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CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL INEQUALITIES—ETC(U)

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CONVERGENT APPROXIMATIONS IN
PARABOLIC VARIATIONAL INEQUALITIES
II: HAMILTON-JACOBI INEQUALITIES

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13 CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL INEQUALITIES, II. HAMILTON-JACOBI INEQUALITIES.

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ABSTRACT

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In this paper we consider two-sided parabolic inequalities of the form

(li) $\psi_1 \leq u \leq \psi_2$, in Ω ;

(lii) $[-\frac{\partial u}{\partial t} + A(t)u + H(x,t,u,Du)]e \geq 0$, in Ω ,

for all e in the convex support cone of the solution given by

$K(u) = \{\lambda(v - u) : \psi_1 \leq v \leq \psi_2, \lambda > 0\}$;

(liii) $\frac{\partial u}{\partial \nu} \Big|_{\Sigma} = 0$, $u(\cdot, T) = \bar{u}$,

where

$\Omega = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$.

Such inequalities arise in the characterization of saddle-point payoffs u in two person differential games with stopping times as strategies. In this case, H is the Hamiltonian in the formulation. A numerical scheme for approximating u is obtained by the continuous time, piecewise linear, Galerkin approximation of a so-called penalized equation. A rate of convergence to u of order $O(h^{1/3})$ is demonstrated in the $L^2(0, T; H^1(\Omega))$ norm, where h is the maximum diameter of a given triangulation.

AMS(MOS) Subject Classification: 65N30, 90D05

Key Words: Parabolic variational inequalities, Hamilton-Jacobi inequalities, Penalization, Finite element method

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SIGNIFICANCE AND EXPLANATION

This paper studies the two-sided differential inequality which characterizes the optimal payoff in a two-person differential game, when the strategies available to the players are stopping times. By converting the inequality to an equation via a so-called penalization method, we are able to apply a standard numerical method for approximating solutions of nonlinear equations. We obtain new rates of convergence for the procedure.

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CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL
INEQUALITIES II: HAMILTON-JACOBI INEQUALITIES

Joseph W. Jerome

§1. Introduction.

In this note we continue the investigation of the discretization of parabolic variational inequalities begun in [3]. The problem is decidedly more general here and our approach correspondingly different. As distinct from the one-sided inequalities defined by linear differential operators such as the heat operator, studied in [3], we consider here two-sided inequalities defined by nonlinear operators. Thus we seek a function u satisfying, on a space-time domain $Q = \Omega \times (0, T)$:

$$\begin{aligned}
 & \text{(i)} \quad \psi_1 \leq u \leq \psi_2 \quad , \quad \text{in } Q \quad ; \\
 & \text{(ii)} \quad - \frac{\partial u}{\partial t} + A(t)u + H(x, t, u, Du) = 0 \quad , \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \text{if } \psi_1 < u < \psi_2 \quad ; \\
 \text{(1.1)} \quad & \text{(iii)} \quad - \frac{\partial u}{\partial t} + A(t)u + H(x, t, u, Du) \geq 0 \quad , \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \text{if } u = \psi_1 \quad ; \\
 & \text{(iv)} \quad - \frac{\partial u}{\partial t} + A(t)u + H(x, t, u, Du) \leq 0 \quad , \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \text{if } u = \psi_2 \quad .
 \end{aligned}$$

Adjoined to (1.1i, ii, iii, iv) is a standard homogeneous Neumann boundary condition and a terminal condition at $t = T$:

$$\text{(1.1v)} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma = \partial\Omega \times (0, T) \quad ; \quad u(\cdot, T) = \bar{u} \quad .$$

The precise hypotheses are stated in section two, where we define a penalized problem with solution u_ϵ and demonstrate a rate of convergence of order $O(\epsilon^{1/4})$ in the $L^2(0, T; H^1(\Omega))$ norm (cf. Theorem 2.2). In section three we define the Faedo-Galerkin approximation u_ϵ^h of u_ϵ and demonstrate convergence to u with the rate of $O(h^{1/3})$ in the $L^2(0, T; H^1(\Omega))$ norm, if piecewise linear elements of maximal diameter h are employed with $h = \epsilon^{3/4}$ (cf. Theorem 3.3).

The formulation (1.1) arises in stochastic control and two-player differential game theory, where it is realized as a (double) minimax value of a stochastic functional (cf. Bensoussan and Lions [2]). In contradistinction to the Hamilton-Jacobi equation, which arises as a (single) minimax value associated with the Legendre transformation, the inequality (1.1) involves the introduction of an optimal stopping time. For the sake of brevity, we omit the details and instead refer the reader to [2]. Nonetheless, we should observe the nonstochastic applications of (1.1), such as certain Stefan problems (cf. Bensoussan and Friedman [1]). The methods of this paper are obviously applicable to the simpler model discussed in [3]. We have chosen a (natural) Neumann boundary condition rather than a (forced) Dirichlet boundary condition to simplify the exposition of §3, since piecewise linear elements cannot vanish on an arbitrary surface Σ . We mention finally that higher rates of convergence are to be expected if $H = 0$, viz., $O(h^{1/2})$ convergence of the solutions of the penalized equations and $O(h^{1/2})$ convergence of the finite element approximations, with $h \rightarrow 0$.

§2. The Penalization.

Let Ω be a smoothly bounded domain in \mathbb{R}^N and let $A(t)$ be a uniformly coercive, elliptic operator in divergence form:

$$(2.1) \quad A(t) = - \sum_{i,j} D_i a_{ij}(\cdot, t) D_j + \sum_k b_k(\cdot, t) D_k + c(\cdot, t) ,$$

where $D_i = \frac{\partial}{\partial x_i}$, $a_{ij} \in C^1(\bar{Q})$ and $b_k, c \in C^1([0, T] ; C(\bar{\Omega}))$ and

$$(2.2) \quad a_t(u, u) \geq c_0 \|u\|_{H^1(\Omega)}^2 , \quad c_0 > 0 ,$$

for all $u \in H^1(\Omega)$. Here

$$(2.3) \quad a_t(u, u) = \|u\|^2 = \sum_{i,j} (D_i u, a_{ij} D_j u)_{L^2(\Omega)} + \sum_k (b_k D_k u, u)_{L^2(\Omega)} + (cu, u)_{L^2(\Omega)} .$$

Let $H : \bar{\Omega} \times [0, T] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ be a continuous function satisfying, generically,

$$(2.4) \quad \begin{aligned} (i) & \quad |H(x, s, v, Dv) - H(x, t, w, Dw)| \leq C(|s - t| + |v - w| + |Dv - Dw|) , \\ (ii) & \quad |H(x, t, v, Dv)| \leq h(x, t) + |v| + |Dv| , \end{aligned}$$

where $D = (D_1, \dots, D_N)$ and $h(\cdot, \cdot)$ is a bounded measurable function; $|\cdot|$ denotes an appropriate Euclidean norm. Let ψ_1 and ψ_2 be given satisfying

$$(2.5) \quad \begin{aligned} (i) & \quad \psi_i \in H^2(Q) , \quad i = 1, 2, \\ (ii) & \quad \psi_1 \leq \psi_2 , \quad \text{in } Q . \end{aligned}$$

We shall define a class of penalized problems depending on a parameter $\varepsilon > 0$.

These provide both a tool for proving the existence of a solution of (1.1) in the class

$$(2.6) \quad X_1 = L^2(0, T ; H^2(\Omega)) \cap H^1[0, T ; L^2(\Omega)]$$

as well as defining the base equation for the Faedo-Galerkin approximation of the next section. This penalized equation is exactly the one introduced in [2].

We define now the penalized problem, for $\epsilon > 0$,

$$(2.7) \quad \begin{aligned} \text{(i)} \quad & -\frac{\partial u_\epsilon}{\partial t} + A(t)u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - \psi_2)^+ - \frac{1}{\epsilon}(u_\epsilon - \psi_1)^- + H(x, t, u_\epsilon, Du_\epsilon) = 0, \quad \text{in } Q, \\ \text{(ii)} \quad & \frac{\partial u_\epsilon}{\partial \nu} = 0, \quad \text{on } \Sigma, \\ \text{(iii)} \quad & u_\epsilon(\cdot, T) = \bar{u}, \end{aligned}$$

The functions $(\cdot)^+$ and $(\cdot)^-$ are the positive and negative parts of the identity, defined so that

$$t = t^+ - t^-, \quad t \in \mathbb{R}^1; \quad t^+, t^- \geq 0.$$

Remark 2.1. It is known that (2.6) possesses a solution u_ϵ satisfying

$$(2.8) \quad u_\epsilon \in L^2(0, T; H^2(\Omega)) \cap H^2[0, T; L^2(\Omega)] = X_2$$

if say

$$(2.9i) \quad \bar{u} \in H^2(\Omega).$$

If, in addition,

$$(2.9ii) \quad \psi_1(\cdot, T) \leq \bar{u} \leq \psi_2(\cdot, T),$$

then one can use the equations (2.7) to prove the existence of solutions u of (1.1) in the regularity class X_1 of (2.6), under the hypotheses described earlier in this section by the arguments of the proof of Théorème 1.1, p. 449, of [2] (cf. [2, pp. 449-455]). Actually, one can prove the existence of $u \in X_1$ satisfying (1.1i, v) and

$$(2.10) \quad \int_0^T (A(t)u + H(\cdot, t, u, Du) - \frac{\partial u}{\partial t}, v - u)_{L^2(\Omega)} dt \geq 0$$

for all $v \in L^2(\Omega)$, $\psi_1 \leq v \leq \psi_2$, from which (1.1ii, iii, iv) follow if ψ_1 and ψ_2 are coincident only on a set of measure zero:

$$(2.11) \quad \text{meas} \{ \psi_1 = \psi_2 \} = 0 \quad (\text{in } Q).$$

Standard methods also give the characterization of (1) of the abstract.

Remark 2.2. As remarked in [2], there is no loss of generality in assuming that

$H(\cdot, \cdot, \cdot, \cdot)$ defines a monotone mapping from $H^1(\Omega) \subset L^2(\Omega)$ into $L^2(\Omega)$:

$$(2.12) \quad (H(\cdot, \cdot, v, Dv) - H(\cdot, \cdot, w, Dw), v - w)_{L^2(\Omega)} \geq 0$$

for all $v, w \in H^1(\Omega)$. We shall assume (2.1-2.5) and (2.8, 2.9, 2.11, 2.12) for the sequel. The reduction to (2.12) is achieved by an integrating factor and change of variable.

Proposition 2.1. The following 'a priori' estimates hold for the solutions of (2.7):

$$(2.13) \quad \begin{aligned} (i) \quad & \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{L^2(Q)} + \|u_\epsilon\|_{L^2(0,T; H^1(\Omega))} \leq C, \\ (ii) \quad & \|u_\epsilon\|_{L^2(0,T; H^2(\Omega))} \leq C, \\ (iii) \quad & \frac{1}{\sqrt{\epsilon}} \|(u_\epsilon - \psi_2)^+\|_{L^2(Q)} + \frac{1}{\sqrt{\epsilon}} \|(u_\epsilon - \psi_1)^-\|_{L^2(Q)} \leq C, \\ (iv) \quad & \frac{1}{\sqrt{\epsilon}} \left\| \frac{\partial}{\partial t} (u_\epsilon - \psi_2)^+ \right\|_{L^2(Q)} + \frac{1}{\sqrt{\epsilon}} \left\| \frac{\partial}{\partial t} (u_\epsilon - \psi_1)^- \right\|_{L^2(Q)} \leq C. \end{aligned}$$

Proof: The estimates (2.13i, iii, iv) are derived as in [2, pp. 449-455]. Estimate (2.13ii) follows from these and (2.7).

Remark 2.3. These estimates are sufficient to prove the existence of a solution

u of (1.1) in X_1 . Note that the existence of $\frac{\partial^2 u_\epsilon}{\partial t^2}$ was used to obtain the 'a priori' estimate of $\frac{\partial u_\epsilon}{\partial t}$.

Theorem 2.2. The solutions of the penalized problems (2.7) converge to the solution of (1.1) with order $O(\epsilon^{1/4})$ in the norm of $L^2(0,T; H^1(\Omega))$:

$$(2.14) \quad \int_0^T \|u - u_\epsilon\|_{H^1(\Omega)}^2 dt \leq C \epsilon^{1/2}.$$

Proof: We note that the function r_ϵ , defined by

$$(2.15i) \quad r_\epsilon = u_\epsilon - (u_\epsilon - \psi_2)^+ + (u_\epsilon - \psi_1)^- .$$

satisfies

$$(2.15ii) \quad \psi_1 \leq r_\epsilon \leq \psi_2 .$$

Thus, from (2.10) we have, using integration by parts,

$$(2.16) \quad \left(-\frac{\partial u}{\partial t} + H(\cdot, t, u, Du) , r_\epsilon - u \right)_{L^2(\Omega)} + a_t(u_\epsilon, r_\epsilon - u) \geq 0$$

for almost all t , $0 < t < T$. Multiplying (2.7) by $r_\epsilon - u$ and subtracting (2.16) from the resultant integrated equation gives

$$(2.17) \quad \begin{aligned} & -\frac{1}{2} \frac{d}{dt} \|u - u_\epsilon\|_{L^2(\Omega)}^2 + a_t(u - u_\epsilon, u - u_\epsilon) \\ & + (H(\cdot, t, u_\epsilon, Du_\epsilon) - H(\cdot, t, u, Du) , u_\epsilon - u)_{L^2(\Omega)} \\ & \leq (A(t)u - \frac{\partial u}{\partial t} , (u_\epsilon - \psi_1)^- - (u_\epsilon - \psi_2)^+)_{L^2(\Omega)} + (H(\cdot, t, u_\epsilon, Du_\epsilon) - H(\cdot, t, u, Du) , \\ & \quad (u_\epsilon - \psi_1)^- - (u_\epsilon - \psi_2)^+)_{L^2(\Omega)} \end{aligned}$$

where we have used (cf. [2, p. 209])

$$-(u_\epsilon - \psi_1)^- \cdot (r_\epsilon - u) \geq 0 , \quad (u_\epsilon - \psi_2)^+ \cdot (r_\epsilon - u) \geq 0 .$$

Integration of (2.17) over $(0, T)$, together with (1.1v), (2.4i), (2.7iii), (2.12) and the elementary inequality,

$$(2.18) \quad |(f, g)| \leq \frac{1}{2} (\eta \|f\|^2 + \eta^{-1} \|g\|^2) ,$$

yields

$$\begin{aligned} \int_0^T a_t(u - u_\epsilon, u - u_\epsilon) dt & \leq C [\| (u_\epsilon - \psi_1)^- \|_{L^2(\Omega)}^2 + \| (u_\epsilon - \psi_2)^+ \|_{L^2(\Omega)}^2 \\ & + \| (u_\epsilon - \psi_1)^- \|_{L^2(\Omega)}^2 + \| (u_\epsilon - \psi_2)^+ \|_{L^2(\Omega)}^2] \end{aligned}$$

and the proof of (2.14) is concluded by use of (2.13iii).

§3. Continuous Time Finite Element Approximations.

For $h > 0$, let T_h be a triangulation of the given domain Ω . Thus,

$$(3.1) \quad \bar{\Omega} = \bigcup_{\tau \in T_h} \tau$$

where τ is a typical (closed) element in the simplicial decomposition T_h ; in particular, we permit nonsimplicial elements near the boundary. Let M_h denote the linear space of continuous piecewise linear trial functions determined by T_h :

$$(3.2) \quad M_h = \{X \in C(\bar{\Omega}) : X|_{\tau} \text{ is linear } \forall \tau \in T_h\} .$$

Let E_h be the Ritz-Galerkin $H^1(\Omega)$ projection defined by

$$(3.3i) \quad (E_h \varphi, X)_{H^1(\Omega)} = (\varphi, X)_{H^1(\Omega)} , \text{ for all } X \in M_h ,$$

for each fixed $\varphi \in H^1(\Omega)$; here we use

$$(3.3ii) \quad (\varphi, \psi)_{H^1(\Omega)} = (\nabla \varphi, \nabla \psi)_{L^2(\Omega)} + \frac{1}{|\Omega|} \int_{\Omega} \varphi \int_{\Omega} \psi ,$$

which defines a norm equivalent to the standard $H^1(\Omega)$ norm in the usual way

(cf. [4]). Let I_h denote the interpolation operator. We shall assume:

$$(3.4) \quad \psi_1 \leq I_h \psi_1 \leq I_h \psi_2 \leq \psi_2 .$$

Roughly speaking, (3.4) asserts that ψ_1 is smooth and convex and ψ_2 is smooth and concave; indeed, these assumptions guarantee (3.4). We make the standard finite element assumptions (cf. [5])

$$(3.5) \quad \|F_h \varphi - \varphi\|_{H^j(\Omega)} \leq Ch^{2-j} \|\varphi\|_{H^2(\Omega)} , \quad j = 0, 1, \varphi \in H^2(\Omega) ,$$

for $F_h = E_h$ and for $F_h = I_h$.

We are now in a position to define the finite element approximation via a standard Faedo-Galerkin method based upon (2.7).

Definition 3.1. The finite element approximation $u_\epsilon^h : [0, T] \rightarrow M_h$ is the unique element in

$$X_1 = H^1[0, T; L^2(\Omega)] \cap L^2(0, T; H^1(\Omega))$$

satisfying

$$(3.6i) \quad \langle J_\epsilon(u_\epsilon^h), \chi \rangle = 0, \quad \text{for all } \chi \in M_h,$$

$$(3.6ii) \quad u_\epsilon^h(\cdot, T) = I_h \bar{u}.$$

Here, J_ϵ is the map,

$$J_\epsilon : X_1 \rightarrow L^2(0, T; H^{-1}(\Omega))$$

defined by the pointwise relation on $(0, T)$:

$$(3.6iii) \quad \langle J_\epsilon(\varphi), \psi \rangle = \left(-\frac{\partial \varphi}{\partial t} + G(\epsilon, \varphi, \psi_1, \psi_2), \psi \right)_{L^2(\Omega)} + a_t(\varphi, \psi) = 0,$$

where

$$(3.6iv) \quad G(\epsilon, \varphi, \psi_1, \psi_2) = \frac{1}{\epsilon} g(\varphi, \psi_1, \psi_2) + H(\cdot, \cdot, \varphi, D\varphi),$$

$$(3.6v) \quad g(\varphi, \psi_1, \psi_2) = -(\varphi - \psi_1)^- + (\varphi - \psi_2)^+.$$

Remark 3.1. The existence of a unique solution of (3.6) is standard and may be achieved via the theory of pseudomonotone operators.

Proposition 3.1. The following 'a priori' estimates hold for the solutions

$\{u_\epsilon^h\}_{\epsilon, h}$ of (3.5):

$$(3.7) \quad \begin{aligned} (i) \quad & \left\| \frac{\partial u_\epsilon^h}{\partial t} \right\|_{L^2(Q)} \leq C_1 h^2/\epsilon + C_2, \\ (ii) \quad & \|u_\epsilon^h\|_{L^2(0, T; H^1(\Omega))} \leq C, \\ (iii) \quad & \frac{1}{\sqrt{\epsilon}} \|g(u_\epsilon^h, \psi_1, \psi_2)\|_{L^2(Q)} \leq C, \end{aligned}$$

where C, C_1, C_2 are independent of ϵ and h .

Proof: Select $\chi = u_\epsilon^h - I_h \psi_1$ in (3.6i). Then, we have

$$(3.8) \quad \begin{aligned} & -\frac{d}{dt} \|u_\epsilon^h\|_{L^2(\Omega)}^2 + a_t(u_\epsilon^h, u_\epsilon^h) + \frac{1}{\epsilon} \|(u_\epsilon^h - \psi_1)^-\|_{L^2(\Omega)}^2 \\ & \leq \left(-\frac{\partial}{\partial t} u_\epsilon^h, I_h \psi_1\right)_{L^2(\Omega)} + a_t(u_\epsilon^h, I_h \psi_1) + (H(u_\epsilon^h), I_h \psi_1)_{L^2(\Omega)} \end{aligned}$$

where we have used (3.4) after an addition and subtraction, have written $H(u_\epsilon^h)$ for the last term in (3.6iv) and have noted

$$(u_\epsilon^h - I_h \psi_1) \cdot (u_\epsilon^h - \psi_2)^+ \geq 0 .$$

Now integrate over (t, T) . Integrating the first term on the r.h.s. of (3.8) by parts, applying (2.4ii) to the third term and estimating the first, the second and part of the third term by the inequality (2.18) we obtain, for appropriate choices of n ,

$$(3.9) \quad \begin{aligned} & \|u_\epsilon^h(\cdot, t)\|_{L^2(\Omega)}^2 + \int_t^T \{a_t(u_\epsilon^h, u_\epsilon^h) + \frac{1}{\epsilon} \|(u_\epsilon^h - \psi_1)^-\|_{L^2(\Omega)}^2\} dt \\ & \leq C_1 + C_2 \int_t^T \|u_\epsilon^h\|_{L^2(\Omega)}^2 dt . \end{aligned}$$

Gronwall's inequality applied to (3.9) yields, in particular,

$$(3.10) \quad \|u_\epsilon^h\|_{L^2(0, T; H^1(\Omega))}^2 + \frac{1}{\epsilon} \|(u_\epsilon^h - \psi_1)^-\|_{L^2(Q)}^2 \leq C .$$

A parallel argument, with $\chi = u_\epsilon^h - I_h \psi_2$, yields

$$(3.11) \quad \|u_\epsilon^h\|_{L^2(0, T; H^1(\Omega))}^2 + \frac{1}{\epsilon} \|(u_\epsilon^h - \psi_2)^+\|_{L^2(Q)}^2 \leq C .$$

Clearly, it remains only to show that

$$(3.12) \quad \left\| \frac{\partial u_\epsilon^h}{\partial t} \right\|_{L^2(Q)} \leq C_1 h^2/\epsilon + C_2 .$$

Setting $\chi = -\frac{\partial u_\epsilon}{\partial t}$ in (3.6i) we have, after adding and subtracting $\frac{\partial \psi_1}{\partial t}$ and integrating over $(0, T)$,

$$(3.13) \quad \begin{aligned} & \left\| \frac{\partial u_\epsilon^h}{\partial t} \right\|_{L^2(Q)}^2 + a(u_\epsilon(\cdot, 0), u_\epsilon(\cdot, 0)) + \frac{1}{\epsilon} \left\| (u_\epsilon^h(\cdot, 0) - \psi_1(\cdot, 0))^- \right\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{\epsilon} \left((I_h \bar{u} - \psi_1(\cdot, T))^- , \frac{\partial \psi_1}{\partial t} \right)_{L^2(\Omega)} + a(\bar{u}, \bar{u}) + (H(u_\epsilon^h) , \frac{\partial u_\epsilon^h}{\partial t})_{L^2(\Omega)}. \end{aligned}$$

However, by (2.9ii),

$$\begin{aligned} (I_h \bar{u} - \psi_1(\cdot, T))^- &= (I_h \bar{u} - I_h \psi_1(\cdot, T))^- + (I_h \psi_1(\cdot, T) - \psi_1(\cdot, T))^- \\ &= (I_h \psi_1(\cdot, T) - \psi_1(\cdot, T))^- . \end{aligned}$$

Thus, by (2.5i), (3.5), and (2.18) we obtain from (3.13),

$$(3.14) \quad \left\| \frac{\partial u_\epsilon^h}{\partial t} \right\|_{L^2(Q)}^2 \leq C_1 \left(\frac{h^2}{\epsilon} + a(\bar{u}, \bar{u}) + C \right) ,$$

from which (3.12) follows and the proof is concluded.

Corollary 3.2. There is a constant C , independent of ϵ and h , such that

$$(3.15) \quad \left\| J_\epsilon(u_\epsilon^h) \right\|_F \leq C[h^3/\epsilon + h/\sqrt{\epsilon} + h]$$

where F denotes the dual of $H^1(\Omega)$.

Proof: Let $v \in H^1(\Omega)$. Then, by (3.3), we have

$$\begin{aligned} \langle J_\epsilon(u_\epsilon^h), v \rangle &= \langle J_\epsilon(u_\epsilon^h), v - E_h v \rangle \\ &= \left\langle -\frac{\partial u_\epsilon}{\partial t} + G(\epsilon, u_\epsilon^h, \psi_1, \psi_2), v - E_h v \right\rangle \end{aligned}$$

so that

$$\begin{aligned}
(3.16) \quad \|J_\epsilon(u_\epsilon^h)\|_F &= \sup_{\|v\|_{H^1(\Omega)}=1} |\langle J_\epsilon(u_\epsilon^h), v \rangle| \\
&\leq C_1(h^2/\epsilon + 1/\sqrt{\epsilon} + C_2) \sup_{\|v\|_{H^1(\Omega)}=1} \|v - E_h v\|_{L^2(\Omega)}.
\end{aligned}$$

Now (cf. [5]),

$$\sup_{\|v\|_{H^1(\Omega)}=1} \|v - E_h v\|_{L^2(\Omega)} \leq Ch$$

so that (3.15) is immediate from (3.16).

Remark 3.2. We are now prepared to state the major result of the paper.

Theorem 3.3. There is a constant C , independent of ϵ and h , such that

$$(3.17) \quad \|u_\epsilon - u_\epsilon^h\|_{L^2(0,T; H^1(\Omega))} \leq C \left[\int_0^T \|J_\epsilon(u_\epsilon^h)\|_F^2 dt \right]^{1/2}.$$

In particular,

$$(3.18i) \quad \|u - u_\epsilon^h\|_{L^2(0,T; H^1(\Omega))} \leq C[h^3/\epsilon + h/\sqrt{\epsilon} + \sqrt{\epsilon}].$$

If the choice $h = \epsilon^{3/4}$ is made then

$$(3.18ii) \quad \|u - u_\epsilon^h\|_{L^2(0,T; H^1(\Omega))} \leq C h^{1/3}.$$

Proof: In light of Theorem 2.2 and Corollary 3.2, it suffices to prove (3.17).

Using the monotonicity of $g(\cdot, \psi_1, \psi_2)$ and (2.12) we have,

$$\begin{aligned}
a_t(u_\epsilon - u_\epsilon^h, u_\epsilon - u_\epsilon^h) &\leq \langle J_\epsilon(u_\epsilon) - J_\epsilon(u_\epsilon^h), u_\epsilon - u_\epsilon^h \rangle \\
&= \langle J_\epsilon(u_\epsilon^h), u_\epsilon - u_\epsilon^h \rangle \\
&\leq \|J_\epsilon(u_\epsilon^h)\|_F \|u_\epsilon - u_\epsilon^h\|_{H^1(\Omega)},
\end{aligned}$$

so that (3.17) follows.

Remark 3.3. If $H \equiv 0$, then (2.13iii) can be strengthened so that $\sqrt{\epsilon}$ is replaced by ϵ . This leads to a rate of convergence in Theorem 2.2 of order $\sqrt{\epsilon}$ (cf. the proof in the case of one obstacle in [2, p. 224]). The 'a priori' estimates of §3 remain unchanged and the choice $\epsilon = h$ leads to a convergence rate of \sqrt{h} , replacing $\sqrt[3]{h}$ in (3.18ii).

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ABSTRACT (Continued)

$$K(u) = \{ \lambda(v - u) : \psi_1 \leq v \leq \psi_2, \lambda > 0 \} ;$$

$$(liii) \quad \left. \frac{\partial u}{\partial \nu} \right|_{\Sigma} = 0, \quad u(\cdot, T) = \bar{u},$$

where

$$Q = \Omega \times (0, T), \quad \Sigma = \partial\Omega \times (0, T).$$

Such inequalities arise in the characterization of saddle-point payoffs u in two person differential games with stopping times as strategies. In this case, H is the Hamiltonian in the formulation. A numerical scheme for approximating u is obtained by the continuous time, piecewise linear, Galerkin approximation of a so-called penalized equation. A rate of convergence to u of order $O(h^{1/3})$ is demonstrated in the $L^2(0, T; H^1(\Omega))$ norm, where h is the maximum diameter of a given triangulation.