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GENERALIZED EQUATIONS AND THEIR
SOLUTIONS, PART II: APPLICATIONS
TO NONLINEAR PROGRAMMING

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ABSTRACT

We prove that if the second-order sufficient condition and constraint

regularity hold at a local minimizer of a nonlinear programming problem, then

for sufficiently smooth perturbations of the constraints and objective

function the set of local stationary points is nonempty and continuous;

further, if certain polyhedrality assumptions hold (as is usually the case in

applications) then the local minimizers, the stationary points and the

multipliers all obey a type of Lipschitz condition. Through the use of

generalized equations, these results are obtained with a minimum of notational

complexity.

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Key Words: Nonlinear programming, Generalized equations

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SIGNIFICANCE AND EXPLANATION

In practical problems from logistics, structural design, chemical

engineering and other areas, it is often necessary to maximize or minimize a

nonlinear function of several variables subject to nonlinear equation and/or

inequality constraints. Since problem data or functional forms may not be

known exactly, it is of interest to know whether a local solution to such a

problem will persist under small changes in the data or in the problem

functions. In this paper, we show that if two fairly well known "niceness"

conditions are satisfied at a local solution of the unperturbed problem, then

for small perturbations the perturbed problem will have one or more solutions

near the original one. Moreover, these solutions will often display good

continuity properties.

GENERALIZED EQUATIONS AND THEIR SOLUTIONS, PART II:

APPLICATIONS TO NONLINEAR PROGRAMMING

Stephen M. Robinson*

Introduction. This paper deals with the stability of solutions and multipliers of
nonlinear programming problems when the data of the problems are subjected to small
perturbations. The problem with which we shall deal may be formulated by introducing
functions f and g from an open convex set $\Omega \subset R^n$ to R and R^m respectively, a
closed convex set $C \subset R^n$ and a closed convex cone $Q \subset R^m$. The problem of interest is
then

$$\text{minimize} \quad f(x)$$
$$\text{subject to} \quad g(x) \in Q^\circ \, , \tag{1.1}$$
$$x \in C \, ,$$

where Q° denotes the polar cone of Q:

$$Q^\circ := \{y \in R^m \mid \langle q,y \rangle \leq 0 \text{ for each } q \in Q\} \, .$$

If we assume that f and g are Frechet differentiable and that certain regularity
conditions are satisfied, it can be shown [17] that with each local minimizer x of (1.1)
there are associated one or more multipliers $u \in R^m$ such that (x,u) satisfy the
necessary optimality conditions

$$0 \in \mathcal{L}_x(x,u) + \partial\psi_C(x)$$
$$0 \in -\mathcal{L}_u(x,u) + \partial\psi_Q(x) \, , \tag{1.2}$$

where $\mathcal{L}(x,u) := f(x) + \langle u,g(x) \rangle$ is the standard Lagrangian, and where the subscript x

or u denotes partial differentiation with respect to that variable. The notation $\partial \psi$

denotes the <u>normal cone</u> at a point: thus,

$$\partial \psi_C(x) := \begin{cases} \{y \in R^n | \langle y, c - x \rangle \leqq 0 \text{ for each } c \in C\} & \text{if } x \in C \\ \phi & \text{if } x \notin C ; \end{cases}$$

for a convex cone like Ω, the description of $\partial \psi$ is even simpler:

$$\partial \psi_\Omega(u) = \begin{cases} \{v \in \Omega^\circ | \langle v, u \rangle = 0\} & \text{if } u \in \Omega \\ \phi & \text{if } u \notin \Omega . \end{cases}$$

Any point x which, with some u, satisfies (1.2) is called a <u>stationary point</u> of (1.1);

there may well exist stationary points which are not local minimizers.

What we shall show here is that if x_0 is a local minimizer of ('1) at which f

and g are twice continuously Frechet differentiable and at which certain regularity

conditions hold (specifically, the second-order sufficient condition and constraint

regularity), and if f and g are smoothly perturbed, then the set of stationary points

near x_0, regarded as a multifunction (multivalued function) of the perturbations, is

(nonempty and) continuous at x_0. Further, we show that the set of multipliers is upper

semicontinuous, where these continuity properties are as defined by Berge [1]. These

results are proved in Section 3, after a review in Section 2 of the necessary regularity

conditions.

In Section 4, we show that if C and Ω are assumed to be polyhedral (as will be the

case in most applications), then even stronger results can be established: the distance

from a stationary point of the perturbed problem to x_0, or from an associated multiplier

to the set of multipliers associated with x_0, obeys a kind of Lipschitz condition.

Examples are given, both in Section 3 and in Section 4, to show that certain stronger

statements, although plausible, are not generally true.

In several parts of this paper we use as a device for simplifying and motivating

results the concept of a <u>generalized equation</u>. These objects, introduced in Part I of this

paper [19], are relations of the form

$$0 \in F(z) + T(z) ,$$

where $F : R^k \to R^k$ and T is a closed multifunction from R^k to itself (often a normal-cone operator). For example, if we rewrite (1.2) in terms of f and g, and use the fact that $\partial \psi_{C \times Q}(x,u) = \partial \psi_C(x) \times \partial \psi_Q(u)$, we obtain the generalized equation

$$0 \in \begin{bmatrix} f'(x) + g'(x)^*u \\ -g(x) \end{bmatrix} + \partial \psi_{C \times Q}(x,u) , \qquad (1.3)$$

and this relation will be used in several ways in our analysis.

Many papers have already been written about various stability questions connected with (1.1), often in the special case in which $C = R^n$ and $Q = R_+^k \times R^\ell$ (where R_+^k is the non-negative orthant in R^k). This special case formulates the standard nonlinear programming problem

$$
\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0 \quad (i = 1,\ldots,k) \\
& g_i(x) = 0 \quad (i = k + 1,\ldots,k + \ell) .
\end{aligned}
\qquad (1.4)
$$

We shall not attempt to review all of these papers here; rather, we mention only a few which illustrate the different types of assumptions that have been imposed.

Fiacco and McCormick [3, §§2.4, 5.2] formulated a basic technique for analyzing (1.4) under the assumptions of strict complementary slackness, linear independence of the gradients of the binding constraints, and the second-order sufficient condition. This technique was refined, to deal with general perturbations, in [15] and in [4]; the basic tool, in all three cases, was the standard implicit-function theorem.

If the assumption of strict complementary slackness is dropped, then the standard implicit-function theorem can no longer be used. However, if the linear independence assumption and a somewhat strengthened form of the second-order sufficient condition are retained, then one can still show that the stationary point and associated multiplier are (locally) single-valued functions of the perturbations, and that they are locally Lipschitzian if appropriate continuity assumptions are made on the problem functions. This is shown for the problem (1.4) in [20], as a by-product of a general implicit-function

theorem for generalized equations. A similar result, without the Lipschitz continuity, was established by Kojima [10, Th. 6.4] as part of his investigation of "strongly stable" solutions of (1.4).

If one weakens the hypotheses still further by dropping the assumption of linear independence, then the appropriate condition to assume in its place is regularity of the constraints [18] (see Section 2; in the case of (1.4) this is the constraint qualification of Mangasarian and Fromovitz [12]). With this assumption, together with the strengthened form of the second-order sufficient condition previously mentioned, Kojima [10, Th. 7.2] has shown the existence of a locally unique stationary point which is continuous under small perturbations. Under similar hypotheses, Levitin [11, Th. 4] stated that if the minimizer exists for small perturbations then various properties, including Lipschitz continuity, followed. However, in Section 4 below we give an example which appears to satisfy Levitin's hypotheses but for which local Lipschitz continuity does not hold. Finally, Sargent [23] has studied the existence and continuity of local minimizers under constraint regularity; however, his methods are quite different from those of the other papers mentioned here, and it appears that some of the results in [23] may not be completely correct.

We begin our analysis, in the next section, by reviewing a generalized form of the well known second-order sufficient condition, and exploring some of its properties.

2. A review of the second-order sufficient condition. The second-order sufficient condition is a very well known regularity condition in nonlinear programming. It is discussed for the standard nonlinear programming problem (1.4) in [3, §2.3]; versions adapted to problems in more general spaces are given by, e.g., Guignard [6], Maurer and Zowe [13] and Maurer [14].

In this section we exhibit a form of this condition suitable for the problem (1.1). Although we have not seen this particular form in the literature, it is not likely to be very surprising to anyone familiar with the field. What may be less familiar, however, is

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its motivation in terms of generalized equations and its use to prove isolation results like Theorem 2.3.

To develop a second-order sufficient condition for (1.1), we return to the generalized equation (1.3) which formulates the necessary optimality conditions. We have shown in [19,20] that important aspects of the behavior of a generalized equation are captured in its linearization about a given point. To make use of this linearization in the present case, we assume that (x_0,u_0) is a point satisfying (1.2); the linearized form of (1.2) (or equivalently, of (1.3)) at (x_0,u_0) is then

$$
0 \in \begin{bmatrix} \mathcal{L}_x(x_0,u_0) \\ \\ -\mathcal{L}_u(x_0,u_0) \end{bmatrix} + \begin{bmatrix} \mathcal{L}_{xx}(x_0,u_0) & \mathcal{L}_{xu}(x_0,u_0) \\ \\ -\mathcal{L}_{ux}(x_0,u_0) & -\mathcal{L}_{uu}(x_0,u_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ \\ u - u_0 \end{bmatrix} + \partial\psi_{C \times Q}(x,u) .
$$

Simplifying this expression, we obtain

$$
0 \in \begin{bmatrix} \mathcal{L}_x(x_0,u_0) \\ \\ -g(x_0) \end{bmatrix} + \begin{bmatrix} \mathcal{L}_{xx}(x_0,u_0) & g'(x_0)^T \\ \\ -g'(x_0) & 0 \end{bmatrix} \begin{bmatrix} x - x_0 \\ \\ u - u_0 \end{bmatrix} + \partial\psi_{C \times Q}(x_0,u_0) . \qquad (2.1)
$$

Examination of the form of (2.1) leads us to the realization that it formulates the necessary optimality conditions for a certain quadratic programming problem, namely

$$
\begin{aligned}
\text{minimize} \quad & \mathcal{L}'(x_0)(x - x_0) + \frac{1}{2}\langle x - x_0, \mathcal{L}_{xx}(x_0,u_0)(x - x_0)\rangle \\
\text{subject to} \quad & g(x_0) + g'(x_0)(x - x_0) \in Q^\circ , \quad x \in C .
\end{aligned} \qquad (2.2)
$$

The quadratic programming problem (2.2) is not derived by any straightforward approximation of the functions in (1.1) (note that the second-order term in the objective function contains, not $\mathcal{L}''(x_0)$, but $\mathcal{L}_{xx}(x_0,u_0)$). However, it has been found to be an appropriate approximation of (1.1) for numerical purposes [9,25]. Since the derivation through linearization of (1.2) is straightforward, we shall refer to (2.2) as the linearization, or linearized form, of (1.1).

If we are looking for a condition to impose on a stationary point (x_0,u_0) of (1.1) to ensure that x_0 will be a strict local minimizer of (1.1), an obvious approach to take is to impose some such conditions on (2.2). In fact, the simplest way of obtaining a

plausible set of conditions is to consider feasible directions of (2.2) at x_0 and to place some restriction on these. Thus, if we consider vectors $h \in R^n$ such that $x_0 + h \in C$ and $g(x_0) + g'(x_0)h \in Q^\circ$, we know (and can prove easily from the fact that (x_0,u_0) satisfies (1.2); see below) that $f'(x_0)h \geq 0$. If $f'(x_0)h > 0$, then if h is small enough the first-order term in the objective of (2.2) will be dominant and the objective value at $x_0 + h$ will be strictly greater than that at x_0. On the other hand, if $f'(x_0)h = 0$ then the only way to obtain strict increase in the objective function is to resort to the second-order term and to require that $(h, L''(x_0,u_0)h) > 0$.

We can now use these observations to formulate an appropriate set of conditions. To do so, we note that the requirements on h involving the objective function and the Lagrangian of (2.5) are independent of the scale of h, but those involving the constraints (i.e., that $x_0 + h \in C$ and $g(x_0) + g'(x_0)h \in Q^\circ$) are not. To simplify the latter, we enlarge slightly the class of vectors h considered by requiring that $h \in T_C(x_0)$ and $g'(x_0)h \in T_{Q^\circ}(g(x_0))$, where $T_C(x_0)$ denotes the tangent cone to C at x_0, and similarly for Q°. Note that, since C and Q° are convex, any h satisfying the earlier requirements will necessarily satisfy the latter. For small enough h, the converse is true if C and Q are polyhedral, but generally not otherwise.

The steps just described therefore lead us to the following general second-order sufficient condition:

DEFINITION 2.1: Suppose (x_0,u_0) is a point satisfying (1.2). The second-order sufficient condition holds at (x_0,u_0) with modulus $\mu > 0$ if for each $h \in T_C(x_0)$ with

$$g'(x_0)h \in T_{Q^\circ}(g(x_0)), \quad f'(x_0)h = 0 ,$$

one has $(h, L''(x_0,u_0)h) \geq \mu|h|^2$.

We remark that it would not have changed Definition 2.1 if we had written $f'(x_0)h \leq 0$, since that inequality implies $f'(x_0)h = 0$ in the presence of (1.2). To see this, note that h and $-[f'(x_0) + g'(x_0)^*u_0]$ belong respectively to the tangent and normal cones to C at x_0. Thus,

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$$0 \leq (f'(x_0) + g'(x_0)^* u_0, h)$$
$$= f'(x_0)h + (u_0, g'(x_0)h) .$$

However, since $g(x_0)$ belongs to the normal cone to Q at u_0, u_0 also belongs to the normal cone to Q° at $g(x_0)$ [22, Cor. 23.5.4]. As $g'(x_0)h$ belongs to the corresponding tangent cone, we have $(u_0, g'(x_0)h) \leq 0$ and thus $f'(x_0)h \geq 0$. Because of this, in what follows we shall sometimes just show that a vector h satisfies $f'(x_0)h \leq 0$ before applying Definition 2.1. In this form, the condition has been discussed for the standard problem (1.4) by Han and Mangasarian, who have shown [7, Th. 3.5] that for that case it is equivalent to the standard second-order sufficient condition as given in, e.g., [3]. It is thus reasonable to suppose that the condition of Definition 2.1 will enable us to show that some positive definite quadratic form minorizes the objective function of (1.1) on that portion of the feasible set near x_0, and takes a minimum equal to $f(x_0)$ at x_0, since such results have been established for other forms of the condition (cf. [8, Th. 7.3], [24, Th. 4.43], [13, Th. 5.2]). Since we shall need this result in Section 3, we establish it in the following theorem.

THEOREM 2.2: _Suppose that_ (x_0, u_0) _satisfies_ (1.2), _that_ f _and_ g _are twice Frechet differentiable at_ x_0, _and that the second-order sufficient condition holds there with modulus_ ρ. _Then for each_ $\epsilon \in [0, \rho)$ _there exists a neighborhood_ V _of_ x_0 _such that if_ $x \in C \cap V$ _and_ $g(x) \in Q^\circ$, _then_ $f(x) > f(x_0) + \frac{1}{2} \epsilon \|x - x_0\|^2$ _or_ $x = x_0$.

PROOF: Suppose that for some $\epsilon \geq 0$ there is a sequence $\{x_i\} \subset C$ converging to x_0 with $x_i \neq x_0$, $g(x_i) \in Q^\circ$ and $f(x_i) \leq f(x_0) + \frac{1}{2} \epsilon \|x_i - x_0\|^2$ for each i. We shall show that $\epsilon \geq \rho$. We can suppose without loss of generality that $\{(x_i - x_0)/\|x_i - x_0\|\}$ converges to some $h \in R^n$; evidently $\|h\| = 1$ and $h \in T_C(x_0)$. We have for each i

$$Q^\circ \ni g(x_i) = g(x_0) + g'(x_0)(x_i - x_0) + o(\|x_i - x_0\|) , \qquad (2.3)$$

so

$$g'(x_0)h = \lim_{i \to \infty} \|x_i - x_0\|^{-1} [g(x_i) - g(x_0)] \in T_{Q^\circ}(g(x_0)) .$$

We have previously remarked that in such a case $f'(x_0)h \geq 0$, but since for each i we have

$$f(x_0) + \frac{1}{2} \epsilon |x_i - x_0|^2 \geq f(x_i) = f(x_0) + f'(x_0)(x_i - x_0) + o(|x_i - x_0|) , \qquad (2.4$$

it follows upon division by $|x_i - x_0|$ and passage to the limit that also $f'(x_0)h \leq 0$,

and therefore that h satisfies the hypothesis of the second-order sufficient condition.

However, we have from (2.3) and (2.4) that since $(u_0, g(x_i)) \leq 0$,

$$f(x_0) + \frac{1}{2} \epsilon |x_i - x_0|^2 \geq f(x_i) \geq \mathcal{L}(x_i, u_0)$$

$$= \mathcal{L}(x_0, u_0) + \mathcal{L}_x(x_0, u_0)(x_i - x_0)$$

$$+ \frac{1}{2} (x_i - x_0, \mathcal{L}_{xx}(x_0, u_0)(x_i - x_0)) + o(|x_i - x_0|^2)$$

$$\geq f(x_0) + \frac{1}{2} (x_i - x_0, \mathcal{L}_{xx}(x_0, u_0)(x_i - x_0)) + o(|x_i - x_0|^2).$$

Subtracting $f(x_0)$, dividing by $\frac{1}{2} |x_i - x_0|^2$ and passing to the limit, we find that

$$\epsilon \geq (h, \mathcal{L}_{xx}(x_0, u_0)h) \geq \rho |h|^2 = \rho ,$$

and the theorem now follows by contraposition.

Theorem 2.2 shows that a stationary point of (1.1) satisfying the second-order

sufficient condition is a strict local minimizer. A reasonable question to ask is whether

it is also an _isolated_ local minimizer: that is, whether there is some neighborhood of

x_0 containing no other local minimizer of (1.1). As we shall now see, this is not so,

even for very simple problems. Consider, for instance, the problem

$$\text{minimize} \quad \frac{1}{2} x^2$$

$$\qquad (2.5)$$

$$\text{subject to} \quad x^6 \sin(1/x) = 0 . \qquad [\sin(1/0) := 0.]$$

Here $C = R = Q$; the feasible region is $\{0\} \cup \{(n\pi)^{-1} | n = \pm 1, \pm 2, \dots\}$. The second-order

sufficient condition is satisfied at the origin; however, the origin is a cluster point of

the feasible region and every feasible point is a local minimizer.

One might reply that such bad behavior is not very surprising since (2.5) is a bad

problem. This is quite true, and we shall see that a result of the type we seek can indeed

be proved if we exclude bad problems: that is, if we impose a constraint qualification.

The qualification we shall use is the requirement that the constraints

$$g(x) \in Q^\circ$$

$$x \in C$$

of (1.1) be _regular_ at x_0 in the sense of [18]; that is, that

$$0 \in \text{int}\{g(x_0) + g'(x_0)(C - x_0) - Q^0\} .\qquad (2.6)$$

We have shown in [17] that (2.6) is a sufficient qualification for the derivation of optimality conditions, and in fact for the standard problem (1.4), the condition (2.6) is equivalent to the well-known constraint qualification of Mangasarian and Fromovitz [12].

We shall now show that if the constraints of (1.1) are regular at a point x_0 which, together with some u_0, satisfies (1.2), then x_0 is locally unique as a solution of (1.2).

THEOREM 2.3: _Assume that_ f _and_ g _are twice Frechet differentiable at_ x_0, _and suppose that the regularity condition (2.6) holds there. Define_ $U_0 := \{u \in R^m \mid (x_0, u)$ _satisfies (1.2)\} (possibly empty), and suppose that for each_ $u_0 \in U_0$ _the second-order sufficient condition is satisfied at_ (x_0, u_0) _with some positive modulus. Then there exists a neighborhood_ W _of_ x_0 _such that if_ (x, u) _satisfies (1.2) and_ $x \in W$ _then_ $x = x_0$.

PROOF: By contraposition. Suppose that there is a sequence $\{(x_i, u_i)\}$ such that $\{x_i\}$ converges to x_0, and such that for each n, $x_i \neq x_0$ and (x_i, u_i) solves (1.2). Assume without loss of generality that $\{(x_i - x_0)/\|x_i - x_0\|\}$ converges to h.

It follows from Theorem 3.2 with $P = \{p_0\}$ that the set U_0 is bounded (note: the applicable portion of the proof of Theorem 3.2 makes no use of the present theorem nor of any result depending on it). Accordingly, with no loss of generality we can suppose that $\{u_i\}$ converges to some u_0. Taking the limit in (1.3) with (x_i, u_i) in place of (x, u), and using continuity and the closedness of $\partial \psi_C$ and $\partial \psi_Q$, we conclude that (x_0, u_0) satisfy (1.3). Thus $u_0 \in U_0$.

Now from (1.3) we find that for each i, $g(x_i) \in Q^*$ and $x_i \in C$; reasoning as in the proof of Theorem 2.2 we can establish that $h \in T_C(x_0)$, $g'(x_0)h \in T_{Q^*}(g(x_0))$ and $f'(x_0)h \geq 0$.

Since (x_i, u_i) satisfies (1.3) for each i, we have

$$0 = (u_i, g(x_i)) = (u_i, g(x_0)) + (u_i, g'(x_0)(x_i - x_0)) + o(\|x_i - x_0\|) ,$$

where we have used the fact that the u_i are uniformly bounded. However, as $u_i \in Q$

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and $g(x_0) \in Q^\circ$ we have $\langle u_i, g(x_0) \rangle \leqq 0$ and therefore

$$0 \leqq \langle u_i, g'(x_0)(x_i - x_0) \rangle + o(\|x_i - x_0\|) .$$

Dividing by $\|x_i - x_0\|$ and passing to the limit, we obtain

$$\langle u_0, g'(x_0)h \rangle \geqq 0 . \tag{2.7}$$

Now from (1.3) we again have for each i

$$f'(x_i) + g'(x_i)^* u_i \in - \partial \psi_C(x_i) ,$$

so that

$$0 \geqq f'(x_i)(x_i - x_0) + \langle u_i, g'(x_i)(x_i - x_0) \rangle ,$$

and upon division by $\|x_i - x_0\|$ and passage to the limit we find, using (2.7), that $f'(x_0)h \leqq 0$ and therefore that h satisfies the hypothesis of the second-order sufficient condition.

We know that

$$\partial \psi_C(x_i) \ni -\mathcal{L}_x(x_i, u_i) = -\mathcal{L}_x(x_0, u_i) - \mathcal{L}_{xx}(x_0, u_i)(x_i - x_0) + o(\|x_i - x_0\|) , \tag{2.8}$$

and

$$\partial \psi_C(x_0) \ni -\mathcal{L}_x(x_0, u_0) . \tag{2.9}$$

Also,

$$\partial \psi_Q(u_i) \ni g(x_i) = g(x_0) + g'(x_0)(x_i - x_0) + g''(x_0)(x_i - x_0)^2 + o(\|x_i - x_0\|^2), \tag{2.10}$$

and

$$\partial \psi_Q(u_0) \ni q(x_0) . \tag{2.11}$$

Using (2.8)-(2.11) together with the monotonicity of $\partial \psi_C$ and $\partial \psi_Q$, we find that

$$0 \leqq \langle -g'(x_0)^*(u_i - u_0) - \mathcal{L}_{xx}(x_0, u_i)(x_i - x_0), x_i - x_0 \rangle + o(\|x_i - x_0\|^2) , \tag{2.12}$$

and

$$0 \leqq \langle u_i - u_0, g'(x_0)(x_i - x_0) + g''(x_0)(x_i - x_0)^2 + o(\|x_i - x_0\|^2) \rangle . \tag{2.13}$$

Adding (2.12) and (2.13), we obtain

$$0 \leqq \langle -\mathcal{L}_{xx}(x_0, u_0)(x_i - x_0), x_i - x_0 \rangle + o(\|x_i - x_0\|^2) ,$$

and division by $\|x_i - x_0\|^2$ and passage to the limit yields a contradiction to the second-order sufficient condition. This completes the proof.

We have thus shown that when regularity is added to the second-order sufficient condition, the stationary points x_0 which, with some u_0, satisfy (1.2), are isolated.

It is an easy consequence of this fact that if the conditions of Theorem 2.3 hold in some
compact region of R^n, that region contains only finitely many stationary points (perhaps
none). However, this result says nothing about the behavior of stationary points, or
minimizers, of problems which are "near" (1.1) in the sense that their problem functions
are slight perturbations of those of (1.1). In the next section, we use the second-order
sufficient condition and the regularity condition to analyze the behavior of such problems.

3. **Local solvability of perturbed nonlinear programming problems.** In this section we
shall be concerned with the problem

$$\text{minimize} \quad f(x,p)$$
$$\text{subject to} \quad g(x,p) \in \Omega^\circ \tag{3.1}$$
$$x \in C ,$$

where p is a perturbation parameter belonging to a topological space P, and x is the
variable in which the minimization is done. The functions f and g are defined from
$\Omega \times P$ to R and R^m respectively, where Ω is an open set in R^n; Ω and C are as
previously. We shall identify (1.1) with the particular case of (3.1) arising when p is
some fixed $p_0 \in P$; our interest here will be in predicting, from information in (1.1),
aspects of the behavior of (3.1) when p varies near p_0, such as solvability, location
of minimizers, etc. Throughout Sections 3 and 4 we shall make the blanket assumption that
$f(\cdot,p)$ and $g(\cdot,p)$ are Frechet differentiable on Ω, that f, g, f' and g' are
continuous on $\Omega \times P$, and that $f(\cdot,p_0)$ and $g(\cdot,p_0)$ are twice Frechet differentiable
at x_0, a point in $C \cap \Omega$ which is a stationary point of (3.1) for $p = p_0$. The
stationary-point conditions for (3.1) are

$$0 \in f'(x,p) + g'(x,p)*u + \partial\psi_C(x)$$
$$0 \in -g(x,p) + \partial\psi_\Omega(u) , \tag{3.2}$$

and we define $U_0 := \{u \in R^m \mid (x_0,u) \text{ satisfies (3.2) for } p = p_0\}$. We shall denote the

distance from a point a to a set A by $d[a,A] := \inf\{\|a - a'\| \mid a' \in A\}$ $(-\infty$ if $A = \emptyset)$.

Our first theorem shows that if the second-order sufficient condition holds at a local minimizer at which the constraints are regular, then that local minimizer persists under small perturbations.

THEOREM 3.1: _Suppose that for_ $p = p_0$, _(3.1) satisfies the second-order sufficient condition at_ x_0 _and some_ $u_0 \in U_0$ _and its constraints are regular at_ x_0.

Then for each neighborhood M _of_ x_0 _in_ Ω _there is a neighborhood_ N _of_ p_0 _such that if_ $p \in N$ _then (3.1) has a local minimizer in_ M.

PROOF: By hypothesis, for $p = p_0$ the constraints of (3.1) are regular at x_0. By [18, Th. 1] there are neighborhoods M_0 and N_0 of x_0 and p_0 respectively, and a constant ζ, such that for each $x \in C \cap M_0$ and $p \in N_0$,

$$d[x,F(p)] \leq \zeta d[g(x,p),Q^\circ] ,$$

where $F(p) := \{x \in C \mid g(x,p) \in Q^\circ\}$. Let ε_0 be the modulus for the second-order sufficient condition at (x_0,u_0); let $\varepsilon \in (0,\varepsilon_0)$ and choose a positive δ so that $(x_0 + 2\delta B) \subset M_0 \cap M \cap V$, where V is the neighborhood of x_0 given by Theorem 2.2 for the chosen ε. Select a neighborhood N_0' of p_0 with $N_0' \subset N_0$, and some positive α, such that if $p \in N_0'$ and if $x_1,x_2 \in x_0 + 2\delta B$ with $\|x_1 - x_2\| \leq \alpha$, then $\|f(x_1,p) - f(x_2,p_0)\| < \varepsilon\delta^2/16 =: \beta$. Next, find a neighborhood N of p_0 with $N \subset N_0'$ and such that if $p \in N$ then

$$\zeta \sup\{\|g(x,p) - g(x,p_0)\| \mid x \in x_0 + \delta B\} \leq \min\{\alpha,\delta/2\} .$$

Denote $f(x_0,p_0)$ by ϕ_0.

Now choose any $p \in N$. If $F(p) \cap \{x \mid \|x - x_0\| = \delta\}$ is not empty, let x' be any point in it. One has $x' \in C \cap \{x_0 + \delta B\}$ and $g(x',p) \in Q^\circ$, so

$$d[x',F(p_0)] \leq \zeta d[g(x',p_0),Q^\circ] \leq \zeta\|g(x',p_0) - g(x',p)\|$$

$$\leq \min\{\alpha,\delta/2\} .$$

Thus there is some $x_0' \in F(p_0)$ with $\|x' - x_0'\| \leq \min\{\alpha,\delta/2\}$, and therefore with $\|x_0' - x_0\| \geq \|x' - x_0\| - \|x' - x_0'\| \geq \delta/2$. Accordingly, by Theorem 2.2 we have

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$f(x_0', p_0) \geq \phi_0 + \frac{1}{2} \varepsilon(\delta/2)^2 = \phi_0 + 2\beta$. Also, $\|x_0' - x_0\| \leq \|x' - x_0\| +$

$\|x' - x_0'\| \leq \delta + \delta/2 = 3\delta/2$, so $x_0' \in x_0 + 2\delta B$; as $\|x_0' - x'\| \leq \alpha$ we have

$\|f(x', p) - f(x_0', p_0)\| < \beta$. Therefore

$$f(x', p) \geq f(x_0', p_0) - \|f(x', p) - f(x_0', p_0)\|$$

$$> \phi_0 + 2\beta - \beta = \phi_0 + \beta.$$

(3.3)

It is also true that

$$d[x_0, F(p)] \leq \zeta d[g(x_0, p), Q^\circ] < \zeta \|g(x_0, p) - g(x_0, p_0)\|$$

$$\leq \min\{\alpha, \delta/2\} .$$

Accordingly, there is some $x'' \in F(p)$ with $\|x_0 - x''\| \leq \min\{\alpha, \delta/2\}$. Thus

$\|f(x'', p) - f(x_0, p_0)\| < \beta$, so

$$f(x'', p) \leq f(x_0, p_0) + \|f(x'', p) - f(x_0, p_0)\| < \phi_0 + \beta .$$

(3.4)

Putting (3.3) and (3.4) together, we see that a minimizer of the function $f(\cdot, p)$ on the

set $F(p) \cap (x_0 + \delta B)$ exists (by compactness and the fact that x'' belongs to

$F(p) \cap (x_0 + \delta B))$, but that no such minimizer lies on $F(p) \cap \{x \mid \|x - x_0\| = \delta\}$.

Therefore each such minimizer is in fact a local minimizer of $f(\cdot, p)$ on $F(p)$: that is,

a local minimizer of (3.1) which belongs to M. This completes the proof.

Our next results will concern the continuity behavior of local minimizers and of

stationary points of (3.1). For ease of reference, we proceed now to define certain

multifunctions which display these points. Assume that the hypotheses of Theorem 3.1 hold

for (3.1) with $p = p_0$; then it follows from the results of [16] that there are

neighborhoods M_1 of x_0 and N_1 of p_0 such that for any $p \in N_1$ and any $x \in M_1$

satisfying

$$g(x, p) \in Q^\circ$$

$$x \in C ,$$

(3.5)

the system (3.5) is regular at x.

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At this point it will be useful to note a property of the multipliers which we shall shortly use. We shall show that if regularity holds at x_0 for $p = p_0$, then the multipliers in (3.2) are uniformly bounded, and in fact the set of all such multipliers is an upper semicontinuous multifunction of (x,p). This extends a result in [11], also given in [5].

THEOREM 3.2: *If the system (3.5) is regular at* x_0 *for* $p = p_0$, *then there exist neighborhoods* M_2 *of* x_0 *and* N_2 *of* p_0, *such that if* $U : M_2 \times N_2 \rightarrow R^m$ *and* $SP : N_2 \rightarrow M_2$ *are multifunctions defined by*

$$U(x,p) := \{u \in R^m \mid (x,u,p) \ \underline{satisfies \ (3.2)}\} \ \underline{for} \ (x,p) \in M_2 \times N_2 \ ,$$

$$SP(p) := \{x \in M_2 \mid \underline{for \ some} \ u, (x,u,p) \ \underline{satisfies \ (3.2)}\} \ for \ p \in N_2 \ ,$$

then U *and* SP *are upper semicontinuous.*

PROOF: We first show that U is locally bounded at (x_0, p_0). Assume, on the contrary, that there are sequences $\{x_i\} \subset \Omega$ and $\{p_i\} \subset P$ converging to x_0 and p_0 respectively, and a sequence $\{u_i\} \subset R^m$ with $\lim_{i \to \infty} \|u_i\| = +\infty$, such that for each i the triple (x_i, u_i, p_i) satisfies (3.2). With no loss of generality we can suppose that $u_i / \|u_i\|$ converges to some y. For each i, (3.2) implies that $u_i \in Q$ (a cone) and $\langle u_i, g(x_i, p_i) \rangle = 0$; dividing by $\|u_i\|$ and passing to the limit we find that $y \in Q$ and $\langle y, g(x_0, p_0) \rangle = 0$. Again from (3.2), we have for each i,

$$f'(x_i, p_i) + g'(x_i, p_i)^* u_i \in -\partial \psi_C(x_i) \ ;$$

again dividing by $\|u_i\|$ and passing to the limit, using the fact that $\partial \psi_C$ is a closed multifunction whose values are cones, we obtain $g'(x_0, p_0)^* y \in -\partial \psi_C(x_0)$. By regularity, for some $\varepsilon > 0$ there is a point $x_\varepsilon \in C$ with $g(x_0, p_0) + g'(x_0, p_0)(x_\varepsilon - x_0) + \varepsilon y \in Q^o$. Using our information about y, we find that

$$0 \geq \langle y, g(x_0, p_0) + g'(x_0, p_0)(x_\varepsilon - x_0) + \varepsilon y \rangle$$

$$\geq \varepsilon \|y\|^2 > 0 \ ,$$

a contradiction. It follows that there must exist neighborhoods M_2 of x_0 in Ω and

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N_2 of p_0 in P, and a compact set $K \subset R^m$, such that if $(x,p) \in M_2 \times N_2$ then $U(x,p) \subset K$. Without loss of generality we can suppose that M_2 and N_2 are small enough so that M_2 is bounded and that for $(x,p) \in M_2 \times N_2$ the values $\|g(x,p)\|$, $\|f'(x,p)\|$ and $\|g'(x,p)\|$ all satisfy some uniform bound.

Now let

$$h(x,p,u) := \begin{bmatrix} f'(x,p) + g'(x,p)^{*}u \\ -g(x,p) \end{bmatrix} ;$$

we note that there is some compact set L such that if $(x,p,u) \in M_2 \times N_2 \times K$ then $h(x,p,u) \in L$.

Now denote by G the graph of the multifunction $\partial\psi_C \times \partial\psi_Q$, and define

$$H := \{(x,p,u,v)\,|\,(x,p) \in M_2 \times N_2, \ (x,p,u,v) \in \text{graph } h, \ (x,u,-v) \in G\} .$$

The continuity assumptions and the properties of $\partial\psi_C$ and $\partial\psi_G$ imply that H is closed in $M_2 \times N_2 \times R^m \times R^{n+m}$. However, if $(x,p,u,v) \in H$ then $(x,p) \in M_2 \times N_2$ and (x,u,p) satisfies (3.2), so that $u \in K$ and therefore $v \in L$. As K and L are compact it follows that the projection of H on the space of the first three, or of the first two, components of (x,p,u,v) is closed. The first of these projections is the graph of the multifunction U, and the second is that of SP. Thus U and SP are closed; however, as the image of U is contained in the compact set K and that of SP in the precompact set M_2, it follows that U and SP are actually upper semicontinuous. This completes the proof.

Our main result on continuity of local minimizers and stationary points is given in the next theorem. For the multifunction SP and the set M_3 in the theorem, $SP \cap M_3$ denotes the multifunction defined by $(SP \cap M_3)(p) := SP(p) \cap M_3$.

THEOREM 3.3: _Suppose that for_ $p = p_0$, _for some_ $x_0 \in \Omega$ _and each_ $u \in U(x_0,p_0)$, _(3.1) satisfies the second-order sufficient condition at_ (x_0,u), _and that its constraints are regular there._

Then there exist neighborhoods M_3 of x_0 and N_3 of p_0 such that if the multifunction $LM : N_3 \to M_3$ is defined by

$$LM(p) := \{x \quad {}_3 | x \text{ is a local minimizer of (3.1)}\}, \qquad (3.6)$$

then $SP \cap M_3$ is continuous ${}_0$ and for each $p \in N_3$ one has $\phi \neq LM(p) \subseteq SP(p)$.

PROOF: Let M_2 and W be the neighborhoods of x_0 produced by Theorems 3.2 and 3.3 respectively; define $M_3 := M_2 \cap W$; suppose in addition that M_3 is bounded and is small enough so that for some neighborhood N_4 of p_0, any $p \in N_4$ and any $x \in M_3$ satisfying (3.5), the system (3.5) is regular at x. Let N_2 be the neighborhood of p_0 given by Theorem 3.2; let N_5 be the neighborhood obtained by applying Theorem 3.1 to (3.1) with $M = M_3$, and define $N_3 := N_2 \cap N_4 \cap N_5$. Define LM as in (3.6).

We shall first show that LM and $SP \cap M_3$ are lower semicontinuous at p_0. By Theorem 2.3, $SP(p_0) \cap M_3 = \{x_0\}$; also, for any $p \in N_3$ if $x \in LM(p)$ then since regularity of (3.5) is a sufficient constraint qualification [17] we have $x \in SP(p) \cap M_3$. Thus if $p \in N_3$ then $\phi \neq LM(p) \subseteq SP(p) \cap M_3$. However, since $x_0 \in LM(p_0)$ by Theorem 2.2, we actually have $LM(p_0) = SP(p_0) \cap M_3 = \{x_0\}$. Let S be any open set in M_3 with $S \cap LM(p_0) \neq \phi$; then obviously $(SP \cap M_3)(p_0) \subseteq S$. Now by applying Theorem 3.1 to S we can find some neighborhood N_6 of p_0 such that $N_6 \subseteq N_3$ and if $p \in N_6$ then $LM(p) \cap S \neq \phi$. Hence LM is lower semicontinuous at p_0, but since $LM \subseteq SP \cap M_3$ and $LM(p_0) = (SP \cap M_3)(p_0)$, we see that $SP \cap M_3$ is also lower semicontinuous at p_0. But SP is upper semicontinuous from N_2 to M_2, so $SP \cap M_3$ is upper semicontinuous from N_3 to M_3, and thus $SP \cap M_3$ is actually continuous at p_0. This completes the proof.

Theorem 3.3 shows that that portion of the set of stationary points near x_0 is continuous at p_0, but it gives no measure of how it depends on p. In the next section, we show that if the sets C and Q are polyhedral then the dependence of SP and U upon p can be measured by the dependence of the problem functions and their derivatives upon p: in particular, if the latter are Lipschitzian then SP is "upper Lipschitzian". In the meantime, however, we present an example, taken from [20], to show

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that LM may not be single-valued near p_0 and that the inclusion of LM in SP may be strict.

Consider the quadratic programming problem

$$\text{minimize} \quad \frac{1}{2}(x_1^2 - x_2^2) - \eta x_1$$

$$\text{subject to} \quad -x_1 + 2x_2 \leq 0$$

$$-x_1 - 2x_2 \leq 0 \ ,$$

where η is the parameter. For $\eta = 0$ this problem has a unique minimizer at the origin; for any positive η it has local minimizers at $\frac{2}{3}\eta(2,\pm1)$ and a saddle point at $(\eta,0)$. Thus in this case LM is multivalued and is strictly contained in SP whenever $\eta > 0$.

4. <u>Upper Lipschitz continuity in the polyhedral case.</u> Throughout this section we shall assume that C and Q are polyhedral convex sets, and we shall see that in this case we can prove stronger results than we did in Section 3. Specifically, we shall prove that the solutions and multipliers of (3.1), regarded as multifunctions of p, are locally upper Lipschitzian at p_0 under appropriate continuity assumptions on the problem functions. We recall that if F is a multifunction from R^n to R^m, F is said to be locally upper Lipschitzian at $z_0 \in R^n$ if there are a neighborhood Z of z_0 and a constant such that for each $z \in Z$,

$$F(z) \subset F(z_0) + \lambda \mathbf{I} z - z_0 \mathbf{I} B \ ,$$

where B is the unit ball in R^m. We showed in [21] that any polyhedral multifunction (one whose graph is the union of finitely many polyhedral convex sets) is locally upper Lipschitzian everywhere with a uniform constant.

We begin by formulating a general continuity result for nonlinear generalized equations. It says that if a certain condition is satisfied by the linearizations of the nonlinear problem, then the latter has, at least locally, an upper Lipschitzian inverse (which, of course, may not be single-valued and might even be empty). As in Section 3, for a set S and a multifunction F, we define the multifunction $F \cap S$ by

$(F \cap S)(x) := F(x) \cap S$. The norm on a product space is taken to be the maximum of the norms of the component spaces.

THEOREM 4.1: Let F be a Frechet differentiable function from an open set $\Omega \subset R^n$ to R^n. Let R be any polyhedral multifunction from R^n to R^n, and define $H := F + R$. For $x_0 \in \Omega$ and $x \in R^n$ define

$$LF_{x_0}(x) := F(x_0) + F'(x_0)(x - x_0) .$$

Suppose there is some compact set $X_0 \subset \Omega$ such that

i) For each x_0 and x_1 in X_0, the restrictions of LF_{x_0} and LF_{x_1} to X_0 are the same.

ii) For some positive γ and each $x_0 \in X_0$,

$$X_\gamma \cap (LF_{x_0} + R)^{-1}(0) = X_0 ,$$

where $X_\gamma := X_0 + \gamma B$.

iii) F' is continuous on X_0, and there is a function $\theta : R_+ \to R_+$ with $\lim_{t \downarrow 0} \theta(t) = 0 = \theta(0)$, such that for each $x \in X_\gamma$ and each $x_0 \in X_0$,
$\|F(x) - LF_{x_0}(x)\| \le \|x - x_0\| \theta(\|x - x_0\|)$.

Then there is some positive δ such that the multifunction $H^{-1} \cap X_\delta$ is locally upper Lipschitzian at 0, with $(H^{-1} \cap X_\delta)(0) = X_0$.

We remark that (iii) is satisfied in particular if Ω is convex and F' is Lipschitzian there.

PROOF: For $x_0 \in X_0$, define T_{x_0} to be $(LF_{x_0} + R)^{-1}$. Let $\varepsilon := \frac{1}{2}\gamma$. We first show that there exist a constant λ and some positive η such that for any $x_0 \in X_0$, $T_{x_0} \cap X_\varepsilon$ is upper Lipschitzian on ηB with modulus λ. Choose any $x_0 \in X_0$; then the sum $LF_{x_0} + R$ is polyhedral and by [21, Prop. 1] there are a constant $\lambda(x_0)$ and a positive $\eta(x_0)$ such that T_{x_0} is upper Lipschitzian on $\eta(x_0)B$ with modulus $\lambda(x_0)$. If we take $\eta(x_0)$ small enough, it follows from the fact that $(T_{x_0} \cap X_\varepsilon)(0) = X_0$ that the multifunction $T_{x_0} \cap X_\varepsilon$ is upper Lipschitzian on $\eta(x_0)B$ with modulus $\lambda(x_0)$. Now, using the continuity assumptions, choose a neighborhood N_0 of x_0 small enough that for any $x \in X_0 \cap N_0$,

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a) $\lambda(x_0)\|F'(x) - F'(x_0)\| < \frac{1}{2}$,

b) For any $x' \in X_\varepsilon$, $\|LF_{x_0}(x') - LF_x(x')\| \leq \frac{1}{2} \eta(x_0)$.

Choose any $x \in X_0 \cap N_0$, and any $q \in \frac{1}{2} \eta(x_0)B$. If $w \in (T_x \cap X_\varepsilon)(q)$ then $w \in X_\varepsilon$ and also

$$q \in LF_x(w) + R(w) ,$$

so

$$q + LF_{x_0}(w) - LF_x(w) \in LF_{x_0}(w) + R(w)$$

and thus

$$w \in (T_{x_0} \cap X_\varepsilon)\{q + LF_{x_0}(w) - LF_x(w)\} .$$

However,

$$\|q + LF_{x_0}(w) - LF_x(w)\| \leq \|q\| + \|LF_{x_0}(w) - LF_x(w)\|$$

$$\leq \frac{1}{2} \eta(x_0) + \frac{1}{2} \eta(x_0) = \eta(x_0) ,$$

so by the upper Lipschitz continuity we have

$$w \in X_0 + \lambda(x_0)\|q + LF_{x_0}(w) - LF_x(w)\|B .$$

Now let x_1 be a point of X_0 with $d[w,X_0] = \|w - x_1\|$. By hypothesis, since x_0, x and x_1 are in X_0, we have $LF_{x_0}(x_1) = LF_x(x_1)$. Therefore

$$\|LF_{x_0}(w) - LF_x(w)\| = \|LF_{x_0}(x_1) - LF_x(x_1) + [F'(x_0) - F'(x)](w - x_1)\|$$

$$\leq \|F'(x_0) - F'(x)\|\|w - x_1\| ,$$

and thus

$$d[w,X_0] \leq \lambda(x_0)\|q\| + \lambda(x_0)\|F'(x_0) - F'(x)\|\|w - x_1\|$$

$$\leq \lambda(x_0)\|q\| + \frac{1}{2} d[w,X_0] .$$

Therefore

$$d[w,X_0] \leq 2\lambda(x_0)\|q\| \ ,$$

and so

$$(T_x \cap X_\varepsilon)(q) \subset (T_x \cap X_\varepsilon)(0) + 2\lambda(x_0)\|q\|B \ ,$$

which proves that $T_x \cap X_\varepsilon$ is upper Lipschitzian on $\frac{1}{2}\eta(x_0)B$ with modulus $2\lambda(x_0)$. An elementary compactness argument now shows that for some λ and some $\eta > 0$, for any $x \in X_0$, $T_x \cap X_\varepsilon$ is upper Lipschitzian on ηB with modulus λ.

Choose a positive $\delta \leq \min\{\varepsilon, \eta\}$ so that if $\beta \leq \delta$ then $\max\{1,\lambda\}\theta(\delta) \leq \frac{1}{2}$. Now let $q \in \frac{1}{2}\eta B$ and let $x \in (H^{-1} \cap X_\delta)(q)$. Then $q \in F(x) - R(x)$, so if $x_1 \in X_0$ with $\|x - x_1\| = d[x,X_0]$ we have

$$q + LF_{x_1}(x) - F(x) \in LF_{x_1}(x) + R(x) \ ,$$

and therefore $x \in (T(x_1) \cap X_\varepsilon)[q + LF_{x_1}(x) - F(x)]$. However,

$$\|q + LF_{x_1}(x) - F(x)\| \leq \|q\| + \theta(\|x - x_1\|)\|x - x_1\|$$

$$\leq \frac{1}{2}\eta + \frac{1}{2}\eta \leq \eta \ ,$$

so by upper Lipschitz continuity of $T(x_1) \cap X_\varepsilon$,

$$d[x_1,X_0] \leq \lambda\|q\| + \frac{1}{2}d[x_1,X_0]$$

implying

$$d[x_1,X_0] \leq 2\lambda\|q\| \ .$$

Hence

$$(H^{-1} \cap X_\delta)(q) \subset X_0 + 2\lambda\|q\|B \tag{4.1}$$

for each $q \in \frac{1}{2}\eta B$.

Now let $x_0 \in X_0$. By hypothesis

$$0 \in (LF_{x_0} + R)(x_0) = F(x_0) + R(x_0) = H(x_0) \ ,$$

so $X_0 \subset (H^{-1} \cap X_\delta)(0)$. On the other hand, by applying (4.1) with $q = 0$ we see that

$(H^{-1} \cap X_\delta)(0) \subseteq X_0$, so that $(H^{-1} \cap X_\delta)(0) = X_0$ and $H^{-1} \cap X_\delta$ is upper Lipschitzian on $\frac{1}{2} nB$ with modulus 2λ. This completes the proof.

This result will allow us to measure the displacement in a stationary point x and multipliers u in terms of perturbations in the problem functions and their derivatives. We show this next, and then proceed to develop a simpler measure of displacement for the case in which the functions involved are Lipschitzian in p. In what follows, we write U_0 for $U(x_0, p_0)$.

THEOREM 4.2: Assume the hypotheses of Theorem 3.3, and suppose also that C and Q are polyhedral. Then there are a neighborhood N_7 of p_0 and a constant μ such that for each $p \in N_7$, each $x \in SP(p)$ and each $u \in U(x,p)$,

$$d[(x,u), \{x_0\} \times U_0] \leq \mu |F(x,u,p_0) - F(x,u,p)| , \qquad (4.2)$$

where

$$F(x,u,p) := \begin{bmatrix} f'(x,p) + g'(x,p)^*u \\ -g(x,p) \end{bmatrix} .$$

PROOF: We shall apply Theorem 4.1 to the function $F(x,u)$ given by $F(x,u,p_0)$. We take R to be $\partial\psi_{C\times Q}$, and as a candidate for X_0 we take the set $\{x_0\} \times U_0$. Because of the special form of X_0, it is clear that F' is continuous there; the rest of hypothesis (iii) follows from the differentiability assumptions and the fact that U_0 is bounded. For (i), we have to show that if $u_0, u_1, u_2 \in U_0$ then

$$LF_{(x_0,u_0)}(u_2) = LF_{(x_0,u_1)}(u_2) .$$

However, an algebraic computation shows that each of these quantities is equal to

$$\begin{bmatrix} f'(x_0,p_0) + g'(x_0,p_0)^*u_2 \\ -g(x_0,p_0) \end{bmatrix} ,$$

so that (i) is satisfied. For hypothesis (ii), we suppose that $u_0 \in U_0$ and that for some x and u, $0 \in (LF_{(x_0,u_0)} + \partial\psi_{C\times Q})(x,u)$. Then x is a stationary point, and u is an associated multiplier, for the quadratic programming problem (cf. (2.2)):

$$\text{minimize} \qquad \ell'(x_0,p_0)(x - x_0) + \frac{1}{2}\ell''(x_0,u_0,p_0)(x - x_0)^2 \qquad\qquad (4.3)$$

$$\text{subject to} \quad g(x_0,p_0) + g'(x_0,p_0)(x - x_0) \in Q^\circ$$

$$x \in C .$$

However, (4.3) satisfies the second-order sufficient condition (at x_0 and any multiplier, the modulus being independent of the multiplier), and its constraints are regular; both of these properties follow from the corresponding properties of the nonlinear problem (3.1). Applying Theorem 2.3 to (4.3) we conclude that there is some neighborhood M_4 of x_0 such that $x \notin M_4$ or $x = x_0$. If we take γ to be small enough that $x_0 + \gamma B \subseteq M_4$, then if $(x,u) \in X_\gamma$ we must have $x \in M_4$, hence $x = x_0$, hence $u \in U_0$ by inspection. Thus hypothesis (ii) of Theorem 4.1 is satisfied, and we can conclude that for some positive δ, and some neighborhood W of the origin in R^{n+m}, the multifunction

$$(F(\cdot,\cdot,p_0) + \partial\psi_{C\times Q})^{-1} \cap X_\delta$$

is upper Lipschitzian on W, with

$$[(F(\cdot,\cdot,p_0) + \partial\psi_{C\times Q})^{-1} \cap X_\delta](0) = \{x_0\} \times U_0 .$$

Let the upper Lipschitz modulus be μ.

Since SP and U are upper semicontinuous at p_0 by Theorem 3.2, we can find neighborhoods $N_7(p_0)$ and $M_5(x_0)$ such that $M_5 \subseteq x_0 + \delta B$ and

(a) For $x \in M_5$ and $p \in N_7$, $U(x,p) \subseteq U_0 + \delta B$;

(b) For $p \in N_7$, $SP(p) \subseteq M_5$;

(c) For $x \in M_5$ and $p \in N_7$, and any $u \in U(x,p)$,

$$F(x,u,p_0) - F(x,u,p) \in W .$$

Let $p \in N_7$; let $x \in SP(p)$ and $u \in U(x,p)$. Then $(x,u) \in X_\delta$ by (a) and (b). Also, we have by (3.2)

$$0 \in F(x,u,p) + \partial\psi_{C\times Q}(x,u) ,$$

so that

$$F(x,u,p_0) - F(x,u,p) \in F(x,u,p_0) + \partial\psi_{C\times Q}(x,u) ;$$

that is,

$$(x,u) \in [(F(\cdot,\cdot,p_0) + \partial\psi_{C\times Q})^{-1} \cap X_\delta](w) ,$$

where

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$$w := F(x,u,p_0) - F(x,u,p) \ .$$

However, $w \in W$ by (c), and therefore by upper Lipschitz continuity

$$d[(x,u),(x_0) \times U_0] \leq u|F(x,u,p_0) - F(x,u,p)| \ ,$$

which completes the proof.

The bound (4.2) can be simplified if we assume more structure in the perturbations.

COROLLARY 4.3: <u>Assume the hypotheses of Theorem 4.2. Suppose that</u> P <u>is a subset of</u> <u>a linear space, and that</u> f', g <u>and</u> g' <u>are Lipschitzian on</u> $\Omega \times P$. <u>Then for some</u> <u>constant</u> λ <u>the bound (4.2) can be replaced by</u>

$$d[(x,u),(x_0) \times U_0] \leq \lambda|p - p_0| \ . \tag{4.4}$$

PROOF: We have, for the quantities in (4.2),

$$F(x,u,p_0) - F(x,u,p) = \begin{bmatrix} [f'(x,p_0) - f'(x,p)] + [g'(x,p_0) - g'(x,p)]^*u \\ -[g(x,p_0) - g(x,p)] \end{bmatrix} \ .$$

As $x \in \Omega$ and as u is uniformly bounded for p near p_0, we find that for some L, $|F(x,u,p_0) - F(x,u,p)| \leq L|p - p_0|$, and (4.4) then follows from (4.2).

The bound (4.4) applies to the distance from x to the point x_0, and from u to the set U_0. One might wonder whether, if additional hypotheses were added to guarantee that SP would be a single-valued function on a neighborhood of p_0, one could establish some stronger property such as local Lipschitz continuity. If we are willing to replace the assumption of regularity by a considerably stronger one (corresponding in the case of the standard nonlinear programming problem to linear independence of the gradients of the binding constraints), then this can indeed be done, and a framework for analyzing such problems was developed in [20]. However, in the present case we cannot establish local Lipschitz continuity because it need not be true. This is shown by the following example, which settles in the negative a conjecture of Daniel [2], and also appears to contradict a result stated by Levitin [11, Th. 4].

Let $C = R^2$, $Q = R_+^2$, and $P = \{(p_1, p_2) \in R^2 \mid 0 \leq p_1 \leq 1, |p_2| \leq \frac{1}{6} p_1^2\}$. Define

$$f(x, p) := \frac{1}{2} \|x\|^2$$

and

$$g(x, p) := -(A + p_1 I)x + \frac{1}{2}(2 + p_1)a + p_2 b ,$$

where

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad a := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

It is easy to check that the hypotheses of Corollary 4.3 are satisfied. However, for any $p \in P$ the unique minimizer of the problem (3.1) with the above data is given by

$$x(p) = \begin{cases} \frac{1}{2} a & \text{if} \quad p = (0,0) \\[2mm] \frac{1}{2} a + p_1^{-1} p_2 b & \text{if} \quad p \neq (0,0) , \end{cases}$$

and $x(\cdot)$ is obviously not locally Lipschitzian at $(0,0)$, although it does satisfy the bound (4.4). This example also shows that the continuity result of Kojima [10, Th. 7.2] for the case of regular constraints cannot be strengthened to prove Lipschitz continuity.

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regularity hold at a local minimizer of a nonlinear programming problem, then for
sufficiently smooth perturbations of the constraints and objective function the
set of local stationary points is nonempty and continuous; further, if certain
polyhedrality assumptions hold (as is usually the case in applications) then the
local minimizers, the stationary points and the multipliers all obey a type of
Lipschitz condition. Through the use of generalized equations, these results
are obtained with a minimum of notational complexity.

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