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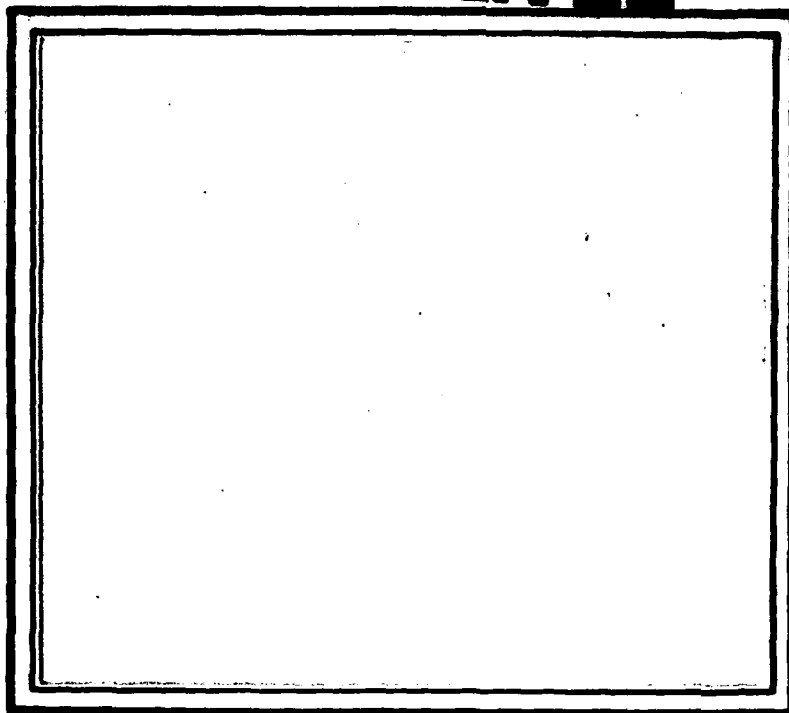
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In this paper we shall indicate some of the current directions in the study of the analytic continuation of solutions to partial differential equations. Our intended audience is not only those mathematicians working in the area of partial differential equations, but also complex analysists interested in seeing how a major topic in the theory of analytic functions can be extended to include certain classes of linear partial differential equations. Our specific aim in this paper is to provide a survey of the general area of the analytic continuation of solutions to partial differential equations and their applications.

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Analytic Continuation of Solutions to
Linear Partial Differential Equations*

by

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I. *Introduction*.

In this paper we shall indicate some of the current directions in the study of the analytic continuation of solutions to partial differential equations. Our intended audience is not only those mathematicians working in the area of partial differential equations, but also complex analysists interested in seeing how a major topic in the theory of analytic functions can be extended to include certain classes of linear partial differential equations. In order for such a generalization to be worthwhile it seems to us that ideally two conditions should be met: The extension should disclose new phenomena that are peculiar to partial differential equations and not characteristic of analytic functions, and the investigation of such new phenomena should illuminate areas of physical application having as their mathematical model the partial differential equations being studied. Our specific aim in this paper is to therefore provide a survey of the general area of the analytic continuation of solutions to partial differential equations and to show how such a study elegantly unites both analytic function theory and the theory of partial differential equations and their applications. For the sake of brevity we shall concentrate on three main topics: the unique continuation of solutions to partial differential equations, reflection principles for equations of elliptic and parabolic type, and, as an example of an area of application, the inverse scattering problem for acoustic waves.

II. *Unique Continuation.*

We first recall that a family of functions is said to have the unique continuation property with respect to a surface S if any two functions of this family which are defined in a neighborhood containing a portion of S in its interior, and agree on one side of the surface S, are in fact equal to each other in the entire neighborhood. Of particular interest in the theory of partial differential equations is the case when the family of functions is the class of all solutions to a given partial differential equation defined in a domain containing the surface S. The unique continuation property in this case clearly depends on both the equation and the surface, and for a linear homogeneous partial differential equation is equivalent to the uniqueness of the solution to Cauchy's problem with data prescribed on S. The classical result in this direction is Holmgren's uniqueness theorem which states that classical solutions of linear partial differential equations with analytic coefficients possess the unique continuation property with respect ot any smooth non-characteristic surface (c.f. [2]). The case of partial differential equations with non-analytic coefficients is only partly solved (c.f. [4] and the references contained in [2]).

Now assume that the linear partial differential equation under consideration is uniformly elliptic with smooth coefficients, and write it as

$$L[u] = 0. \qquad (2.1)$$

A remarkable result established by Lax ([14]), Malgrange ([17]) and Browder ([3]) is that the unique continuation property for solutions of L[u] = 0 is equivalent to a Runge approximation property for the adjoint equation:

Runge Approximation Property: Solutions of L[u] = 0 are said to have the Runge approximation property if, whenever D_1 and D_2 are two bounded simply connected domains, D_1 a subset of D_2, any solution in D_1 can be approximated uniformly on compact subsets of D_1 by a sequence of solutions which can be extended as solutions to D_2.

The theorem of Lax, Malgrange and Browder can now be formulated as follows: solutions of L[u] = 0 have the Runge approximation property if and only if solutions of the adjoint equation have the unique continuation property with respect to any smooth surface S. As a simple consequence of this result we have, using Holmgren's uniqueness theorem, that the harmonic polynomials are complete (with respect to the maximum norm over compact subsets) in the space of solutions of Laplace's equation defined in a bounded simply connected domain. This follows by choosing D_2 to be a sphere in the definition of the Runge approximation property.

It would be desirable to extend the result of Lax, Malgrange and Browder to the case of parabolic equations defined in domains with moving boundaries. Some partial results in this direction can be found in [5]. For example, let $D = \{(x_1 t): s_1(t) < x < s_2(t), 0 \le t \le t_o\}$ where t_o is a positive constant and $s_1(t)$, $s_2(t)$ are

analytic functions of t for $0 \le t \le t_o$. Then it can be shown that
the polynomials

$$h_n(x,t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k} t^k}{(n-2k)!\, k!}$$ (2.2)

are complete in the space of solutions to the heat equation

$$u_{xx} = u_t$$ (2.3)

defined in D.

III. *Reflection Principles*.

Given the possibility of unique continuation, the problem
which naturally arises is how to carry out this continuation
in specific cases. Occasionally one has an explicit integral
representation of a solution in terms of analytic functions with
known singularity manifolds, and in this case the continuation
can often be preformed by using classical results from analytic
function theory (c.f. [10], [18]). However in general one is
given a solution of a partial differential equation defined in
a domain D and satisfying known boundary data f on a portion σ
of the boundary of D, and from a knowledge only of D and f it
is desired to continue the solution across σ. In order to solve
this problem it is necessary to develop reflection principles
for partial differential equations. In particular assume that
a preliminary change of variables has been made such that σ is
a portion of a hyperplane. Then the partial differential equation

is said to have a reflection principle with respect to σ if, under suitable conditions on the boundary data f, it is possible to continue the solution u across σ into the domain D*, where D* is the mirror image of D with respect to σ. The classical example of such behavior is the Schwarz reflection principle for solutions of Laplace's equation that vanish on a portion of a hyperplane, and in this case an explicit reflection rule is provided by the formula

$$u(-x_1, x_2, \ldots, x_n) = -u(x_1, x_2, \ldots x_n) \qquad (3.1)$$

where we have taken σ to be $x_1 = 0$. For example of other classes of partial differential equations with constant coefficients admitting an explicit reflection rule for solutions having vanishing Dirichlet data on a hyperplane see [13].

The case of partial differential equations with constant coefficients and solutions having vanishing Dirichlet data on a portion of the boundary is rather limited and hence efforts have been made to extend such results to linear equations with analytic coefficients and nonhomogeneous boundary data. The somewhat surprising result is that in general this is only possible for partial differential equations in two independent variables. To state the results in the case of two independent variables we assume, according to our convention, that σ is a portion of one of the coordinate axis. For second order elliptic equations in two independent variables with analytic coefficients a reflection principle was established by Lewy ([16]) under the condition that

the boundary data f prescribed on σ is the restriction to σ
of an analytic function f(z) for zεD∪σ∪D*. The case for second
order parabolic equations in two independent variables with
analytic coefficients has been studied by Colton ([5]) with the
result that a reflection principle is valid provided f is analytic
on σ, where now we must insist that σ is the t axis so that it is
not characteristic. Both of these results have been extended to
the case where the solution satisfies a first order linear boundary
condition on σ ([5], [16]).

The problem of when a reflection principle is valid for
partial differential equations in more than two independent
variables is considerably more complicated than the case of
equations in two independent variables. In the case of equations
with constant coefficients and σ a hyperplane we refer the reader
to the previously cited work of John. Reflection principles are
also possible for solutions of the Laplace or Helmholtz equation
defined in a ball and vanishing on a portion of the spherical
boundary (The case of Laplace's equation is due to Schwarz and
is classical, whereas the reflection principle for the Helmholtz
equation has only recently been obtained by Colton ([5])). On
the other hand, Lewy has given an example of a solution of
Laplace's equation satisfying a first order linear homogeneous
boundary condition with constant coefficients along a hyperplane
such that the solution cannot be continued across this hyperplane
into the mirror image of its original domain of definition ([16]).
The problem of when solutions of higher dimensional elliptic
equations can be reflected across analytic boundaries was taken

up by Garabedian ([9]) who showed that the domain of dependence
associated with a solution of a n-dimensional elliptic equation
at a point on one side of an analytic surface is in general
a whole n-dimensional ball on the other side. Only in exceptional
circumstances does some kind of degeneracy occur which causes
the domain of dependence to collapse onto a lower dimensional
subset, thus allowing a continuation into a larger region than
that afforded in general. Such is the situation for example in
the case of the Schwarz reflection principle for harmonic functions
defined in a ball or half space (where the domain of dependence is
a point) and Colton's reflection principle for solutions of the
Helmholtz equation defined in a ball (where the domain of
dependence is a one dimensional line segment). Such a collapsing
of the domain of dependence can be viewed as a Huygen's principle
for reflection, analogous to the classical Huygen's principle for
hyperbolic equations.

The analysis discussed above still leaves open the possibility
of obtaining a reflection principle for parabolic equations in
two space variables with analytic coefficients. In the case of
analytic solutions of such equations having analytic Dirichlet
data on a portion of a plane boundary, a reflection principle has
been established by Colton ([7]) and the possibility of obtaining
explicit reflection formulas has been investigated by Hill([11]).
In this case the domain of dependence is a one dimensional line
segment, in conformity with Huygen's principle for reflection.
An open problem is to consider the case of first order boundary
conditions and to remove the assumption of analyticity of the

solutions. This last problem is intimately connected with the regularity of solutions to initial value problems for equations having the operator

$$L[u] = u_{xy} - u_t \qquad (3.2)$$

as principal part ([7]).

IV. *The Inverse Scattering Problem.*

In recent years there has been a rapidly growing interest in various classes of <u>inverse problems</u>, in particular those in which from measured data it is desired to either determine the domain of definition of the solution or, given the domain, to determine the boundary conditions satisfied by the solution. Typical examples of this type of problem are the backwards heat equation, the inverse Stefan problem, and the inverse scattering problem (c.f. [5]). A broad class of these problems (in particular the ones just listed) can be characterized as problems in the analytic continuation of solutions to partial differential equations, but where now the solutions are not known exactly but only in some approximate sense. An interesting characteristic of such problems is that they are in general improperly posed, i.e. the solution either does not exist, is not unique, or does not depend continuously on the initial data. Due to what now appears to be an inexplicable bias, the area of improperly posed problems in applied mathematics was left almost untouched by mathematicians until the late 1950's when the beginning of a theory was initiated. Thus in 1961 Courant

wrote ([8]): "The stipulations about existence, uniqueness and stability of solutions dominate classical mathematical physics. They are deeply inherent in the ideal of a unique, complete and stable determination of physical events by appropriate conditions at the boundaries, at infinity, at time t=0, or in the past. Laplace's vision of the possibility of calculating the whole future of the physical world from complete data of the present state is an extreme expression of this attitude. However, this rational ideal of casual-mathematical determination was gradually eroded by confrontation with physical reality. Nonlinear phenomena, quantum theory, and the advent of powerful numerical methods have shown that "properly posed" problems are by far not the only ones which appropriately reflect real phenomena. So far, unfortunately, little mathematical progress has been made in the important task of solving or even identifying and formulating such problems which are not "properly posed" but still are important and motivated by realistic situations". Since the time of these observations by Courant, there has been a virtual explosion of interest in improperly posed problems, and we refer the reader to the recent conferences by Anger ([1]) and Nashed ([20]) for current developments in this area. However, as the interest has risen, the problems have multiplied, and many of the most important improperly posed problems arising in applications are still waiting for a solution that is suitable for practitioners.

In order to investigate a problem that is improperly posed, we must answer two basic questions: (1) What do we mean by a solution? and (2) How do we construct this solution? The answers to these questions are by no means trivial. For example, as initially posed a solution may not even exist in the classical sense, or if it does exist, may not be defined in a large enough domain to be of practical use (This is the place where analytic continuation often comes into play). A classic example of such a situation arises in the study of the inverse scattering problem, a subject which has already been the topic of numerous research papers and monographs. Here we restrict ourselves to the inverse scattering problem for acoustic waves, i.e. we consider the scattering of a time harmonic plane wave by a rigid bounded obstacle, and from a knowledge of the asymptotic behavior of the scattered wave we want to determine either the location of the so-called "equivalent sources" generating the field, or the shape of the scattering obstacle.

We first consider the problem of the location of the equivalent sources, i.e. we regard the scattered field u as being generated by a set of sources all of which are contained within the true scattering obstacle. These sources are not in general unique, although they are confined to some finite region of space, and one of the fundamental problems in scattering theory is to determine the extent and location of this region (c.f. [5], [19], [22]). The sharpest results in this direction to date are due to Colton ([5]) and are concerned with the case when the scattered field is axially symmetric. To describe these results,

assume the wave number is normalized to be one and that the
scattered field is axially symmetric, i.e. in cylindrical
coordinates (r,z,ϕ), u is independent of ϕ. Then we have that
u behaves asymptotically like

$$u \sim \frac{e^{iR}}{R} f(\cos\theta); \quad R \to \infty \tag{4.1}$$

where $z=R\cos\theta$, $r=R\sin\theta$, the time dependency has been factored
out, and f is the <u>far field pattern</u>. Given f, our problem is to
determine the location of the sources which generate u. To this
end we note that f is an entire function of $\cos\theta$ and expand f
in a Legendre series

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta). \tag{4.2}$$

If we now define the analytic function h by

$$h(\zeta) = \sum_{n=0}^{\infty} a_n (2i\zeta)^n \tag{4.3}$$

it can be shown that h is an entire function of order one and
type C where C is the radius of the smallest ball containing
the scattering obstacle (and hence the equivalent sources). Now
let G be the <u>indicator diagram</u> of h and G* its conjugate (For
information on the indicator diagram and its role in the theory
of entire functions, see Levin ([15])). Consider the complex ζ
plane as superimposed on the (R,θ) plane. The theorem of Colton
states that the equivalent sources are contained in the rotation
of G∪G* about the axis of symmetry. The proof of this result

is based on using the reflection principle for the Helmholtz
equation to analytically continue u across the sphere of radius
C; for details see [5].

We now consider the problem of determining the shape of
the scattering obstacle from <u>measurements</u> of the far field
pattern. We note that the analysis above is not applicable
to this problem since it is not possible to detect from
experimental data whether or not the function being measured is
entire. Furthermore, the problem is clearly improperly posed
since if the function chosen to approximate the far field
pattern is not entire of a certain order and type, no solution
to the problem can exist. Our problem is basically one of
analytic continuation from infinity where the data at infinity
is only known in an approximate sense. Hence we can only hope
to reconstruct the shape of the scattering obstacle in some
approximate manner. It is worthwhile to note this point that
the word "approximation" has little meaning unless some acknow-
ledgment is given to the concept of error estimates.

To fix our ideas, we consider the case when the scattering
obstacle is an infinite cylinder having a bounded simply connected
cross section D with smooth boundary ∂D. In this case the problem
is two dimensional and it can be shown that the scattered field
u behaves asymptotically like

$$u \sim \frac{e^{ikR}}{\sqrt{R}} \, F(\theta;k) \quad ; \quad R \to \infty \qquad (4.4)$$

where (R,θ) are polar coordinates, k is the wave number, and F is the far field pattern (We have again factored out the time dependency). Given F for $-\pi \leq \theta \leq \pi$ and small values of the wave number k, our problem is to determine ∂D, i.e. the shape of D. To this end we expand F in a Fourier series

$$F(\theta,k) = \sum_{-\infty}^{\infty} a_n(k)e^{in\theta} \qquad (4.5)$$

and let f be the (unique) conformal mapping taking the exterior of the unit disk in the w plane onto the exterior of D in the z plane such that for $|w|>1$, f has the Laurent expansion

$$f(w) = \frac{w}{a} + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \ldots \quad ; \ a>0 \qquad (4.6)$$

where a^{-1} is the <u>mapping radius</u> of D. Then it can be shown ([5]) that form a knowledge of the behavior of the Fourier coefficients $a_n(k)$, n=0, 1,2,...,N, for k small, it is possible to determine the area of D and the Laurent coefficients $b_0, b_1, \ldots, b_n$ of the conformal mapping f, module the mapping radius a^{-1}. To determine the mapping radius it is necessary to move the transmitter and measure the far field pattern a second time. Having done this we can now construct an approximation to f defined by

$$f_n(w) = \frac{w}{a} + b_0 + \frac{b_1}{w} + \ldots + \frac{b_n}{w^n} , \qquad (4.7)$$

and by evaluating f_n on $|w|=1$ we have an approximation to ∂D.

We now need to estimate the error made in approximating f by f_n on $|w|=1$. To this end we define the mean square error by

$$E(f-f_N) = \int_{-\pi}^{\pi} |f(e^{i\theta})-f_N(e^{i\theta})|^2 d\theta$$

$$= \sum_{n=N+1}^{\infty} |b_n|^2. \tag{4.8}$$

Then from the Area Theorem in univalent function theory ([21]) we have

$$E(f-f_N) \leq \frac{1}{N+1} \left[\frac{1}{a^2} - \frac{A}{\pi}\right] \tag{4.9}$$

where A is the (measured) area of D. It is of course desirable to improve this error estimate, and one approach for doing this is to utilize "a priori" information one has on the shape of D. The problem of course is how to incorporate this a priori information into the mathematical model. As an example of how this can be done consider the case when it is known a priori that D is convex. Then from known coefficient estimates for univalent functions ([21], p. 50) we have that

$$|b_n| \leq \frac{2}{an(n+1)} \tag{4.10}$$

and hence from (4.8) and a short calculation using the integral test for infinite series we have

$$E(f-f_N) \leq \frac{4(N+2)^2}{3a^2(N+1)^5} \tag{4.11}$$

We note that (4.11) is a considerable improvement on (4.9), except in the case when D is a small perturbation of a disk and we have $\pi a^{-2} \approx A$.

A complete and satisfactory treatment of the inverse scattering problem for acoustic waves still lies in the (hopefully not too distant!) future. As steps in this direction it would be worthwhile to investigate the inverse scattering problem for intermediate values of the wave number (i.e. away from the low frequency and high frequency limits) as well as the full three dimensional inverse scattering problem. Partial progress in these directions can be found in [12] and [6] respectively.

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