,H}) + P_{j-1,H} \qquad \text{(II.9)}$$

where $1 \leqslant j \leqslant H$. Using equations (II.8) for $1 \leqslant j \leqslant H$,

$$P_{j,H} - P_{j-1,H} = (-1)^{H-j} \sum_{r=0}^{j} a_{r,H} \binom{H-r}{j-r} - (-1)^{H+1-j} \sum_{r=0}^{j-1} a_{r,H} \binom{H-r}{j-1-r}$$

$$= (-1)^{H-j} \sum_{r=0}^{j} a_{r,H} \binom{H+1-r}{j-r}$$

since $\binom{r+1}{j} = \binom{r}{j} + \binom{r}{j-1}$. Substituting this in equation (II.9)

$$P_{j,H+1} = p_{H+1} (-1)^{H-j} \sum_{r=0}^{j} a_{r,H} \binom{H+1-r}{j-r} + (-1)^{H+1-j} \sum_{r=0}^{j-1} a_{r,H} \binom{H-r}{j-1-r}$$

Using the result of Lemma 1

$$P_{j,H+1} = (-1)^{H+1-j} \sum_{r=0}^{j} a_{r,H+1} \binom{H+1-r}{j-r}$$

for $1 \leqslant j \leqslant H$. For $j = 0$

$$P_{0,H+1} = p_1 \; p_2 \; \ldots \; p_{H+1}$$

$$= (-1)^{H+1} a_{0,H+1}$$

For $j = H+1$,

$$P_{H+1,H+1} = P_{H,H} (1 - p_{H+1})$$

$$= \sum_{r=0}^{H} a_{r,H} (1 - p_{H+1})$$

$$= \sum_{r=0}^{H} a_{r,H} - p_{H+1} \sum_{r=0}^{H} a_{r,H}$$

Using Lemma 1 again

$$P_{H+1,H+1} = -P_{H+1}\, a_{0,H} + \sum_{r=1}^{H} a_{r,H+1} + a_{H,H}$$

$$= \sum_{r=0}^{H+1} a_{r,H+1} \quad \text{since } a_{H,H} = 1 \text{ for all } H.$$

Lemma 3: If $a_{r,K}$ and $P_{j,K}$ are defined as in Lemma 2 then

$$a_{r,K} = (-1)^{K-r} \sum_{j=0}^{r} P_{j,K} \binom{K-j}{r-j} \tag{II.10}$$

Proof: Notice that equations (II.8) can be manipulated to give a recursive set of equations for $a_{r,K}$ in terms of $P_{j,K}$ as

$$a_{0,K} = (-1)^{K} P_{0,K} \tag{II.11}$$

$$a_{r,K} = (-1)^{K-r} P_{r,K} - \sum_{j=0}^{r-1} a_{j,K} \binom{K-j}{r-j} \tag{II.12}$$

The proof of equation (II.10) is inductive. Obviously equation (II.10) is true for $r = 0$. It is shown that if equation (II.10) is true for $r < v$ then it is true for $r = v$. Thus, from equation (II.12) for $v > 1$

$$a_{v,K} = (-1)^{K-v} P_{v,K} - \sum_{j=0}^{v-1} a_{j,K} \binom{K-j}{v-j} \tag{II.13}$$

In the summation of equation (II.13) the following substitution can be made

$$a_{j,K} = (-1)^{K-j} \sum_{m=0}^{j} P_{m,K} \binom{K-m}{j-m}$$

which results in

$$a_{v,K} = (-1)^{K-v} P_{v,K} - \sum_{j=0}^{v-1} (-1)^{K-j} \sum_{m=0}^{j} P_{m,K} \binom{K-m}{j-m}\binom{K-j}{v-j}$$

$$= (-1)^{K-v} \left\{ P_{v,K} - \sum_{j=0}^{v-1} (-1)^{v-j} \sum_{m=0}^{j} P_{m,K} \binom{K-m}{j-m}\binom{K-j}{v-j} \right\}.$$

Using the fact that

$$\sum_{j=0}^{v} \sum_{i=0}^{j} g_{ij}\, b_i = \sum_{j=0}^{v} b_j \sum_{i=j}^{v} g_{ji}$$

$$a_{v,K} = (-1)^{K-v} \left\{ P_{v,K} - \sum_{j=0}^{v-1} P_{j,K} \sum_{m=j}^{v-1} (-1)^{v-m} \binom{K-j}{m-j}\binom{K-m}{v-m} \right\}$$

Now

$$\sum_{m=j}^{v-1} (-1)^{v-m} \binom{K-j}{m-j}\binom{K-m}{v-m} = \binom{K-j}{v-j} \sum_{m=j}^{v-1} (-1)^{v-m} \binom{v-j}{m-j}$$

since

$$\binom{K-j}{m-j}\binom{K-m}{v-m} = \frac{(K-j)!\ (K-m)!}{(K-m)!\ (m-j)!\ (K-v)!\ (v-m)!}$$

$$= \frac{(K-j)!}{(v-j)!\ (K-v)!}\ \frac{(v-j)!}{(v-m)!\ (m-j)!}$$

$$= \binom{K-j}{v-j}\binom{v-j}{m-j}$$

If it is shown that

$$\sum_{m=j}^{v-1} (-1)^{v-m} \binom{v-j}{m-j} = -1$$

the proof is complete. In this summation put m = b + j and the summation becomes

$$(-1)^{v-j} \sum_{b=0}^{v-j-1} (-1)^b \binom{v-j}{b}$$

Using the binomial theorem, $(a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$, with $a = -1$, $b = 1$ gives

$$\sum_{j=0}^{n-1} \binom{n}{j} (-1)^j = (-1)^{n-1}$$

Thus, the summation is equal to $(-1)^{2(v-j)-1} = -1$.

TABLE 1. STATE PROBABILITIES FOR FIXED PRI EMITTERS

Gate time (τ) limits W = 0,1,2...	Possible number of pulses	States of Model	Model state probabilities
Wz,	W	0	$p_0 = \frac{(W+1)z - \tau}{z}$
$(W+1)z$	W+1	1	$p_1 = \frac{\tau - Wz}{z}$

TABLE 2. STATE PROBABILITIES FOR TWO POSITION STAGGER EMITTERS, $z_2 > z_1$

Case	Gate time (τ) limits W = 0,1,2...	Possible number of pulses	States of model	Model state probabilities
A	$W(z_1+z_2)$,	2W	0	$p_0 = \frac{(2W+1)(z_1+z_2) - 2\tau}{z_1+z_2}$
	$W(z_1+z_2) + z_1$	2W+1	1	$p_1 = \frac{2\tau - 2W(z_1+z_2)}{z_1+z_2}$
B	$W(z_1+z_2) + z_1$,	2W	0	$p_0 = \frac{W(z_1+z_2) + z_2 - \tau}{z_1+z_2}$
		2W+1	1	$p_1 = \frac{2z_1}{z_1+z_2}$
	$W(z_1+z_2) + z_2$	2W+2	2	$p_2 = \frac{\tau - z_1 - W(z_1+z_2)}{z_1+z_2}$
C	$W(z_1+z_2) + z_2$,	2W+1	0	$p_0 = \frac{(2W+2)(z_1+z_2) - 2\tau}{z_1+z_2}$
	$(W+1)(z_1+z_2)$	2W+2	1	$p_1 = \frac{2\tau - (2W+1)(z_1+z_2)}{z_1+z_2}$

TABLE 3(a). STATE PROBABILITIES FOR THREE POSITION STAGGER EMITTERS, $z_3 > z_2 > z_1$, $z_1 + z_2 > z_3$

Case	Gate time (τ) limits W = 0,1,2...	Possible number of pulses	States of model	Model state probabilities
A	$W(z_1+z_2+z_3)$,	3W	0	$p_0 = \frac{(3W+1)(z_1+z_2+z_3)-3\tau}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1$	3W+1	1	$p_1 = \frac{3\tau-3W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
B	$W(z_1+z_2+z_3)+z_1$,	3W	0	$p_0 = \frac{z_3+z_2-2\tau+2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+1	1	$p_1 = \frac{\tau+2z_1-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_2$	3W+2	2	$p_2 = \frac{\tau-z_1-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
C	$W(z_1+z_2+z_3)+z_2$,	3W	0	$p_0 = \frac{z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+1	1	$p_1 = \frac{2(z_1+z_2)-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_3$	3W+2	2	$p_2 = \frac{2\tau-z_1-z_2-2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
D	$W(z_1+z_2+z_3)+z_3$,	3W+1	0	$p_0 = \frac{(3W+2)(z_1+z_2+z_3)-3\tau}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1+z_2$	3W+2	1	$p_1 = \frac{3\tau-(3W+1)(z_1+z_2+z_3)}{z_1+z_2+z_3}$
E	$W(z_1+z_2+z_3)+z_1+z_2$,	3W+1	0	$p_0 = \frac{2(z_3-\tau)+z_1+z_2+2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+2	1	$p_1 = \frac{\tau-z_3+z_1+z_2-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1+z_3$	3W+3	2	$p_2 = \frac{\tau-z_1-z_2-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
F	$W(z_1+z_2+z_3)+z_1+z_3$,	3W+1	0	$p_0 = \frac{z_2+z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+2	1	$p_1 = \frac{3z_1+z_2+z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_2+z_3$	3W+3	2	$p_2 = \frac{2(\tau-z_1)-z_2-z_3-2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
G	$W(z_1+z_2+z_3)+z_2+z_3$,	3W+2	0	$p_0 = \frac{(3W+3)(z_1+z_2+z_3)-3\tau}{z_1+z_2+z_3}$
	$(W+1)(z_1+z_2+z_3)$	3W+3	1	$p_1 = \frac{3\tau-(3W+2)(z_1+z_2+z_3)}{z_1+z_2+z_3}$

TABLE 3(b). STATE PROBABILITIES FOR THREE POSITION STAGGER EMITTERS, $z_3 > z_2 > z_1$, $z_1 + z_2 < z_3$

Case	Gate time (τ) limits W = 0,1,2...	Possible number of pulses	States of model	Model state probabilities
A	$W(z_1+z_2+z_3)$,	3W	0	$p_0 = \frac{(3W+1)(z_1+z_2+z_3)-3\tau}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1$	3W+1	1	$p_1 = \frac{3\tau-3W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
B	$W(z_1+z_2+z_3)+z_1$,	3W	0	$p_0 = \frac{z_3+z_2-2\tau+2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+1	1	$p_1 = \frac{\tau+2z_1-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_2$	3W+2	2	$p_2 = \frac{\tau-z_1-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
C	$W(z_1+z_2+z_3)+z_2$,	3W	0	$p_0 = \frac{z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+1	1	$p_1 = \frac{2(z_1+z_2)-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1+z_2$	3W+2	2	$p_2 = \frac{2\tau-z_1-z_2-2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
D	$W(z_1+z_2+z_3)+z_1+z_2$,	3W	0	$p_0 = \frac{z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+1	1	$p_1 = \frac{z_1+z_2}{z_1+z_2+z_3}$
		3W+2	2	$p_2 = \frac{z_1+z_2}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_3$	3W+3	3	$p_3 = \frac{\tau-z_1-z_2-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
E	$W(z_1+z_2+z_3)+z_3$,	3W+1	0	$p_0 = \frac{2(z_3-\tau)+z_1+z_2+2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+2	1	$p_1 = \frac{\tau-z_3+z_1+z_2-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_1+z_3$	3W+3	2	$p_2 = \frac{\tau-z_1-z_2-W(z_1+z_2+z_3)}{z_1+z_2+z_3}$

TABLE 3(b)(CONTD.).

Case	Gate time (τ) limits W = 0,1,2...	Possible number of pulses	States of model	Model state probabilities
F	$W(z_1+z_2+z_3)+z_1+z_3$,	3W+1	0	$p_0 = \frac{z_2+z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
		3W+2	1	$p_1 = \frac{3z_1+z_2+z_3-\tau+W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
	$W(z_1+z_2+z_3)+z_2+z_3$	3W+3	2	$p_2 = \frac{2(\tau-z_1)-z_2-z_3-2W(z_1+z_2+z_3)}{z_1+z_2+z_3}$
G	$W(z_1+z_2+z_3)+z_2+z_3$,	3W+2	0	$p_0 = \frac{(3W+3)(z_1+z_2+z_3)-3\tau}{z_1+z_2+z_3}$
	$(W+1)(z_1+z_2+z_3)$	3W+3	1	$p_1 = \frac{3\tau-(3W+2)(z_1+z_2+z_3)}{z_1+z_2+z_3}$

TABLE 4. STATE PROBABILITIES FOR EMITTERS WITH VARIABLE PRI

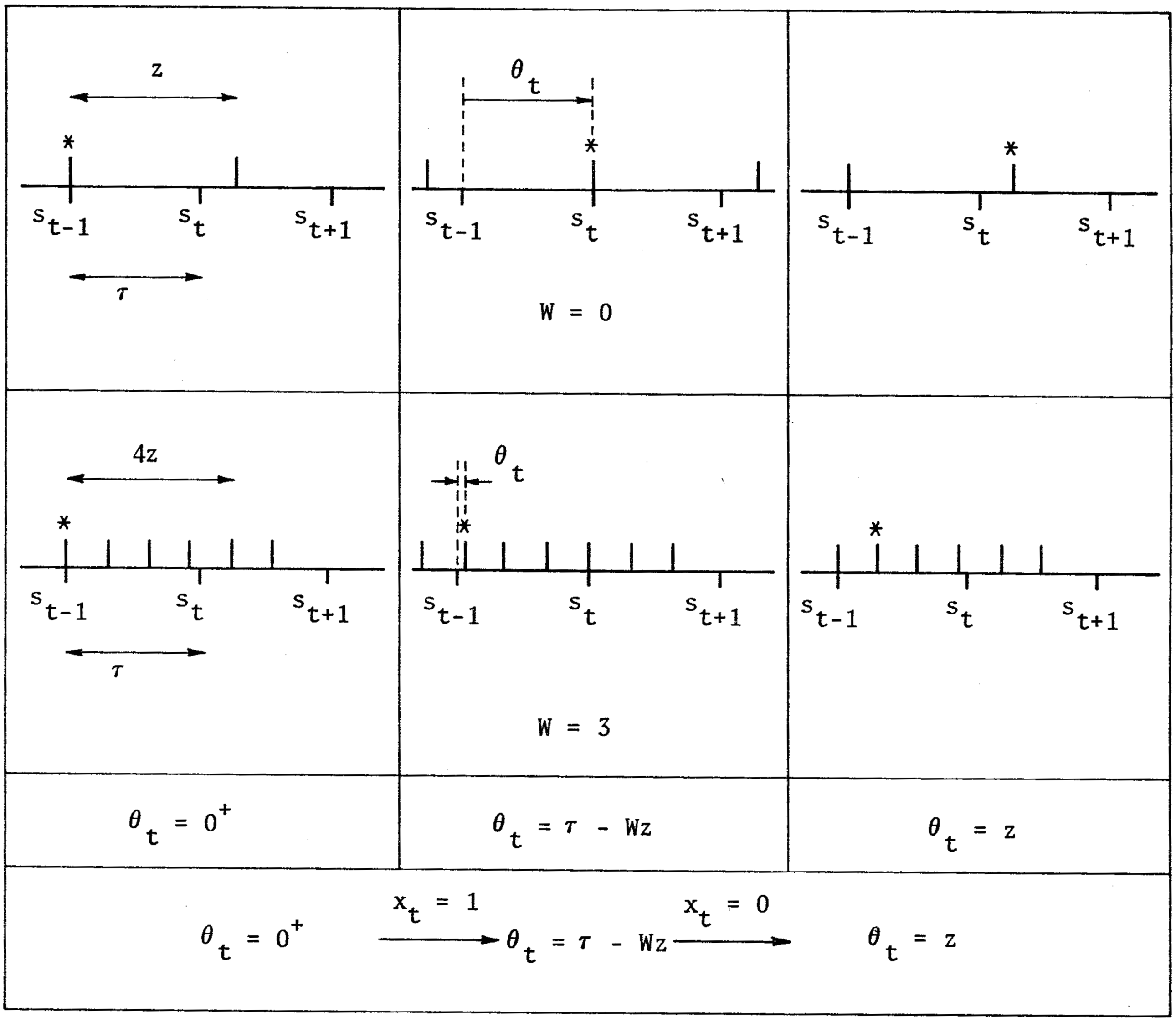
Gate time (τ) limits	Possible number of pulses	States of model	Model state probabilities
$0 < \tau \leqslant a$	0	0	$p_0 = (\tau_1 - \tau)/\tau_1$
	1	1	$p_1 = \tau/\tau_1$
$a \leqslant \tau \leqslant b$	0	0	$p_0 = \tau_1^{-1} \int_{\tau}^{b} p(y)(y-\tau)\,dy$
	1	1	$p_1 = \tau_1^{-1} \left\{ \int_{a}^{\tau} p(y)(2y-\tau)\,dy + \int_{\tau}^{b} p(y)\tau\,dy \right\}$
	2	2	$p_2 = \tau_1^{-1} \int_{a}^{\tau} p(y)(\tau - y)\,dy$
$b \leqslant \tau \leqslant 2a$	1	0	$p_0 = (2\tau_1 - \tau)/\tau_1$
	2	1	$p_1 = (\tau - \tau_1)/\tau_1$
$2a \leqslant \tau \leqslant 2b$	1	0	$p_0 = \tau_2^{-1} \int_{a}^{b} \int_{\tau-y_1}^{b} p(y_1)p(y_2)[2(y_1+y_2)-2\tau]\,dy_1\,dy_2$
	2	1	$p_1 = \tau_2^{-1} \left\{ \int_{a}^{b} \int_{\tau-y_1}^{b} p(y_1)p(y_2)[2\tau-(y_1+y_2)]\,dy_1\,dy_2 + \int_{a}^{b} \int_{a}^{\tau-y_1} p(y_1)p(y_2)[3(y_1+y_2)-2\tau]\,dy_1\,dy_2 \right\}$
	3	2	$p_2 = \tau_2^{-1} \int_{a}^{b} \int_{a}^{\tau-y_1} p(y_1)p(y_2)[2\tau-2(y_1+y_2)]\,dy_1\,dy_2$
$2b \leqslant \tau \leqslant 3a$	2	0	$p_0 = (3\tau_1 - \tau)/\tau_1$
	3	1	$p_1 = (\tau - 2\tau_1)/\tau_1$

TABLE 5. STATE PROBABILITIES FOR EMITTERS WITH UNIFORM PRI

Gate time (τ) limits	Possible number of pulses	States of model	Model state probabilities
$0 < \tau \leqslant a$	0	0	$p_0 = \dfrac{a+b-2\tau}{a+b}$
	1	1	$p_1 = \dfrac{2\tau}{a+b}$
$a \leqslant \tau \leqslant b$	0	0	$p_0 = \dfrac{(\tau-b)^2}{(b-a)(a+b)}$
	1	1	$p_1 = \dfrac{2\tau b-2\tau a-2a^2-2\tau^2}{(b-a)(a+b)}$
	2	2	$p_2 = \dfrac{(\tau-a)^2}{(b-a)(a+b)}$
$b \leqslant \tau \leqslant 2a$	1	0	$p_0 = \dfrac{2(a+b)-2\tau}{a+b}$
	2	1	$p_1 = \dfrac{2\tau-a-b}{a+b}$
$2a \leqslant \tau \leqslant 2b$	1	0	$p_0 = \dfrac{(2b-\tau)^3-(b+a-\tau)^3}{3(b^2-a^2)(b-a)}$
	2	1	$p_1 = \dfrac{4\tau b+4\tau a-2\tau^2-8ab/3-5b^2/3-11a^2/3}{3(b^2-a^2)}$
	3	2	$p_2 = \dfrac{(b+a-\tau)^3-(2a-\tau)^3}{3(b^2-a^2)(b-a)}$
$2b \leqslant \tau \leqslant 3b$	2	0	$p_0 = \dfrac{3(a+b)-2\tau}{a+b}$
	3	1	$p_1 = \dfrac{2\tau-2(a+b)}{a+b}$

TABLE 6. PROBABILITIES FOR PRI TYPE RECOGNITION AND PARAMETER ESTIMATION

Type of PRI variation	$\tau < a$	$\tau = \tau_1 = (a+b)/2$	$\tau = 2\tau_1 = a+b$
Fixed PRI	$p_0 = (z-\tau)/z$ $p_2 = 0$	$p_0 = 0$ $p_2 = 0$	$p_0 = 0$ $p_2 = 0$
Two position stagger	$p_0 = \frac{z_1+z_2-2\tau}{z_1+z_2}$ $p_2 = 0$	$p_0 = \frac{z_2-z_1}{2(z_1+z_2)}$ $p_2 = p_0$	$p_0 = 0$ $p_2 = 0$
Uniform	$p_0 = \frac{a+b-2\tau}{a+b}$ $p_2 = 0$	$p_0 = \frac{b-a}{4(a+b)}$ $p_2 = p_0$	$p_0 = \frac{b-a}{3(a+b)}$ $p_2 = p_0$
Sinusoidal	$p_0 = \frac{a+b-2\tau}{a+b}$ $p_2 = 0$	$p_0 = \frac{b-a}{\pi(a+b)}$ $p_2 = p_0$	$p_0 = \frac{4(b-a)}{\pi^2(a+b)}$ $p_2 = p_0$



∗ denotes the first pulse after start of gate interval t

Figure 1. Graphical derivation of state probability formulae

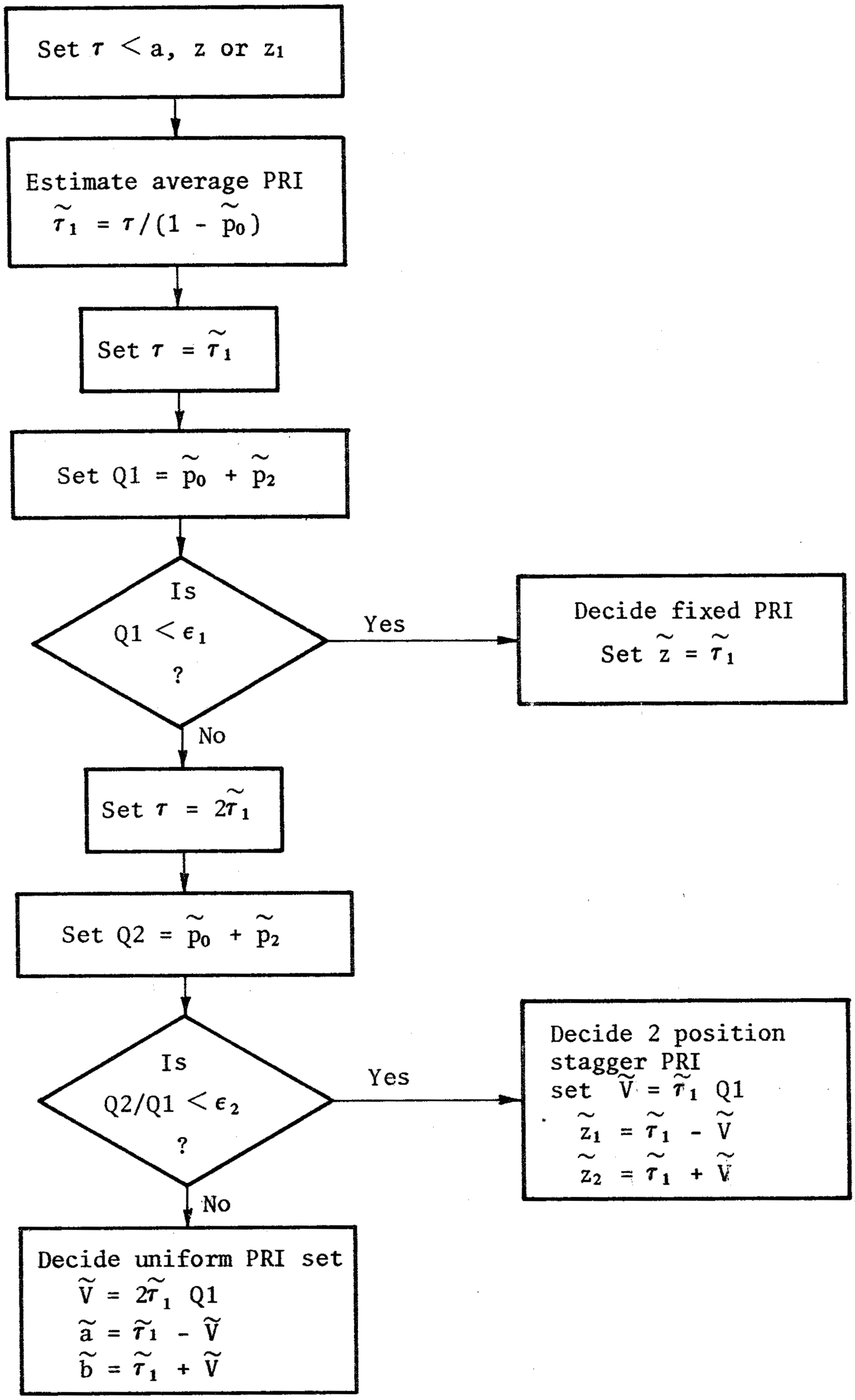


Figure 2. Algorithm for PRI recognition and parameter estimation

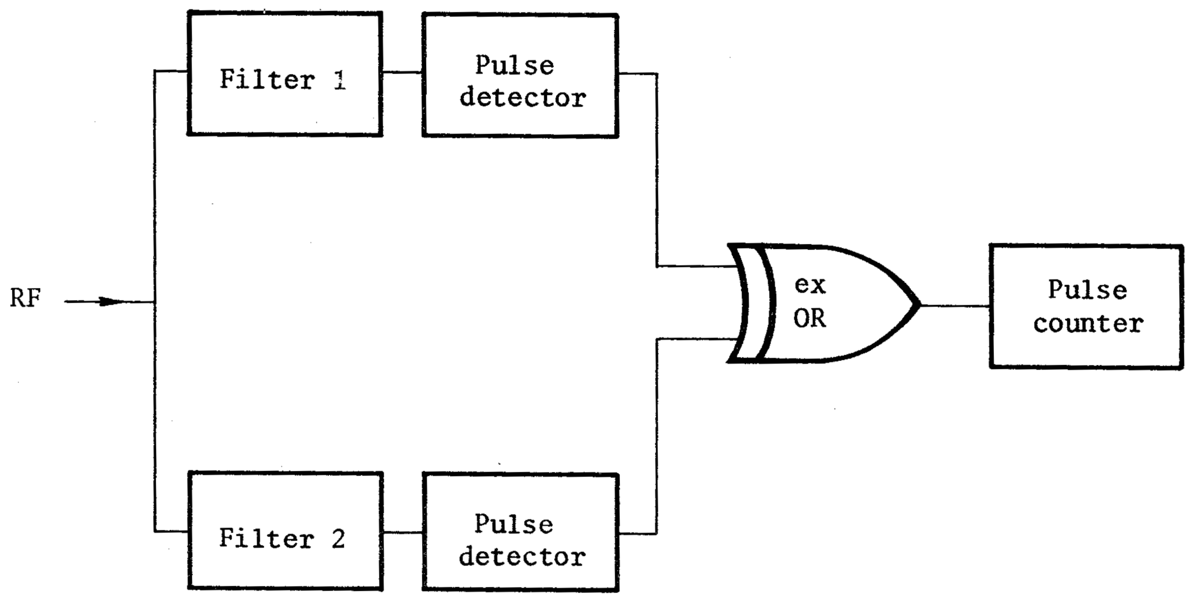
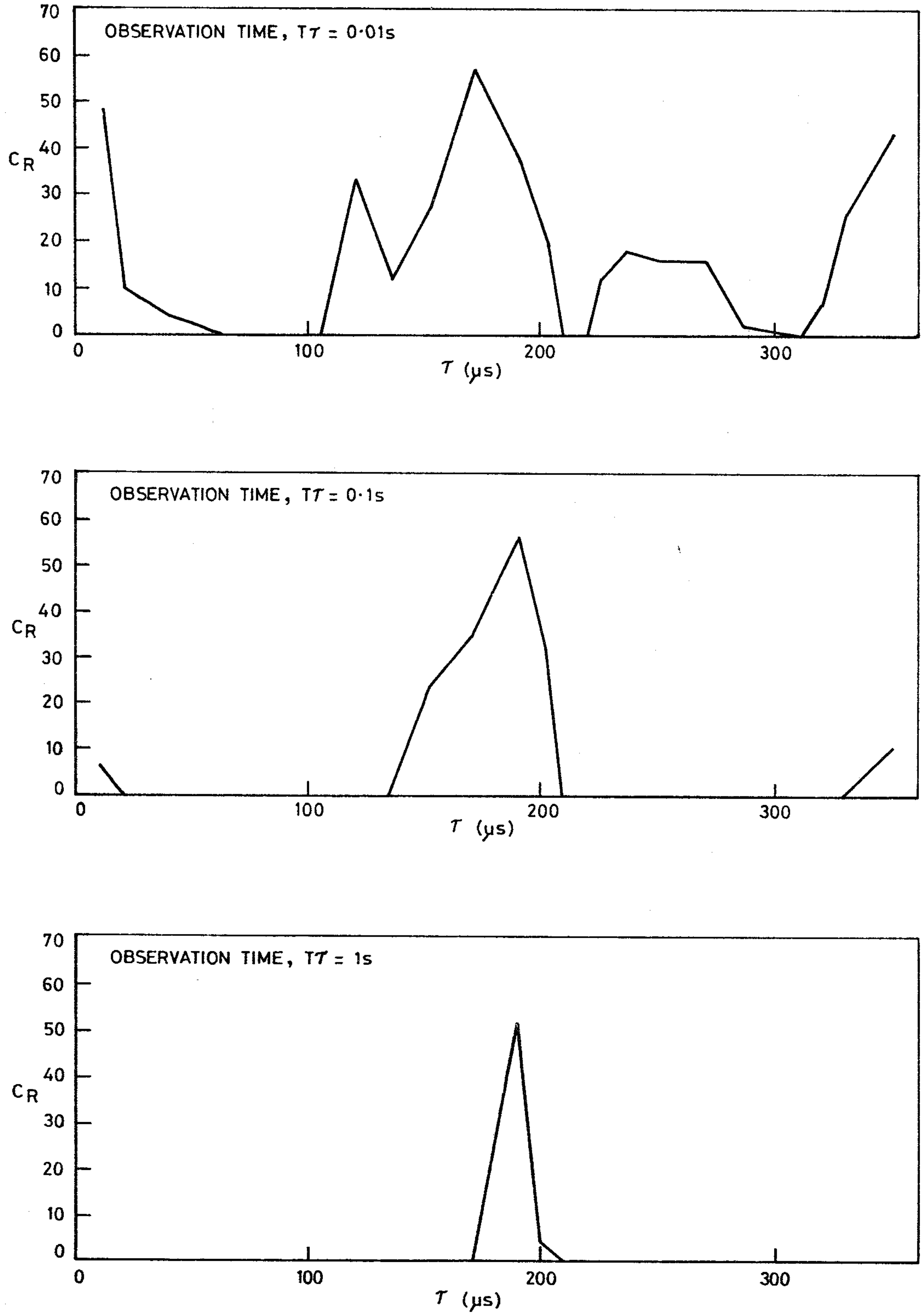
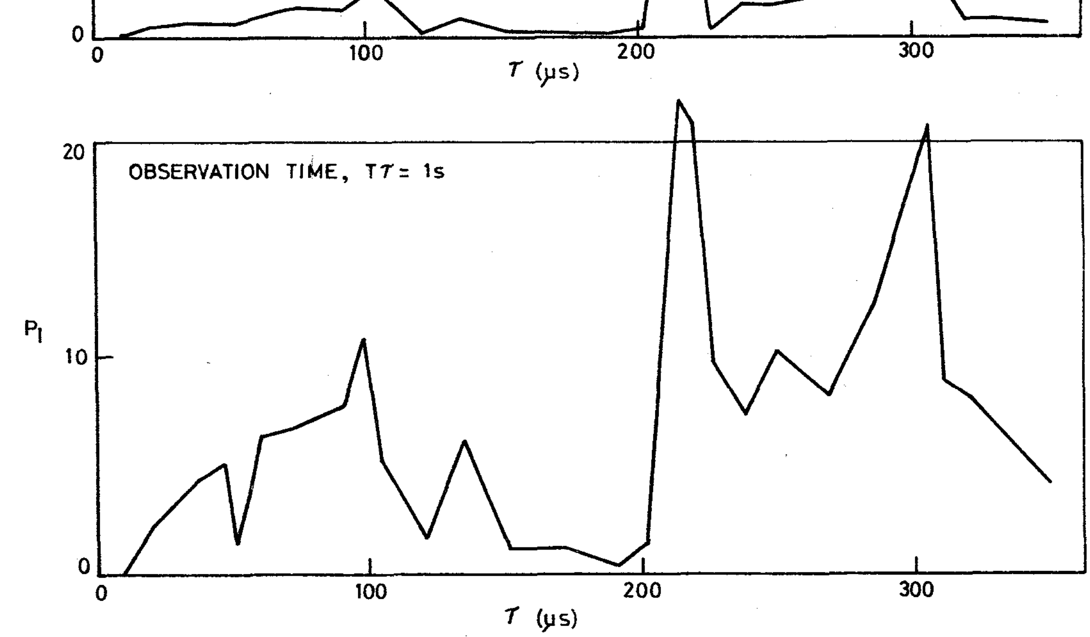


Figure 3. Detecting overlapping pulses



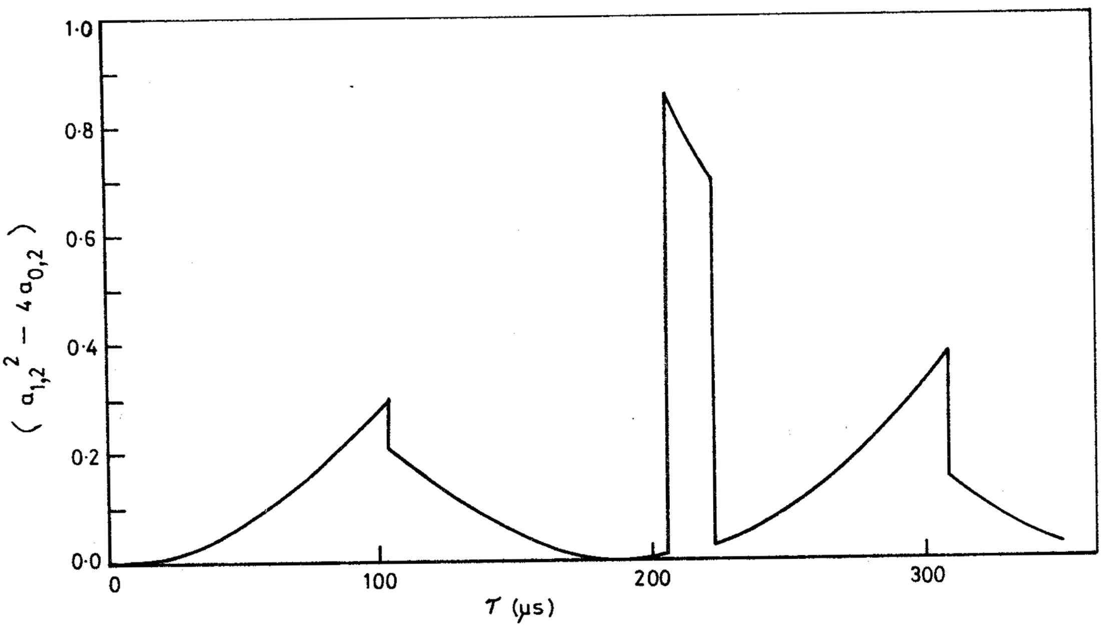
Emitter PRI's are z^1 = 103 μs and z^2 = 223 μs (nominally)

Figure 4. Effect of τ and ($T\tau$) on occurrence of complex zeroes



Emitter PRI's are z^1 = 103 μs and z^2 = 223 μs (nominally)

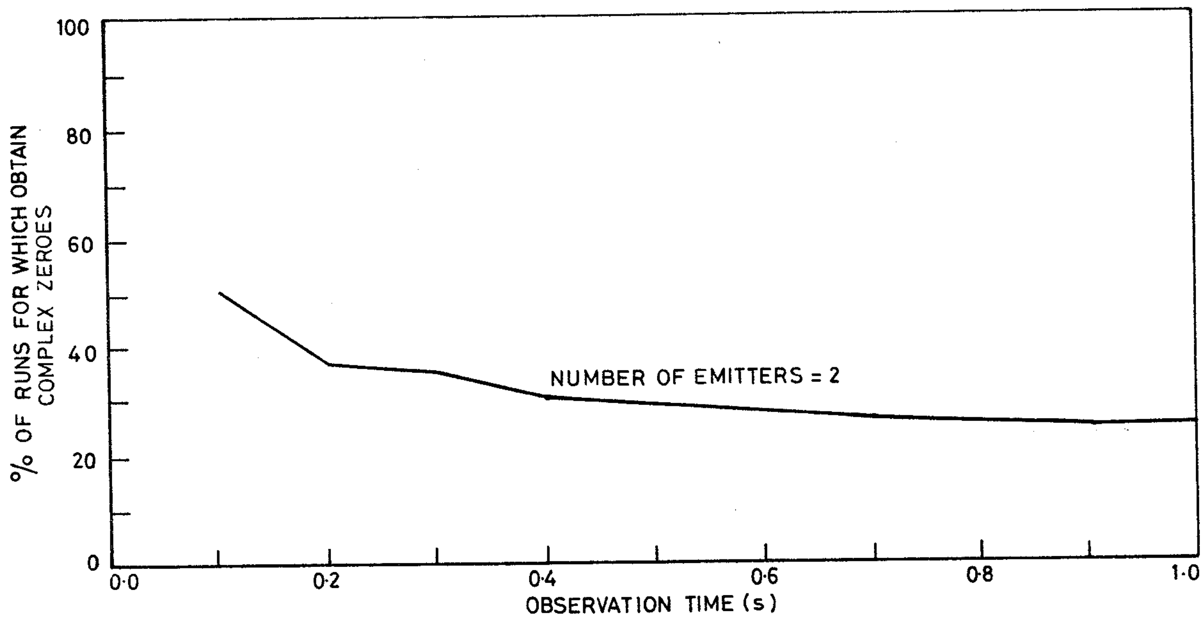
Figure 5. Effect of τ and ($T\tau$) on PRI estimate error



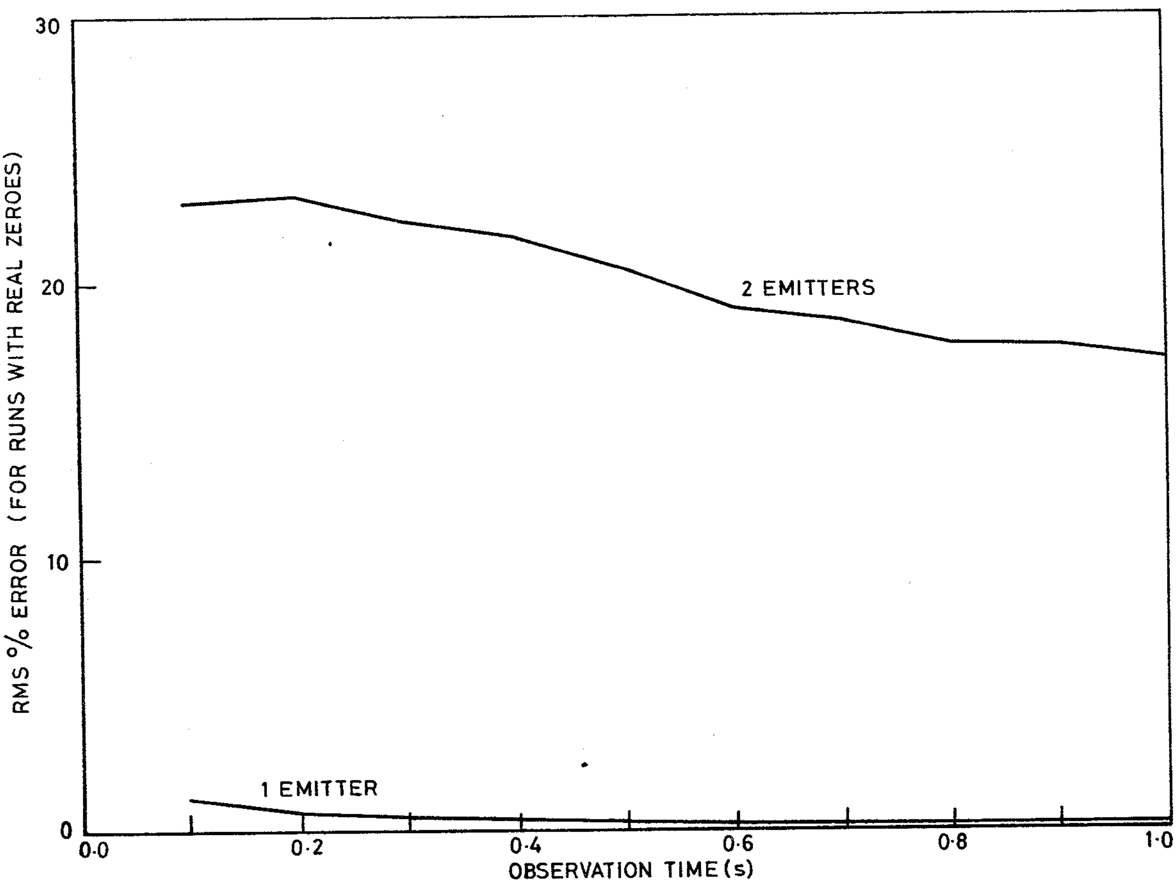
Emitter PRI's are z^1 = 103 μs and z^2 = 223 μs (nominally)

Figure 6. Disciminant of $f_2(\lambda)$ for simulated environment

PRI range is 200 - 5000 μs
Number of runs is 1000 - Gate time, τ = 100 μs



(a)



(b)

Figure 7. Performance of nonadaptive PRI estimates

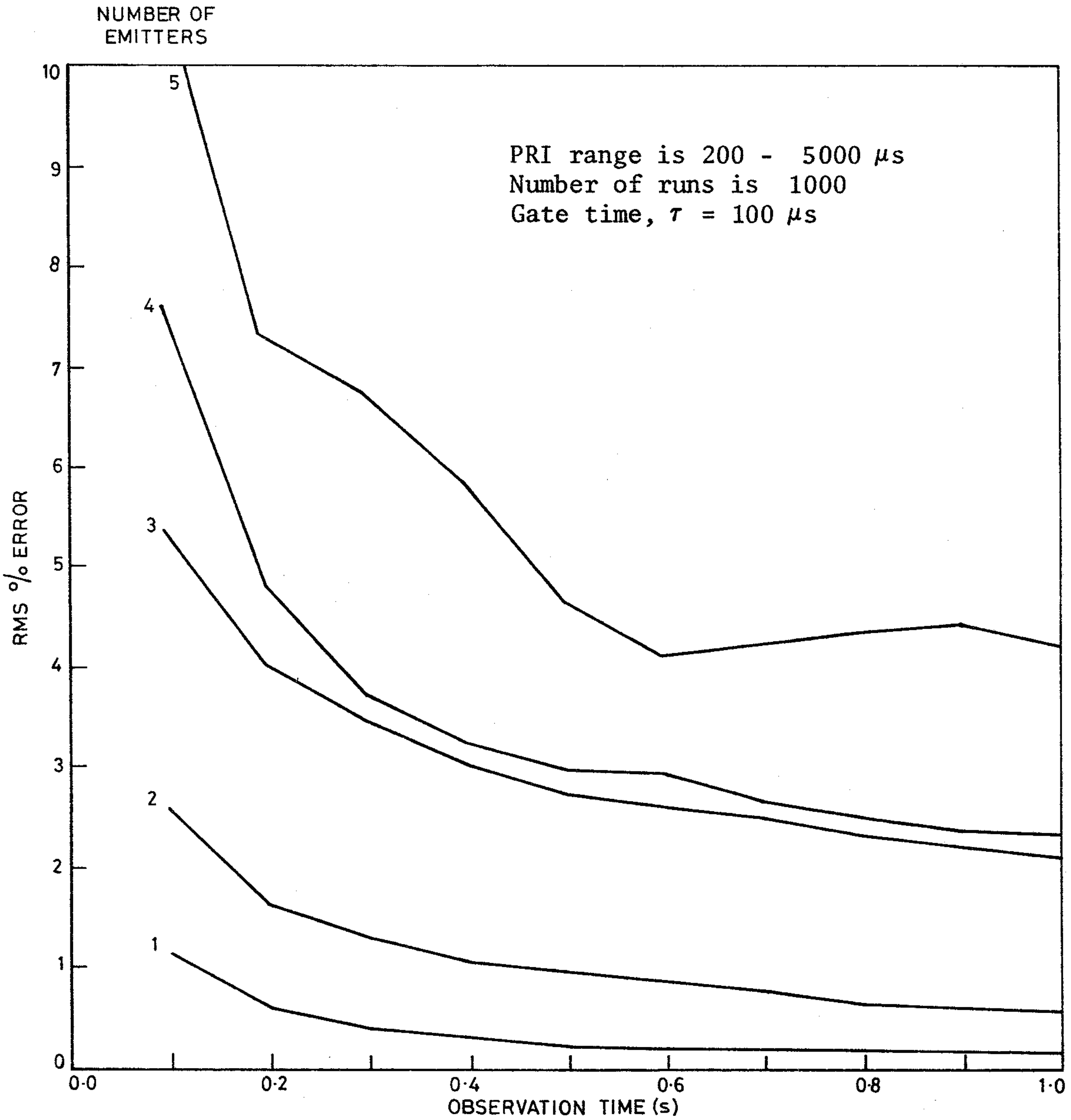


Figure 8. Performance of adaptive estimate of equations (49), (50)

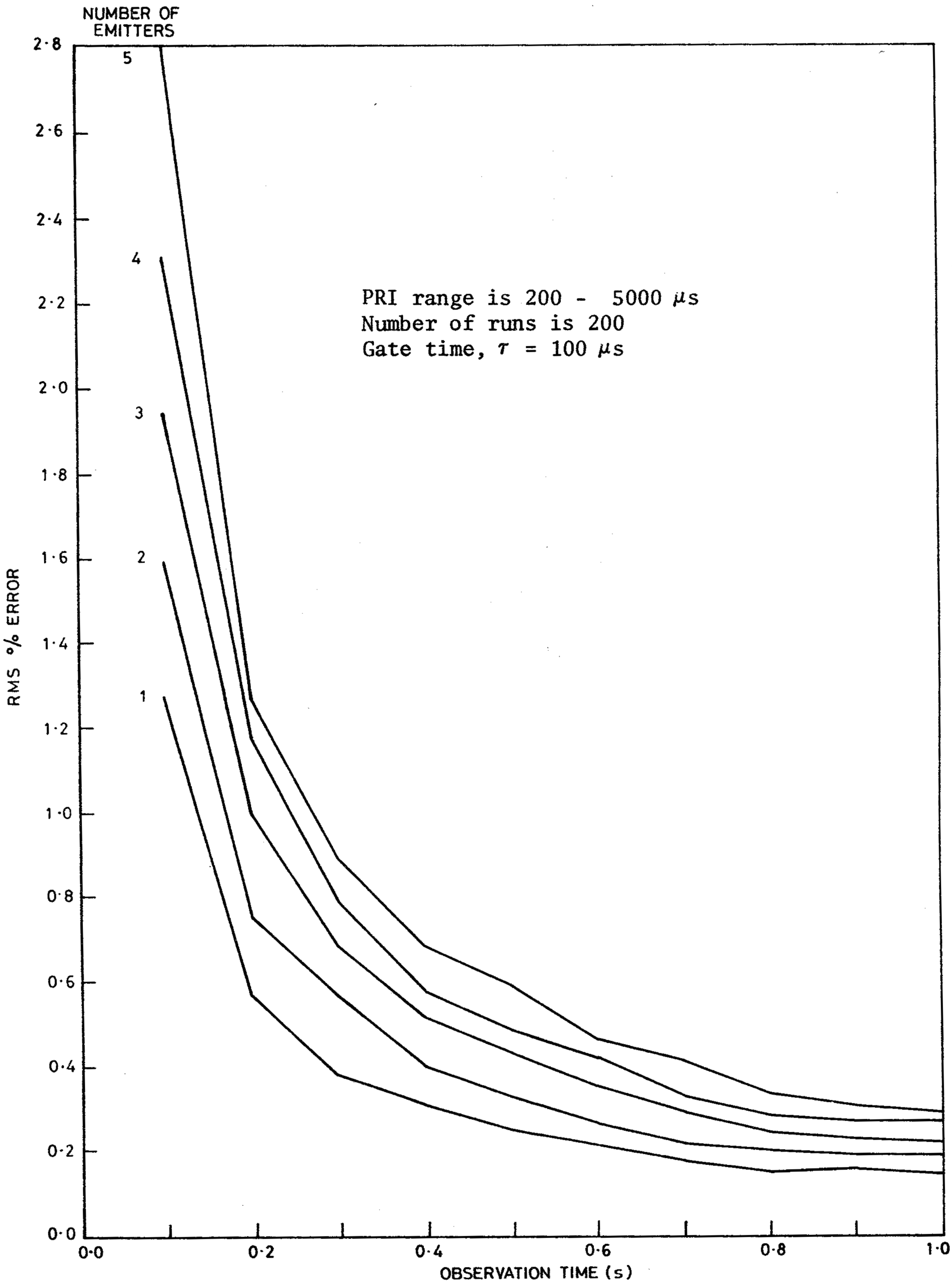
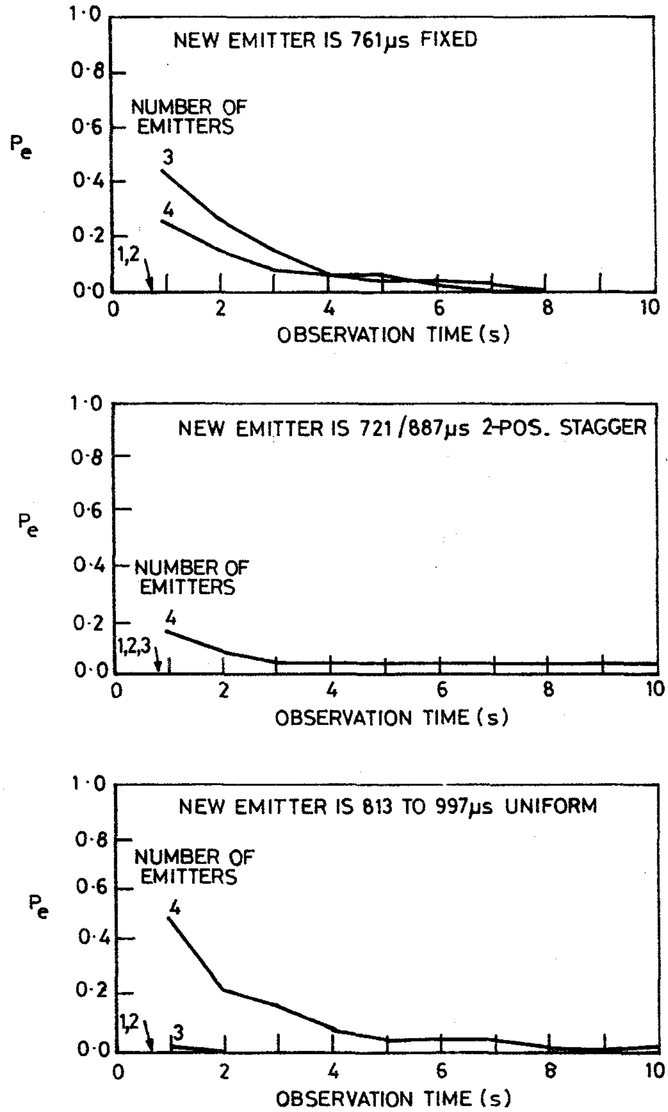


Figure 9. Performance of adaptive estimate of equations (49), (51)



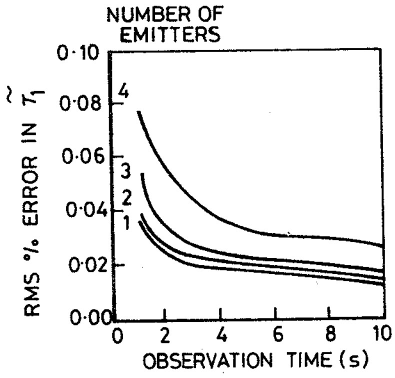
Probability of error, P_e, versus observation time obtained from simulation (100 runs)

Background emitters:

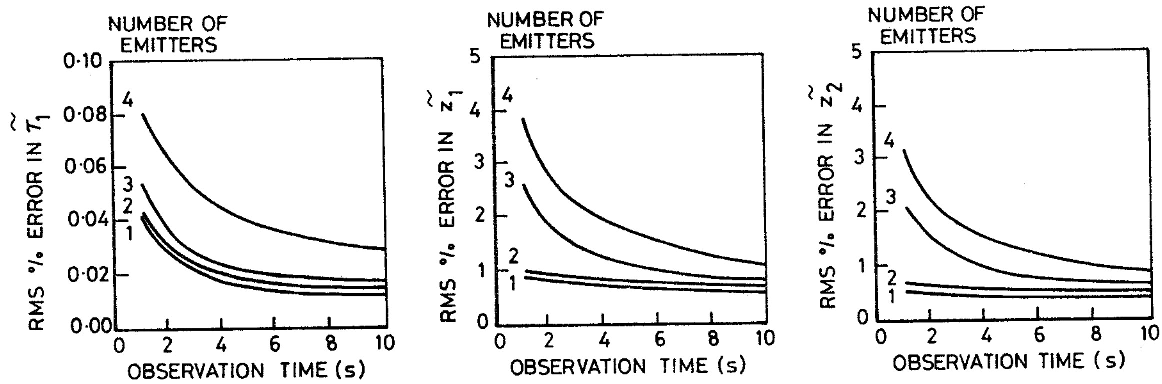
1. 614 μs fixed
2. 401/491 μs 2-pos.stag.
3. 950 to 1050 μs uniform

		NUMBER OF BACKGROUND EMITTERS			
		0	1	2	3
DECISION THRESHOLDS	ϵ_1	0·0001	0·001	0·003	0·015
	ϵ_2	0·1	0·2	0·5	0·9

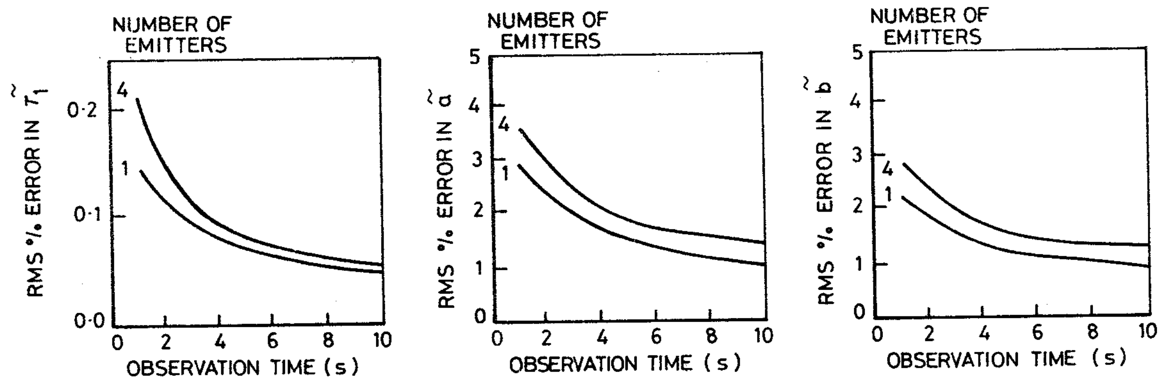
Figure 10. Probability of error for adaptive classification of emitters



New emitter is 761 μs fixed



New emitter is 721/887 μs 2 position stagger



New emitter is 813 to 997 μs uniform

Figure 11. Performance of adaptive parameter estimates

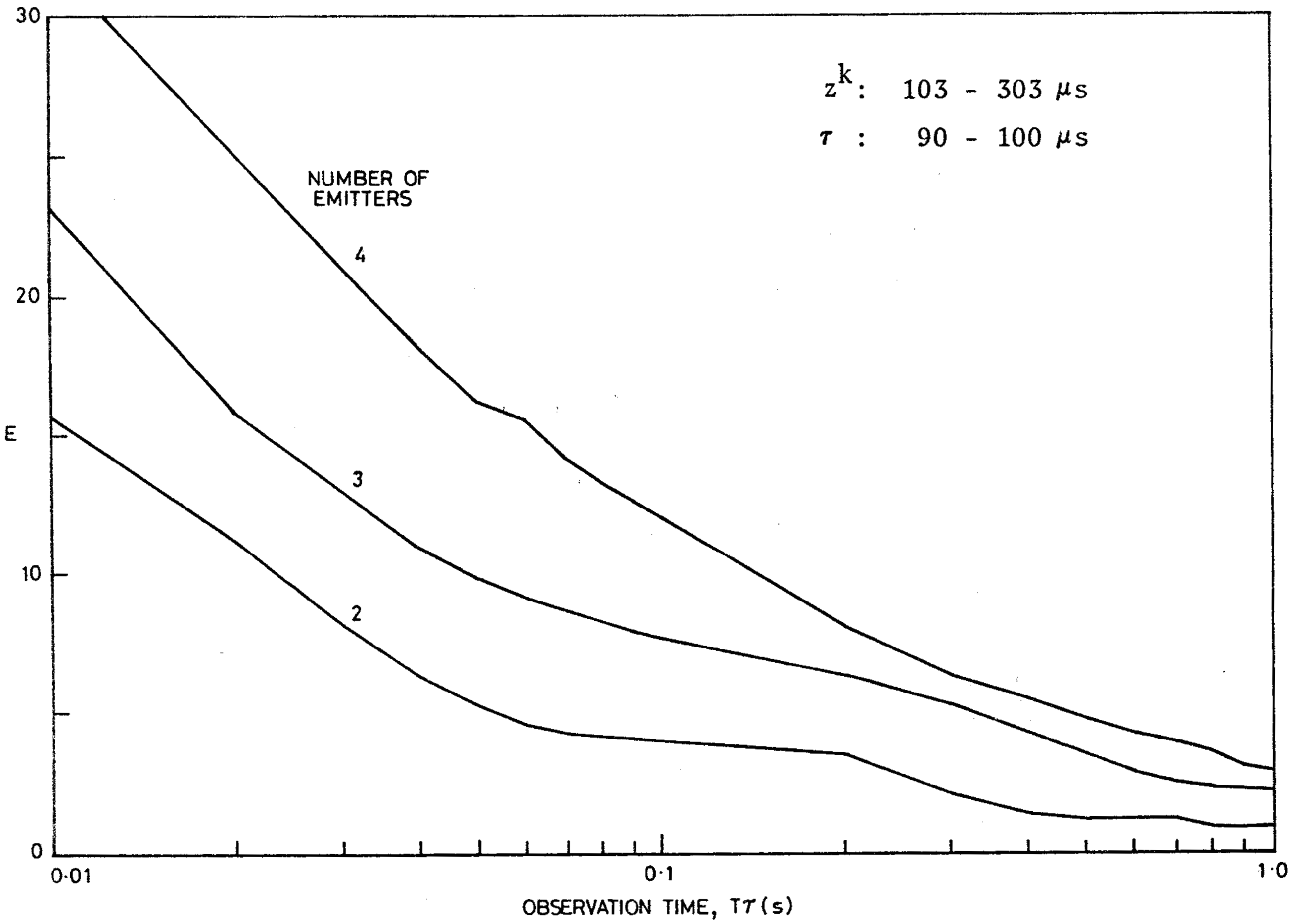


Figure 12. Model verification results

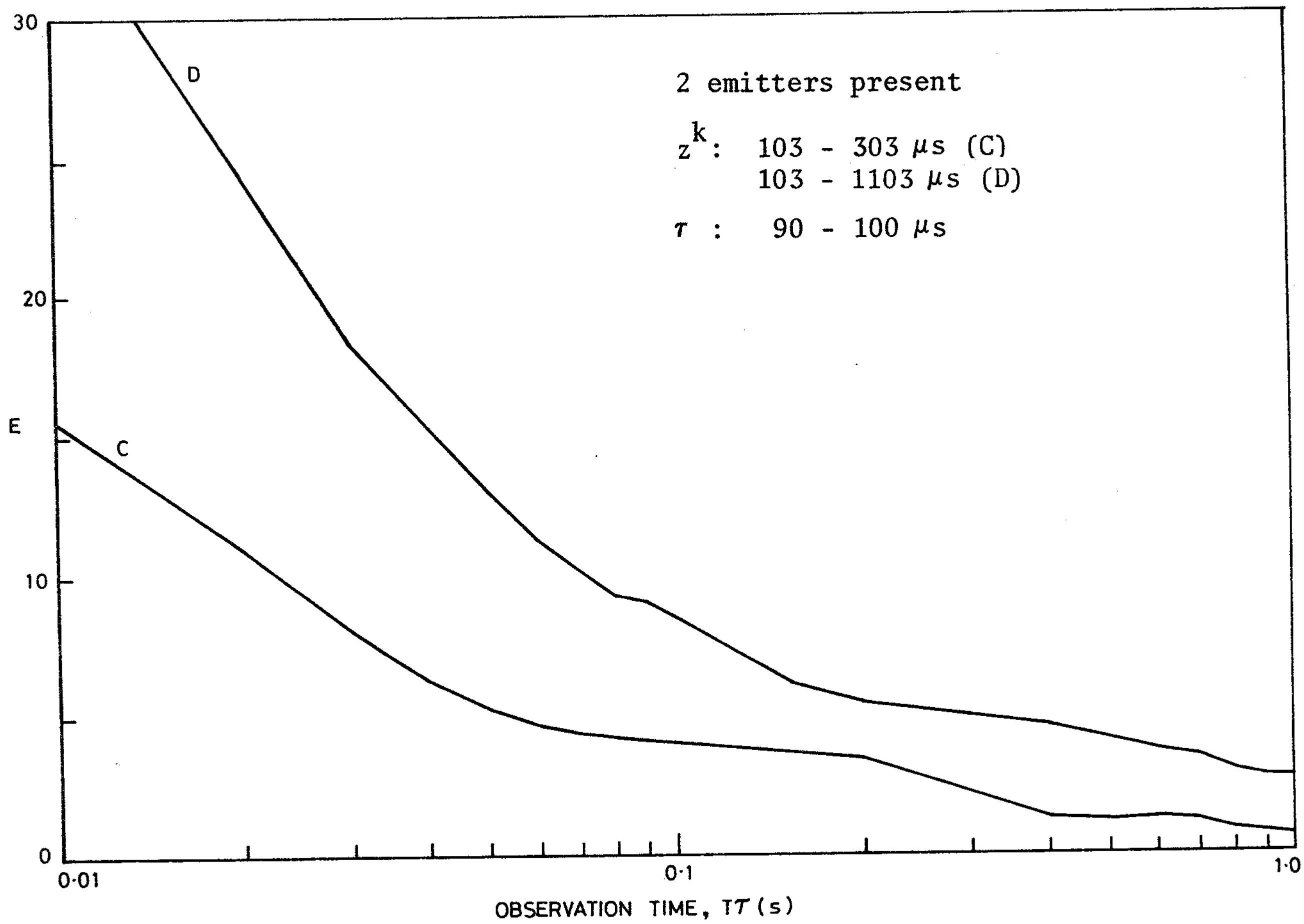


Figure 13. Model verification for greater PRI range

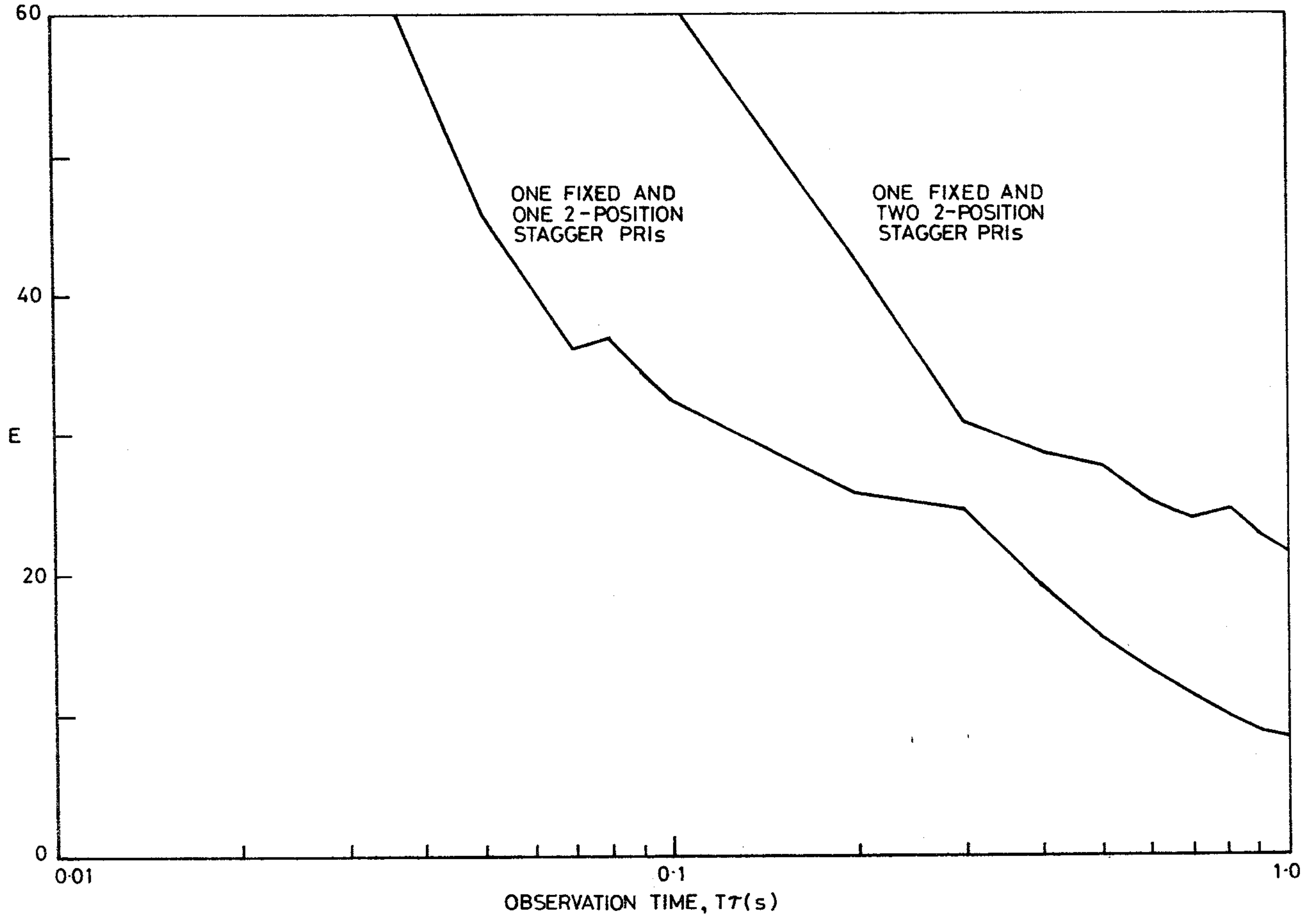


Figure 14. Model verification for two position stagger PRI

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A method for estimating radar emitter pulse repetition interval parameters from intercepted emissions is presented. Adaptive parameter estimates based on certain measurements are derived. The measurements are obtained by a readily implementable observation procedure. The approach can deal with emitters with variable pulse repetition interval in a dense environment. Computer simulation results for the technique are included.

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