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THE STABILITY AND STABILIZABILITY OF INFINITE DIMENSIONAL LINEAR SYSTEMS VIA LIAPUNOV'S DIRECT METHOD

A. P. ROSS

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This dissertation attempts to extend to infinite dimensional linear systems in a Hilbert space some of the stability and stabilizability results that have been obtained for finite dimensional systems using the direct method of Liapunov. → next page

A finite dimensional linear system has the representation:

$$\dot{x} = Ax + Bu$$

where the state of the system x is an n-vector, and the input $u(\cdot):R^+ \to R^m$. A and B are nxn and nxm dimensional matrices, respectively. When the state of a physical system is represented by an infinite dimensional Hilbert space H, we have the following abstract differential equation describing the system:

$$\dot{x} = Ax + Bu \qquad (*)$$

where x is the state vector and is ϵ H. The input $u:R^+ \to U$, where U is the Hilbert space of input vectors. The linear operator A is the infinitesimal generator of a C_0 (strongly continuous at the origin) semigroup. The linear operator $B \in B(U;H)$, implying the control is distributed.

The stability problem is one of finding a criterion to be satisfied by the infinitesimal generator A which will imply that the "solution" of (*) (for a given initial state and $u(\cdot) = 0$) converges, in some sense, to the origin as $t \to \infty$. A related problem to that of stability is the problem of stabilizability. That is, the problem of finding a linear operator $F \in B(H;U)$ such that the unforced system

$$x = (A + BF)x \qquad (**)$$

is, in some sense, stable.

The familiar finite dimensional applications of Liapunov's direct method to the stability and stabilizability of linear systems involve the existence of certain positive matrices which satisfy some form of algebraic Riccati equation. Former extensions of these results to infinite dimensional systems in Hilbert space ~~(c.f. [6], [7], [8], [9], and [33])~~ apply exclusively to exponential (uniform asymptotic) stability. Recently, recognizing that exponential stability is a very strong property to expect of some physical systems, some attention has been paid to weaker forms of stability. ~~(c.f. [2]~~, [20], [21] and [27]).

This thesis generalizes the results of Liapunov's direct method to infinite dimensional systems in a manner that addresses these weaker forms of stability. This is accomplished by developing the distinct concepts of nonnegative, positive, and strictly positive operators. The resulting stability and stabilizability theorems are stated in terms of weak stability, but they are shown to be applicable, in many cases, to strong and exponential stability as well. These theorems are applied to examples which cannot be handled by the existing infinite dimensional Liapunov theorems developed in [8] and [10].

The different concepts of positive operators are further exploited by redefining exact and approximate controllability in terms of the degree of positivity of a controllability operator. The controllability operator is shown to be positive if the system is approximately controllable, strictly positive if the system is exactly controllable, and compact if the input operator B or the semigroup generated by A is.

The above results on controllability and stabilizability are then combined in a single inequality from which Benchimole's main result on weak stabilizability and approximate controllability [2], and an extension of Slemrod's results on exponential stabilizability and exact controllability [27] follow immediately. The final result is an extension of Benchimole's main theorem to semigroups which are not contractions.

UNIVERSITY OF CALIFORNIA

Los Angeles

The Stability and Stabilizability of Infinite Dimensional Linear Systems Via Liapunov's Direct Method

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering

by

Alan Paul Ross

1979

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ABSTRACT OF THE DISSERTATION

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Via Liapunov's Direct Method

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This thesis generalizes the results of Liapunov's direct method to infinite dimensional systems in a manner that addresses these weaker forms of stability. This is accomplished by developing the distinct concepts of nonnegative, positive, and strictly positive operators. The resulting stability and stabilizability theorems are stated in terms of weak stability, but they are shown to be applicable, in many cases, to strong and exponential stability as well. These theorems are applied to examples which cannot be handled by the existing infinite dimensional Liapunov theorems developed in [8] and [10].

The different concepts of positive operators are further exploited by redefining exact and approximate controllability in terms of the degree of positivity of a controllability operator. The controllability operator is shown to be positive if the system is approximately controllable, strictly positive if the system is exactly controllable, and compact if the input operator B or the semigroup generated by A is.

The above results on controllability and stabilizability are then combined in a single inequality from which Benchimole's main result on weak stabilizability and approximate controllability [2], and an extension of Slemrod's results on exponential stabilizability and exact controllability [27] follow immediately. The final result is an extension of Benchimole's main theorem to semigroups which are not contractions.

CHAPTER 1

PROBLEM FORMULATION AND PRELIMINARIES

1.1 MATHEMATICAL BACKGROUND AND NOTATION

This paper will investigate the stabilizability of the system (A,B) which is described by the abstract equation

$$\dot{x} = Ax + Bu , \quad x(0) = x_0 \in D(A)$$

where A is the generator of a strongly continuous semigroup of bounded linear operators on a separable Hilbert space H; T(t), $t \geq 0$, and B is a bounded linear operator from a Hilbert space U into H. H is called the state space and U is the input space, and $u(\cdot) : [0,\infty] \to U$. By analogy with the finite dimensional case, we expect

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)ds$$

to be the "solution" to the above equation under some appropriate definition. The form of this definition will, in general, depend on what restrictions are placed on $u(\cdot)$.

For our purposes, it will be convenient to look at the homgeneous equation

$$\dot{x}(t) = Ax(t) , \quad 0 < t < \infty$$

$$x(0) \in D(A).$$

This equation has the unique solution $x(t)$, where (see [1])

a) $x(t) \in D(A), \quad t \geq 0,$

b) $x(t)$ is absolutely continuous for $t > 0$,

c) $\|x(t) - x(0)\| \to 0$ as $t \to \infty$,

d) $x(t) = T(t)x(0).$

Now suppose the infinitesimal generator A is subject to a bounded perturbation of the form BF, where B is as defined above and F is a bounded operator, $F : H \to U$. We can associate this problem with the abstract differential equation

$$\dot{x} = (A + BF)x \ , \quad x(0) \in D(A + BF) = D(A)$$

This equation is equivalent to the original system (A,B) being subject to the feedback control $BFx(t)$. The resulting equation has the solution

$$x(t) = T(t)x(0) + \int_0^t T(t-z)BFx(z)dz.$$

If we designate the C_0 semigroup generated by $A + BF$ as $S(t)$, we get

$$S(t)x(0) = T(t)x(0) + \int_0^t T(t-z)BFS(z)x(0)dz$$

where the above are Bochner integrals.

A property of C_0 semigroups that will be useful in what follows is the "Exponential Growth Property." This states that there exists a positive real number w_0 such that for each $w > w_0$

$$T(t) \leq M(w)e^{wt} \tag{1.1.1}$$

for some positive real number $M(w) \geq 1$. If $\|T(t)\| \leq 1$, then $T(t)$ is a contraction semigroup. Contraction semigroups have the following properties which will be used in the sequel:

1. If A is the infinitesimal generator of a contraction semigroup, then

$$2\mathrm{Re}(Ax,x) \leq 0 \ , \ \forall x \in D(A).$$

2. $T^*(t)$ is also a contraction semigroup with infinitesimal generator A^*.

1.2 STABILITY DEFINITIONS

Suppose the system described in Section 1.1 has $B = 0$; i.e., the system is uncontrolled. In this condition, the behavior of the solution $T(t)x$, for a given initial state $x \in D(A)$, may tend to the origin as $t \to \infty$. Obviously, this "stability" property is very desirable for physical systems. However, the question remains as to the manner in which $T(t)x$ approaches 0. For instance, the system may tend to the origin in the weak, strong, or uniform topology. In this section, we define these different stability concepts.

<u>Definition 1.2.1.</u>

A semigroup is <u>weakly stable</u> if and only if $x, y \in H$ imply

$$\lim_{t \to \infty} (T(t)x,y) = 0.$$

<u>Definition 1.2.2.</u>

A semigroup is <u>strongly stable</u> if and only if for all $x \in H$

$$\lim_{t \to \infty} \|T(t)x\| = 0.$$

<u>Definition 1.2.3.</u>

A semigroup is <u>uniformly stable</u> if and only if

$$\lim_{t \to \infty} \|T(t)\| = 0.$$

<u>Definition 1.2.4.</u>

A semigroup is <u>exponentially stable</u> if and only if there exists two real numbers $M \geq 1$ and $w > 0$ such that

$$\|T(t)\| \le Me^{-wt}, \quad t > 0.$$

In reference [10], Datko shows that uniform stability is equivalent to exponential stability. In the following chapters, this strongest form of stability will be referred to according to the latter definition.

1.3 THE STABILIZABILITY PROBLEM

In solving the stabilizability problem, we are trying to take the system (A,B), represented by the abstract differential equation

$$\dot{x} = Ax + Bu, \quad x(0) \in D(A),$$

which has been defined in the previous sections, and bring an arbitrary initial state $x(0) \in D(A)$ to the origin via a feedback control of the form $u(t) = Fx(t)$, $F \in B(H,U)$. Using the development of Section 1.1, we are asking that

$$S(t)x(0) = T(t)x(0) + \int_0^t T(t-z)BFS(z)x(0)dz \to 0$$

as $t \to \infty$. Note that $S(t)$ is the semigroup generated by $A + BF$. Hence, this is the same as asking that $S(t)$ be stable.

If we want the trajectory starting at $x(0)$ to approach the origin in the weak sense, then we want the semigroup $S(t)$ to be weakly stable. Similarly for the strong limit and strong stability. If $S(t)x(0)$ is to tend to zero uniformly in $x(0)$, then we need exponential stability of the $S(t)$ semigroup.

1.4 CONTROLLABILITY

Consider the following nonhomogeneous equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \quad x(0) \in D(A) \tag{1.4.1}$$

where A and B are as described in Section 1.1, and $u(t) \in L_2[(0,a);U]$, i.e., $u(t)$ is weakly measurable and

$$\int_0^a \|u(z)\|^2 dz < \infty \ , \quad \forall a \in (0,\infty).$$

Then, if $x(t)$ is strongly continuous at the origin, equation (1.4.1) has the "solution"

$$x(t) = T(t)x(0) + \int_0^t T(t-z)Bu(z)dz$$

in the sense that

$$\frac{d}{dt}(x(t),y) = (x(t),A^*y) + (Bu(t),y) \quad \text{a.e.}$$

and

$$\lim_{t \to 0} (x(t),y) = (x(0),y) \ , \quad \forall y \in D(A^*)$$

(see [1]).

Definition 1.4.1.

The system (A,B), representing equation 1.4.1, is said to be approximately controllable if

$$\overline{\bigcup_{a\geq 0} R}\left(\int_0^a T(a-z)Bu(z)dz\right) = H$$

where $\int_0^a T(t-z)Bu(z)dz$ is a map from $L_2((0,a);U) \to H$.

In other words, the union of the range of this mapping for all positive values of a is dense in H. Two other equivalent definitions of approximate controllability are provided by the following theorem and its corollary (Balakrishnan [1]).

Theorem 1.4.1.

The system (A,B) is approximately controllable if and only if

$$\overline{\bigcup_{t\geq 0} R}(T(t)B) = H.$$

Corollary 1.4.1.

The system (A,B) is approximately controllable if and only if

$$\int_0^a T(z)BB^*T^*(z)x\,dz = 0$$

for some $x \in H$ and every $a > 0$ implies $x = 0$.

Corresponding to the definition of approximate controllability given by Corollary 1.4.1, we have the more restrictive property of exact controllability.

Definition 1.4.2.

A system (A,B) is exactly controllable if and only if

$$\bigcup_{a \geq 0} R\left(\int_0^a T(a-z)Bu(z)dz\right) = H$$

Therefore, the union of the range of the above mapping for all positive values of a is not just dense in H, but is equal to the whole space. In this case, the following theorem applies (Balakrishnan [1]).

Theorem 1.4.3.

The system (A,B) is exactly controllable if and only if for some $t_f > 0$

$$R\left(\int_0^{t_f} T(t_f-z)Bu(z)dz\right) = H.$$

Analogous to the above property, which is equivalent to exact controllability, we have

Definition 1.4.3.

A system (A,B) is approximately controllable in finite time if and only if for some $t_f > 0$

$$\overline{R}\left(\int_0^{t_f} T(t_f - z)Bu(z)dz\right) = H.$$

Note that this is a stronger property than approximate controllability (see Doleki reference [11]).

1.5 POSITIVE OPERATORS

In the literature on linear operators, there is frequent reference to nonnegative/positive operators. Unfortunately, there is no general agreement on the terminology employed in designating various degrees of positivity. For the purpose of this paper, three different types of positive operators will be considered. The following definitions will serve to fix our terminology.

Definition 1.5.1.

A bounded self-adjoint operator P on a Hilbert space H is said to be a nonnegative operator if and only if

$$(Px,x) \geq 0$$

for all $x \in H$.

Definition 1.5.2.

A bounded self-adjoint operator P on a Hilbert space H is said to be a positive operator if and only if

$$(Px,x) > 0$$

for all $x \in H$ and not equal to zero.

Definition 1.5.3.

A bounded self-adjoint operator P on a Hilbert space H is said to be strictly positive if there exists a positive number m such that

$$(Px,x) \geq m\|x\|^2.$$

It follows from Definition 1.5.3 that if P is strictly positive, then it has a bounded inverse. If P is positive, then it still has an inverse; however, it may not be bounded.

Theorem 1.5.1.

If P is positive, then there exists a closed operator P^{-1}, such that

$P^{-1}Px = x$ for all $x \in H$, and $PP^{-1}x = x$ for all $x \in D(P^{-1})$.

<u>PROOF</u>: Since $Px_1 = Px_2$ implies $P(x_1 - x_2) = 0$, P is one-one onto its range. Therefore, there exists an inverse linear mapping P^{-1} defined on $R(P)$. But $H = N(P) \oplus \overline{R(P)}$, and $N(P) = \{0\}$ by Definition 1.5.2. Hence, $\overline{R(P)} = \overline{D(P^{-1})} = H$. Consequently, P^{-1} has a dense domain. Now, let $x_n \in D(P^{-1}) = R(P)$, $x_n \to x$ and $P^{-1}x_n \to y$. Then there exists y_n such that $Py_n = x_n \to x$. It follows that $P^{-1}x_n = P^{-1}Py_n = y_n \to y$. This implies that $x_n = Py_n \to Py$ and $Py = x$ or $P^{-1}x = y$. Therefore, the linear operator P^{-1} is closed, and the theorem is proved.

CHAPTER 2

SURVEY OF EXTANT RESULTS

2.1 EQUIVALENCE OF STABILITY CONCEPTS

In Section 1.2, essentially three different degrees of stability are defined, ranging from weak to exponential. Most of the theorems in later chapters are stated in terms of weak stability. However, the following theorems show that for many important types of semigroups, weak stability is equivalent to strong or exponential stability.

Theorem 2.1.1.

If $T(t)$ has a compact resolvent $R(\lambda,A)$, then it is strongly stable if and only if it is weakly stable (see [2]).

Theorem 2.1.2.

If $T(t)$ is a self-adjoint semigroup ($T^*(t) = T(t)$), it is strongly stable if and only if it is weakly stable.

PROOF: To show that in this case weak stability implies strong stability, simply note that

$$\|T(t)x\|^2 = (T(t)x,T(t)x) = (T(2t)x,x) \to 0$$

as $t \to \infty$.

Theorem 2.1.3.

If $T(t)$ is compact for some $t_0 > 0$, then $T(t)$ is exponentially stable if and only if it is weakly stable.

PROOF: See [2].

2.2 THE DIRECT METHOD OF LIAPUNOV

We now establish the basic definitions and notation associated with the Liapunov technique in infinite dimensions. The following theorems are generalizations of the familiar finite dimensional results, and are stated in references [14], [16] and [31]. Unfortunately, some of the proofs are not given in these references. Proofs will be provided here when they are not available in the literature.

Definition 2.2.1.

Let $T(t)$ be a C_0 semigroup, $t \geq 0$, on a Hilbert space H. A continuous functional $V:H \to R^+$ is said to be a Liapunov functional for the semigroup $T(t)$ if $V(0) = 0$ and $\dot{V} \leq 0 \ \forall x \in H$. $\dot{V}$ is defined by the equation

$$\dot{V}(x) = \lim_{\delta \to 0^+} \sup_{t \in (0,\delta)} \frac{1}{t}(V(T(t)x) - V(x)).$$

In what follows, let $W:H \to R$ be a continuous functional such that $\dot{V} \leq -W(x) \leq 0$ for all $x \in H$.

Lemma 2.2.1: Let V be a Liapunov functional for a semigroup $T(t)$ on a Hilbert space H. If $x \in H$, then

$$V(T(t)x) - V(x) \le - \int_0^t W(T(s)x)ds \quad t \ge 0.$$

PROOF: (see [31], Lemma 3.4)

Theorem 2.2.1.

Let A be the infinitesimal generator of the C_0 semigroup $T(t)$ on the Hilbert space H. Let $V:H \to R$ and $W:H \to R^+$ be continuous on H, and let V be Frêchet differentiable on D(A) such that

$$\delta V(x;Ax) \le -W(x) \le 0 \quad \text{for all } x \in D(A).$$

Then:

(a) if $V(0) = 0$, V is a Liapunov functional on H with

$$\dot{V}(x) \le -W(x) \le 0 \quad \text{for all } x \in H,$$

(b) for $x \in D(A)$, $\dot{V}(x) = \delta V(x;Ax)$,

$$V(T(t)x) - V(x) = \int_0^t \delta V(T(s)x;AT(s)x)ds,$$

(c) for $x \in H$, $\dot{V}(x) \le -W(x)$,

$$V(T(t)x) - V(x) \le - \int_0^t W(T(s)x)ds.$$

PROOF: (Theorem 3.9, [31])

Theorem 2.2.2.

In addition to the assumptions of Theorem 2.2.1, let us assume that

(i) there exists a continuous nondescreasing scalar function α such that $\alpha(0) = 0$ and, for $x \neq 0$,

$$0 < \alpha(\|x\|) \leq V(x) \; ;$$

(ii) there exists a continuous scalar function γ such that $\gamma(0) = 0$ and $W(x)$ satisfies, for all $x \neq 0$,

$$-W(x) \leq -\gamma(\|x\|) < 0 \; ;$$

(iii) there exists a continuous nondescreasing scalar function β such that $\beta(0) = 0$ and

$$V(x) \leq \beta(\|x\|) \; ;$$

(iv) $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$.

Then the C_0-semigroup $T(t)$ is strongly stable.

PROOF: Take any $x_0 \in H$, then there exists $C_1(\|x_0\|) > 0$ such that $\beta(\|x_0\|) < \alpha(C_1)$ by assumption iv. Now take any $0 < \epsilon < \|x_0\|$. There exists $\nu(\epsilon) > 0$ such that $\beta(\nu) < \alpha(\epsilon)$, because β is continuous and $\beta(0) = 0$. Denote by $C_2(\epsilon, \|x_0\|) > 0$ the minimum of the continuous function γ on the compact set $[\nu(\epsilon), C_1(\|x_0\|)]$. Let $\tau(\epsilon, \|x_0\|) = \beta(\|x_0\|)/C_2(\epsilon, \|x_0\|)$.

Suppose now that $\|T(t)x_0\| > \nu$ over the interval $0 \le t \le \tau$, by i, ii and Theorem 2.2.1, we have

$$
\begin{aligned}
0 < \alpha(\nu) &\le V(T(t_1)x_0) \\
&\le V(x_0) - \int_0^\tau W(T(s)x_0)ds \\
&\le V(x_0) - \int_0^\tau \gamma(\|T(s)x_0)ds \; .
\end{aligned}
$$

But $\|T(s)x_0\| \ge \nu$ by assumption, and

$$
\begin{aligned}
\alpha(\|T(s)x_0\|) \le V(T(s)x_0) &\le V(x_0) \\
&\le \beta(\|x_0\|) \le \alpha(C_1)
\end{aligned}
$$

which implies

$$\nu \le \|T(s)x_0\| \le C_1, \; s \in [0,\tau].$$

We then have

$$0 < \alpha(\nu) \leq \beta(\|x_0\|) - \tau C_2 = 0,$$

a contradiction. Since

$$\alpha(\nu) \leq \beta(\nu) < \alpha(\epsilon) \leq \alpha(\|x_0\|) \quad \text{implies} \quad \nu < \|x_0\|,$$

it must be true that for some $t \in (0,\tau)$ say t_1, we have $\|T(t_1)x_0\| = \nu$. Therefore,

$$\begin{aligned}\alpha(\|T(t - t_1)T(t_1)x_0\|) &\leq V(T(t - t_1)T(t_1)x_0) \\ &\leq V(T(t_1)x_0) \\ &\leq \beta(V) < \alpha(\epsilon)\end{aligned}$$

for all $t \geq t_1$. Hence

$$\|T(t)x_0\| < \epsilon \quad \text{for all} \quad t \geq \tau \geq t_1$$

which proves strong stability.

2.3 THE INVARIANCE PRINCIPAL

In order to employ the direct method of Liapunov, as presented in Theorem 2.2.1, it was assumed that the Liapunov functional $V(x)$ satisfies

$$\dot{V}(x) \leq -\gamma(\|x\|) < 0,$$

where γ is a continuous scalar function such that $\gamma(0) = 0$. This condition may in some cases be too restrictiv~ In this section, the invariance principal (also known as Liapunov's second method) will be used to weaken this assumption. However, stronger conditions on the nature of the trajectory $\{T(t)x_0,\ t \geq 0\}$ will be required.

Before stating the invariance principal, the following definitions are needed.

Definition 2.3.1.

Let $T(t)$ be a C_0 semigroup on the Hilbert space H. The positive orbit $0^+(x)$ though $x \in H$ is defined to be

$$0^+(x) = \bigcup_{t\geq 0} T(t)x$$

Definition 2.3.2.

A set M^+ is a positive invariant set for the C_0 semigroup $T(t)$ if

$$x \in M^+ \text{ implies } T(t)x \in M$$

for all $t \in R^+$.

We must also define the set

$$s = \{x \in H;\ \dot{V}(x) = 0\}$$

where V is a Liapunov functional for T(t).

Definition 2.3.3.

The positive limit set $\Gamma^+(x)$ of the orbit through x is the set of $p \in H$ such that there is a nondecreasing sequence $\{t_N\}$, $t_N > 0$, $t_N \to \infty$ such that

$$\|T(t_N)x - p\| \to 0 \quad \text{as} \quad n \to \infty .$$

In other words,

$$\Gamma^+(x) = \bigcap_{\tau \geq 0} \left(\overline{\bigcup_{t \geq \tau} T(t)x} \right) .$$

We can now state and prove the invariance principal.

Theorem 2.3.1.

Suppose $T(t)$ is a C_0-semigroup on H. If V is a Liapunov functional, $H \to R^+$, and the positive orbit $O^+(x_0)$ is in a compact set of H, then

$$T(t)x_0 \to M^+ \quad \text{as} \quad t \to \infty,$$

where M^+ = the largest positive invariant set in s.

PROOF: We know that $V(x(t))$ is monotonically nonincreasing and bounded from below. Therefore, $V(x(t)) \to c$ as $t \to \infty$. Now if $p \in \Gamma^+(x_0)$, then there exists $\{t_N\}$, $t_N > 0$, $t_N \to \infty$ such that

$$\|x(t_N) - p\| \to 0 \quad \text{as} \quad n \to \infty \ ;$$

this implies $V(x(t_N)) \to V(p) = c$. In addition, $\Gamma^+(x_0)$ is invariant because $T(t_N)x_0 \to p$ implies $T(s + t_N)x_0 \to T(s)p$. Therefore, $\dot{V}(p) = 0$ for all $p \in \Gamma^+(x_0)$. Also, $T(t)x_0 \to \Gamma^+(x_0)$. If this were not true, then there exists $\{t_N\}$, $t_N \to \infty$ such that dist $(\Gamma^+(x_0), T(t_N)x_0) \geq \epsilon > 0$. But some subsequence of $\{T(t_N)x_0\}$ must converge to a limit p_0 because of the compactness assumption, giving us a contradiction. In the same way, we show that $\Gamma^+(x_0)$ is not empty. This establishes the theorem.

An obvious limitation of this theorem is the fact that for each $x \in H$, $T(t)x$ must remain in a compact subset of H. Also, the set s must be determined in such a manner as to yield useful stability information.

2.4 THE QUADRATIC LIAPUNOV FUNCTIONAL

For linear time invariant systems, the Liapunov functional invariably takes on a quadratic form; i.e., $V(x) = (Dx, x)$ where D is a positive operator on H. Hence,

$$\dot{V}(x) = (DAx, x) + (x, DAx)$$

for all $x \in D(A)$. In order to conclude some form of stability, it is required that

$$(DAX, x) + (x, DAx) = -(Cx, x)$$

for all $x \in D(A)$ where the self-adjoint operator C is at least non-negative. In the finite dimensional case, we have

Theorem 2.4.1.

A necessary and sufficient condition for the origin 0 of an autonomous linear system $\dot{x} = Ax$ to be exponentially stable is the existence of strictly positive matrices C and Q satisfying the equation

$$A^*Q + QA = -C.$$

PROOF: (See [16], Section 9.4.)

This theorem has been generalized to the infinite dimensional case by Datko [8].

Theorem 2.4.2.

Let $T(t)$ be a strongly continuous C_0 semigroup on H with the infinitesimal generator A. A necessary and sufficient condition for the semigroup $T(t)$ to be exponentially stable is the existence of a strictly positive operator C, and a positive self-adjoint operator Q such that

$$2\mathrm{Re}(Qx,x) = -(Cx,x), \qquad \forall x \in D(A).$$

2.5 SOME STABILIZABILITY RESULTS

The stabilizability problem has been approached, in the finite dimensional case, in the context of an optimal control problem with quadratic cost functional. Assuming that the resulting Riccati equation has a strictly positive steady state solution, we end up with the following algebraic Riccati equation for the finite dimensional linear system (A,B):

$$DA + A^*D - DBB^*D = -C. \tag{2.5.1}$$

In this equation, C is a positive definite $n \times n$ matrix and the feedback control is of the form $-B^*Dx(t)$. This leaves us with the stabilizability result

Theorem 2.5.1.

If the Riccati equation (2.5.1) has a strictly positive solution D, then the system (A,B) is exponentially stabilized by the feedback matrix $-B^*D$.

PROOF: This is easily seen by applying Theorem 2.4.1 to

$$\dot{x} = (A - BB^*D)x.$$

For the infinite dimensional case, we can modify equation (2.5.1) to get

$$2\mathrm{Re}(DAx,x) - (DBB^*Dx,x) = -(Cx,x) \qquad (2.5.1')$$

for all $x \in D(A)$.

In this case, Theorem 2.5.1 generalizes to

Theorem 2.5.2.

If, for the linear system (A,B) described in Section 1.1, the Riccati equation (2.5.1) (where C is now a strictly positive operator) has a self-adjoint positive solution D, then the system (A,B) is exponentially stabilized by the feedback operator $-B^*D$.

<u>PROOF</u>: Apply Theorem 2.4.2 to $\dot{x} = (A - BB^*D)x$.

The next question that comes to mind after looking at the above theorems is: when does equation (2.5.1) have a solution? One answer to this question involves controllability and will be discussed in Section 2.7. Another answer was given by Datko [9]. The results will be presented in the remainder of this section.

First, we define a cost functional for the system (A,B) on $L_2(I;H)$:

$$C(u,I,x_0) = \int_I ((Wx_u(s),x_u(s)) + (u(s),u(s)))ds \qquad (2.5.2)$$

where $I = (0,t)$, and

$$x_u(t) = T(t)x_0 + \int_0^t T(t - s)Bu(s)ds \, . \qquad (2.5.3)$$

W is a strictly positive operator on H.

<u>Theorem 2.5.3.</u>

Let $I_n = (0,t_n)$ be a monotonically increasing sequence of intervals such that $t_n \to \infty$. If for each $x \in H$

$$\lim_{n\to\infty} \min_{u\in L_2(I_n)} C(u,I_n,x) < +\infty ,$$

then there exists a self-adjoint endomorphism D such that for each $x_0 \in H$

$$(Dx_0,x_0) = \int_0^\infty ((W + DBB^*D)x(s,x_0),x(s,x_0))ds$$

where

$$x(t,x_0) = T(t)x_0 - \int_0^t T(t - s)BB^*Dx(s,x_0)ds .$$

PROOF: Theorem 2, Lemma 1, and Theorem 4 - Datko [9].

It is easily seen that D is a solution to equation (2.5.1') where $C = W$.

2.6 STATE SPACE DECOMPOSITION

In Section 2.7, we will examine the application of the concept of controllability to the stabilizability problem. In this effort, some authors have found it convenient to decompose the Hilbert space H into two subspaces, $H = H_1 \oplus H_2$, that reduce the semigroup, $T(t) = T(t)|_{H_1} \oplus T(t)|_{H_2}$. Usually the restriction of the semigroup to one of these reducing subspaces already exhibits the desired stability properties. Hence, it is sufficient to stabilize the system on the remaining unstable subspace. In this section, two such decomposition theorems will be presented.

The first decomposition result is a combination of the Nagy-Foias conical decomposition of contractive semigroups ([24], p. 145), and Foguel's Theorem ([17], p. 723). It applies to contraction semigroups and weak stability.

Theorem 2.6.1.

Let $T(t)$ be a contractive semigroup with infinitesimal generator A. Then there exists subspaces $H_{c.n.u.}(T)$ and $H_u(T)$ such that $H = H_{c.n.u.}(T) \oplus H_u(T)$, and $T(t) = T_{c.n.u.}(t) \oplus T_u(t)$, $t \geq 0$. $T_{c.n.u.}(t) = T(t)|_{H_{c.n.u.}(T)}$ and $T_u(t) = T(t)|_{H_u(T)}$ are completely nonunitary and unitary semigroups, respectively, on their associated subspaces. Moreover, $\overline{D(A) \cap H_u(T)} = H_u(T)$ and $H_u(T) = H_u(T^*)$.

PROOF: See Levan reference [20], p. 722.

Corollary 2.6.1.

A completely nonunitary semigroup is weakly stable.

PROOF: See Levan, reference [20], p. 723.

The second decomposition theorem establishes sufficient conditions for decomposing a C_0 semigroup into an exponentially stable part and an "unstable" part. Before presenting this theorem, it will be necessary to introduce the spectrum determined growth assumption. Recall from Section 1.1 that for a C_0 semigroup $T(t)$, generated by A, there exists a positive real number w_0 such that for each $w > w_0$

$$T(t) \leq Me^{wt}$$

for some positive real number $M \geq 1$. Now, if $w_0 = \sup \mathrm{Re}\sigma(A)$, we say that A satisfies the spectrum determined growth assumption. Clearly, if A satisfies the spectrum determined growth assumption and $\sup \mathrm{Re}\sigma(A) < 0$, then $T(t)$ is exponentially stable.

Theorem 2.6.2.

Let $\delta > 0$, and consider the following partitions $\sigma_u(A)$ and $\sigma_s(A)$ of the spectrum $\sigma(A)$ of A;

$$\sigma_u(A) = \sigma(A) \cap \{\lambda : \mathrm{Re}\lambda \geq -\delta\}$$

$$\sigma_s(A) = \sigma(A) \cap \{\lambda : \mathrm{Re}\lambda < -\delta\} .$$

Assume that the set $\sigma_u(A)$ is bounded and is separated from the set $\sigma_s(A)$ in such a way that a rectifiable simple closed curve can be drawn so as to enclose an open set containing $\sigma_u(A)$ in its interior and $\sigma_s(A)$ in its exterior. Then

(i) The operator A may be decomposed according to the decomposition $H = H_u \oplus H_s$; i.e., H_u and H_s are invariant under A.

(ii) $\sigma(A_s) = \sigma_s(A)$ and $\sigma(A_u) = \sigma_u(A)$, where $A_s = A|_{H_s}$ and $A_u = A|_{H_u}$.

(iii) $T(t) = T_u(t) \oplus T_s(t)$, where $T_u(t) = T(t)|_{H_u}$ and $T_s(t) = T(t)|_{H_s}$ are C_0 semigroups on their associated subspaces, generated by A_u and A_s, respectively.

PROOF: See reference [19], p. 178.

Corollary 2.6.2.

If A_s satisfies the spectrum determined growth assumption, then $T_s(t)$ is exponentially stable on H_s.

PROOF: Triggiani [30], p.392.

Note the semigroups with compact self-adjoint resolvents satisfy the conditions of both Theorem 2.6.2 and Corollary 2.6.2.

2.7 EXACT CONTROLLABILITY AND EXPONENTIAL STABILIZABILITY

Theorem 2.5.3 establishes the following condition for exponential stabilizability:

Definition 2.7.1.

The system (A,B) satisfies the stabilizability condition if for every $x_0 \in H$ there exists a control function $u \in L_2([0,\infty);V)$ such that

$$\int_0^\infty \|x_u(t;x_0)\|^2 dt < +\infty \tag{2.7.1}$$

where

$$x_u(t;x_0) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds .$$

From the form of the stabilizability condition, one might expect the concepts of controllability and stabilizability to be related. In fact, for the finite dimensional case, Kalman [17] has shown that controllability implies inequality 2.7.1. In attempting to generalize this result to infinite dimensions, it should be noted that for infinite dimensional systems, the properties of exact and approximate controllability are not equivalent. Triggiani [28] has provided several examples of approximately controllable systems which are not exponentially stabilizable. However, Zabczyk [33] has shown that an exactly controllable system satisfies inequality 2.7.1 for some $\bar{u} \in L_2([0,\infty);H)$. The proof given by Zabczyk is quite terse. We will, therefore, state the theorem and give a more complete proof here.

Theorem 2.7.1.

If the system (A,B) of Section 1.1 is exactly controllable (see Section 1.4), then inequality (2.7.1) is satisfied. Hence, there exists a $D \in B(H,H)$ such that the system (A,B) is exponentially stabilizable by the feedback operator $-B^*D$.

PROOF: Using the definition of exact controllability given in Section 1.4, we know that

$$\int_0^{\bar{t}} T(s)BB^*T^*(s)ds = C$$

is a self-adjoint operator with a bounded inverse. We therefore let $I_n > \bar{t}$ in equation (2.5.2), and substitute the following control function into equations (2.5.3) and (2.5.2):

$$\bar{u}(s) = \begin{cases} B^*T(\bar{t} - s)C^{-1}T(\bar{t})x_0 & 0 \leq s \leq \bar{t} \\ 0 & \bar{t} < s \end{cases}$$

This gives us

$$\min_{u \in L_2(I_n)} C(u,I_n,x_0) \leq \int_0^{\bar{t}} (Wx_{\bar{u}}(s),x_{\bar{u}}(s))ds + \int_0^{\bar{t}} (\bar{u}(s),\bar{u}(s))ds$$

$$< +\infty ,$$

for all $n \geq$ some n_0 and $x_0 \in H$.

The above, along with Theorems 2.5.2 and 2.5.3, gives us our result.

It follows from the above theorem that exact controllability is a sufficient condition for exponential stabilizability. However, it is not a necessary condition. Triggiani [30] shows that a semigroup T(t), with infinitesimal generator A satisfying the conditions of Theorem 2.6.2 and Corollary 2.6.2, is exponentially stabilizable provided it is exactly controllable on the unstable subspace H_u. In order to see this, the system (A,B) is decomposed as follows.

Let P be the projection operator onto the subspace H_u. Then applying P and (I - P) to both sides of equation 1.4.1 yields

$$\dot{x}_u = A_u x_u + PBu \,, \quad x_{0u} = Px_0 \in D(A_u)$$

$$\dot{x}_s = A_s x_s + (I - P)Bu \,, \quad x_{0s} = (I - P)x_0 \in D(A_s)$$

Operator C of Theorem 2.7.1 becomes

$$C = \int_0^{\bar{t}} \begin{bmatrix} T_u(t) & 0 \\ 0 & T_s(t) \end{bmatrix} \begin{bmatrix} PB \\ (I - P)B \end{bmatrix}$$

$$[B^*P, B^*(I - P)] \begin{bmatrix} T_u^*(t) & 0 \\ 0 & T_s^*(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} = \int_0^{\bar{t}} T_u(t)PBB^*PT_u^*(t)dt$$

$$C_{12} = \int_0^{\bar{t}} T_u(t)PBB^*(I - P)T_s^*(t)dt$$

$$C_{21} = C_{12}^*$$

$$C_{22} = \int_0^{\bar{t}} T_s(t)(I - P)BB^*(I - P)T_s^*(t)dt$$

It is now possible to state and prove

Theorem 2.7.2. (Triggiani [30])

Let the semigroup $T(t)$ and its infinitesimal generator A satisfy the conditions of Theorem 2.6.2 and Corollary 2.6.2. Further, let the projection of (A,B) onto H_u (A_u, PB) be exactly controllable. Then the system (A,B) is exponentially stabilizable.

PROOF: The fact that (A_u, PB) is exactly controllable implies that the operator $C_{11} : H_u \to H_u$, defined above, has a bounded inverse for some $\overline{t} > 0$. Therefore, for each $x_0 \in D(A)$, let

$$\overline{u}(s) = \begin{cases} -B^*T^*(\overline{t} - s)\begin{bmatrix} C_{11}^{-1}T_u(t)Px_0 \\ 0 \end{bmatrix} & 0 \leq s \leq \overline{t} \\ 0 & \overline{t} < s \end{cases}$$

It follows that

$$\int_0^\infty \|x_{\overline{u}}(t;x_0)\|^2 dt = \int_0^{\overline{t}} \|x_{\overline{u}}(t;x_0)\|^2 dt +$$

$$\int_{\overline{t}}^\infty \Bigg\| \begin{bmatrix} T_u(t - \overline{t}) & \\ 0 & T_s(t - \overline{t}) \end{bmatrix} \Bigg\{ \begin{bmatrix} T_u(\overline{t})Px_0 \\ T_s(\overline{t})(I - P)x_0 \end{bmatrix} -$$

$$\int_0^{\overline{t}} \begin{bmatrix} T_u(\overline{t} - s)PBB^*PT_u^*(\overline{t} - s) & T_u(\overline{t} - s)PBB^*(I - P)T_s^*(\overline{t} - s) \\ T_s(\overline{t} - s)(I - P)BB^*PT_u^*(\overline{t} - s) & T_s(\overline{t} - s)(I - P)BB^*(I - P)T_s^*(\overline{t} - s) \end{bmatrix}$$

$$\begin{bmatrix} C_{11}^{-1}T_u(\overline{t})Px_0 \\ 0 \end{bmatrix} ds \Bigg\} \Bigg\|^2 dt$$

$$= \int_0^{\overline{t}} \|x_{\overline{u}}(t;x_0)\|^2 dt +$$

$$\int_0^\infty \left\| \begin{bmatrix} T_u(t-\bar{t}) & 0 \\ 0 & T_s(t-\bar{t}) \end{bmatrix} \left\{ \begin{bmatrix} T_u(\bar{t})Px_0 \\ T_s(\bar{t})(I-P)x_0 \end{bmatrix} - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} C_{11}^{-1}T_u(t)Px_0 \\ 0 \end{bmatrix} \right\} \right\|^2 dt$$

$$= \int_0^{\bar{t}} \|x_u(t;x_0)\|^2 dt + \int_{\bar{t}}^\infty \|T_s(t-\bar{t})\bar{x}\|^2 dt$$

where

$$\bar{x} = T_s(\bar{t})(I-P)x_0 - C_{21}C_{11}^{-1}T_u(\bar{t})Px_0 . \qquad (2.7.2)$$

By the corollary on page 615 of reference [8], and the fact that $T_s(t)$ is exponentially stable on H_s,

$$\int_{\bar{t}}^\infty \|T_s(t-t)x\| dt < +\infty .$$

Hence, inequality 2.7.1 is satisfied and the system (A,B) is exponentially stabilizable. This completes the proof.

Remark 2.7.1.

In both Theorem 2.7.1 and Theorem 2.7.2, the controllability assumptions are used in establishing the existence, for each $x_0 \in D(A)$, of a $\bar{u}(t;x_0)$ which drives the initial state x_0 to a "exponentially stable state" $\bar{x}$. The term exponentially stable state is used here to denote a state $\bar{x}$ for which

$$\int_0^\infty \|T(t)\bar{x}\|^2 dt < +\infty .$$

In Theorem 2.7.1 $\bar{x} = 0$, where as in Theorem 2.7.2, $\bar{x}$ is given by equation 2.7.2. Note that the exponentially stable states of a semigroup $T(t)$ form an invariant subspace M. Hence, one possible criterion for exponential stabilizability is that the set

$$M \cap \Big(\bigcup_{t \geq 0} \bigcup_{u \in L_2([0,\infty);U)} \Big(T(t)x_0 + \int_0^t T(t-s)Bu(s)ds\Big)$$

be non-empty for each $x_0 \in D(A)$.

2.8 APPROXIMATE CONTROLLABILITY AND WEAK STABILIZABILITY

The exponential stabilizability results of the last section require that the system (A,B) (or at least the unstable part of it) be exactly controllable. Unfortunately, many systems of interest are not exactly controllable. Consequently, some authors have investigated the stability implications of approximate controllability. As pointed out in Section 2.7, approximate controllability does not imply exponential stabilizability. However, as we will see below, approximate controllability does in some special cases imply weak stabilizability.

The first results obtained in this regard are those of Slemrod [27]. Although these were improved upon by Benchimole [2], whose main theorem will also be presented in this section, Slemrod's weak stabilizability theorem provides an excellent application of the invariance principal to linear infinite dimensional time invariant systems. Both theorems require that A generate a contraction semigroup. The reason for this assumption will be clarified in Chapter 4.

Theorem 2.8.1. (Slemrod [27], Theorem 3.2)

Let A be the infinitesimal generator of a C_0 contraction semigroup $T(t)$ on H for $t \geq 0$, and let $S(t)$ be the C_0 contraction semigroup generated by $A - BB^*$. If

(i) for every $y \in H$, $S^*(t)y$ remains in some compact set of H (which may depend on y) for $t \geq 0$, and

(ii) the system (A,B) is completely controllable,

then $S(t)y \to 0$ weakly as $t \to \infty$ for all $y \in H$; i.e., (A,B) is weakly stabilizable.

PROOF: Let $C = A - BB^*$. Then C^* is the infinitesimal generator of $S^*(t)$. Consider the functional on H given by $V(x) = 1/2 \|x\|^2$. Since $\|T(t)\| \leq 1$, we have $(x,A^*x) \leq 0$ for all $x \in D(A^*)$ (see Section 1.1). Thus, a simple computation shows

$$\dot{V}(S^*(t)x_0) \leq -\|B^*S^*(t)x_0\|^2$$

for $x_0 \in D(C^*) = D(A^*)$. Now let $y \in H$. Since $D(C^*)$ is dense in H, there exists $\{y_n\} \subset D(C^*)$ so that $y_n \to y$ in H. Then

$$\begin{aligned}
\dot{V}(y) &= \overline{\lim_{t\to0+}} \frac{1}{t} [V(S^*(t)y) - V(y)] \\
&= \overline{\lim_{t\to0+}} \lim_{n\to\infty} \frac{1}{t} [V(S^*(t)y_n) - V(y_n)] \\
&= \overline{\lim_{t\to0+}} \lim_{n\to\infty} \frac{1}{t} [\int_0^t \dot{V}(S^*(s)y_n)ds] \\
&\leq \overline{\lim_{t\to0+}} \lim_{n\to\infty} \frac{1}{t} [-\int_0^t \|B^*S^*(s)y_n\|^2 ds] \\
&\leq -\|B^*y\|^2 \quad \text{for} \quad y \in H.
\end{aligned}$$

Now we can apply Theorem 2.3.1 to the semigroup $S^*(t)$. Hypothesis (i) implies that for the semigroup $S^*(t)$, $O^+(y)$ belongs to a compact set of H. Theorem 2.3.1 then implies that $S^*(t)y \to M^+$ as $t \to \infty$, where M^+ is the largest positive invariant set in $\{y \in H;\ \dot{V}(y) = 0\}$.

We now show that $M^+ = \{0\}$. Let $m \in M^+$ and define

$$Z(t) = \int_0^t S^*(s)m\,ds.$$

Since C^* is closed, it follows that $Z(t) \in D(C^*)$ and

$$\dot{Z}(t) = Az(t) - BB^*z(t) + m \quad \text{for } t \geq 0 \tag{2.8.1}$$

From the definition of M^+, it follows that $B^*S^*(t)m = 0$ for all $t \geq 0$.

Thus, $B^*Z(t) = 0$ for $t \geq 0$ and we see, using (2.8.1), that

$$Z(t) = \int_0^t T^*(s)m\,ds.$$

But $B^*Z(t) = 0$, so

$$\int_0^t B^*T^*(s)m\,ds = 0 \quad \text{for } t \geq 0.$$

This in turn implies $B^*T^*(t)m = 0$ for all $t \geq 0$ and, employing the approximate controllability hypothesis, we see that $m = 0$. Applying Theorem 2.3.1, we conclude that $S^*(t)y \to 0$ as $t \to \infty$. Thus, $S(t)y \to 0$ weakly and we have proved weak stabilizability.

Benchimole [2] was able to extend this theorem to include general contraction semigroups by employing the decomposition results of Theorem 2.6.2.

Theorem 2.8.2. (Benchimole [2], Theorem 3.4.1.)

If the system (A,B) is such that A generates a C_0 contraction semigroup, then the system is weakly stabilized by the feedback operator $-B^*$.

PROOF: Let $-B^*$ be the feedback gain. Then $A - BB^*$ generates a contraction semigroup $S(t)$. Applying Theorem 2.6.1 to $S(t)$, we obtain a decomposition of H into two subspaces; $H_u(s)$, reducing $S(t)$ to a unitary group, and $H_{cnu}(s)$, reducing $S(t)$ to a completely nonunitary semigroup. By Corollary 2.6.1, we know that for all $x \in H_{cnu}(s)$ $S(t)x \to 0$ weakly as $t \to +\infty$.

Therefore, we can establish weak stability if we can show that $H_u(s) = \{0\}$. Let $x \in D(A^*) \cap H_u(s)$, then for all $t \geq 0$

$$\frac{d}{dt}\|S^*(t)x\|^2 = ((A^* - BB^*)S^*(t)x, S^*(t)x)$$

$$+ (S^*(t)x, (A^* - BB^*)S^*(t)x) = 0 ,$$

since $S^*(t)$ is unitary on $H_u(S) = H_u(S^*)$ by Theorem 2.6.1. But

$$(A^*y, y) + (Y, A^*y) \leq 0$$

for all $y \in D(A^*)$, as A^* is a contraction semigroup. This implies that $B^*S^*(t)x = 0 \ \forall t \geq 0$. It follows from approximate controllability of the system that x = 0. Since $\overline{D(A^*) \cap H_u} = H_u$, we have $H_u = \{0\}$, and the theorem is proved.

In Section 4.2, we will obtain the results of this theorem without using the Nagy-Foias conical decomposition of contraction semigroups, or Foguel's theorem. In addition, our method will yield some nice exponential stabilizability results as well.

CHAPTER 3

STABILITY AND STABILIZABILITY

3.1 SOME QUADRATIC LIAPUNOV THEOREMS

In studying stability and stabilizability of linear systems in Hilbert space, we use quadratic Liapunov functionals of the type described in Section 2.4. In this section, we present two such Liapunov stability theorems for strongly continuous semigroups. The first concerns exponential (uniform asymtotic) stability. It is simply a restatement of Theorem 2.4.2, provided here for comparison with the second Liapunov theorem which is in some sense a generalization of this result. This second Liapunov theorem has the advantage of being applicable to the weaker forms of stability defined in Section 1.2 which cannot be determined by the use of Theorem 3.1.1.

Theorem 3.1.1.

Let $T(t)$ be a strongly continuous C_0 semigroup on H with the infinitesimal generator A. A necessary and sufficient condition for the semigroup $T(t)$ to be exponentially stable is the existence of a positive operator $D \in B(H,H)$, and a strictly positive operator $C \in B(H,H)$ such that

$$\text{i)} \quad V(x) = (Dx,x) \qquad x \in H,$$

$$\text{ii)} \quad \dot{V}(x) = 2\mathrm{Re}(DAx,x) \leq -(Cx,x) \qquad x \in D(A).$$

This is nothing more than a slight modification of Datko's result in [3], although it is a vast improvement over Theorem 4.7, [31].

Theorem 3.1.2.

Let $T(t)$ be a strongly continuous C_0 semigroup on H with the infinitesimal generator A. A sufficient condition for weak stability is the existence of a strictly positive operator D and a positive operator C such that

$$\text{i)}\quad V(x) = (Dx,x) \qquad x \in H,$$

$$\text{ii)}\quad \dot{V}(x) = 2\mathrm{Re}(DAx,x) \le -(Cx,x) \qquad x \in D(A).$$

PROOF: Let $\bar{x} \in D(A)$, then

$$\begin{aligned}\frac{dV}{dt}(T(t)\bar{x}) &= 2\mathrm{Re}(DAT(t)\bar{x},T(t)\bar{x}) \\ &\le -(CT(t)\bar{x},T(t)\bar{x}) .\end{aligned}$$

Therefore,

$$\begin{aligned}0 \le V(T(t)\bar{x}) \le V(\bar{x}) &- \int_0^t (CT(s)\bar{x},T(t)\bar{x})ds \\ &\le V(\bar{x}) .\end{aligned}$$

This series of inequalities implies:

$$\text{a)}\quad (DT(t)\bar{x},T(t)\bar{x}) \le V(\bar{x}). \tag{3.1.1}$$

Note if D is strictly positive, then there exists a $m > 0$, such that

$$m\|T(t)\bar{x}\|^2 \le V(\bar{x}) \qquad \text{for all } x \in D(A) \tag{3.1.2}$$

But D(A) is dense in H, therefore,

$$\|T(t)x\|^2 \le \frac{1}{m}V(x) \qquad \text{for all } x \in H.$$

Using the uniform boundedness principle, this implies

$$\|T(t)\| \le M.$$

b) $$\int_0^t (CT(s)\bar{x},T(s)\bar{x})ds \le (D\bar{x},\bar{x}) \qquad \text{for all } \bar{x} \ge D(A).$$

Once again, D(A) being dense in H implies this inequality holds for all $x \in H$.

Since C is positive, there exists a positive square root $G = \sqrt{C}$. We can now rewrite inequality b)

$$\int_0^t \|GT(s)x\|^2 ds \le (Dx,x) \qquad \text{for all } t \ge 0.$$

This implies

$$\int_0^\infty \|GT(s)x\|^2 ds \le (Dx,x).$$

Let us now consider $\|GT(s)x\|^2$. We know that $\underline{\lim}\ \|GT(s)x\| = 0$. If $\overline{\lim}\ \|GT(s)x\| \neq 0$, then there exists a constant $\epsilon > 0$ and a sequence of disjoint closed intervals $I_i = [a_i, b_i]$, $i = 1, 2, \ldots$, such that for each i:

$$\text{i)} \quad \|GT(a_i)x\| = \epsilon,$$

$$\text{ii)} \quad \epsilon \leq \|GT(s)x\| \leq 2\epsilon, \quad s \in [a_i, b_i],$$

$$\text{iii)} \quad GT(b_i)x = 2\epsilon.$$

But,

$$\begin{aligned} \epsilon = \|GT(b_i)x\| - \|GT(a_i)x\| &\leq \|GT(b_i)x - GT(a_i)x\| \\ &\leq \|GT(a_i)\|\ \|T(b_i - a_i)x - x\| \quad (3.1.4) \\ &\leq M\|G\|\ \|T(b_i - a_i)x - x\|. \end{aligned}$$

This implies that $|b_i - a_i| \not\to 0$. Therefore, there exists a subsequence $\{|b_k - a_k|\}$, and some $\delta > 0$ such that $|b_k - a_k| > \delta$ for all k. We then have

$$+\infty = \lim_{k\to\infty} k\delta\epsilon^2 \leq \epsilon^2 \sum_{k=1}^{\infty} |b_k - a_k| \leq \int_0^{\infty} \|GT(s)x\|^2 ds,$$

which is a contradiction. Hence, $\lim_{s\to\infty} GT(s)x = 0$, and for any $y \in H$, $\lim_{s\to\infty} (GT(s)x,y) = \lim_{s\to\infty} (T(s)x,Gy) = 0$.

Now for any $x \in R(G)$, $\lim_{s\to\infty} (T(s)x,z) = 0$, and $R(G)$ is dense in H because G is positive. Hence, for $z \in H$, there exists $z_m \in R(G)$ such that $z_m \to z$, and since $T(t)x$ is bounded

$$
\begin{aligned}
(T(t)x,z) &= (T(t)x,z - z_m) + (T(t)x,z_m) \\
&\leq M\|x\| \, \|z - z_m\| + (T(t)x,z_m).
\end{aligned}
$$

Taking the limit with respect to t first and then with respect to m, we have $\lim (T(t)x,z) = 0$ as $t \to \infty$, for all $x,z \in H$. This completes the proof.

The above theorem generalizes, in a form different from Datko [8], the familiar finite dimensional result.

Note that in the statement of Theorem 3.1.2, we required the operator D to be strictly positive. This property is used in establishing the uniform boundedness of the C_0 semigroup $T(t)$. For some important classes of infinitesimal generators, this assumption can be weakened so that D need only be a positive operator.

Corollary 3.1.1.

In Theorem 3.1.2, if the infinitesimal generator A is the generator of a uniformly bounded semigroup, then D need only be a positive operator.

PROOF: The only use made of D being strictly positive in Theorem 3.1.2 was in part a) where we used this property to establish the uniform boundedness of the semigroup.

The most important class of semigroups which satisfy the uniform boundedness hypothesis of Corollary 3.1.1 are contraction semigroups. This is the case, as we mentioned in Chapter 1, if the infinitesimal generator A is dissipative, i.e.,

$$2\mathrm{Re}(Ax,x) \leq 0 \qquad x \in D(A).$$

In fact, almost all semigroups met with in practice are contraction semigroups.

Now, suppose we have an integro-differential equation of Volterra type such as

$$\frac{\partial W(t,\xi)}{\partial t} = \int_{\Omega} R(\xi,s)W(t,s)ds,$$

where (Ω,β,μ) is a σ-finite measure space, and $R(\xi,s)$ is an m x m matrix function defined on $\Omega \times \Omega$, measureable $\beta \times \beta$, and such that

$$\|R\|^2 = \int_{\Omega\times\Omega} \mathrm{Tr.}\ R(\xi,s)R(\xi,s)^* \, d(\mu \times \mu) < \infty .$$

Then, if $R(\xi,s)$ commutes with its adjoint, the generator A defined by

$$Af = g; \quad g(t) = \int_{\Omega} R(t,s)f(s)d\mu , \quad t \in \Omega ,$$

mapping $H = L_2(\Omega,\mathcal{B},\mu)$ into itself, is a compact normal linear operator. In this case, the operator D in Theorem 3.1.2 need only be positive.

Corollary 3.1.2.

In Theorem 3.1.2, let the *infinitesimal* generator A be compact and normal. Then, it is sufficient that the operator D be positive for the semigroup to be weakly stable.

PROOF: Suppose in the hypothesis of Theorem 3.1.2, the operator D is positive. Since

$$2\mathrm{Re}(DAx,x) \leq -(Wx,x) \qquad \forall x \in D(A)$$

the null space of A must be $\{0\}$. Therefore, $\overline{R(A)} = H$ and the assumption that A is a compact normal operator on the Hilbert space H implies that the eigenvectors of A, $\{\phi_k\}_1^\infty$, form an orthonormal basis for H. But, as in the proof of Theorem 3.1.2 equation 3.1.1, it can be shown that

$$(DT(t)x,T(t)x) \leq (Dx,x) \qquad \forall x \in H.$$

Letting λ_k be the eigenvalue associated with the eigenvector ϕ_k, we have

$$(DT(t)\phi_k, T(t)\phi_k) = (D\phi_k e^{\lambda_k t}, \phi_k e^{\lambda_k t})$$

$$= e^{(2\mathrm{Re}\lambda_k)t} (D\phi_k, \phi_k)$$

$$\leq (D\phi_k, \phi_k)$$

for all $t \geq 0$.

Since D is a positive operator, and ϕ_k is an eigenvector, it follows that

$$(D\phi_k, \phi_k) > 0.$$

Hence

$$e^{2\mathrm{Re}\lambda_k t} \leq 1 \qquad \text{for all } t \geq 0.$$

Therefore, $\mathrm{Re}\lambda_k \leq 0$, and letting

$$x = \sum_{k=1}^{\infty} (x, \phi_k)\phi_k$$

it is easily seen that

$$(Ax,x) + (x,Ax) = 2\sum_{k=1}^{\infty} \mathrm{Re}\lambda_k (x,\phi_k)^2$$

$$\leq 0.$$

Consequently, T(t) is a contraction semigroup and Corollary 3.1.1 can be used to conclude weak stability.

For many applications in physics and engineering, the systems encountered have an infinitesimal generator with compact and normal resolvent. As in the above examples, we may again weaken the hypothesis of Theorem 3.1.2 and allow D to be positive.

Corollary 3.1.3.

In Theorem 3.1.2, let the infinitesimal generator A have a compact and normal resolvent. Then, it is sufficient that the operator D be positive for the semigroup to be weakly stable.

PROOF: The proof of this corollary is similar to that of Corollary 3.1.2. The fact that there exists a complex number λ_0 in the resolvent set of the infinitesimal generator A for which $(\lambda_0 I - A)^{-1}$ is compact and normal, implies the existence of an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ in H and a sequence of complex numbers $\{\lambda_1, \lambda_2, \ldots\}$ such that

$$x = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n \text{ , and}$$

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x,\phi_n)\phi_n \qquad \text{for all } x \in D(A).$$

Once again, using equation 3.11 of Theorem 3.1.2, we have

$$(DT(t)\phi_n, T(t)\phi_n) = (D\phi_k e^{\lambda_n t}, \phi_k e^{\lambda_n t}) \leq (D\phi_k, \phi_k).$$

This, just as in the proof of Corollary 3.1.2, implies $\mathrm{Re}\lambda_n \leq 0$.
Therefore,

$$(Ax,x) + (x,Ax) = 2\sum_{n=1}^{\infty} \mathrm{Re}\lambda_n (x,\phi_n)^2$$

$$\leq 0.$$

Hence, $T(t)$ is a contraction semigroup. Weak stability follows from Collary 3.1.1.

3.2 EXTENSION TO STRONGER FORMS OF STABILITY

The results of the last section were stated in terms of weak stability. However, for many important types of semigroups, the conditions of Theorem 3.1.2 imply strong or even exponential stability. In this section, we will prove several corollaries which extend the results of the previous section to these stronger forms of stability. This will be accomplished by making further assumptions on the type of semigroup we are considering, and employing the results of Section 2.1. For instance, in almost all applications of partial differential equations with bounded domains, the semigroups turn out to be compact. In this case, we have the following corollary to Theorem 3.1.2.

Corollary 3.2.1.

In Theorem 3.1.2, if the semigroup $T(t)$ is compact, then it is exponentially stable.

PROOF: Use Theorem 2.1.3.

Further, if A generates a semigroup and the resolvent $R(\lambda,A)$ is compact and self-adjoint for some $\lambda > w_0$, then the Hilbert space H is separable and the semigroup is compact and self-adjoint. In this case, we can weaken the assumption on D from strict positivity to positivity and obtain a necessary and sufficient condition for exponential stability.

Corollary 3.2.2.

Let $T(t)$ be a strongly continuous C_0 semigroup on H with the infinitesimal generator A. If the resolvent of A is compact and self-adjoint for some $\lambda > w_0$, then a necessary and sufficient condition for the semigroup $T(t)$ to be exponentially stable is the existence of a positive operator D and a positive operator C such that

$$2\mathrm{Re}(DAx,x) \le -(Cx,x) \qquad \text{for all } x \in D(A).$$

PROOF: Sufficiency: Apply Corollary 3.1.2 and Corollary 3.2.1. Necessity: Since $R(\lambda_0,A)$ is compact and self-adjoint for some $\lambda_0 > \omega_0$, there exists an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ in H and a sequence of real numbers $\{\lambda_1, \lambda_2, \ldots\}$ such that

$$x = \sum_{n=1}^{\infty} (x, \phi_n)\phi_n ,$$

and

$$Ax = \sum_{n=1}^{\infty} \lambda_n(x,\phi_n)\phi_n \qquad \text{for all } x \in D(A).$$

Also, the real numbers $\{\lambda_1, \lambda_2, \ldots\}$ cannot have an accumulation point in the finite part of the complex plane, i.e., $|\lambda_n| \to \infty$. From the exponential stability of the semigroup, it follows that

$$\|T(t)\phi_n\| = e^{\lambda_n t} \le Me^{-wt}$$

for some M, $w > 0$. This implies $\lambda_n < 0$ for all n. Now let

$$Dx = \sum_{n=1}^{\infty} \lambda_n^{-2}(x, \phi_n)\phi_n,$$

and

$$Cx = \sum_{n=1}^{\infty} |\lambda_n|^{-1} (x,\phi_n)\phi_n .$$

Then,

$$2\mathrm{Re}(DAx,x) = \sum_{n=1}^{\infty} -|\lambda_n|^{-1} (x,\phi_n)^2$$

$$= -(Cx,x).$$

Since D and C are clearly positive, the hypothesis of the theorem is satisfied. This completes the proof.

We now consider,

<u>Example 3.2.1.</u>

Let $T(t)$ be the compact self-adjoint semigroup

$$T(t)x = \sum_{1}^{\infty} e^{-nt}(x,\phi_n)\phi_n ,$$

where $\{\phi_n\}$ is an orthonormal basis. If

$$Dx = \sum_{1}^{\infty} \frac{1}{n^2} (x,\phi_n)\phi_n ,$$

then D is a positive operator and

$$2\text{Re } DAx,x \quad = \quad -2\sum_{1}^{\infty} \frac{1}{n} (x,\phi_n)^2.$$

Define W such that

$$Wx \quad = \sum_{1}^{\infty} 2/n(x,\phi_n)\phi_n$$

Then, clearly W is a positive operator also, and

$$2\text{Re}(DAx,x) \quad = \quad -(Wx,x) \qquad \forall x \in D(A).$$

Therefore, by Corollary 3.2.2, T(t) should be exponentially stable. This is indeed the case since

$$\|T(t)x\| \quad = \left(\sum_{1}^{\infty} e^{-2nt}(x,\phi_n)^2\right)^{1/2}$$

$$\leq \; e^{-t}\left(\sum_{1}^{\infty} e^{-2(n-1)t}(x,\phi_n)^2\right)^{1/2}$$

$$\leq \; e^{-t}\|x\|.$$

If the infinitesimal generator A is not self-adjoint but does have a compact resolvent, then the conclusion of Theorem 3.1.2 can still be extended to strong stability of the semigroup.

Corollary 3.2.3.

If the infinitesimal generator A of the semigroup T(t) has a compact resolvent $R(\lambda,A)$, for some $\lambda > \omega_0$, then Theorem 3.1.2 can be stated in terms of strong stability.

PROOF: Use Theorem 2.1.1.

Theorem 3.1.2 can also be strengthened to strong stability of the semigroup if the generator A does not have a compact resolvent but is self-adjoint.

Corollary 3.2.4.

In Theorem 3.1.2, if the semigroup T(t) has a self-adjoint infinitesimal generator A, then it is strongly stable.

PROOF: Use Theorem 2.1.2.

In Section 3.1, we discuss semigroups generated by operators which are compact and normal. If it is further assumed that the generator A is self-adjoint, then necessary and sufficient conditions for strong stability can be established.

Corollary 3.2.5.

Let $T(t)$ be a strongly continuous C_0 semigroup on a Hilbert space H with compact self-adjoint infinitesimal generator A. A necessary and sufficient condition for strong stability is the existence of a strictly positive operator D and a positive operator C such that

$$2\mathrm{Re}(DAx,x) \leq -(Cx,x) \qquad \text{for all } x \in H.$$

PROOF: Sufficiency: This follows from Corollary 3.2.3.

Necessity: Since A is compact and self-adjoint, we can again conclude the existence of a orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ in H and a sequence of real numbers $\{\lambda_1, \lambda_2, \ldots\}$ such that

$$x = \sum_{n=1}^{\infty} (x,\phi_n)\phi_n, \text{ and}$$

$$Ax = \sum_{n=1}^{\infty} \lambda_n(x,\phi_n)\phi_n \qquad \text{for all } x \in D(A).$$

Similar to Corollary 3.2.3,

$$\|T(t)\phi_n\| = e^{\lambda_n t} \to 0 \qquad \text{as } t \to \infty$$

since $T(t)$ is strongly stable. Hence, once again, this implies $\lambda_n < 0$

for all n. Let $D = I$ and

$$Cx = \sum_{n=1}^{\infty} |\lambda_n| (x,\phi_n)\phi_n .$$

So that

$$2\mathrm{Re}(DAx,x) = -\sum_{n=1}^{\infty} |\lambda_n| (x,\phi_n)^2$$

$$\leq -(Cx,x) .$$

This satisfies the necessary condition, and completes the proof.

In reference [3], Datko gives the following example of a "stable" semigroup where Theorem 3.1.1 does not apply:

<u>Example 3.2.2.</u>

Let $H = \ell_2$ and

$$T(t)x = (e^{-t}x_1, e^{-t/2}x_2, \ldots, e^{-t/n}x_n, \ldots).$$

Clearly, $T(t)$ is strongly continuous of class C_0 with infinitesimal generator

$$Ax = (-x_1, -1/2x_2, \ldots, -1/nx_n, \ldots).$$

It is easily seen that A is both self-adjoint and compact. Letting $D = I$ and

$$Cx = 2(x_1, x_2/2, x_3/3, \ldots, x_n/n, \ldots)$$

we have

$$2\mathrm{Re}(Ax,x) \leq -(Cx,x).$$

Hence, by Corollary 3.2.5, the semigroup $T(t)$ is strongly stable. To see that this is actually the case, first note that

$$\|T(t)\| = \sup_{\|x\|=1} \|T(t)x\|$$

$$= \lim_{n\to\infty} e^{-1/nt} = 1$$

for all $t \geq 0$. Hence, $T(t)$ is a contraction semigroup. Now for any $x \in H$, there exists a sequence $\{x_1, x_2, x_3, \ldots\}$ in H such that each x_m has finitely many nonzero elements and $x_m \to x$ as $m \to \infty$. Therefore,

$$\|T(t)x\| \leq \|T(t)\ (x - x_m)\| + \|T(t)x_m\|$$

$$\leq \|x - x_m\| + \|T(t)x_m\| .$$

Taking the limit as $t \to \infty$, and then letting $m \to \infty$ in the above inequality, it follows that $\|T(t)x\| \to 0$ as $t \to \infty$, and the semigroup $T(t)$ is strongly stable.

In both Corollary 3.2.2 and Corollary 3.2.5, assumptions are made on the infinitesimal generator A which imply the existence of an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ for the Hilbert space H for which

$$T(t)\phi_n = e^{\lambda_n t}\phi_n .$$

In general, if there exists such a basis for H, then for each $x \in H$

$$x = \sum_{i=1}^{\infty} \alpha_i \phi_i .$$

Denote by F_i the subspace of H consisting of those elements having all of their coefficients $\{\alpha_1, \alpha_2, \ldots\}$ with subscripts larger than i equal to zero. Then, in both the above corollaries,

$$T(t) : F_i \to F_i \qquad \text{for all } t \geq 0.$$

It can be shown that if the semigroup $T(t)$ of Theorem 3.1.2 satisfies the above condition, or more generally, if for each $x \in F_i$, for some i, there exists a j such that $T(t)x \in F_j$ for all $t \geq 0$, then weak stability of the semigroup implies strong stability of the semigroup.

We state the above assumption formally as

Hypothesis 3.2.1.

Let H have an orthonormal basis, and let T(t) be a semigroup defined on H. If $x \in F_i$, for some i, then there exists a j such that $T(t)x \in F_j$, for all $t \geq 0$.

Under Hypothesis 3.2.1, it is possible to prove

Theorem 3.2.1.

Let T(t) be a strongly continuous C_0 semigroup on H satisfying Hypothesis 3.2.1. Then weak stability of T(t) implies strong stability.

PROOF: Let F denote the subspace in H of vectors having finitely many nonzero coefficients with respect to the orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of H. Then for each $\overline{x} \in F$, there exists an m such that

$$\|T(t)\overline{x}\| = \left(\sum_{i=1}^{m} (T(t)\overline{x},\phi_i)^2\right)^{1/2}.$$

Since T(t) is weakly stable, $T(t)\overline{x} \to 0$ as $t \to \infty$. Now for each $x \in H$, there exists a sequence $\{\overline{x}_1, \overline{x}_2, \ldots\} \in F$ such that $\overline{x}_n \to x$ as $n \to \infty$. Also, since T(t) is weakly stable, $\|T(t)\| \leq M$. Hence

$$\|T(t)x\| \le \|T(t)\ (x - \bar{x}_n)\| + \|T(t)\bar{x}_n\|$$

$$\le M\|x - \bar{x}_n\| + \|T(t)\bar{x}_n\|.$$

Taking the limit with respect to t and then with respect to n of each side of the above inequality, we see that $T(t)x \to 0$ as $t \to \infty$. The semigroup is therefore strongly stable. This completes the proof.

As a consequence of this theorem and Theorem 3.1.2, we have the following,

Corollary 3.2.6.

Let $T(t)$ be a strongly continuous C_0 semigroup on H with infitesimal generator A. If $T(t)$ satisfies the conditions of Theorem 3.1.2 and Hypothesis 3.2.1, then $T(t)$ is strongly stable.

PROOF: Use Theorem 3.1.2 and Theorem 3.2.1.

In [10], Datko gives another example of a "stable" (meaning strongly stable) semigroup for which Theorem 3.1.1 does not apply. We shall now show that Corollary 3.2.6 does apply to his example.

Example 3.2.3.

Let ℓ_2 be the Hilbert space of all real sequences $x = (x_1, x_2, . . .)$ such that

$$\sum_{i=1}^{\infty} x_i^2 < +\infty .$$

On ℓ_2, define the bounded linear operator A by the equation

$$Ax = (-x_1 + x_2, -x_2 + x_3, \ldots, -x_n + x_{n+1}, \ldots)$$

and the infinite system of differential equations

$$\dot{x}_i = -x_i + x_{i+1}, \qquad i = 1, 2, \ldots$$

The conditions of Theorem 3.1.2 are satisfied for this system by letting D = I, and C = -(A + A*). That C is positive follows from the fact that

$$\begin{aligned}(Cx,x) &= x_1^2 + x_1^2 - 2x_2x_1 + x_2^2 \\ &+ x_2^2 - 2x_2x_3 + x_3^2 + \\ &\ldots x_n^2 - 2x_nx_{n+1} + x_{n+1}^2 \\ &= x_1^2 + \sum_{i=1}^{\infty}(x_i - x_{i+1})^2 .\end{aligned}$$

Therefore,

$$2\mathrm{Re}(Ax,x) = ((A + A^*)x,x) = -(Cx,x),$$

for all $x \in H$. From the definition of the infinitesimal generator A and the fact that A is bounded, it is easily seen that T(t) must satisfy Hypothesis 3.2.1. Consequently, T(t) is strongly stable. To verify this, observe that

$$x_i(t) = e^{-t}\sum_{n=0}^{\infty} x_{i+n}t^n/n! \ , \qquad (3.2.1)$$

where $x_i(t)$ is the ith component of T(t)x. We then observe that

$$1/2\frac{d}{dt}(\|T(t)x\|^2) = -\sum_{i=1}^{\infty} x_i^2(t) + \sum_{i=1}^{\infty} x_i(t)x_{i+1}(t).$$

Hence, by the Schwarz inequality,

$$d(\|T(t)x\|)/dt \leq 0 \qquad \text{for all } t \geq 0.$$

This means that T(t) is a contraction. From the form of equation 3.2.1, it is easily seen that for the elements $\bar{x}$ in ℓ_2 with only a finite number of nonzero coordinates $\|T(t)\bar{x}\| \to 0$ as $t \to \infty$. Thus, applying the same method used in the proof of Theorem 3.2.1, we have strong stability of the semigroup.

Two classes of semigroups which automatically satisfy Hypothesis 3.2.1 are those with compact self-adjoint infinitesimal generators, and those with generators having compact normal resolvents. Therefore, it is possible to prove

Corollary 3.2.7.

Let $T(t)$ be a strongly continuous semigroup on H whose infinitesimal generator A

i) is compact and self-adjoint, or

ii) has compact self-adjoint resolvent.

Then a sufficient condition for strong stability is the existence of a positive operator D and a positive operator C such that

$$(DAx,x) + (x,DAx) \leq (Cx,x) \qquad \text{for all } x \in D(A).$$

PROOF: Use Corollaries 3.1.2, 3.1.3, and Theorem 3.2.1.

3.3 STABILIZABILITY AND THE ALGEBRAIC RICCATI EQUATION

In the previous two sections of this chapter, we have given sufficient conditions for a strongly continuous C_0 semigroup to be stable. In this section, we turn our attention to the stabilizability problem. Recall from Chapter 1 that the stabilizability problem involves the abstract differential equation

$$\dot{x} = Ax + Bu \tag{3.3.1}$$

where A is the infinitesimal generator of a C_0 semigroup on the Hilbert space H, $u \in L_2([0,\infty);U)$, and B is a bounded linear operator; $B : U \to H$. In order to stabilize this system, we must find a bounded linear operator $F : H \to U$ such that the semigroup generated by $A + BF$ is in some sense stable. Using the results of Section 3.1, we can state the following rather obvious stabilizability theorem,

Theorem 3.3.1.

The linear control system described by equation 3.3.1 is weakly (exponentially) stabilizable if there exists a bounded linear operator $F : U \to H$, a strictly positive (positive) operator D, and a positive (strictly positive) operator W such that

$$2\mathrm{Re}(D(A + BF)x,x) \leq -(Wx,x)$$

for all $x \in D(A)$.

PROOF: If A generates a strongly continuous C_0 semigroup, then so does $A + BF$, since BF is a bounded operator. Also, $D(A + BF) = D(A)$. Now apply Theorem 3.1.2 (3.1.1) to the semigroup generated by $A + BF$.

If we can satisfy the conditions of the above theorem for weak stability, it may be possible to conclude strong or exponential stability by showing that the semigroup generated by $A + BF$ satisfies one of the corollaries of Section 3.2. The sufficient condition of Theorem 3.3.1 can be put in a more familiar form by letting $F = -B^*D$. We then have

Corollary 3.3.1.

The system described by equation 3.3.1 is weakly (exponentially) stabilizable by the feedback operator $-B^*D$ if we can find a strictly positive (positive) operator D and a positive (strictly positive) operator P such that

$$(DAx,x) + (x,DAx) - (DBB^*Dx,x) = -(Px,x)$$

for all $x \in D(A)$.

PROOF: Let $F = -B^*D$ and $W = P + DBB^*D$ in Theorem 3.1.1.

We are now faced with the task of finding a solution to the well-known algebraic Riccati equation

$$(DAx,x) + (x,DAx) - (DBB^*Dx,x) = -(Px,x) \tag{3.3.2}$$

for all $x \in D(A)$. If P is strictly positive and a positive operator D exists which satisfies this Riccati equation, then the system of equation 3.1.1 is exponentially stabilized by the feedback operator $-B^*D$. This is Datko's result presented in Section 2.5. What is new here is that if W is only positive and we can find a strictly positive solution D, then the system is <u>weakly</u> stabilized by the feedback operator -B*D.

<u>Remark 3.3.1</u>: We note that in the finite dimensional case, exponential stability and weak stability are equivalent. Also, a self-adjoint matrix is positive if and only if it is strictly positive. Therefore, both the above cases reduce to the stable regulator problem where we are trying to find a strictly positive matrix D which solves the algebraic Riccati equation for a strictly positive matrix W. The stable regulator is then given by the feedback control $u(t) = -B^*Dx(t)$.

In [28], Triggiani gives several examples of systems with the form of equation 3.3.1. In each of these examples both A and B are compact operators. Hence, for any bounded operator $F : H \rightarrow U$, the spectrum of the compact operator $A + BF$ always contains the origin. Triggiani therefore concludes that the system is not exponentially stabilizable. However, we shall present here a modification of one of Triggiani's examples where A and B are compact but the system is strongly stabilizable.

<u>Example 3.3.1.</u>

Let the system $\dot{x} = Ax + Bu$ be defined on $H = \ell_2$, $x = (x_1, x_2, \ldots)$. Let A be the operator whose matrix representation, with respect to the usual basis, has the following entries: $1/2^i$, $i = 1, 2, \ldots$ on the diagonal immediately below the main diagonal and zero elsewhere. The operator B; $H \to H$ is defined such that $Bx = (b_1x_1, b_2x_2, \ldots)$ where $b_1^2 = 1$, $b_2^2 = (1/2 + 1/4), \ldots, b_n^2 = (1/2^{n-1} + 1/2^n), \ldots$ If we let D = I in Corollary 3.3.1, then

$$
\begin{aligned}
(DAx,x) + (x,DAx) - (DBB^*Dx,x) &= ((A + A^*)x,x) \\
&\quad -(B^*x,B^*x) \\
&= 2\sum_{i=1}^{\infty} 1/2^i x_i x_{i+1} \\
&\quad -\sum_{i=2}^{\infty} (1/2^{i-1} + 1/2^i)x_i^2 \\
&\quad -x_1^2 \\
&= -1/2x_1^2 - \sum_{i=1}^{\infty} 1/2^i(x_i - x_{i+1})^2.
\end{aligned}
$$

Setting $P = -(A + A^*) + BB^*$, it follows that the system (A,B) satisfies the conditions of Corollary 3.3.1. Therefore, it is weakly stable. Also, from the form of the infinitesimal generator and the [illegible] that it is bounded, it is clear that this semigroup satisfies

Hypothesis 3.2.1. Hence, we can conclude strong stability by invoking Corollary 3.2.6. Noting that $\Sigma(1/2^i)^2 < \infty$, and $\Sigma 1/2^i < \infty$, it follows that A and B are not only compact, they are in fact Hilbert-Schmidt.

In order to use Corollary 3.3.1 to stabilize the system (A,B), a strictly positive operator D must be found which satisfies Equation (3.3.2). It would be useful to derive a sufficient condition for the existence of such an operator.

Recall Datko's results for the exponential stabilizability problem described in Section 2.5. There we defined a cost functional on $L_2(I;U)$ by the expression

$$C(u,I,x_0) = \int_I [(Px_u(s),x_u(s)) + (u(s),u(s))]ds \tag{3.3.3}$$

where $I = [0,\bar{t}]$,

$$x_u(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \tag{3.3.4}$$

and P is a nonnegative operator on H. We then let $I_n = [0,t_n]$ be a monotonically increasing sequence of intervals such that $t_n \to \infty$, and assumed that for each $x \in H$

$$\lim_{n\to\infty} m(I_n,x) < +\infty \tag{3.3.5}$$

where

$$m(I_n,x) = \min_{u \in L_2(I_n;U)} C(u,I_n,x). \qquad (3.3.6)$$

Under the assumption that the system described by equation 3.3.1 satisfies inequality 3.3.5, it was possible to conclude the existence of a positive operator D such that

$$(Dx,x) = \lim_{t \to \infty} \int_0^t ((P + DBB^*D)S(s)x_0, S(s)x_0)dS \qquad (3.3.7)$$

where $S(t)$ is the semigroup generated by the infinitesimal generator $A - BB^*D$.

For the exponential stabilizability case, the requirement that the system satisfy inequality 3.3.5 for some strictly positive P is sufficient to insure stabilizability of system 3.3.1 by the feedback operator $-B^*D$. However, suppose P is only positive. Can we still conclude some form of stabilizability? It turns out that if the positive operator D, whose existence is guaranteed by the satisfaction of inequality 3.3.5, is strictly positive, then system 3.3.1 is strongly stabilized by the feedback operator $-B^*D$.

Theorem 3.3.2.

Let the system described by equation 3.3.1 satisfy inequality 3.3.5 for some positive operator P. Then if the positive operator D satisfying equation 3.3.7 is strictly positive, the system 3.3.1 is strongly stabilized by the feedback operator -B*D, and D satisfies equation 3.3.2.

PROOF: Since D is strictly positive and satisfies equation 3.3.7, there exists a positive number m such that

$$m\|S(\tau)x\|^2 \le (DS(\tau)x, S(\tau)x)$$

$$= \lim_{t\to\infty} \int_0^t ((P + DBB^*D)S(\tau + s)x, S(\tau + s)x)ds$$

$$= \lim_{t\to\infty} \int_\tau^{t+\tau} ((P + DBB^*D)S(s)x, S(s)x)ds$$

$$= \int_0^\infty ((P + DBB^*D)S(s)x, S(s)x)ds$$

$$- \int_0^\tau ((P + DBB^*D)S(s)x, S(s)x)ds$$

for each $x \in H$ and all $\tau \geq 0$. Dividing by m and taking the limit as $\tau \to \infty$ gives

$$\lim_{\tau \to \infty} \|S(\tau)x\|^2 = 0.$$

Hence, S(t) is strongly stable. This comples the first part of the proof.

Letting $G = (P + DBB^*D)^{1/2}$, it follows that $GS(t)x \to 0$ as $t \to \infty$. We now show that D satisfies the conditions of Corollary 3.3.1. By equation 3.3.7,

$$(D(A - BB^*D)x,x) + (x,D(A - BB^*D)x)$$

$$= \lim_{t \to \infty} \int_0^t \frac{d}{ds}((P + DBB^*D)S(s)x,S(s)x)ds$$

$$= \lim_{t \to \infty} \|GS(t)x\|^2 - \|Gx\|^2 \qquad (3.3.7)$$

for all $x \in D(A)$.

Since $GS(t)x \to 0$ as $t \to \infty$, we have

$$(D(A - BB^*D)x,x) + (x,D(A - BB^*D)x)$$

$$= -((P + DBB^*D)x,x), \tag{3.3.8}$$

or

$$(DAx,x) + (x,DAx) - (DBB^*Dx,x) = -(Px,x) \tag{3.3.9}$$

for all $x \in D(A)$. Thus, D satisfies equation 3.3.2, and the theorem is proved.

Remark 3.3.2: We note that in Example 3.3.1, the operator A is bounded. This example is a reply to Triggiani's nonexponentially stabilizable examples of [28] where the infinitesimal generators are all compact, and therefore bounded. In Section 4.2 of the next chapter, we will present an example with A unbounded and show that it is stabilizable using the results of this chapter.

CHAPTER 4

CONTROLLABILITY AND STABILIZABILITY

4.1 SOME CONTROLLABILITY RESULTS

In finite dimensions, the concepts of stabilizability and controllability are tightly connected. Clearly if a system (A,B) is stabilizable, then it is approximately controllable. It has also been shown that approximate controllability implies stabilizability for the finite dimensional case. In an attempt to investigate the implications of controllability with respect to stabilizability for the infinite dimensional case, it will be useful to examine the two concepts of controllability presented in Chapter 1 in greater detail.

Benchimole ([2], Theorem 1.3.5) attempts to give a symmetrical definition of approximate and exact controllability. This theorem is not correct as stated, however. We will state and prove such a theorem here, as it will be most useful in the next section.

Theorem 4.1.1.

The system $\dot{x} = Ax + Bu$, where A generates a C_0 semigroup T(t), is:

i) Approximately Controllable if and only if the self-adjoint operator C defined by

$$\int_0^\infty e^{-2\lambda s} T(s)BB^*T^*(s)x\,ds = Cx, \quad \lambda > w_0 \text{ [1]}.$$

1. See equation 1.1.1.

is positive.

ii) Exactly Controllable if and only if C is strictly positive.

PROOF:

i) To see that C is well defined, consider for each $t > 0$

$$C(t) = \int_0^t e^{-2\lambda s} T(s)BB^*T^*(s)x\,ds.$$

Then

a) $(C(t)x,y) = (x,C(t)y)$,

b) $0 \le (C(t_1)x,x) \le (C(t_2)x,x) \qquad t_1 \le t_2$, and

c) $$\left(\int_0^t e^{-2\lambda s} T(s)BB^*T^*(s)x\,ds, y\right)^2$$

$$\le \left(\int_0^t e^{-\lambda s}|B^*T^*(s)x|\,e^{-\lambda s}|B^*T^*(s)y|\,ds\right)^2$$

$$\le \int_0^t e^{-2\lambda s}|B^*T^*(s)x|^2 ds \int_0^t e^{-2\lambda s}|B^*T^*(s)y|^2 ds.$$

Now if $\lambda > w_0$, there exists w, $w_0 < w < \lambda$, and M such that

$$|T^*(s)| = |T(s)| \leq Me^{ws}.$$

Therefore, the last expression is

$$\leq (M|B|\ |x|)^2/2(\lambda - w))((M|B|\ |y|)^2/2(\lambda - w)).$$

By Lemma 4 in [8], it follows that there exists a self-adjoint positive operator C on H such that for each $x \in H$, $\lim |C(t)x - Cx| \to 0$, as $t \to \infty$.

Now assume (A,B) is approximately controllable. Then $(Cx,x) = 0$ implies $|B^*T^*(t)x| = 0$ for all $t > 0$, which implies $x = 0$. In other words, C is positive. But, if C is positive, then $|B^*T^*(t)x| = 0$ for all $t > 0$ implies $(Cx,x) = 0$. This implies that $x = 0$, and therefore (A,B) is approximately controllable.

ii) Necessity: If (A,B) is exactly controllable, then for some $\bar{t} > 0$, the range of

$$C_{\bar{t}} \triangleq \int_0^{\bar{t}} T(\bar{t} - s)Bu(s)ds : L_2([0,\bar{t}];U) \to H$$

is the whole space H (see [11], prop. 2).

Since $C_{\bar{t}}$ is onto

$$\hat{C}_{\bar{t}} \triangleq \int_0^{\bar{t}} e^{-\lambda(\bar{t}-s)} T(\bar{t} - s)Bu(s)ds : L_2([0,\bar{t});U) \to H$$

where $\lambda > w_0$ must also be onto. But the bounded linear operator $\hat{C}_{\bar{t}}$ from $L_2([0,\bar{t});U)$ into H is onto H if and only if $\hat{C}_{\bar{t}}^*$ has a bounded inverse, i.e.,

$$\hat{C}_{\bar{t}}^* x\|_{L_2} \geq m\|x\|_H$$

for some $m > 0$. This implies that

$$\int_0^{\bar{t}} e^{-2\lambda(\bar{t}-s)} \|B^*T^*(\bar{t} - s)x\|^2 ds \geq m^2\|x\|_H{}^2.$$

Letting $\tau = \bar{t} - s$, we have

$$\int_0^{\bar{t}} e^{-2\lambda\tau} \|B^*T^*(\tau)x\|^2 d\tau \geq m^2\|x\|_H{}^2$$

It follows that

$$\int_0^\infty e^{-2\lambda s}\|B^*T^*(s)x\|^2 ds = (Cx,x) \geq m^2\|x\|^2,$$

and C is strictly positive.

Sufficiency: Given that C is strictly positive, we know that for some $\gamma > 0$

$$\int_0^\infty e^{-2\lambda s}\|B^*T^*(s)x\|^2 ds \geq \gamma\|x\|^2$$

where $\lambda > w_0$. Therefore, there exists ϵ, $M > 0$ such that $\|e^{-\lambda s}T(s)\| \leq e^{-\epsilon s}M$. This implies that for $\bar{t}$ sufficiently large and some $\bar{\gamma} < \gamma$

$$\int_0^{\bar{t}} e^{-2\lambda s}\|B^*T^*(s)x\|^2 ds \geq \bar{\gamma}\|x\|^2.$$

Letting $\tau = \bar{t} - s$, we have

$$\int_0^{\bar{t}} e^{-2\lambda(\bar{t}-\tau)}\|B^*T^*(\bar{t} - \tau)x\|^2 d\tau \geq \bar{\gamma}\|x\|^2$$

and, consequently, we have the range of

$$\int_0^{\bar{t}} e^{-2\lambda(\bar{t}-\tau)} T(\bar{t} - \tau)BB^*T^*(\bar{t} - \tau)d\tau$$

equal to the whole space H. Since $\{e^{-\lambda(\bar{t}-\tau)}B^*T^*(\bar{t}-\tau)x;$ $0 \le \tau \le \bar{t}\} \in L_2([0,\bar{t});U)$ for any $x \in H$, $C_{\bar{t}}$ is also onto, and the system (A,B) is exactly controllable.

This completes the proof.

For some systems (A,B), it has been shown that exact controllability implies the state space H is actually finite dimensional. Two such situations where this is true are when B is compact, and when the semigroup generated by A is compact. Triggiani [29] provides a proof for the case where B is compact and the state space is a Banach space, but it is fairly complicated. The controllability operator of Theorem 4.1.1 allows a very simple proof of these facts for Hilbert space.

Theorem 4.1.2.

If the system (A,B) has a compact input operator B, or if the semigroup T(t) generated by A is compact, then the controllability operator C defined in Theorem 4.1.1 is compact.

PROOF: Note that for fixed $t > 0$, the set

$$\{T(t)Bx, \|x\| = 1\}$$

has a compact closure by definition of compactness of $T(t)$ or B. Given $\varepsilon > 0$, it can be covered by a finite number of spheres $s(x_k;\varepsilon)$, $k = 1, . . ., n$, with center at x_k and radius ε. For any x such that $\|x\| = 1$,

$$\begin{aligned}(T(t + \delta) - T(t))Bx &= (T(\delta) - I)T(t)Bx \\ &= (T(\delta) - I)x_k + (T(\delta) - I)(T(t)Bx - x_k)\end{aligned}$$

where $T(t)Bx \in S(x_k;\varepsilon)$, and δ is small enough so that $\|(T(\delta)-I)x_k\| < \varepsilon$ for $k = 1, . . ., n$. Since $\|(T(\delta) - I)(T(t)Bx - x_k)\| \leq \|T(\delta) - I\|\varepsilon$, it follows that $T(t)B$ is uniformly continuous for $t > 0$. But this implies that $T(t)BB^*T^*(t)$ is also uniformly continuous, so we can define for each $\varepsilon > 0$, $L > 0$,

$$\int_{\varepsilon}^{L} e^{-2\lambda t}T(t)BB^*T^*(t)dt$$

as a Riemann integral in the topology of $L(H,H)$, i.e., the uniform operator topology. Since $\|T(t)\|$ is bounded on bounded intervals, the integral converges in this topology as $\varepsilon \to 0$, and if $\mathrm{Re}\lambda > w_0$, it converges in the same topology as $L \to \infty$. Hence

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$$C = \int_0^{\infty} e^{-2\lambda t} T(t)BB^*T^*(t)dt$$

is compact. This completes the proof.

Corollary 4.1.1.

If the system (A,B) is as described in the hypothesis of Theorem 4.1.2, then exact controllability implies that H is finite dimensional.

PROOF: Use Theorem 4.1.1 and Theorem 4.1.2 to conclude that C is a compact, invertible operator mapping H into H. This implies that H is finite dimensional.

Remark 4.1.1: In Section 2.7 (Theorem 2.7.2), it was shown that if a C_0 semigroup satisfies the Spectrum Decomposition Assumption [30], and if the stable part of the decomposed semigroup satisfies the Spectrum Determined Growth Assumption, then exact controllability of the unstable subsystem (A_u, PB) is sufficient to insure exponential stabilizability. With the developments in this section, it can be easily shown that (as Triggiani mentions in [30]) approximate controllability, in this case, implies exponential stabilizability providing the unstable subspace is finite dimensional.

To see this, simply express the controllability operator C in terms of the stable and unstable subspaces, i.e.,

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} = \int_0^\infty e^{-2\lambda t} T_u(t)PBB^*PT_u^*(t)dt$$

$$C_{12} = \int_0^\infty e^{-2\lambda t} T_u(t)PBB^*(I - P)T_s^*(t)dt$$

$$C_{21} = C_{12}^*$$

$$C_{22} = \int_0^\infty e^{-2\lambda t} T_s(t)(I - P)BB^*(I - P)T_s^*(t)dt.$$

If (A,B) is approximately controllable, then by Theorem 4.1.1, C must be positive. But if C is positive, then C_{11} must also be positive, and since C_{11} is finite dimensional, it is strictly positive. Again, by Theorem 4.1.1, this implies that the system (A_u, PB) is exactly controllable. Hence, by Theorem 2.7.2, the system (A,B) is exponentially stabilizable.

It should also be noted here that approximate and exact controllability are invariant under feedback. That is, if (A,B) is approximately (exactly) controllable, then $(A + BF, B)$ is approximately (exactly) controllable for any $F \in B(H,U)$.

4.2 CONTROLLABILITY AND STABILIZABILITY FOR CONTRACTION SEMIGROUPS

In this section, we apply the theorems that we have developed so far by establishing Theorem 2.8.2 — the main result of Benchimole [2] — in a much simpler and direct manner. To do this, we must first derive a very important inequality.

Consider the system (A,B), defined in Section 1.1, and let A be the generator of a C_0 contraction semigroup. Then $A - BB^*$ and $A^* - BB^*$ also generate contraction semigroups, since $A - BB^*$ and $A^* - BB^*$ are closed and accretive (see [13], Chapter 9). Therefore, we can look at the stability of the contraction semigroup $S^*(t)$ (generated by $A^* - BB^*$) by letting

$$Dx = \int_0^\infty e^{-2\lambda t} S(t)S^*(t)x\,dt; \quad \lambda > 0,$$

in Theorem 3.1.1 and Corollary 3.1.1. We know that D is well defined from the development in Theorem 4.1.1, and we have for $x \in D(A)$

$$2\mathrm{Re}(D(A^* - BB^*)x,x)$$

$$= 2\mathrm{Re}(\int_0^\infty e^{-2\lambda t}S(t)S^*(t)(A^* - BB^*)x\,dt,x)$$

$$= 2\mathrm{Re}\int_0^\infty (e^{-2\lambda t}S(t)S^*(t)(A^* - BB^*x,x)dt$$

$$= 2\mathrm{Re}\int_0^\infty e^{-2\lambda t}((A^* - BB^*)S^*(t)x,S^*(t)x)dt$$

$$= \int_0^\infty e^{-2\lambda t}2\mathrm{Re}(A^*S^*(t)x,S^*(t)x)dt$$

$$- \int_0^\infty 2e^{-2\lambda t}|B^*S^*(t)x|^2 dt.$$

But A^* is accretive, therefore

$$2\mathrm{Re}(D(A^* - BB^*)x,x) \leq -\int_0^\infty 2e^{-2\lambda t}|B^*S^*(t)x|^2 dt. \qquad (4.2.1)$$

Now, if we note that exponential (weak) stability of $S^*(t)$ implies exponential (weak) stability of $S(t)$, we have the following theorems.

Theorem 4.2.1.

If the system (A,B) is exactly controllable and if A generates a contraction semigroup, then the system is exponentially stabilizable by the feedback $-B^*$.

PROOF: Use inequality (4.2.1), Theorem 4.1.1(ii), and Theorem 3.1.1.

Thus, we have an improvement over Theorem 2.1 of reference [27] where Slemrod requires that $T(t)$ be a C_0 group.

As we have mentioned in Section 2.8, in most situations we do not have exact controllability. We therefore present a more useful theorem for the case where we have only approximate controllability.

Theorem 4.2.2.

If the system (A,B) is approximately controllable and A generates a contraction semigroup, then the system is weakly stabilized by the feedback operator $-B^*$.

PROOF: Use inequality (4.2.1), Theorem 4.1.1(i), and Theorem 3.1.2.

Corollary 4.2.1.

If the system (A,B) is approximately controllable and A generates a contraction semigroup, and if

1) T(t) is self-adjoint, or

2) A has a compact resolvent,

then the system is strongly stabilized by the feedback operator -B*.

PROOF:

1) If A is self-adjoint, then so is A - BB*. Now use Corollary 3.2.4 and Theorem 4.2.2.

2) If A has a compact resolvent, then so does A - BB* (Slemrod [27], Lemma 2.1). Now use Corollary 3.2.3 and Theorem 4.2.2.

Corollary 4.2.2.

If the system (A,B) is approximately controllable and if A generates a compact contraction semigroup, then the system is exponentially stabilizable by the feedback operator -B*.

PROOF: By Theorem 4.2.2, we know that the semigroup S(t) generated by A - BB* is weakly stable. Also, Corollary 4.12.2 [1] implies that S(t) is a compact semigroup. Hence, it follows from Corollary

3.2.1 that the S(t) is exponentially stable.

As we have seen in Section 4.1 when T(t) is compact, it is impossible for the system (A,B) to be exactly controllable. However, it is perfectly reasonable (as we shall see in Example 4.2.1) to require that the system be approximately controllable. Hence, by Corollary 4.2.2, it is exponentially stabilized by the feedback operator $-B^*$, providing T(t) is a contraction. In this sense, Corollary 4.2.2 can be viewed as the alternative to Theorem 4.2.1 for the compact contraction semigroup case.

We now present some illustrative examples.

<u>Example 4.2.1.</u>

Consider the following constant coefficient diffusion equation with distributed control:

$$\frac{\partial x}{\partial t}(r,t) = \frac{\partial^2 x}{\partial r^2}(r,t) + u(r,t) \qquad 0 \le r \le 2\pi,\ t > 0,$$

subject to the boundry conditions

$$x(0) = x(2\pi); \quad x'(0) = x'(2\pi).$$

For this system $A = \frac{\partial^2}{\partial r^2}$, $D(A) = [x; x, x'$ absolutely continuous and

$x'(\cdot)$, $x''(\cdot) \in L_2[0,2\pi]]$, and $B = I$ with input space $U = H$. Therefore, A generates a compact self-adjoint contraction semigroup $T(t)$ (see [1], p. 195) which is given by

$$T(t)x = \sum_{-\infty}^{\infty} e^{-n^2 t}(x,\phi_n)\phi_n, \quad \phi_n = e^{-inr}/\sqrt{2\pi},$$

$t \geq 0$ and $x \in H = L_2[0,2\pi]$.

Since $B = I$, it is easily seen that $B^*T^*(t)x = T(t)x = 0$, $t \geq 0$ implies $x = 0$, and the system is approximately controllable. Using the feedback operator $-B^* = -I$, the semigroup generated by the stabilized system is given by

$$S(t)x = \sum_{-\infty}^{\infty} e^{-(1+n^2)t}(x,\phi_n)\phi_n$$

which is clearly exponentially stable.

<u>Example 4.2.2.</u>

For our second example, we have the system

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial r} + bu(t), \; r \in [0,2\pi], \; t > 0 \; .$$

As in example 4.2.1, $H = L_2[0,2\pi]$, $A = -\frac{\partial}{\partial r}$ and $D(A) = [x \in H;\ x$ absolutely continuous, $x' \in H$ and $x(0) = x(2\pi)]$. Thus, $A = -A^*$ and $T(t)$ is unitary. $T(t)$ is given by

$$T(t)x = \sum_{-\infty}^{\infty} e^{int}(x,\phi_n)\phi_n, \quad \phi_n = e^{inr}/\sqrt{2\pi}$$

$t \geq 0$ and $x \in H$.

Let $b = \sum_{-\infty}^{\infty} b_n\phi_n$, where $b_n \neq 0$. Since $b^*T(t)x = \sum_{-\infty}^{\infty} b_n(x,\phi_n)e^{int} = 0$ for all $t \geq 0$ implies $(x,\phi_n) = 0$ for all n, we have an approximately controllable system. Hence, the semigroup $S(t)$ generated by $A - bb^*$ is weakly stable. But A has a compact resolvent, therefore, $S(t)$ is strongly stable by Corollary 4.2.1.

4.3 CONTROLLABILITY AND STABILIZABILITY FOR NONCONTRACTION SEMIGROUPS

Considering the results of the last section, one might ask if approximate controllability implies weak stabilizability for general C_0 semigroups. As it turns out, this is not the case. Benchimole (p. 69; [2]) gives an appropriate counterexample. However, we know from Theorem 3.3.1 that if there exists a strictly positive operator D and a positive operator P such that

$$(DAx,x) + (x,DAx) - (DBB^*Dx,x) \leq -(Px,x) \tag{4.3.1}$$

for all $x \in D(A)$, the the system (A,B) is weakly stabilizable by the feedback operator $-B^*D$. In this section, we will essentially "combine" this result with the approximate controllability assumption in order to extend Theorem 4.2.2 to a larger class of systems.

In Section 4.2, the adjoint of the infinitesimal generator A^* and the adjoint of the stabilized semigroup $S^*(t)$ were used extensively in proving that approximate controllability implies stabilizability for contraction semigroups. It should be no surprise that similar adjoint operators will play an important role in extending these results. Specifically, we have the following lemma:

Lemma 4.3.1.

Let the system (A,B) be such that for some strictly positive operator D

$$(DA^*x,x) + (x,DA^*x) \leq (BB^*x,x) \tag{4.3.2}$$

for all $x \in D(A)$. Then

(i) there exists a nonnegative operator $\overline{C}$ defined for $x \in H$ by

$$\overline{C}x = \lim_{t\to\infty} \int_0^t T_{\overline{A}}(s)BB^*T_{\overline{A}}^*(s)x\,ds$$

where $\bar{A} = A - BB^*D$.

(ii) If the system (A,B) is approximately controllable, $\bar{C}$ is positive.

PROOF:

(i) Add $-2(BB^*x,x)$ to both sides of inequality (4.3.2) to get

$$(D(A^* - D^{-1}BB^*)x,x) + (x,D(A - D^{-1}BB^*)x) \le -(BB^*x,x) \quad (4.3.3)$$

for all $x \in D(A^*)$. Let $\bar{A}^* = A^* - D^{-1}BB^*$ be the infinitesimal generator of the semigroup $T_{\bar{A}}^*(t) = T_{\bar{A}}^*(t)$. Then

$$0 \le (DT_{\bar{A}}^*(t)x,T_{\bar{A}}^*(t)x) = (Dx,x) + \int_0^t \frac{d}{ds}(DT_{\bar{A}}^*(s)x,T_{\bar{A}}^*(s)x)ds$$

for all $x \in D(A^*)$. But for $x \in D(A^*)$

$$\frac{d}{ds}(DT_{\bar{A}}^*(s)x,T_{\bar{A}}^*(s)x) = (D\bar{A}^*T_{\bar{A}}^*(s)x,T_{\bar{A}}^*(s)x) + (T_{\bar{A}}^*(s)x,DA^*T_{\bar{A}}^*(s)x).$$

It follows from inequality (4.3.3) that

$$(DT_{\bar{A}}^*(t)x, T_{\bar{A}}^*(t)x) \leq (Dx,x) - \int_0^t \|B^*T_{\bar{A}}^*(s)x\|^2 ds.$$

Hence,

$$\int_0^t \|B^*T_{\bar{A}}^*(s)x\|^2 ds \leq (Dx,x)$$

for all [illegible], and all $x \in D(A^*)$. As in the proof of Theorem 3.1.2, this implies that

$$\int_0^\infty \|B^*T_{\bar{A}}^*(s)x\|^2 ds \leq (Dx,x)$$

for all $x \in H$. The existence of $\bar{C}$ follows from Lemma 2 of [8].

(ii) Suppose $(\bar{C}x,x) = 0$. Then by the definition of $\bar{C}$, this implies that $B^*T_{\bar{A}}^*(t)x = 0$ for all $t \geq 0$. But this is impossible, unless $x = 0$, because the system $(A - BB^*D^{-1},B)$ is approximately controllable if (A,B) is. Hence, $(\bar{C}x,x) > 0$ for $x \neq 0$, and $\bar{C}$ is positive. This completes the proof.

We are now able to prove the main result of this section.

Theorem 4.3.1.

Let the system (A,B) be such that for some strictly positive D

$$(DA^*x,x) + (x,DA^*x) \le (BB^*x,x) \qquad (4.3.4)$$

for all $x \in D(A^*)$. Then approximate controllability of the system implies weak stabilizability by the feedback operator $-B^*D^{-1}$.

PROOF: In the proof of Lemma 4.3.1, it was shown that

$$(DT_{\bar{A}}^*(t)x, T_{\bar{A}}^*(t)x) \le (Dx,x)$$

for all $x \in D(A^*)$. $T_{\bar{A}}^*(t)$ is, therefore, a uniformly bounded semigroup (see the proof of Theorem 3.1.2). Now, let $\bar{C}$ be as described in the previous lemma. Then

$$(\bar{C}T_{\bar{A}}^*(t)x, T_{\bar{A}}^*(t)x) = \int_t^{\infty} (T_{\bar{A}}(s)BB^*T_{\bar{A}}^*(s)x,x)ds$$

$$= (\bar{C}x,x) - \int_0^t (T_{\bar{A}}(s)BB^*T_{\bar{A}}^*(s)x,x)ds.$$

Hence

$$\lim_{t\to\infty}(\overline{C}T_{\overline{A}}^*(t)x,T_{\overline{A}}^*(t)x) = 0. \tag{4.3.5}$$

Since $T_{\overline{A}}^*(t)$ is bounded and $\overline{C}$ is positive, Equation (4.3.5) implies that $T_{\overline{A}}^*(t)$ is weakly stable. But if $T_{\overline{A}}^*(t)$ is weakly stable, then so is $T_{\overline{A}}(t)$. This completes the proof.

As usual, the results of Theorem 4.3.1 can be strengthened for some special classes of semigroups.

Corollary 4.3.1.

Let the system (A,B) satisfy Inequality (4.3.4) for some strictly positive operator D, and all $x \in D(A^*)$. Further, let the system (A,B) be approximately controllable. Then if A generates a compact semigroup, the system (A,B) is exponentially stabilizable by the feedback operator $-B^*D^{-1}$.

PROOF: By Corollary 4.12.2 [1], $A - BB^*D^{-1}$ also generates a compact semigroup. Exponential stability follows from Theorem 2.1.3.

Corollary 4.3.2.

Let (A,B) satisfy the conditions of Corollary 4.3.1. Then if A has a compact resolvent, the system (A,B) is strongly stabilized by the feedback operator $-B^*D^{-1}$.

PROOF: If A has a compact resolvent, then so does $A - BB^*D^{-1}$. Strong stability follows from Theorem 2.2.1.

Note in Theorem 4.3.1 that if A is the generator of a contraction semigroup, then so is A^*. Consequently,

$$(A^*x,x) + (x,A^*x) \leq 0$$

for all $x \in D(A^*)$. Since BB^* is a nonnegative operator, we see that inequality (4.3.4) is trivially satisfied by $D = I$, and the system (A,B) is weakly stabilized by $-BB^*D^{-1} = -BB^*$. This is precisely Theorem 4.2.2.

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