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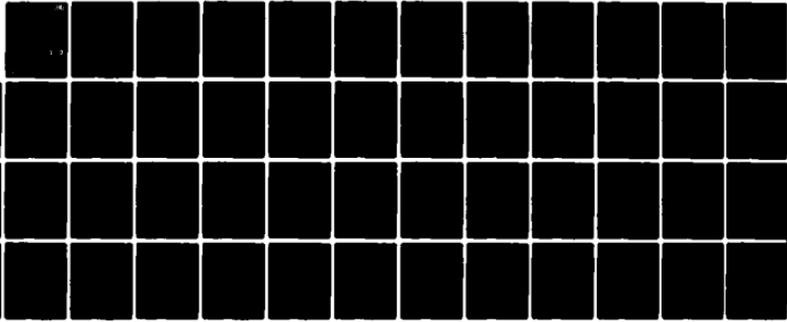
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OPTIMAL EXIT PROBABILITIES AND DIFFERENTIAL GAMES.(U)
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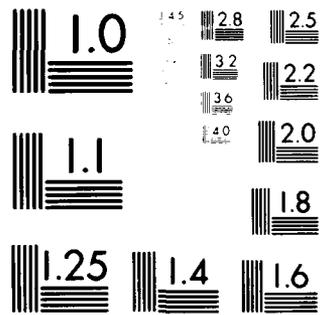
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OPTIMAL EXIT PROBABILITIES AND DIFFERENTIAL GAMES

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Optimal Exit Probabilities and Differential Games

by

Wendell H. Fleming and Chun-Ping Tsai

Abstract: The problem is to control the drift of a Markov diffusion process in such a way that the probability that the process exits from a given region D during a given finite time interval is minimum. An asymptotic formula for the minimum exit probability when the process is nearly deterministic is given. This formula involves the lower value of an associated differential game. It is related to a result of Venttsel and Freidlin for nearly deterministic, uncontrolled diffusions.

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OPTIMAL EXIT PROBABILITIES AND DIFFERENTIAL GAMES

Wendell H. Fleming and Chun-Ping Tsai

1. Introduction. Let ξ be an n -dimensional stochastic process with continuous sample paths, defined for times $t \geq 0$. Let D be a given bounded open subset of n -dimensional E^n . If $\xi(0) \in D$, the exit time is the first time τ such that $\xi(\tau) \in \partial D$. For fixed $T > 0$ the exit probability is $P(\tau \leq T)$.

In this paper, we suppose that ξ is a controlled process, which obeys the stochastic differential equation

$$(1.1) \quad d\xi = b[\xi(t), y(t)]dt + \epsilon^{1/2}dw,$$

where $y(t)$ is a control applied at time t , $\epsilon > 0$ a parameter, and $w(t)$ an n -dimensional brownian motion. We assume that $y(t) \in Y$, where $Y \in E^m$ is a given compact set. Moreover, the control processes y admitted in (1.1) have the feedback form

$$y(t) = \underline{y}(t, \xi(t))$$

where \underline{y} is any Borel measurable function from $[0, T] \times E^n$ into Y . As initial data we have $\xi(0) = x$, for given vector $x \in D$. For general background on these concepts see [FR1, Chap. VI].

Let us denote the exit probability by $\tau_{\underline{y}}^\epsilon$, to indicate dependence on the feedback control law \underline{y} and on ϵ . Let $q_{\underline{y}}^\epsilon = P(\tau_{\underline{y}}^\epsilon \leq T)$. We seek a control law which minimizes the exit

probability $q_{\underline{y}}^{\epsilon}$. From the point of view of applications, this is a reasonable criterion of performance of the control system, if D is considered as a region of states in which the system operates in an "acceptable" way. The minimum exit probability is denoted by

$$(1.2) \quad q^{\epsilon} = \min_{\underline{y}} q_{\underline{y}}^{\epsilon}.$$

In §2 we shall replace the initial time 0 by any initial time s , $0 \leq s < T$. Then $q^{\epsilon} = q^{\epsilon}(s, x)$ is a function of the initial time s and initial state $x = \xi(s)$. The function q^{ϵ} satisfies the dynamic programming equation with suitable boundary conditions (see (2.4), (2.5) below). In special cases, this boundary value problem was solved numerically by Dorato and Van Melaert [DVM]. However, it is generally difficult to get effective information about q^{ϵ} and the optimal control law in this way. Instead, we seek an asymptotic formula for q^{ϵ} valid for small ϵ . Our main results (Theorems 4.1 and 4.2) assert that

$$(1.3) \quad -\lim_{\epsilon \rightarrow 0} \epsilon \log q^{\epsilon} = I,$$

where I is the lower value of a certain differential game. The proofs involve some technical complications. However, two different heuristic arguments can be given to suggest the validity of (1.3). One of these is as follows (a second heuristic derivation of (1.3) based on the dynamic programming equation for q^{ϵ} is given in §2.) Given a feedback control \underline{y} the drift coefficient in the stochastic differential equation (1.1) is

$$(1.4) \quad b_{\underline{y}}(t, \xi) = b(\xi, \underline{y}(t, \xi)).$$

If $b_{\underline{y}}$ is Lipschitz, then the Ventcel-Freidlin estimates [Fr 2, Chap. 14] imply

$$(1.5) \quad -\lim_{\epsilon \rightarrow 0} \epsilon \log q_{\underline{y}}^{\epsilon} = I_{\underline{y}},$$

$$(1.6) \quad I_{\underline{y}} = \min_{\phi \in \mathcal{F}} \frac{1}{2} \int_0^T |b_{\underline{y}}(t, \phi(t)) - \dot{\phi}(t)|^2 dt,$$

and where \mathcal{F} is the class of all $\phi \in C^1([0, T]; E^n)$ such that $\phi(0) = x$ and $\phi(\theta) \in \partial D$ for some $\theta \in [0, T]$. A stochastic control proof of (1.6) is given in [F4, §7]. If we set

$$(1.7) \quad \dot{\phi}(t) = z(t), \quad 0 \leq t \leq T,$$

then we can regard $z(t)$ as a new control associated with the minimum problem (1.6). Intuitively, minimizing $q_{\underline{y}}^{\epsilon}$ corresponds in the limit as $\epsilon \rightarrow 0$ to choosing \underline{y} to maximize $I_{\underline{y}}$. This leads us to consider a differential game, described formally as follows. Let

$$(1.8) \quad L(x, y, z) = \frac{1}{2} |b(x, y) - z|^2.$$

In (1.6) we may replace the upper limit T by the first time θ such that $\phi(\theta) \in \partial D$, and take $\dot{\phi}(t) = b_{\underline{y}}(t, \phi(t))$ for $t > \theta$. The integral in (1.6) is then

$$(1.9) \quad \int_0^{\theta} L(\phi(t), y(t), z(t)) dt,$$

where $y(t) = \underline{y}(t, \phi(t))$ is chosen by the original controller and $z(t)$ is chosen by a new second controller who knows \underline{y} . The second controller then knows $y(t)$ as well as $\phi(t)$.

In the formal description of the differential game, $\phi(t)$ is the state of the game at time t , with initial data $\phi(0) = x$, $x \in D$. The game dynamics are (1.7), subject to the restriction that $\phi(t)$ must reach ∂D at some time $\theta \in (0, T]$. The original controller seeks to maximize the payoff (1.9), and the second controller seeks to minimize (1.9). Note that $y(t)$ appears in (1.9) but not in (1.7). Thus the role of the maximizing controller is a passive one. The term "lower value" refers to the advantage in information which the minimizing controller has (see [Fr 1]).

The intuitive description above does not suffice to define rigorously a differential game with lower value I . In the rigorous treatment we shall obtain I as the limit of values of corresponding discrete-time games, following the method of [F1]. We have been unable to obtain the limit in (1.3) directly from the Ventcel-Freidlin estimates (1.5). The restriction by Lipschitz under which (1.5) is derived is too restrictive for the problem of optimal exit probability; and there is no guarantee of uniformity with respect to \underline{y} of the limit in (1.5).

The outline of the paper is as follows. In §2 we introduce the function $I^\epsilon = -\epsilon \log q^\epsilon$. Note that (1.3) asserts that $I^\epsilon \rightarrow I$ as $\epsilon \rightarrow 0$. This logarithmic transformation changes the

dynamic programming equation for q^ϵ into the nonlinear parabolic partial differential equation (2.7) for I^ϵ . As $\epsilon \rightarrow 0$, this equation degenerates into a first-order equation, which is just the Isaacs equation for the lower value I^ϵ of the differential game described intuitively above. The function $I^\epsilon(s,x)$ tends to $+\infty$ as $s \rightarrow T$. We introduce a penalty function method, in which I^ϵ is replaced by a solution I_M^ϵ of (2.7) with terminal data $M\phi(x)$ when $s = T$. As $M \rightarrow \infty$, I_M^ϵ increases to I^ϵ . In §2 we give a priori estimates for I^ϵ , based on standard estimates for brownian motion and on the maximum principle for parabolic equations.

In §3 we introduce penalized differential games, by a discretization procedure similar to [F1], [F2]. For technical reasons, we impose a bound $|z| \leq c$ for the second controller. It turns out that for sufficiently large $c \geq c(M)$, I_M^ϵ is the lower value of the penalized stochastic differential game. As $\epsilon \rightarrow 0$, $I_M^\epsilon \rightarrow I_M$ where I_M is the lower value of the corresponding penalized deterministic differential game.

As $M \rightarrow \infty$, I_M increases to a limit I . In §4 we show that $I^\epsilon \rightarrow I$ as $\epsilon \rightarrow 0$, and identify I as the lower value of the differential game described formally above.

2. Preliminary Results. We make the following assumptions throughout the paper. D is a bounded, open subset of E^n , with boundary ∂D a manifold of class C^3 . The control space Y is a compact subset of E^n . We consider times in the interval $[0,T]$. The vector function b in (1.1) is of class $C^1(E^n \times Y; E^n)$.

Moreover, for some constants B_1, B_2

$$|b(x,y)| \leq B_1,$$

$$|b(x,y) - b(\bar{x},y)| \leq B_2|x-\bar{x}|.$$

This is actually no restriction since b is C^1 and we stop the solution ξ to (1.1) when $\xi(t)$ leaves the bounded set D or when $t = T$. The function L in (1.8) then satisfies

$$(2.1) \quad |L(x,y,z) - L(\bar{x},y,z)| \leq \Lambda(1+|z|)|x-\bar{x}|.$$

$$(2.2) \quad c_1|z|^2 - c_2 \leq L,$$

for some constants $\Lambda, c_1 > 0, c_2$.

Let \mathcal{Y} be the class of Borel measurable functions γ from $[0,T] \times E^n$ into Y . For a feedback control law $\gamma \in \mathcal{Y}$, the stochastic differential equations (1.1) take the form

$$(2.3) \quad d\xi = b_\gamma(t, \xi(t))dt + \epsilon^{1/2}dw$$

with b_γ as in (1.4). We consider (2.3) with initial data $\xi(s) = x$, where $0 \leq s < T, x \in D$. (Later, we specialize by taking $s = 0$.) This is a problem of completely observed controlled diffusions [FR1, Chap. VI]. However, instead of the kind of performance criterion considered in [FR1] we wish to minimize the exit probability

$$q_Y^\epsilon(s, x) = P(\tau_Y^\epsilon \leq T),$$

with τ_Y^ϵ the first time $t > s$ such that $\xi(t) \in \partial D$. Let

$$Q = (0, T) \times D, \quad \bar{Q} = [0, T] \times \bar{D}, \quad \bar{Q}_0 = \bar{Q} - \{T\} \times \partial D.$$

Given $A \subset E^{n+1}$ and $0 < \beta \leq 1$ we say that a function $q(s, x)$ has finite β -norm on A if there exists a constant K such that

$$|q(s, x) - q(s', x')| \leq K(|x - x'|^\beta + |s - s'|^{\beta/2})$$

for all $(s, x), (s', x') \in A$. We say that $q \in C_\beta^{0,1}(A)$ if q and the gradient q_x in the variables $x = (x^1, \dots, x^n)$ have finite β -norm on A . If $q, q_x, q_s, q_{x^i x^j}, i, j = 1, \dots, n$ have finite β -norm on A , then we say that $q \in C_\beta^{1,2}(A)$.

The minimum exit probability $q^\epsilon(s, x)$ belongs to $C_\beta^{1,2}(A)$ for any compact $A \subset \bar{Q}_0$ and $\beta \in (0, 1)$. Moreover, in \bar{Q}_0 the following dynamic programming equation holds:

$$(2.4) \quad 0 = q_s^\epsilon + \frac{\epsilon}{2} \Delta_x q^\epsilon + \min_{y \in Y} q_x^\epsilon \cdot b(x, y),$$

where Δ_x is the Laplacean. See [FR1, p. 161], also the Remark below. The boundary data are

$$(2.5) \quad q^\epsilon(s, x) = 1 \text{ for } s < T, x \in \partial D; \quad q^\epsilon(T, x) = 0, \quad x \in D.$$

Let

$$(2.6) \quad I^\epsilon(s, x) = -\epsilon \log q^\epsilon(s, x).$$

By elementary calculus I^ϵ satisfies the nonlinear parabolic partial differential equation

$$(2.7) \quad 0 = I_s + \frac{\epsilon}{2} \Delta_x I^\epsilon + H(x, I_x^\epsilon),$$

$$(2.8) \quad H(x, p) = -\frac{1}{2} |p|^2 + \max_{y \in Y} p \cdot b(x, y).$$

Here $p = (p_1, \dots, p_n)$ denotes a row vector.

From (2.5) the boundary data are

$$(2.9) \quad I^\epsilon(s, x) = 0 \text{ for } s < T, x \in \partial D; I^\epsilon(T, x) = +\infty, x \in D.$$

The function L in (1.8) is strictly convex and quadratic as a function of z . The function dual to L , in the sense of duality for concave and convex functions, is

$$(2.10) \quad H(x, y, p) = \min_z [L(x, y, z) + p \cdot z],$$

where the min is over all $z \in E^n$. An easy calculation shows that

$$H(x, y, p) = -\frac{1}{2} |p|^2 + p \cdot b(x, y),$$

and therefore by (2.8), (2.10)

$$(2.11) \quad H(x,p) = \max_{y \in Y} \min_z [L(x,y,z) + p \cdot z].$$

Equation (2.7) then takes the form

$$(2.7') \quad 0 = I_S^\epsilon + \frac{\epsilon}{2} \Delta_x I^\epsilon + \max_{y \in Y} \min_z [L(x,y,z) + I_x^\epsilon \cdot z].$$

This is the dynamic programming equation associated with a stochastic differential game. Similar equations were considered in [F2]. When $\epsilon = 0$, (2.7') reduces to a first order partial differential equation, which is the Isaacs equation for the non-stochastic differential game described formally in §1. This provides a second heuristic argument for considering this differential game, in addition to the heuristic argument given in §1.

We shall approximate the infinite terminal condition $I^\epsilon(T,x) = +\infty$ by a large, but finite, terminal condition as follows. Let $\phi(x)$ be a function of class $C^2(\bar{D})$ satisfying

$$(2.12) \quad \gamma_1 \text{dist}(x, \partial D) \leq \phi(x) \leq \gamma_2 \text{dist}(x, \partial D) \quad \text{for } x \in D$$

$$\phi(x) = 0 \quad \text{for } x \notin D,$$

where γ_1, γ_2 are positive constants and $\text{dist}(x, \partial D)$ is the distance between a point x and the boundary ∂D . For each $M > 0$ let

$$(2.13) \quad q_M^\varepsilon(s, x) = \min_{\underline{Y}} E \exp\left\{-\frac{M\Phi[\xi(\tau_Y^\varepsilon \wedge T)]}{\varepsilon}\right\}$$

where as before ξ satisfies (2.3) with initial data $\xi(s) = x$ and $\tau_Y^\varepsilon \wedge T = \min[\tau_Y^\varepsilon, T]$. The minimum is taken among all feedback controls $\underline{Y} \in \mathcal{Y}$. This is a stochastic control problem of the kind considered in [FR1, Chap VI, §4,6]. According to results given there, q_M^ε satisfies the dynamic programming equation (2.4). The function q_M^ε belongs to $C_\beta^{0,1}(\bar{Q}) \cap C_\beta^{1,2}(A)$ for any compact $A \subset \bar{Q}_0$ and $\beta \in (0,1)$. Moreover, $q_M^\varepsilon(s, x)$ is also the minimum of the expectation in (2.13), taken among nonanticipative Y -valued control processes [FR1, pp. 162-163].

Remark. By estimates for parabolic equations, and the fact that q_M^ε is uniformly bounded, the partial derivatives $(q_M^\varepsilon)_s, (q_M^\varepsilon)_{x^i}, (q_M^\varepsilon)_{x^i x^j}$ satisfy Hölder conditions on any compact subset of \bar{Q}_0 which are uniform with respect to M for fixed $\varepsilon > 0$. See [LSU], also [FR1, Appendix E]. The boundary data are

$$q_M^\varepsilon(s, x) = 1, \quad \text{for } s < T, x \in \partial D; \quad q_M^\varepsilon(T, x) = \exp\left[-\frac{M\Phi(x)}{\varepsilon}\right], \quad x \in D.$$

Moreover, $0 < q_{M+1}^\varepsilon \leq q_M^\varepsilon < 1$. The limit $\tilde{q}^\varepsilon(s, x)$ of $q_M^\varepsilon(s, x)$ as $M \rightarrow \infty$ is a solution of (2.4) belonging to $C_\beta^{1,2}(A)$ for any compact $A \subset \bar{Q}_0$. Moreover, \tilde{q}^ε has the boundary data (2.5). A Verification Theorem [FR1, p. 159] implies that $\tilde{q}^\varepsilon = q^\varepsilon$ and that $q^\varepsilon(s, x)$ is the minimum exit probability. Actually, the proof of the Verification Theorem in [FR1] must be modified slightly since

q^ϵ is discontinuous at points (T, x) , $x \in \partial D$. This causes no difficulty, since the random variable $\xi(T)$ is absolutely continuous with respect to Lebesgue measure in E^n . Hence

$$P(\tau^\epsilon = T) \leq P(\xi(T) \in \partial D) = 0.$$

Let

$$(2.14) \quad I_M^\epsilon(s, x) = -\epsilon \log q_M^\epsilon(s, x).$$

Since $q_M^\epsilon \geq q^\epsilon$, and $q_M^\epsilon \downarrow q^\epsilon$ as $M \rightarrow \infty$,

$$(2.15) \quad I_M^\epsilon \leq I^\epsilon, \quad I_M^\epsilon \uparrow I^\epsilon \quad \text{as } M \rightarrow \infty.$$

Moreover, I_M^ϵ is a solution of (2.7) (or equivalently (2.7')) with the boundary data

$$(2.9_M) \quad I_M^\epsilon(s, x) = 0 \quad \text{for } s < T, \quad x \in \partial D; \quad I_M^\epsilon(T, x) = M\phi(x), \quad x \in D.$$

Let us now give some bounds for I^ϵ and I_M^ϵ . In these lemmas we write $I^\epsilon(s, x; T)$, $I_M^\epsilon(s, x; T)$ to emphasize dependence on the final time T . Since $H(x, p)$ does not depend on time, we have

$$(2.16) \quad I^\epsilon(s, x; T) = I^\epsilon(0, x; T-s), \quad I_M^\epsilon(s, x; T) = I_M^\epsilon(0, x; T-s).$$

Lemma 2.1. Let $T' < T$. There exists U such that $I^\epsilon(s, x; T) \leq U$ for $0 \leq s \leq T'$, $x \in D$, and for small ϵ . The

constant U depends on upper and lower bounds for $T - T'$, but
not on ϵ .

Proof. By (2.16) we may take $s = 0$ and impose the bounds
 $0 < k \leq T \leq K$. Since $|b| \leq B_1$, (2.3) implies

$$\xi^1(T) - x^1 \geq -B_1 T + \epsilon^{1/2} (w^1(T) - w^1(0))$$

where x^1 is the first component of x etc. Since D is
bounded, there exists a constant a such that $\xi^1(T) - x^1 \geq a$
implies $\tau_Y^\epsilon \leq T$. Hence

$$P(T^{-1/2} w^1(T) \geq (\epsilon T)^{-1/2} (a + B_1 T)) \leq P(\tau_Y^\epsilon \leq T) = q_Y^\epsilon.$$

Since $T^{-1/2} w^1(T)$ has a standard normal distribution, we have
[Fe1, p. 166]

$$\lim_{\lambda \rightarrow \infty} \lambda^{-2} \log P(T^{-1/2} w^1(T) \geq \lambda) = -\frac{1}{2}.$$

Hence, for small ϵ we have for all Y

$$-\epsilon \log q_Y^\epsilon \leq \frac{(a + B_1 T)^2}{T}.$$

By (1.2), $I^\epsilon = -\epsilon \log q^\epsilon$ satisfies the same inequality. In
Lemma 2.1 let $U = k^{-1} (a + B_1 K)^2$.

Lemma 2.2. Let $T' < T$. There exists a positive constant M_0 (depending on T' but not on ϵ) such that $I^\epsilon(s,x;T) \leq M_0 \text{dist}(x, \partial D)$ for $0 \leq s \leq T'$ and for small ϵ .

Proof. By Lemma 2.1 it suffices to verify such an estimate in a neighborhood of any $x_1 \in \partial D$. Let z_1 be the exterior unit normal to ∂D at x_1 ; and let

$$D_1 = \{x \in D: \text{dist}(x, x_1) < \rho\}$$

where ρ is to be chosen later. By (2.7')

$$I_s^\epsilon + \frac{\epsilon}{2} \Delta_x I_x^\epsilon + \max_{y \in Y} L(x, y, z_1) + I_x^\epsilon \cdot z_1 \geq 0.$$

The maximum occurs at some $\underline{y}^*(s, x)$, measurable in (s, x) [FR1, p. 199].

Choose any $T'' \in (T', T)$. For $s < T''$, $x \in D$, define the process ψ by

$$d\psi = z_1 dt + \epsilon^{1/2} dw, \quad \psi(s) = x.$$

Let $\theta^\epsilon = \min\{T'', \text{exit time of } \psi(t) \text{ from } D\}$.

From Itô's formula

$$I^\epsilon(s, x; T) \leq E \int_s^{\theta^\epsilon} L(\psi(t), \underline{y}^*, z_1) dt + I^\epsilon(\theta^\epsilon, \psi(\theta^\epsilon); T)$$

where $\underline{y}^* = \underline{y}^*(t, \psi(t))$. Moreover, $I^\epsilon(T'', x; T) \leq U$ by Lemma 2.1,

and $I^\epsilon(s, x; T) = 0$ for $x \in \partial D$. Since $L(x, y^*, z_1)$ is bounded by some C^*

$$I^\epsilon(s, x; T) \leq C^* E(\theta^\epsilon - s) + UP(\theta^\epsilon = T''),$$

$$P(\theta^\epsilon = T'') = P(\theta^\epsilon - s \geq T'' - s) \leq (T'' - s)^{-1} E(\theta^\epsilon - s).$$

To prove Lemma 2.2 it then suffices to show that

$$(2.17) \quad E(\theta^\epsilon - s) \leq C \text{ dist}(x, \partial D)$$

for $x \in D_1$, $0 \leq s \leq T'$. This is done as in proving [F3, Lemma 4.2], as follows.

Let $\theta_1^\epsilon = \min(T'', \text{exit time of } \psi(t) \text{ from } D_1)$, and

$$g^\epsilon(s, x) = E(\theta^\epsilon - s), \quad g_1^\epsilon(s, x) = E(\theta_1^\epsilon - s).$$

Then g^ϵ, g_1^ϵ both satisfy the parabolic equation

$$(2.18) \quad u_s + \frac{\epsilon}{2} \Delta_x u + u_x \cdot z_1 + 1 = 0$$

in $[0, T'') \times D_1$ with $g^\epsilon = g_1^\epsilon = 0$ for $x \in \partial D$ or $s = T''$. For $x \in \partial D_1 - \partial D$, $g_1^\epsilon = 0$, $0 \leq g^\epsilon \leq T$.

Let

$$\phi(t) = x + z_1(t-s)$$

and θ^0 the exit time of $\phi(t)$ from D . We choose $\rho < \frac{1}{2}(T-T'')$ and small enough that $\theta^0 - s \leq 2 \text{ dist}(x, \partial D)$ for all $x \in D_1$. The function $g^0(s, x) = \theta^0 - s$ satisfies in $[0, T''] \times D_1$ the first order equation

$$(2.19) \quad g_s^0 + g_x^0 \cdot z_1 + 1 = 0,$$

with $g^0 = 0$ for $x \in \partial D$ and $g^0 \geq 0$ for $x \in \partial D_1 - \partial D$ or $s = T''$. Let $\mu > 0$ and $g = (1+\mu)g^0$. By (2.18) and (2.19), for small ϵ

$$g_s + \frac{\epsilon}{2} \Delta_x g + g_x \cdot z_1 + 1 \leq 0.$$

Hence, by the maximum principle for parabolic equations $g_1^\epsilon \leq g$ on $[0, T''] \times D_1$. By Itô's formula

$$g^\epsilon(s, x) = E \int_s^{\theta_1^\epsilon} 1 dt + E g^\epsilon(\theta_1^\epsilon, \psi(\theta_1^\epsilon)).$$

Since the first term on the right is $g_1^\epsilon(s, x)$ and $g^\epsilon \leq T$,

$$(2.20) \quad g^\epsilon(s, x) \leq g(s, x) + TP(\psi(\theta_1^\epsilon) \in \partial D_1 - \partial D).$$

By reasoning as in [F4, p. 488], for $s \leq T'$, $x \in D_1$

$$P(\psi(\theta_1^\epsilon) \in \partial D - \partial D_1) \leq \epsilon B g_1^\epsilon(s, x)$$

for some B . Since $g_1^\epsilon \leq g^\epsilon$, (2.20) implies

$$g^\epsilon(s, x) \leq \frac{1+\mu}{1-\epsilon BT} g^0(s, x) \leq \frac{2(1+\mu)}{1-\epsilon BT} \text{dist}(x, \partial D).$$

For small ϵ , we then have (2.17) with $C = 3(1+\mu)$. This proves Lemma 2.2.

Lemma 2.3. For $T' < T$, there exists a constant M_1 such that $I^\epsilon(s, x; T) \leq I_M^\epsilon(s, x; T')$ for all $M \geq M_1$, $0 \leq s \leq T'$, $x \in D$, and for small ϵ .

Proof. Both $I^\epsilon(s, x; T)$ and $I_M^\epsilon(s, x; T')$ satisfy (2.7).

By Lemma 2.2, for small ϵ

$$I^\epsilon(T'; x, T) \leq M_0 \text{dist}(x, \partial D),$$

and by (2.12)

$$I_M^\epsilon(T', x; T') = M\phi(x) \geq M\gamma_1 \text{dist}(x, \partial D).$$

For $x \in \partial D$, $I^\epsilon = I_M^\epsilon = 0$. Let $M_1 = M_0\gamma_1^{-1}$. By the maximum principle for parabolic equations, $I^\epsilon \leq I_M^\epsilon$ in the cylinder $Q' = (0, T') \times D$ for any $M \geq M_1$. This proves Lemma 2.3.

The minimum in (2.11) is taken among all $z \in E^n$. For technical reasons we wish to consider equations corresponding to (2.7) when bounds are imposed on $|z|$.

For $c \geq 1$ let

$$Z^c = \{z \in E^n: |z| \leq c\}.$$

Let I_{Mc}^ϵ be the solution in Q of

$$(2.21) \quad (I_{Mc}^\epsilon)_s + \frac{\epsilon}{2} \Delta_x I_{Mc}^\epsilon + H^c(x, (I_{Mc}^\epsilon)_x) = 0,$$

$$(2.22) \quad H^c(x, p) = \max_{y \in Y} \min_{z \in Z^c} [L(x, y, z) + p \cdot z]$$

with the same boundary data (2.9_M) as for I_M^ϵ . The solution I_{Mc}^ϵ belongs to $C_\beta^{0,1}(\bar{Q}) \cap C_\beta^{1,2}(A)$ for any compact $A \subset \bar{Q}_0$ and $\beta \in (0,1)$. Since $H^c \geq H$, we always have $I_{Mc}^\epsilon \geq I_M^\epsilon$. In section 3 we will show that for c large enough ($c \geq c(M)$), $I_M^\epsilon = I_{Mc}^\epsilon$. See (3.15).

Lemma 2.4. There exists a constant $B = B(M)$ such that
 $I_{Mc}^\epsilon(s, x) \leq B \text{ dist}(x, \partial D)$, for all $(s, x) \in Q$ and for small ϵ .

Proof. Since ∂D is of class C^3 , D has the exterior sphere property. Following [F5, p. 275], given $x_1 \in \partial D$ let A be an n -dimensional spherical ball with center x_2 , radius $|x_1 - x_2|$ intersecting \bar{D} in the single point x_1 . Let $\psi(x) = |x - x_2| - |x_1 - x_2|$ and let $\underline{z}(x) = -\psi_x(x)$. For small ϵ

$$\frac{\epsilon}{2} \Delta_x \Psi + \underline{z} \cdot \Psi_x \leq -\frac{1}{2}$$

for all $x \in D$. As in the proof of Lemma 2.2 the function $J^\epsilon = I_{Mc}^\epsilon$ satisfies

$$J_s + \frac{\epsilon}{2} \Delta_x J^\epsilon + J_x^\epsilon \cdot \underline{z} + L^*(x) \geq 0,$$

$$L^*(x) = \max_{y \in Y} L(x, y, \underline{z}(x)).$$

Moreover, $L^*(x) \leq L_1$, for some L_1 . For $B \geq 2L_1$

$$(B\Psi - J^\epsilon)_s + \frac{\epsilon}{2} \Delta_x (B\Psi - J^\epsilon) + (B\Psi - J^\epsilon)_x \cdot \underline{z} \leq 0.$$

For $x \in \partial D$, $B\Psi(x) - J^\epsilon(s, x) = B\Psi(x) \geq 0$. For $B \geq B(M) \geq 2L_1$ we have

$$B\Psi(x) - J^\epsilon(T, x) = B\Psi(x) - M\phi(x) \geq 0,$$

for all $x \in D$. The maximum principle for parabolic equations implies $B\Psi - J^\epsilon \geq 0$ in Q .

Given $x \in D$, let x_1 be a point of ∂D nearest x . Then $\text{dist}(x, \partial D) = \Psi(x)$, which implies Lemma 2.4.

3. Penalized Differential Games. In the differential game formally described in §1, the minimizing controller is required to bring the game state $\phi(t)$ to ∂D by some time $\theta \leq T$. We

now omit that restriction and instead impose a penalty if ∂D is not reached. Let us begin by describing formally the penalized differential game, and the corresponding penalized stochastic differential game for $\epsilon > 0$. The penalized game has the same dynamics (1.7) as before, but instead of (1.9) the payoff is

$$(3.1) \quad \int_0^\theta L[\phi(t), y(t), z(t)] dt + M\phi[\phi(\theta)],$$

where now $\theta = \min\{T, \text{exit time of } \phi(t) \text{ from } D\}$. The penalty $M\phi(\theta) = 0$ if $\theta < T$ by (2.12). For $\epsilon > 0$ the game state $\psi(t)$ obeys a white-noise perturbation of (1.7):

$$d\psi = z(t)dt + \epsilon^{1/2}dw.$$

The payoff is

$$(3.2) \quad E \left\{ \int_0^{\theta^\epsilon} L[\psi(t), y(t), z(t)] dt + M\phi[\psi(\theta^\epsilon)] \right\},$$

with $\theta^\epsilon = \min\{T, \text{exit time of } \psi(t) \text{ from } D\}$. For technical reasons, we impose a bound $|z(t)| \leq c$ for the minimizing controller. Moreover, we replace the initial time 0 in (3.1) or (3.2) by any initial time $s \in [0, T]$, and consider the initial data $\phi(s) = \psi(s) = x$. The Isaac's equation for the penalized stochastic differential game is (2.21); for the deterministic game the Isaac's equation is the first-order equation corresponding to (2.21) when $\epsilon = 0$. We shall show below that I_{Mc}^ϵ is the lower value (in a suitably defined sense) of the penalized stochastic

differential game and that its limit I_{Mc}^ϵ as $\epsilon \rightarrow 0$ is the lower value of the penalized deterministic game. We also show that $I_{Mc}^\epsilon = I_M^\epsilon$ for large enough $c \geq c(M)$.

In order to treat these games on a rigorous basis, we discretize the game dynamics and payoff in the same way as [F1], [F2]. A somewhat different discretization procedure was used in [Fr1].

In the discussion to follow we fix M, c , and T . For $N = 1, 2, \dots$ let

$$\delta = \frac{T}{N}, \quad t_k = \frac{(k-1)T}{N}, \quad k = 1, \dots, N + 1.$$

For simplicity we do not indicate the dependence of δ and t_k on N . Let η_k^i , $k = 1, \dots, N$, $i = 1, \dots, n$, be mutually random variables which assume the values 1 and -1 each with probability $\frac{1}{2}$. In order to avoid certain analytical questions of integrability and the existence of a value, we shall arrange that each of these games is finite. Let

$$Y_1 \subset Y_2 \subset \dots \subset Y, \quad Z_1 \subset Z_2 \subset \dots \subset E^n,$$

where Y_N, Z_N are finite sets for each N , the union of the sets Y_N is dense in Y , and the union of the sets Z_N is dense in E^N . For $c \geq 1$ let

$$Z^c = \{z: |z| \leq c\}, \quad Z_N^c = Z^c \cap Z_N.$$

We shall define for initial data (s, x) with $s = t_\ell$, $\ell = 1, \dots, N$, $x \in D$, a game with $2(N-\ell+1)$ moves. At move $2k-1$, the first controller chooses some $y_k \in Y_N$, and at move $2k$ the second controller chooses some $z_k \in Z_N^C$, $k = \ell, \dots, N$. Both choices are obtained using strategies, as described below. The state ψ_{k+1} after move $2k$ is determined from the system of difference equations

$$(3.3) \quad \psi_{k+1} = \psi_k + \delta z_k + \delta^{1/2} \epsilon^{1/2} \eta_k, \quad \psi_\ell = x.$$

At each move $2k-1$ or $2k$ the controllers know all previous moves and also $\eta_\ell, \dots, \eta_{k-1}$. The game stops at the first step F^ϵ when either $\psi_{F^\epsilon+1} \notin D$ or $F^\epsilon = N$. We call $t_{F^\epsilon} = (F^\epsilon-1)\delta$ the exit time for ψ_k . After play stops, the first controller receives the payoff

$$(3.4) \quad \pi^\epsilon(s, x) = E \left\{ \sum_{k=\ell}^{F^\epsilon} \delta L(\psi_k, y_k, z_k) + M\Phi(\psi_{F^\epsilon+1}) \right\}.$$

When $\epsilon = 0$ we consider the corresponding deterministic game whose states ϕ_k obey

$$(3.5) \quad \phi_{k+1} = \phi_k + \delta z_k, \quad \phi_\ell = x,$$

with payoff

$$(3.6) \quad \pi(s, x) = \sum_{k=\ell}^F \delta L(\phi_k, y_k, z_k) + M\Phi(\phi_{F+1})$$

where t_F is the exit time for ϕ_k .

The game is described more precisely by introducing the idea of strategy (also called a δ -strategy). For the deterministic game a strategy for the first controller is the $(N-\ell+1)$ -vector $\Gamma = (\Gamma^\ell, \dots, \Gamma^N)$ such that Γ^ℓ is any point $y_\ell \in Y_N$ and Γ^k , $k = \ell + 1, \dots, N$ is any function of z_m , $\ell \leq m < k$ taking values in Y_N . A strategy for the second controller is the $(N-\ell+1)$ -vector $\Delta = (\Delta^\ell, \dots, \Delta^N)$ such that Δ^k , $k = \ell, \dots, N$, is any function of y_m, y_k , $\ell \leq m < k$ with values in Z_N^C . For the stochastic game ($\varepsilon > 0$) strategies for both players are defined similarly except that the functions Γ^k, Δ^k , $k = \ell + 1, \dots, N$, can depend on η_m , $m = \ell, \dots, k - 1$ too. A strategy Γ for the deterministic game defines a strategy for the stochastic game in the obvious way. On the other hand, given the random inputs $\zeta = (\eta_\ell, \dots, \eta_N)$ a strategy Γ for the stochastic game induces a strategy (denoted by Γ/ζ) for the deterministic game. Similar remarks apply to the second controller. Given a pair of strategies (Γ, Δ) , and a vector of random inputs ζ in the stochastic game, the successive moves are found by taking y_ℓ at move $2\ell - 1$ and setting

$$z_\ell = \Delta^\ell(y_\ell), y_{\ell+1} = \Gamma^{\ell+1}(y_\ell, \eta_\ell), \dots$$

Let us indicate explicitly dependence on the strategies Γ, Δ by writing the payoff in (3.4) as $\pi^\varepsilon(s, x, \Gamma, \Delta)$. Similarly, we write $\pi(s, x, \Gamma, \Delta)$ in (3.6) when $\varepsilon = 0$. Note that the game is biased

in favor of the minimizing controller, who knows y_k before choosing z_k . The concept of strategy used in [F2] is slightly different, but equivalent to the present one.

Given the initial data (s,x) and N , each of these games has a finite number of possible positions. There are a finite number of control choices for each controller at each move. From the theory of positional games, the stochastic difference game has a value $W_N^\epsilon(s,x)$ and optimal strategies Γ^*, Δ^* exist:

$$\pi^\epsilon(s,x,\Gamma,\Delta^*) \leq W_N^\epsilon(s,x) \leq \pi^\epsilon(s,x,\Gamma^*,\Delta)$$

for every Γ, Δ , with equality when $\Gamma = \Gamma^*, \Delta = \Delta^*$. In fact, $W_N^\epsilon(s,x)$ can be defined by backward induction by the functional equation

$$(3.7) \quad W_N^\epsilon(s,x) = \max_{Y_N} \min_{Z_N^C} E[\delta L(x,y,z) + W_N^\epsilon(s+\delta,\psi)], \quad s = t_k < T, x \in D,$$

$$(3.8a) \quad W_N^\epsilon(s,x) = 0, \quad x \notin D$$

$$(3.8b) \quad W_N^\epsilon(T,x) = M\phi(x),$$

where

$$\psi = x + \delta z + \delta^{1/2} \epsilon^{1/2} \eta$$

and the components η^i of η are independent random variables, each with values ± 1 with probability $\frac{1}{2}$.

When $\epsilon = 0$ the deterministic difference game has a value $W_N(s, x)$ satisfying the functional equation corresponding to (3.7) and the conditions (3.8). We shall show that as $N \rightarrow \infty$, $\epsilon \rightarrow 0$, W_N and the function I_{Mc}^ϵ defined at the end of §2 tend to the same limit. Moreover, this limit I_M does not depend on c , for large enough c . See Theorem 3.3 and formula (3.15).

We begin with a series of lemmas which give uniform estimates for $W_N^\epsilon - W_N$ and for $W_N^\epsilon(s, x') - W_N^\epsilon(s, x)$.

Lemma 3.1. There exist constants $C_1 = C_1(M)$ and N_1 such that for $N \geq N_1$ and small ϵ

$$W_N^\epsilon(s, x) \leq C_1 \text{dist}(x, \partial D) + C_1(\delta + n(\delta\epsilon)^{1/2}), \quad s = t_\ell, \quad \ell = 1, \dots, N, \quad x \in D.$$

Proof. Given $x_0 \in D$, let x_1 be a point of ∂D nearest x_0 . Choose x_2 and $\Psi(x) = |x - x_2| - |x_1 - x_2|$ as in the proof of Lemma 2.4. Let the minimizing controller use the control z_k^* chosen as follows. Given N , divide $Z^1 = \{|z| \leq 1\}$ into nonoverlapping Borel measurable sets A_1, \dots, A_ν such that each A_i contains exactly one point $z^i \in Z_N^1$ and $\max_{1 \leq i \leq \nu} \text{diam } A_i \rightarrow 0$ as $N \rightarrow \infty$. Let $\underline{z}(x) = z^i$ if $-\Psi_x(x) \in A_i$. Since $|\Psi_x| = 1$ there exists N_1 such that

$$\underline{z}(x) \cdot \Psi_x(x) \leq -\frac{1}{2}, \quad x \in D, \quad N \geq N_1.$$

We take $z_k^* = \underline{z}(\psi_k)$; since we always take $c \geq 1$, $z_k^* \in Z_N^C$. In (3.3)

$$\psi_{k+1} = \psi_k + \delta z_k^* + (\delta\epsilon)^{1/2} \eta_k.$$

Let the maximizing controller choose y_k to maximize $L(\psi_k, y, z_k^*)$, and let

$$M_k = C_1 \Psi(\psi_k) - W_N(t_k, \psi_k), \quad \text{if } \psi_k \in D,$$

where C_1 is chosen below. If $\psi_k \in D$ (and hence $k > F^\varepsilon$) we let

$$M_k = M_{F^\varepsilon+1} = C_1 \Psi(\psi_{F^\varepsilon+1}).$$

For $\psi_k \in D$, $k \leq N$, we have by (3.7) and choice of y_k

$$\begin{aligned} W_N^\varepsilon(t_k, \psi_k) &\leq E[\delta L(\psi_k, y_k, z_k^*) + W_N^\varepsilon(t_k + \delta, \psi_{k+1}) | \zeta_k] \\ &\leq \delta L_1 + E[W_N^\varepsilon(t_k + \delta, \psi_{k+1}) | \zeta_k], \end{aligned}$$

where $|L(x, y, z(x))| \leq L_1$ and $\zeta_k = (\eta_k, \dots, \eta_{k-1})$. By expanding $\Psi(\psi_{k+1})$ by Taylor's formula about ψ_k ,

$$E[\Psi(\psi_{k+1}) | \zeta_k] = \Psi(\psi_k) + \delta z_k^* \cdot \Psi_x(\psi_k) + O((\delta + (\delta\varepsilon)^{1/2})^2).$$

Since $z(x) \cdot \Psi_x(x) \leq -\frac{1}{2}$ for all $x \in D$, we have for small ε and $\psi_k \in D$

$$E[\Psi(\psi_{k+1}) | \zeta_k] \leq \Psi(\psi_k) - \frac{\delta}{4},$$

$$E[M_{k+1} | \zeta_k] \leq M_k - \frac{\delta C_1}{4} + \delta L_1 \leq M_k$$

if $4L_1 \leq C_1$. If $\psi_k \in D$, then $M_{k+1} = M_k$. Therefore,
 $E[M_{k+1} | \zeta_k] \leq M_k$, which implies by iterating

$$E(M_{N+1}) \leq M_0.$$

Choose $C_1 \geq 4L_1$ large enough that $C_1\psi(x) - M\phi(x) \geq 0$ on D .

If $F^e = N$ and $\psi_{N+1} \in D$,

$$M_{N+1} = C_1 \psi(\psi_{N+1}) - M\phi(\psi_{N+1}) \geq 0.$$

Otherwise,

$$M_{N+1} = C_1 \psi(\psi_{F^{\epsilon+1}}) \geq C_1 [\psi(\psi_{F^{\epsilon+1}}) - \psi(\psi_{F^\epsilon})]$$

since $\psi_{F^\epsilon} \in D$ and $\psi(x) \geq 0$ on D . Since $|\psi_x| = 1$ we have in the second case

$$M_{N+1} \geq -C_1 |\psi_{F^{\epsilon+1}} - \psi_{F^\epsilon}| \geq -C_1 (\delta + n(\delta\epsilon)^{1/2}).$$

Since $M_\ell = C_1 \psi(x_0) - W_N^\epsilon(s, x_0)$,

$$-C_1 (\delta + n(\epsilon\delta)^{1/2}) \leq C_1 \psi(x_0) - W_N^\epsilon(s, x_0)$$

Since $\psi(x_0) = \text{dist}(x_0, \partial D)$ and $x_0 \in D$ is arbitrary, this proves Lemma 3.1.

Remark. Similarly, when $\epsilon = 0$, an estimate $W_N(s, x) \leq C_1 [\text{dist}(x, \partial D) + \delta]$ holds.

Lemma 3.2. There exists C_2 depending on M and c such that for small ϵ and $N \geq N_1$

$$|W_N^\epsilon(s, x) - W_N(s, x)| \leq C_2 [\epsilon^{1/2} (T-s)^{1/2} + \delta + (\delta\epsilon)^{1/2}],$$

$s = t_\ell$, $\ell = 1, \dots, N$, $x \in D$.

Proof. Given (s,x) let Γ be an optimal strategy for the maximizing controller in the stochastic game and Δ an optimal strategy for the minimizing controller in the deterministic game. Let $G = \min(F^\epsilon, F)$ with F^ϵ, F as in (3.4), (3.6). The strategy Δ is played against Γ in the stochastic game, giving by dynamic programming arguments

$$(*) \quad W_N^\epsilon(s,x) \leq E\left\{ \sum_{k=\ell}^G \delta L(\psi_k, y_k, z_k) + W_N^\epsilon(t_{G+1}, \psi_{G+1}) \right\}.$$

Given a vector $\zeta = (\eta_\ell, \dots, \eta_N)$ of random inputs to (3.3), the strategy Γ/ζ is played against Δ in the deterministic game. We have

$$(**) \quad W_N(s,x) \geq \sum_{k=\ell}^G \delta L(\phi_k, y_k, z_k) + W_N(t_{G+1}, \phi_{G+1}),$$

for each ζ . Hence, the same inequality holds if we take the expectation of the right side.

By (3.3), (3.5)

$$\psi_{k+1} - \phi_{k+1} = \epsilon^{1/2} w_k$$

where

$$w_k = \delta^{1/2} \sum_{m=\ell}^k \eta_m.$$

By (2.1), $L(\cdot, y, z)$ is Lipschitz with some constant K_1 , for $|z| \leq c$. Moreover, ϕ is Lipschitz with some constant K_2 .

Let $||w|| = \max\{|w_k|: k = \ell, \dots, N+1\}$. Then

$$\sum_{k=\ell}^G \delta |L(\psi_k, y_k, z_k) - L(\phi_k, y_k, z_k)| \leq K_1(T-s)\epsilon^{1/2} ||w||,$$

$$|M\phi(\psi_{N+1}) - M\phi(\phi_{N+1})| \leq MK_2\epsilon^{1/2} ||w||.$$

We subtract the expectation of (**) from (*), and recall that $W_N(t_{G+1}, \phi_{G+1}) \geq 0$ together with (3.8):

$$\begin{aligned} W_N^\epsilon(s, x) - W_N(s, x) &\leq (K_1(T-s) + MK_2)\epsilon^{1/2} E||w|| \\ &\quad + E\{W_N^\epsilon(t_{G+1}, \psi_{G+1}); G < N, \psi_{G+1} \in D\}. \end{aligned}$$

Now $G < N, \psi_{G+1} \in D$ imply $\phi_{G+1} \notin D$, and

$$\text{dist}(\psi_{G+1}, \partial D) \leq |\psi_{G+1} - \phi_{G+1}| \leq \epsilon^{1/2} ||w||.$$

By Lemma 3.1 we have

$$\begin{aligned} E\{W_N^\epsilon(t_{G+1}, \psi_{G+1}); G < N, \psi_{G+1} \in D\} \\ \leq C_1(\epsilon^{1/2} E||w|| + \delta + n(\delta\epsilon)^{1/2}). \end{aligned}$$

Since the sequence of components $w_k^i, i = 1, \dots, n$, form a martingale with $w_{\ell-1}^i = 0$ we have [D, p. 311]

$$(3.9) \quad E||w|| \leq 2nE|w_N^i| \leq 2n(E|w_N^i|^2)^{1/2} = 2n(T-s)^{1/2}.$$

Therefore, for some constant C_2

$$W_N^\epsilon(s,x) - W_N(s,x) \leq C_2 \epsilon^{1/2} (T-s) + C_2 (\delta + (\delta\epsilon)^{1/2}).$$

The opposite inequality is proved in the same way, by taking Γ optimal for the deterministic game and Δ optimal for the stochastic game, and using the Remark after Lemma 3.1.

Lemma 3.3. There exists a constant R depending on M ,
such that for small ϵ and $N \geq N_1$

$$|W_N^\epsilon(s,x) - W_N^\epsilon(s,x')| \leq R|x-x'| + R(\delta + (\delta\epsilon)^{1/2}),$$

$s = t_\ell, \ell = 1, \dots, N, x, x' \in D.$

Proof. Let Γ be an optimal strategy for the maximizing controller, for initial data (s,x) ; and let Δ be an optimal strategy for the minimizing controller, for initial data (s,x') , in the stochastic game. We use the strategies Γ, Δ both in the game with initial data (s,x) and in the game with initial data (s,x') . Let ψ_k, ψ'_k be the solutions to (3.3) with $\psi_\ell(s) = x, \psi'_\ell(s) = x'$. [Recall that strategies are expressed in terms of past control choices and random inputs, but not in terms of the states ψ_k .] Let $G = \min(F^\epsilon, F^{\epsilon'})$ where $F^\epsilon, F^{\epsilon'}$ are the corresponding stopping times in (3.4). Then

$$W_N^\varepsilon(s, x) \leq E\left\{ \sum_{k=\ell}^G \delta L(\psi_k, y_k, z_k) + W_N^\varepsilon(t_{G+1}, \psi_{G+1}) \right\},$$

$$W_N^\varepsilon(s, x') \geq E\left\{ \sum_{k=\ell}^G \delta L(\psi'_k, y_k, z_k) + W_N^\varepsilon(t_{G+1}, \psi'_{G+1}) \right\}.$$

By (3.3), $\psi_k - \psi'_k = x - x'$. We then have as in the proof of Lemma 3.2

$$\begin{aligned} (*) \quad W_N^\varepsilon(s, x) - W_N^\varepsilon(s, x') &\leq E \sum_{k=\ell}^G \delta |L(\psi_k, y_k, z_k) - L(\psi'_k, y_k, z_k)| \\ &\quad + MK_2 |x - x'| + E\{W_N^\varepsilon(t_{G+1}, \psi_{G+1}); G < N, \psi_{G+1} \in D\}. \end{aligned}$$

The terms on the right side of (*) are estimated as follows. By (2.1) and (2.2), for some Λ_1

$$|L(\psi_k, y_k, z_k) - L(\psi'_k, y_k, z_k)| \leq \Lambda_1 (1 + L(\psi'_k, y_k, z_k)) |x - x'|.$$

Since Δ is optimal for initial data (s, x') ,

$$E \sum_{k=\ell}^G \delta L(\psi'_k, y_k, z_k) \leq W_N^\varepsilon(s, x').$$

If $G < N$, $\psi_{G+1} \in D$, then $\psi'_{G+1} \notin D$ and

$$\text{dist}(\psi_{G+1}, \partial D) \leq |\psi_{G+1} - \psi'_{G+1}| = |x - x'|.$$

On the right side of (*), we have:

$$\text{1st term} \leq \Lambda_1 (T - s + W_N^\varepsilon(s, x')) |x - x'|.$$

By Lemma 3.1

$$\text{3rd term} \leq C_1 |x' - x| + C_1 (\delta + n(\delta\epsilon)^{1/2}).$$

Since $\text{dist}(x, \partial D) \leq \text{diam } D < \infty$, another application of Lemma 3.1 gives a bound on $W_N^\epsilon(s, x')$. Hence, for some constant R

$$W_N^\epsilon(s, x) - W_N^\epsilon(s, x') \leq R|x - x'| + R(\delta + (\delta\epsilon)^{1/2}).$$

Since the roles of x and x' can be exchanged, this proves Lemma 3.3.

We will next show that, for fixed ϵ , $W_N^\epsilon(s, x)$ tends to I_{Mc}^ϵ as $N \rightarrow \infty$. The main step in doing this is Lemma 3.5. To make the backward induction in that Lemma, we use the following slightly different payoff for the discrete-time stochastic difference game. Let D_ρ be the ρ -neighborhood of D . Let J^ϵ be an extension of I_{Mc}^ϵ to a cylinder $[0, T] \times D_\rho$, for some $\rho > 0$, as follows:

$$J^\epsilon(s, x) = I_{Mc}^\epsilon(s, x), \quad \text{if } x \in \bar{D},$$

$$J^\epsilon(s, x_1 + rz(x_1)) = J^\epsilon(s, x_1 - rz(x_1)) + 2rJ_v^\epsilon(s, x_1)$$

for $x_1 \in \partial D$, $z(x_1)$ the exterior unit normal to D at x_1 , J_v^ϵ

the normal derivative, $0 < r < \rho$. The function J^ϵ belongs to $C_{\beta}^{0,1}([0,T] \times \bar{D}_{\rho})$ and to $C_{\beta}^{1,2}([0,T'] \times \bar{D}_{\rho})$ for any $T' < T$ and $\beta \in (0,1)$. Moreover, if $\text{dist}(x, \partial D) < \rho$,

$$(3.11) \quad |J^\epsilon(s,x)| \leq S\rho$$

where $S = 3B$ with $B = B(M)$ as in Lemma 2.4. Instead of (3.4) we take the payoff

$$\tilde{\pi}^\epsilon(s,x) = E\left\{ \sum_{k=\ell}^{F^\epsilon} \delta L(\psi_k, y_k, z_k) + J^\epsilon(t_{F^\epsilon+1}, \psi_{F^\epsilon+1}) \right\}.$$

Let $\hat{W}_N^\epsilon(s,x)$ denote the value of the stochastic difference game with payoff $\tilde{\pi}^\epsilon(s,x)$, $x \in D$. For $x \in D_\rho - D$ let $\tilde{W}_N^\epsilon(s,x) = J^\epsilon(s,x)$. If $F^\epsilon = N$ and $\psi_{N+1} \in D$, the last term is $M\phi(\psi_{N+1})$ as in (3.4). Otherwise, $\psi_{F^\epsilon} \in D$, $\psi_{F^\epsilon+1} \notin D$. By (3.3) and the fact that $|\eta_k^i| = 1$, $|z_k| \leq c$,

$$\text{dist}(\psi_{F^\epsilon+1}, \partial D) \leq |\psi_{F^\epsilon+1} - \psi_{F^\epsilon}| < \rho$$

provided $N \geq N_1$ where N_1 is large enough that

$$(3.12) \quad c\delta + n\epsilon^{1/2}\delta^{1/2} < \rho.$$

By (2.12) $\phi(\psi_{F^\epsilon+1}) = 0$ if $\psi_{F^\epsilon+1} \notin D$. By (3.11) we have for $N \geq N_1$, $|\tilde{\pi}^\epsilon - \pi^\epsilon| \leq S\rho$ for every pair of strategies Γ, Δ . Hence

$$(3.13) \quad |\tilde{W}_N^\epsilon(s, x) - W_N^\epsilon(s, x)| \leq S\rho,$$

for all $s = t_\ell$, $\ell = 1, \dots, N$, $x \in D$.

For fixed ϵ and N let

$$(3.14) \quad q_\ell = \sup_{x \in D} |\tilde{W}_N^\epsilon(t_\ell, x) - J^\epsilon(t_\ell, x)|.$$

By choice of $\tilde{\pi}^\epsilon$, $q_{N+1} = 0$. In Lemmas 3.4 and 3.5, M, c, ϵ are fixed. The constants appearing in those lemmas may depend on M, c, ϵ .

Lemma 3.4. There exist S_1, N_2 such that

$$|\tilde{W}_N^\epsilon(s, x) - \phi(x)| \leq S_1(T-s)^{1/2} + S_1\rho$$

for $N \geq N_2$, $0 < T - s \leq 1$, $x \in D$.

Proof. Consider any pair of strategies Γ, Δ for a stochastic game with initial data $s = t_\ell$, $x = \psi_\ell$. We have, for some B , $0 \leq L \leq B$ on $Q \times Y \times Z^C$. Choose N_2 such that (3.12) holds for $N \geq N_2$. Then

$$\begin{aligned} |\tilde{\pi}^\epsilon(s, x) - \phi(x)| &\leq B(T-s) + ME|\phi(\psi_{F^{\epsilon+1}}) - \phi(x)| \\ &\quad + E|J^\epsilon(t_{F^{\epsilon+1}}, \psi_{F^{\epsilon+1}}) - M\phi(\psi_{F^{\epsilon+1}})| \end{aligned}$$

The last term is 0 if $F^\epsilon = N$, $\psi_{N+1} \in D$; otherwise $\phi(\psi_{F^{\epsilon+1}}) = 0$

and $\text{dist}(\psi_{F^{\epsilon+1}}, \partial D) < \rho$ by (3.12). By (3.3), (3.9) and the fact that $|z_k| \leq c$,

$$E|\psi_{F^{\epsilon+1}} - x| \leq c(T-s) + 2n\epsilon^{1/2}(T-s)^{1/2}.$$

Let K be a Lipschitz constant for ϕ . Then, for $0 < T - s \leq 1$,

$$|\tilde{\pi}^\epsilon(s, x) - \phi(x)| \leq B(T-s) + MK(c + 2n\epsilon^{1/2})(T-s) + S\rho$$

with S as in (3.11). We take

$$S_1 = \max[B + MK(c + n\epsilon^{1/2}), S].$$

Since the above inequality holds for any Γ, Δ we get Lemma 3.4.

Since J^ϵ is Hölder continuous on \bar{Q} , $|J^\epsilon(s, \cdot) - J^\epsilon(T, \cdot)|$ is uniformly small on D if $T - s$ is small. Since $J^\epsilon(T, x) = \phi(x)$ for $x \in D$, from Lemma 3.4 we have the:

Corollary. Given $a > 0$ there exist $k > 0$ and N_2 such that
 $q_\ell < a$ if $0 < T - s < k$, $s = t_\ell$, $N \geq N_2$.

Lemma 3.5. Given $T' < T$ there exist constants $A > 0$,
 $0 < \alpha < \frac{1}{2}$ such that the following is true: given $\mu > 0$
there exists N_0 such that for $N \geq N_0$

$$q_\ell \leq (\mu + A\delta^\alpha)(T' - s) + q_m$$

for $s = t_\ell \leq t_m \leq T'$.

Proof. The function J^ϵ is of class $C_\beta^{1,2}([0,T] \times \bar{D}_\rho)$ for any $\beta \in (0,1)$. Let $\alpha = \beta/2$. As above we take $N \geq N_2$ large enough that, for any $x \in D$, $|z| \leq c$, $\eta = (\eta^1, \dots, \eta^n)$ with $\eta^i = \pm 1$,

$$\psi = x + \delta z + \delta^{1/2} \epsilon^{1/2} \eta$$

belongs to D_ρ . By Taylor's formula

$$J^\epsilon(s+\delta, \psi) = J^\epsilon(s, \psi) + \delta J_s^\epsilon(s, \psi) + \textcircled{H}_1,$$

$$\begin{aligned} J^\epsilon(s, \psi) &= J^\epsilon(s, x) + J_x^\epsilon(s, x) \cdot (\psi - x) \\ &+ \frac{1}{2} \sum_{i,j=1}^n J_{x_i x_j}^\epsilon(s, x) (\psi^i - x^i) (\psi^j - x^j) + \textcircled{H}_2, \end{aligned}$$

where for suitable constants A_1, A_2

$$|\textcircled{H}_1| \leq A_1 \delta^{1+\alpha}, \quad |\textcircled{H}_2| \leq A_2 |\psi - x|^{2+2\alpha}, \quad \beta = 2\alpha.$$

Since

$$|J_s^\epsilon(s, \psi) - J_s^\epsilon(s, x)| \leq A_3 |\psi - x|^{2\alpha},$$

we have

$$\begin{aligned} J^\epsilon(s+\delta, \psi) &= J^\epsilon(s, x) + \delta J_s^\epsilon(s, x) + J_x^\epsilon(s, x) \cdot (\psi - x) \\ &+ \frac{1}{2} \sum_{i,j=1}^n J_{x_i x_j}^\epsilon(s, x) (\psi^i - x^i) (\psi^j - x^j) + \textcircled{H}_3, \end{aligned}$$

where

$$|\textcircled{H}_3| \leq A_4 (\delta^{1+\alpha} + \delta |\psi-x|^{2\alpha} + |\psi-x|^{2+2\alpha}).$$

By taking expectations, we have

$$\begin{aligned} EJ^\varepsilon(s+\delta, \psi) &= J^\varepsilon(s, x) + \delta [J_s^\varepsilon(s, x) + J_x^\varepsilon(s, x) \cdot z \\ &\quad + \frac{\varepsilon}{2} \Delta_x J^\varepsilon] + \textcircled{H}, \\ \textcircled{H} &= E\textcircled{H}_3 + \frac{\delta^2}{2} \sum_{i,j=1}^n z^i z^j J_{x_i x_j}^\varepsilon. \end{aligned}$$

Now L is uniformly continuous on $\bar{D} \times Y \times Z^c$ and J_x^ε is uniformly continuous on \bar{Q} . By (2.22), given $\mu > 0$ there exist $N_0 \geq N_1$ and a function y^* from \bar{Q} into the finite set Y_{N_0} , such that

$$L(x, \underline{y}^*(s, x), z) + J_x^\varepsilon(s, x) \cdot z \geq H^c(x, J_x^\varepsilon(s, x)) - \mu$$

for all $(s, x) \in \bar{Q}$, $|z| \leq c$.

If $\gamma \leq 3$

$$|\psi-x|^\gamma = |\delta z + \varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}} \eta|^\gamma \leq A_5 (\delta^\gamma + (\varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}} |\eta|)^\gamma)$$

and $E|\eta^i|^\gamma = 1$, $i = 1, \dots, n$. We have

$$E|\textcircled{H}_3| \leq A_6 [\delta^{1+\alpha} + \delta^{1+2\alpha} + \delta^{2+2\alpha}].$$

For N_0 large enough, $\delta < 1$ and

$$|\textcircled{H}| \leq A\delta^{1+\alpha}$$

for suitable A .

Since $J^\epsilon = I_{MC}^\epsilon$ is a solution of (2.21) in Q , we have for $x \in D$, $\underline{y}^* = \underline{y}^*(s, x)$

$$E\{\delta L(x, \underline{y}^*, z) + J^\epsilon(s+\delta, \psi)\} \geq J^\epsilon(s, x) - \delta\mu - A\delta^{1+\alpha}.$$

Let us now estimate q_ℓ in terms of $q_{\ell+1}$. By (3.14)

$$J^\epsilon(s+\delta, \psi) \leq q_{\ell+1} + \tilde{W}_N^\epsilon(s+\delta, \psi)$$

if $\psi \in D$. This inequality holds also for $\psi \notin D$, since $J^\epsilon = \tilde{W}_N^\epsilon$ by definition in that case. Hence, for all $z \in Z^C$

$$E\{\delta L(x, \underline{y}^*, z) + \tilde{W}_N^\epsilon(s+\delta, \psi)\} \geq J^\epsilon(s, x) - \delta\mu - A\delta^{1+\alpha} - q_{\ell+1}.$$

Since \tilde{W}_N^ϵ satisfies the functional equation (3.7), this implies

$$\tilde{W}_N^\epsilon(s, x) \geq J^\epsilon(s, x) - \delta\mu - A\delta^{1+\alpha} - q_{\ell+1}.$$

A similar argument, choosing $\underline{z}^* = \underline{z}^*(s, x, y)$ with values in $Z_{N_0}^C$ such that

$$L(x, y, \underline{z}^*) + J_x^\epsilon \cdot \underline{z}^* \leq H^C(x, J_x^\epsilon(s, x)) + \mu$$

for all $(s,x) \in \bar{Q}$, $y \in Y$, shows that

$$\tilde{W}_N^\varepsilon(s,x) \leq J^\varepsilon(s,x) + \delta\mu + A\delta^{1+\alpha} + q_{\ell+1}.$$

Therefore,

$$q_\ell \leq \delta(\mu + A\delta^\alpha) + q_{\ell+1}.$$

Since $\delta(m-\ell) = t_m - t_\ell \leq T' - s$, this proves Lemma 3.5.

By (3.12) the number ρ in (3.13) can be chosen arbitrarily small if N is large enough. By (3.13), Lemma 3.5, and the Corollary to Lemma 3.4 we have:

Theorem 3.1. For fixed $\varepsilon, \tilde{W}_N^\varepsilon(s,x) - I_{MC}^\varepsilon(s,x)$ and $W_N^\varepsilon(s,x) - I_{MC}^\varepsilon(s,x)$ tend to 0 as $N \rightarrow \infty$, uniformly for $x \in D$, $s = t_\ell$, $\ell = 1, \dots, N$.

As in [F2] we call I_{MC}^ε the lower value of the penalized stochastic differential game, for initial data $(s,x) \in Q$. From Lemma 3.3 and Theorem 3.1 we have the following uniform Lipschitz condition for $I_{MC}^\varepsilon(s, \cdot)$.

Theorem 3.2. There exists a constant $R = R(M)$, such that for small ε

$$|I_{MC}^\varepsilon(s,x) - I_{MC}^\varepsilon(s,x')| \leq R|x-x'|, \quad x, x' \in D, \quad 0 \leq s \leq T.$$

Theorem 3.2 implies the a priori estimate on the gradient

$$|(I_{Mc})_x| \leq R.$$

By (1.8) the minimum over $z \in E^n$ of $L + p \cdot z$ is attained at $z = b + p$. Since $|b(x,y)| \leq B_1$, $|p| \leq R$ implies $|b+p| \leq B_1 + R$. Hence, by (2.11), (2.22)

$$H^c(x,p) = H(x,p), \quad |p| \leq R, \quad c \geq B_1 + R.$$

Let $p = (I_{Mc}^\epsilon)_x$, $c(M) = B_1 + R(M)$. By (2.7), (2.9_M) and (2.21)

$$(3.15) \quad I_{Mc}^\epsilon = I_M^\epsilon, \quad c \geq c(M).$$

Let us now take $s = 0$. Since $b = b(x,y)$ the problem is autonomous, and hence this is no real restriction (see, in particular (2.16)). Lemma 3.2 gives a uniform estimate for $W_N^\epsilon - W_N$, which tends to 0 as $N \rightarrow \infty$, $\epsilon \rightarrow 0$. Since $J^\epsilon = I_{Mc}^\epsilon$ on Q for $c \geq c(M)$, we have

Theorem 3.3. As $\epsilon \rightarrow 0$, $I_M^\epsilon(0,x)$ tends to a limit $I_M(0,x)$. Moreover, $W_N(0,x) - I_M(0,x)$ tends to 0 as $N \rightarrow \infty$, uniformly for $x \in D$.

As in [F2] we call $I_M(0,x)$ the lower value of the deterministic penalized game starting at $x \in D$. As in §2 let us also denote this lower value by $I_M(0,x;T)$ to indicate dependence on the final time T . By (2.15), I_M is nondecreasing in M , and by Lemma 2.1, I_M is bounded above for fixed $T > 0$. Hence,

$I_M(0,x;T)$ is nondecreasing in M and bounded above. Let

$$(3.16) \quad I(0,x;T) = \lim_{M \rightarrow \infty} I_M(0,x;T), \quad x \in D.$$

4. Main Results. In this section, we show that $I^\epsilon = -\epsilon \log q^\epsilon$ tends to a limit I as $\epsilon \rightarrow 0$, and interpret I as the lower value of the differential game described formally in §1. In view of (2.16) it suffices to consider initial time $s = 0$, as already noted at the end of §3. The discrete time games in §3 then start with $\ell = 1$. We recall that the value $W_N(0,x)$ of the deterministic penalized game in §3 depends also on M and c . For $T' < T$, let $a = T'T^{-1}$. In the following lemma, $W'_N(0,x)$ denotes the value of the corresponding game in which T is replaced by T' and c by $a^{-1}c$.

Lemma 4.1. Let $\frac{1}{2}T < T' < T$. There exist constants R_1, N_1, c^* (depending on T) such that $|W_N(0,x) - W'_N(0,x)| \leq R_1(T-T')$ for any $M, N \geq N_1, c \geq c^*$.

Proof. Let $\delta = TN^{-1}$, $\delta' = a\delta = T'N^{-1}$. For any pair of strategies (Γ, Δ) for the discrete time deterministic game in §3 with time-step δ , we consider the corresponding strategies (Γ', Δ') for a game with time-step δ' , in such a way that

$$\phi'_1 = \phi_1 = x, \quad y'_k = y_k, \quad z'_k = a^{-1}z_k, \quad k = 1, \dots, N,$$

where ϕ'_k denotes the state and y'_k, z'_k the controls for the latter game. By (3.5) we have $\phi'_k = \phi_k$ for all k . We wish to

estimate the difference of the respective payoffs π, π' . By (1.8), (3.6)

$$\pi - \pi' = \frac{1}{2} \sum_{k=1}^F \delta [(1-a)|b_k|^2 + (1-a^{-1})|z_k|^2]$$

where $b_k = b(\phi_k, y_k)$. Since $0 < a < 1$, the first term in the sum is positive, and the second term is negative. Since $|b_k| \leq B_1$,

$$\pi \leq \pi' + \frac{1}{2} B_1 T (1-a).$$

Since this is true for any Γ, Δ , and since $Ta = T'$,

$$(*) \quad W_N(0, x) \leq W'_N(0, x) + \frac{1}{2} B_1 (T - T').$$

On the other hand, from (2.2), (3.6), and $\phi \geq 0$

$$\pi' \leq \pi + \frac{a^{-1}-1}{2} \sum_{k=1}^F \delta |z_k|^2 \leq \pi + \frac{a^{-1}-1}{2c_1} (\pi + c_2 T),$$

$$W'_N(0, x) \leq W_N(0, x) + \frac{T-T'}{2c_1 T'} (W_N(0, x) + c_2 T).$$

Let us show that $W_N(0, x) \leq K$ for $N \geq N_1$, $c \geq c^*$ where K, N_1 and c^* depend only on T . For this purpose, consider any $z_0 \in Z_N^1$ with $|z_0| \geq 1/2$. Let the minimizing controller use the constant control $z_k = c^* z_0$, where $\frac{1}{2} c^* T > \text{diam } D$. With this choice, there exists N_1 such that exit occurs by step $F < N$ for $N \geq N_1$. Moreover, the payoff π satisfies $\pi \leq \delta F L^* \leq T L^*$ where L^* is a bound for $L(x, y, z)$

when $|z| \leq c^*$. Let $K = TL^*$. Since $T' < T < 2T'$

$$(**) \quad W'_N(0,x) \leq W_N(0,x) + \frac{(L^*+c_2)}{c_1} (T-T')$$

By (*) and (**), $|W'_N(0,x) - W_N(0,x)| \leq R_1(T-T')$ where R_1 does not depend on M . This proves Lemma 4.1.

We have by Lemma 4.1, Theorem 3.3, and (3.16):

$$\begin{aligned} \text{Corollary 4.1.} \quad |I_M(0,x;T') - I(0,x;T)| &\leq R_1(T-T') \\ |I(0,x;T') - I(0,x;T)| &\leq R_1(T-T'). \end{aligned}$$

By combining results above we obtain the first main theorem.

Theorem 4.1. $I(0,x;T) = \lim_{\epsilon \rightarrow 0} I^\epsilon(0,x;T)$ for every $x \in D$
and $T > 0$.

Proof. Consider $T' < T$. By Lemma 2.3 and Theorem 3.3,
for large enough M

$$\limsup_{\epsilon \rightarrow 0} I^\epsilon(0,x;T) \leq I_M(0,x;T').$$

Since $I_M^\epsilon \leq I^\epsilon$, for each M

$$\liminf_{\epsilon \rightarrow 0} I^\epsilon(0,x;T) \geq \lim_{\epsilon \rightarrow 0} I_M^\epsilon(0,x;T) = I_M(0,x;T).$$

Hence by (3.16) and Corollary 4.1

$$\limsup_{\epsilon \rightarrow 0} I^\epsilon(0, x; T) \leq I(0, x; T) + R_1(T - T')$$

$$\liminf_{\epsilon \rightarrow 0} I^\epsilon(0, x; T) \geq I(0, x; T).$$

Since $T - T'$ can be made arbitrarily small, this proves Theorem 4.1.

It remains to characterize I as a lower value. Let us discretize time as in §3, and define a deterministic game as follows. The dynamics are again (3.5), and strategies Γ, Δ are as in §3. However, instead of (3.6) the payoff is now

$$(4.1) \quad \pi_\infty(0, x; \Gamma, \Delta) = \sum_{k=1}^F \delta L(\phi_k, y_k, z_k) + \chi$$

where $\chi = +\infty$ if $F = N$ and $\phi_{N+1} \in D$ and $\chi = 0$ otherwise.

Let

$$(4.2) \quad V_N^c(0, x; T) = \inf_{|\Delta| \leq c} \sup_{\Gamma} \pi_\infty(0, x; \Gamma, \Delta).$$

The notation $|\Delta| \leq c$ means $|\Delta^k| \leq c$ for each k ; this is the same bound imposed in §3. We now write $W_N(0, x; T)$ for the value $W_N(0, x)$ for the penalized deterministic game in §3; recall that W_N depends also on M and c .

Lemma 4.2. Let $0 < \beta < \frac{1}{2}$. There exist c^*, M_1, N_1 (depending on T and β) and a constant L_1 such that

$$W_N(0, x; T) \leq V_N^c(0, x; T) \leq W_{\hat{N}}(0, x; \hat{T}) + L_1(T - \hat{T})$$

for all $M \geq M_1$, $\hat{T} = \hat{N}\delta$, $N_1 \leq \hat{N} < N$, $T - \hat{T} \geq \beta T$, $c \geq c_1^*$.

Proof. The payoff $\pi (= \pi_M)$ in (3.6) satisfies $\pi \leq \pi_\infty$. Hence, we have the left-hand inequality in Lemma 4.2. To obtain the right-hand inequality, let the maximizing controller choose any strategy Γ . We define a strategy Δ for the minimizing controller as follows. Let $\hat{\Delta}$ be an optimal strategy for the penalized game with \hat{N} moves. The strategy Δ agrees with $\hat{\Delta}$ on moves $2k$, $k = 1, \dots, \hat{N}$. Let \hat{F} be the first step when $\phi_{\hat{F}+1} \notin D$ or $\hat{F} = \hat{N}$. If $\phi_{\hat{F}+1} \in D$, the choices z_k for $k = \hat{N} + 1, \dots, N$ are arbitrary. If $\hat{F} = \hat{N}$ and $\phi_{\hat{N}+1} \in D$, then

$$\sum_{k=1}^{\hat{N}} \delta L(\phi_k, y_k, z_k) + M\phi_{\hat{N}+1} \leq W_{\hat{N}}(0, x; \hat{T})$$

since $\hat{\Delta}$ is optimal. Let $c_1^* = 4T^{-1} \text{diam } D$. For $c \geq c_1^*$, the right side is bounded, as shown in the proof of Lemma 4.1. By (2.12) and the fact that $L \geq 0$,

$$\text{dist}(\phi_{\hat{N}+1}, \partial D) \leq CM^{-1}$$

for some C . Let A be the α -neighborhood of ∂D , for suitable $\alpha < \beta T$. We choose α small enough that, for every $x \in A$, there exists $\underline{z}(x) \in Z_N^2$ such that $x + s\underline{z}(x) \notin D$ for some $s > 0$, $s < \alpha$. Let $M_1 \geq C\alpha^{-1}$. When $\hat{F} = \hat{N}$ and $\phi_{\hat{N}+1} \in D$, then $\phi_{\hat{N}+1} \in A$ for $M \geq M_1$. In this case, let $z_k = \underline{z}(\phi_{\hat{N}+1})$ for $k = \hat{N} + 1, \dots, N$. For N large, the first F when $\phi_{F+1} \notin D$ satisfies $F < N$ since $T - \hat{T} \geq \beta T > \alpha$.

If $\phi_{\hat{F}+1} \notin D$, then $\hat{F} = F$. Otherwise,

$$\delta(F-\hat{F}) \leq \delta(N-\hat{N}) = T - \hat{T}.$$

When $|z| \leq 2$, $L(x,y,z) \leq L_1$ for some L_1 . Then

$$\begin{aligned} \pi_\infty &= \sum_{k=1}^{\hat{F}} L(\phi_k, y_k, z_k) + \sum_{k=\hat{F}+1}^F L(\phi_k, y_k, z(\phi_{\hat{N}+1})) \\ &\leq W_{\hat{N}}(0, x; \hat{T}) + L_1(T - \hat{T}). \end{aligned}$$

For this strategy Δ ,

$$\sup_{\Gamma} \pi_\infty(0, x; \Gamma, \Delta) \leq W_{\hat{N}}(0, x; \hat{T}) + L_1(T - \hat{T}),$$

which gives Lemma 4.2.

The function V_N^c in (4.2) is clearly a nonincreasing function of c . Let $V_N = \lim_{c \rightarrow \infty} V_N^c$. Then

$$(4.3) \quad V_N(0, x; T) = \inf_{\Delta} \sup_{\Gamma} \pi_\infty(0, x; \Gamma, \Delta)$$

where the strategy Δ for the minimizing controller is now chosen without the constraint $|\Delta| \leq c$. Let $y^N = (y_1, y_2, \dots, y_N)$ denote any sequence of control choices for the maximizing controller, $y_k \in Y_N$. Thus y^N is an open loop strategy. Then

$$(4.4) \quad V_N^C(0, x; T) = \inf_{|\Delta| \leq c} \sup_{y^N} \pi_\infty(0, x; y^N, \Delta)$$

$$V_N(0, x; T) = \inf_{\Delta} \sup_{y^N} (0, x; y^N, \Delta).$$

To see this, clearly \geq holds in (4.4). On the other hand, any pair of strategies Γ, Δ and initial state x define sequences y, z with $\Gamma^1 = y_1$ and

$$z_1 = \Delta^1(y_1), y_2 = \Gamma^2(z_1), z_2 = \Delta^2(y_1, y_2), \dots$$

and $\pi_\infty(0, x; y^N, \Delta) = \pi_\infty(0, x; \Gamma, \Delta)$.

We use the following truncation procedure. Let $\pi_N: Z_N \rightarrow Z_N^C$ be such that $\pi_N(z) = z$ if $z \in Z_N^C$ (recall that $Z_N^C = Z_N \cap \{|z| \leq c\}$). Given a sequence $z^N = (z_1, \dots, z_N)$, let $\tilde{z}_k = \pi_N(z_k)$. Given a strategy $\Delta = (\Delta^1, \dots, \Delta^N)$ let $\tilde{\Delta} = (\tilde{\Delta}^1, \dots, \tilde{\Delta}^N)$ where $\tilde{\Delta}^k = \pi_N \circ \Delta^k$. As in (3.5) define $\phi_k, \tilde{\phi}_k$ by

$$\phi_{k+1} = \phi_k + \delta z_k, \quad \tilde{\phi}_{k+1} = \tilde{\phi}_k + \delta \tilde{z}_k, \quad \phi_1 = \tilde{\phi}_1 = x.$$

For $1 \leq F \leq N$, we use the notation

$$\|\phi\|_F = \max_{1 \leq k \leq F+1} |\phi_k|.$$

Lemma 4.3. Given $K > 0$ there exists $B = B(K)$ such that

$$\sum_{k=1}^F \delta |z_k|^2 \leq K \text{ implies } \|\phi - \tilde{\phi}\|_F \leq 2Kc^{-1} \text{ and}$$

$$\sum_{k=1}^F \delta L(\tilde{\phi}_k, y_k, \tilde{z}_k) \leq \sum_{k=1}^F \delta L(\phi_k, y_k, z_k) + Bc^{-1}.$$

Proof. Let $A = \{k: |z_k| > c\}$ and $|A|$ its cardinality. By Cauchy's inequality

$$c\delta|A| \leq \sum_A \delta|z_k| \leq (\delta|A|)^{1/2} K^{1/2},$$

and therefore $\delta|A| \leq Kc^{-2}$,

$$\sum_A \delta|\tilde{z}_k| \leq \sum_A \delta|z_k| \leq Kc^{-1}.$$

Since $|\tilde{z}_k| \leq |z_k|$

$$\begin{aligned} \|\phi - \tilde{\phi}\|_F^2 &\leq \left(\sum_A \delta|z_k - \tilde{z}_k| \right)^2 \leq \delta|A| \sum_A \delta|z_k - \tilde{z}_k|^2 \\ &\leq 4\delta|A|K \leq 4K^2c^{-2}. \end{aligned}$$

This proves the first assertion. From the definition (1.8) of L and $|\tilde{z}_k| \leq |z_k|$,

$$\begin{aligned} &\sum_{k=1}^F [\delta L(\phi_k, y_k, z_k) - \delta L(\phi_k, y_k, \tilde{z}_k)] \\ &\geq \frac{1}{2} \sum_{k=1}^F \delta[|b_k|^2 - |\tilde{b}_k|^2] - \sum_{k=1}^F \delta(b_k z_k - \tilde{b}_k \tilde{z}_k) \end{aligned}$$

where $b_k = b(\phi_k, y_k)$, $\tilde{b}_k = b(\tilde{\phi}_k, y_k)$. Now $b(x, y)$ is bounded and Lipschitz in x . Hence $|b|^2$ is also Lipschitz in x , and for suitable B_1

$$\sum_{k=1}^F \delta(|b_k|^2 - |\tilde{b}_k|^2) \geq -B_1 \|\phi - \tilde{\phi}\|_F \geq -2B_1 Kc^{-1},$$

$$\sum_{k=1}^F \delta b_k (z_k - \tilde{z}_k) \leq B_1 \sum_A \delta |z_k - \tilde{z}_k| \leq 2B_1 Kc^{-1}$$

$$\begin{aligned} \sum_{k=1}^F \delta (b_k - \tilde{b}_k) \tilde{z}_k &\leq B_1 \|\phi - \tilde{\phi}\|_F \sum_{k=1}^F \delta |\tilde{z}_k| \\ &\leq B_1 \cdot 2Kc^{-1} \cdot T^{1/2} \cdot K^{1/2}. \end{aligned}$$

In the last step we have used Cauchy's inequality and $|\tilde{z}_k| \leq |z_k|$.
By combining these inequalities we get Lemma 4.3.

Lemma 4.4. Let $\beta > 0$. There exist N_1, c_1, B (depending on T
and β) and L_1 , such that

$$V_N^c(0, x; \tilde{T}) \leq V_N(0, x; T) + Bc^{-1} + L_1 \beta T$$

if $c \geq c_1$, $N_1 \leq N < \tilde{N}$, $\tilde{T} = \tilde{N}\delta$, $\tilde{T} - T \geq \beta T$.

Proof. By (1.8) given K_1 there exists K such that

$$\sum_{k=1}^F \delta L(\phi_k, y_k, z_k) \leq K_1 \quad \text{implies} \quad \sum_{k=1}^K \delta |z_k|^2 \leq K.$$

Let $K_1 > V_N(0, x; T)$, and consider any strategy Δ for the game with N steps ($2N$ moves) such that for any open loop $y^N = (y_1, \dots, y_N)$

$$\sum_{k=1}^F \delta L(\phi_k, y_k, z_k) = \pi_\infty(0, x; y^N, \Delta) \leq K_1.$$

As before F is the step at which exit occurs (or $F = N$). We must have $\phi_{F+1} \notin D$ even in case $F = N$, since otherwise $\pi_\infty = \infty$. For the game with \tilde{N} steps we define a strategy $\tilde{\Delta}$ with $|\tilde{\Delta}| \leq c$ as follows. For $1 \leq k \leq F$, let $\tilde{\Delta}^k = \pi_N \circ \Delta^k$ be defined by truncation. If $\tilde{\phi}_{F+1} \notin D$, the new game stops after $\tilde{F} \leq F$ steps. Suppose that $\tilde{\phi}_{F+1} \in D$. Since $\phi_{F+1} \notin D$, by Lemma 4.3

$$\text{dist}(\tilde{\phi}_{F+1}, \partial D) \leq \|\tilde{\phi} - \phi\|_F \leq 2Kc^{-1}.$$

As in the proof of Lemma 4.2 let A be the α -neighborhood of ∂D , where $\alpha < \beta T$ is sufficiently small. Then $\tilde{\phi}_{F+1} \in A$ for large enough c . For $k = F + 1, \dots, \tilde{N}$, let $\tilde{z}_k = \underline{z}(\tilde{\phi}_{F+1})$, where $\underline{z}(x)$ is defined as in the proof of Lemma 4.2. This defines the strategy $\tilde{\Delta}$. For large enough N_1 , $\tilde{\phi}_{\tilde{F}+1} \notin D$ where $\delta(\tilde{F}-F) < \beta T$. For each open loop y^N we then have

$$\pi_\infty(0, x; y^N, \tilde{\Delta}) = \sum_{k=1}^F \delta L(\tilde{\phi}_k, y_k, \tilde{z}_k) + \sum_{k=F+1}^{\tilde{F}} \delta L(\tilde{\phi}_k, y_k, \underline{z}(\tilde{\phi}_{F+1})).$$

Let L_1 be as in the proof of Lemma 4.2. By Lemma 4.3

$$\begin{aligned} \pi_\infty(0, x; y^N, \tilde{\Delta}) &\leq \sum_{k=1}^F \delta L(\phi_k, y_k, z_k) + Bc^{-1} + L_1 \beta T \\ &= \pi_\infty(0, x; y^N, \Delta) + Bc^{-1} + L_1 \beta T. \end{aligned}$$

Hence,

$$V_N^C(0,x;\tilde{T}) \leq \sup_{y^N} \pi_\infty(0,x;y^N,\tilde{\Delta}) \leq$$

$$\sup_{y^N} \pi_\infty(0,x;y^N,\Delta) + Bc^{-1} + L_1\beta T.$$

Since the inf over Δ of the first term on the right side is $V_N(0,x;T)$, this gives Lemma 4.4.

By exactly the same change of time scale argument as for Lemma 4.1:

Lemma 4.5. Let $T < \tilde{T}$. Then there exist R_1, N_1, c^* such that

$$|V_N^C(0,x;T) - V_N^C(0,x;\tilde{T})| \leq R_1(\tilde{T}-T)$$

for all $N \geq N_1, c \geq c^*$.

Theorem 4.2. $V_N(0,x;T) \rightarrow I(0,x;T)$ as $N \rightarrow \infty$.

Proof. For each M and $c \geq c(M)$ Theorem 3.3 implies

$$I_M^C(0,x;T) = \lim_{N \rightarrow \infty} W_N(0,x;T) \leq \liminf_{N \rightarrow \infty} V_N^C(0,x;T).$$

Let $\beta > 0$. Choose $\tilde{N}, \tilde{T} = \tilde{N}\delta$ as in Lemma 4.4 such that $\beta T \leq \tilde{T} - T \leq 2\beta T$. For $c \geq c_1, N \geq N_1$

$$V_{\tilde{N}}^C(0,x;\tilde{T}) \leq V_N(0,x;T) + Bc^{-1} + L_1\beta T.$$

By Lemma 4.5, for $c \geq c^*$

$$|V_{\tilde{N}}^c(0, x; \tilde{T}) - V_{\tilde{N}}^c(0, x; T)| \leq R_1(\tilde{T} - T) \leq 2\beta TR_1.$$

Therefore,

$$I_M(0, x; T) \leq \liminf_{N \rightarrow \infty} V_N(0, x; T) + Bc^{-1} + \beta T(L_1 + 2R_1).$$

However, c is arbitrarily large and β arbitrarily small. From (3.16) we then have

$$I(0, x; T) \leq \liminf_{N \rightarrow \infty} V_N(0, x; T).$$

To prove the opposite inequality, let $0 < \beta < \frac{1}{2}$. For N large we can choose $\hat{T} = \delta \hat{N}$ as in Lemma 4.2 with $\beta T \leq T - \hat{T} \leq 2\beta T$. Then

$$V_N(0, x; T) \leq V_N^c(0, x; T) \leq W_{\hat{N}}(0, x; \hat{T}) + L_1(T - \hat{T}).$$

By Lemma 4.1

$$|W_{\hat{N}}(0, x; \hat{T}) - W_{\hat{N}}(0, x; T)| \leq R_1(T - \hat{T})$$

provided $\delta = TN^{-1}$ is small enough that $\hat{N} \geq N_1$. Since $T - \hat{T} \leq 2\beta T$, Theorem 3.3 implies

$$\begin{aligned} \limsup_{N \rightarrow \infty} V_N(0, x; T) &\leq \lim_{N \rightarrow \infty} W_{\hat{N}}(0, x; T) + 2\beta T(R_1 + L_1) \\ &= I_M(0, x; T) + 2\beta T(R_1 + L_1) \end{aligned}$$

for $c \geq c(M)$. Since β is arbitrarily small and $I_M \leq I$,

$$\limsup_{N \rightarrow \infty} V_N(0,x;T) \leq I(0,x;T).$$

This proves Theorem 4.2.

Theorem 4.2 justifies calling $I(0,x;T)$ the lower value.

Concluding remarks. It should be possible to show that $I(s,x;T) = I(0,x;T-s)$ satisfies for almost all (s,x) the Isaacs equation, which is the first order equation obtained from (2.7') when $\epsilon = 0$. However, we have not done so. One could perhaps show that $I(\cdot, \cdot; T)$ is Lipschitz for $s \leq T' < T$, and then use an argument like the proof of [F2, p. 1005].

The lower value $I(0,x;T)$ is a nonincreasing function of T . Let

$$I^*(x) = \lim_{T \rightarrow \infty} I(0,x;T).$$

It would be interesting to investigate various properties of $I^*(x)$. For instance, is $I(0,x;T) = I^*(x)$ for some finite T ? That kind of result is suggested by examples treated by the method of characteristics for the Isaacs equation.

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