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TERMINAL EFFECTIVENESS, VULNERABILITY METHODOLOGY AND FRAGMENTATION WARHEAD OPTIMIZATION. I. A TECHNICAL SURVEY FROM AN HISTORICAL PERSPECTIVE

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (hib) The design of a fragmentation warhead for optimal effectiveness requires a knowledge of fragment formation, launch and flight characteristics as well as penetration ability and target vulnerability. In addition, a knowledge of the expected engagement conditions between warhead and target as well as weight, volume and other design constraints is needed. Present views of the subject have developed over more than forty years, so it is to be expected that an intimate knowledge of this development will prove to be helpful in making further progress. For this

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reason various historical threads are woven together with new material in this report in order to lay the groundwork for future advances.

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I. INTRODUCTION

Kinetic energy projectiles and shaped charge jet warheads require direct hits to inflict target damage. A warhead carrying a self-forging fragment need not hit the target, but its orientation and fuzing must be exactly controlled for its lethal mechanism to hit the target. When direct hits by the warhead are unlikely (as in long range engagements or for partially concealed targets) or not required (as in the case of a collection of widely distributed targets) fragmentation warheads are employed. Since direct hits are not contemplated, the flight and fuzing requirements of such warheads are less stringent. However, the design of the warhead itself requires great care if its effect on the target is to be nearly optimum. The best design for a fragmentation warhead obviously depends on a knowledge of target vulnerability as well as on the engagement conditions and the various options that are available regarding fragment number, size and launch characteristics. The problem of designing such a warhead for optimum effect is rather complicated and is made more difficult by the uncertainty of our knowledge about target vulnerability and fragment characteristics. If each warhead is required to address more than one type of target (as is often the case), the problem is still more difficult. In spite of these difficulties, the problem of optimizing or at least of estimating the effectiveness of a design has received considerable attention over the last half century. However, for a variety of reasons, the trend has been toward the inclusion of more and more detail in model descriptions of effectiveness and away from design optimization.

In current design practice for fragmentation warheads, the focus is usually on the delivery system. Of course, this system is important and inevitably places weight and/or volume constraints on the warhead designer. However, there is a tendency to incorporate traditional warhead designs into new delivery systems, with warhead effectiveness studies performed only after the overall design has been fixed. This tendency is understandable but regrettable. It seems obvious that the goal of optimal warhead effectiveness should drive the overall design of a system, especially when direct hits are not contemplated. The warhead designer should in fact propose configurations of metal and explosive aimed at maximizing target kill probability even if they tax the ingenuity of his colleagues to deliver them to the target. In this way significant improvements might result from compromises between an optimum warhead and an optimum delivery system. However, our methods of describing warhead effectiveness need to be simplified first before attention can be re-directed toward those factors which most influence effectiveness. The eventual object of this report series is to employ simplified descriptions as an aid to design optimization. The object of this report is to review the development of our present methods as a prelude to suggesting improvements.

II. EFFECTIVENESS STUDIES UNTIL THE END OF WWII
A. Methodology Prior to WWII

There seems to be little extant literature on this subject written prior to the second world war. Of course, devices which explosively launched pieces of metal were in use, but early treatises on ballistics emphasized launch and flight techniques. Relatively little attention was paid to measures of effectiveness, either absolute or relative. This was reflected in the organizational structure of the Ballistics Research Laboratory when it was established in 1938. There were Interior and Exterior Ballistics Sections, but no Terminal Ballistics Section until 1943. Instead, a small Effect of Fire Unit within the Interior Ballistics Section conducted fragmentation and penetration studies. Eventually the Terminal Ballistics Section was entitled a Laboratory and gave birth to the Vulnerability/Lethality Division. The need for an overview of all aspects of cost/effectiveness led to the establishment of a Weapons Systems Laboratory within BRL and eventually to the Army Materiel Systems Analysis Activity[1].

The earliest paper on the subject of fragmentation effectiveness appears to be Kent's report in 1933[2]. Kent took as a measure of effectiveness the number of casualties produced (personnel put out of action). He commented on the desirability of measuring effectiveness by the extent of damage produced, especially for materiel targets like aircraft where the number of man-hours needed to repair the damage might be a useful criterion. However, for footsoldiers he choose a simple threshold criterion to distinguish between a casualty and a survivor. He directed his attention to the simple case of a vertical cylindrical shell bursting at ground level in an effectively infinite field of uniformly distributed, unshielded, standing personnel targets. He wrote an expression for the number of casualties, C, namely,

$$C = T_A \pi r_1^2 + T_A \int_{r_1}^{\infty} N_\Omega (A_p/r^2) (2\pi \, r dr) \tag{1}$$

where T_A is the constant number of targets per unit area of ground. The radius, r_1, defines an area (πr_1^2) on the ground, centered at the burst point, within which a target will receive at least one lethal hit. All targets in this area will be casualties, but there is a loss in efficiency since multiple lethal hits are wasteful. Consequently, the distance r_1 was called the radius of overhitting.

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[1] *Ballisticians in War and Peace. A History of the United States Army Ballistic Research Laboratories, Vol. I, 1914-1956*, Edited by John Schmidt, Aberdeen Proving Ground, MD, pp 12, 21, 47 and 98-102.

[2] R. H. Kent, *"Considerations on the Effect of the Size of the Projectile on the Efficiency of its Fragmentation"*, BRL RX-58, Feb 1933.

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A target standing a distance r from the burst point was replaced by an average cross-sectional area normal to r which we call the presented area, A_p. Kent did not discuss this area in detail and merely remarked that each target subtends a solid angle (A_p/r^2) at the burst point. This is equivalent to replacing each target by a spherical cap centered at the burst point of area

$$A_p = \int_0^{2\pi} \int_0^{\theta} r^2 \sin\theta \, d\theta \, d\phi = 2\pi r^2 (1-\cos\theta) = 4\pi r^2 \sin^2(\theta/2)$$

$$= \Omega r^2 \tag{2}$$

where θ is the co-latitude measured from the pole r and ϕ is the equatorial angle. Thus, the solid angle subtended by this area is $\Omega = A_p/r^2 = 4\pi \sin^2(\theta/2)$.

If $A_p \ll r^2$, the spherical cap can be thought of as practically flat, leading to the "poker chip" approximation for the target area. No distinction was made between the presented area and that part of this area which might be vulnerable, since Kent was able to eliminate A_p from Equation (1) by using his definition of r_1. The symbol N_Ω stands for the number of lethal fragments emitted per unit solid angle, so that $N_\Omega \, \Omega$ is the number of lethal hits expected on A_p. In other words, the number of lethal hits on a single target is expected to be

$$N_L = N_\Omega \, \Omega = N_\Omega \, A_p/r^2 \tag{3}$$

which is unity for $r \leq r_1$, by hypothesis and less than unity for $r > r_1$. The number of targets in a ring of radius r and width dr is T_A $(2\pi r dr)$ and N_L times this number is the number of casualties expected in this differential area. If we integrate and add $T_A(\pi r_1^2)$ we obtain Equation (1).

In order to carry out the integration we must make some assumption about the r-dependence of N_Ω. This number is expected to be a decreasing function of r since it is observed that for two fragments of the same shape launched at the same speed, a less massive one will lose speed more rapidly as r increases because of air drag. If both are above some lethality threshold (mass and speed combination) initially, the less massive fragment will fall below this threshold at a smaller

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value of r. Kent considered only the case in which all fragments are identical (same mass, shape and launch speed) so that all are lethal out to a radius r_2 and ineffective for $r > r_2$. To represent this situation he let N_Ω be equal to its value at $r=r_1$, namely, letting $N_L=1$ in Equation (3), $N_\Omega = r_1^2/A_P$ for $r_1 < r < r_2$ with $N_\Omega=0$ for $r > r_2$. This eliminates A_P from Equation (1), which becomes

$$C = T_A \pi r_1^2 \left[1 + 2 \ln (r_2/r_1) \right] . \tag{4}$$

Kent used Equation (4) to compare the effectiveness of two shells with the same geometry, scaled so that the maximum range of the larger is greater than that of the smaller, that is, $r_{2L} = r_L > r_S = r_{2S}$. The value of r_1 was taken to be the same for both since the fragment density is the same. Then a measure of their relative effectiveness is the ratio of Equation (4) for each, namely,

$$
\begin{aligned}
C_L/C_S &= \left[1 + 2 \ln (r_L/r_1) \right] / \left[1 + 2 \ln (r_S/r_1) \right] \\
&= 1 + \left[2 \ln (r_L/r_S) \right] / \left[1 + 2 \ln (r_S/r_1) \right] \\
&< 1 + 2 \ln (r_L/r_S)
\end{aligned} \tag{5}
$$

for $r_S > r_1$. For example, if the larger shell has twice the diameter and twice the height of the smaller, it will weigh about eight times as much. For this case Kent estimated $r_L/r_S = 3.67$, provided both launch speeds = 3,500 ft/sec. Then $C_L/C_S < 3.6$ in Equation (5), and on a weight basis $(C_L/W_L) / (C_S/W_S) < 3.6/8 = 0.45$ where W is the weight , so the larger shell is less than half as efficient per unit weight for $r_L > r_S > r_1$. If $r_1 > r_L > r_S$, then $N_\Omega = 0$ in Equation (1) and $C_L/C_S = (r_{1L}/r_{1S})^2 = (3.67)^2 = 13.47$ and on a weight basis the larger is more efficient by a factor $13.47/8 = 1.68$. The larger will be less efficient only if $C_L < 8C_S$ or $r_{1L} < \sqrt{8}\ r_{1S}$.

In the case just considered, the assumption was made that $r_{2L} > r_{2S}$ since the fragments of the larger shell are more massive but not more numerous. Kent further considered the case in which $r_{2L} = r_{2S} = r_E$, that is, the case in which both shells emit fragments of the same mass so that the larger emits more fragments. Kent explicitly assumed that the larger would emit eight times as many fragments and implicitly assumed as before that the launch speeds would be the same. Consequently

N_Ω will be eight times bigger for the larger shell and its radius of overhitting will be $r_{1L} = \sqrt{8}\, r_{1S}$ from Equation (3) with $N_L = 1$.

Now in Equation (4) $r_2 = r_E$ for both shells while $r_{1L} = \sqrt{8}\, r_{1S} = \sqrt{8}\, r_1$ so

$$
\begin{aligned}
C_L/C_S &= 8\left[1+2\ln\left\{r_E / (\sqrt{8}\, r_1)\right\}\right] / \left[1+2\ln (r_E/r_1)\right] \\
&= 8\left[1 - \ln 8/\left\{1+2\ln (r_E/r_1)\right\}\right] .
\end{aligned}
\tag{6}
$$

The relative efficiency per unit weight is one eighth of this and is necessarily less than unity if $r_E > r_1$. If $r_E < r_1$, $N_\Omega = 0$ and $C_L/C_S = 1$ instead of 8 in Equation (1), since hitting each target more often with lethal fragments does not increase the number of casualties. For each shell the number of casualties will be $T_A\, (\pi r_E^2)$. Thus on a weight basis the relative efficiency is 1/8.

In general then, Kent concluded that smaller shells are more efficient on a weight basis, provided the fragments are not so small that the effective length of the fragment trajectory is less than the radius of overhitting. Kent realized he was considering a highly idealized case and did not concern himself with the absolute accuracy of his estimates. Implicitly he assumed that his errors were about the same for homologous shells so that significant statements could be made about relative efficiencies, at least in the size range considered in practice.

In 1937 Tolch[3] discussed the data available at Aberdeen Proving Ground in terms used at the time, namely, the effective area or effective radius of a burst. The effective area was defined as an area within which each target on average received at least one lethal hit while the effective radius was that of a circle of equivalent area. For a vertical cylinder burst at ground level, the effective radius is the same as Kent's radius of overhitting in Equation (1) above. More generally, this area will be more or less elliptical when the axis of the shell is inclined toward the horizontal because of the dominance of the side spray over nose and base sprays. This remains true when account is taken of the remaining velocity of the shell when it strikes the ground. The main effect of this added velocity component is to throw the side spray forward and render the base spray even less effective. Tolch remarked that the attitude of the target (horizontal or vertical, face or side on) will influence effective areas, but he contented himself with an average value for A_p. For various munitions

[3]N. A. Tolch, "Effective Area of Burst and Fragmentation Efficiency of Hand Grenades, and 37 and 81 MM Projectiles", BRL R90, 22 Dec 1937. (AD#493519)

he estimated effective areas from the number of perforations through wood panels in "silhouette fields". Finally, he compared the number of shells required to neutralize a target field as predicted by the effective area method with that predicted by Kent's method, using $C/(T_A A_p)$ in Equation (1) above as a measure of efficiency with A_p chosen to be one square yard. Multiplying this measure by other values of T_A or A_p would then give values for particular target configurations. Obviously the effective area method predicts that more rounds will be required than Kent's method predicts since it considers only the region of 100% effectiveness, the first term in Equation (1).

A year later Tolch published a report[4] giving details of more elaborate fragmentation and penetration experiments. He used horizontal gun firings of 75MM shells bursting in the center of semi-circular arrays of one-inch thick wood panels placed at various radial distances to measure the number of fragments per unit solid angle as a function of co-latitude, θ (measured from the nose) and distance from the burst point, r. This quantity, $N_\Omega (r_1, \theta)$, was studied as a function of remaining velocity of the shell at the moment of burst. Fragments were classified into nose spray (fuze pieces), side spray (about 30 degree angular width) and base spray as well as into perforating, penetrating and denting (< 1/16 inch penetration) categories. From these panel tests and the axial symmetry of the shell it was estimated that about 5,000 fragments were produced. These could be roughly divided into 700 perforating, 900 penetrating and 3,400 denting fragments. As the remaining velocity increased, the nose spray panel effects became more pronounced (more perforations), the center of the side spray moved forward from θ = 95° (static firing) to θ=60° (for a remaining velocity of 1,100 f/s), and base spray effects decreased, since the remaining velocity reduced the net fragment launch velocity in this direction. Complementary sand pit tests, in which fragments above a certain size were collected and graded by sieves, produced about 780 fragments which accounted for 95% of the metal weight, indicating that most of the metal ends up in the perforating category if one-inch thick spruce boards are used as witness panels. Quarter-inch thick steel plates were found to be much tougher targets allowing only 1/6 the number of perforations at 15 feet and only one perforation at 36 feet. Tolch also published reports on the effectiveness of shrapnel munitions[5,6].

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[4]N. A. Tolch, "Fragmentation Effects of the 75MM H. E. Shell T3 (M48) as Determined by Panel and Pit Fragmentation Tests", BRL R126, 2 Dec 38. (AD#702233)

[5]N. A. Tolch, "Effective Heights of Burst and Patterns of 75MM M37 Schrapnel fired from the 75MM Pack Howitzer M1", BRL R100, 1 Apr 38.

[6]N. A. Tolch, "Patterns of Pack Howitzer Schrapnel at the Optimum Heights of Burst", BRL R102, 16 May 38. (AD#492929)

Two years later Tolch published another report[7] in which he noted a number of differences between the test results of his 1938 report, obtained under symmetrical conditions with particular efforts to avoid the influence of the ground, and the results which obtain under field conditions where asymmetries exist and the ground has a large influence. For example, when a shell bursts close to the ground, many of the fragments are deflected upward by the ground and do not follow straight trajectories even over relatively short distances or they are buried in the ground. In addition, fragments which start on straight trajectories are in flight because of gravity, pass over targets close by and hit targets farther away. Lewy[8] later discussed such fragment trajectories, using the traditional methods of exterior ballistics. He pointed out that the supersonic portion of a trajectory is essentially straight since the gravitational force is negligible compared to the drag force which depends on v^2. He also gave methods for calculating the point of impact and terminal speed for all fragments projected from a burst a height h above the ground. However, he pointed out that fragments initially projected upward will reach a terminal speed in air on the downward part of their flight which is low enough to render them non-lethal for common fragment masses. The situation may differ for very high altitude bursts and targets.

Tolch also mentioned the natural unevenness of terrain which might intercept some parts of a fragment pattern as well as man-made shielding like that afforded by fox-holes. Such shielding has been included in later models. In general, Tolch pointed out that model estimates of fragmentation effectiveness based on idealized panel tests will not accurately represent effectiveness under service conditions. However, valuable insight into at least relative efficiencies of various shell designs under different engagement conditions should be obtainable from such models.

In discussing angle of fall, Tolch distinguished between the shell axis and the tangent to the trajectory at the moment of burst. However, he made no use of this distinction and assumed that they were co-linear for all practical purposes. By using an x, y, z = h coordinate system with origin at the burst point, Tolch wrote an expression for the slant distance from the burst point to a point on the ground, namely,

$$R = \sqrt{x^2 + y^2 + h^2} = \sqrt{r^2 + h^2} \tag{7}$$

[7]N. A. Tolch, "Auxiliary Curves for use in the Computation of Fragment Density as a Function of the Angle of Fall and Height of Burst", BRL R178, 29 Feb 40. (AD#491803)

[8]H. Lewy, "Asymptotic Integration of Fragment Trajectories", BRL R559, 13 Jul 45.

where h is the height of the burst above the ground. Unlike Tolch, we will take the z axis positive in the upward direction for convenience in later comparisons, so that the ground point directly below the burst point is z = -h. Tolch defined the angle of fall, ω, in the x, h plane between the shell axis and the horizontal x-axis, so that the projection of R on the shell axis can be written in terms of the angle θ between R and the shell axis as follows

$$R \cos \theta = x \cos \omega - h \cos (\pi/2-\omega) = x \cos \omega - h \sin \omega, \tag{8}$$

by using the relation for the angle between two lines in terms of their direction cosines. Given h and ω, we may compute R and θ for any x, y pair. If we plot curves of constant R, we obtain circles in the ground plane. Tolch also plotted curves of constant θ, but did not point out their general character, although this is easily done. If we eliminate R between Equations (7) and (8) we obtain the quadratic form

$$\left[\left(\frac{\cos \omega}{\cos \theta}\right)^2 - 1\right] x^2 - 2 \left(\frac{\cos \omega}{\cos \theta}\right)\left(\frac{\sin \omega}{\cos \theta}\right) h x - y^2 + h^2\left[\left(\frac{\sin \omega}{\cos \theta}\right)^2 - 1\right] = 0 \tag{9}$$

which has the discriminant

$$\delta = 4 \left[\left(\frac{\cos \omega}{\cos \theta}\right)^2 - 1\right], \tag{10}$$

If $\delta < 0$, the curve is an ellipse, if $\delta = 0$, it is a parabola and if $\delta > 0$, it is a hyperbola.

If the target area vector $\vec{A}$ makes an angle ϕ with R, then the area normal to R is $A_p = A \cos \phi$ and the total number of lethal hits expected on a target is

$$N_L = \left[N_\Omega (R, \theta) / R^2\right] A_p \tag{11}$$

where $N_\Omega (R, \theta)$ is known from Tolch's controlled experiments, choosing perforations or some other criterion as evidence of lethality. Equation (11) is the three-dimensional analog of Equation (3) and includes a dependence on angle as well.

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In 1938 and 1939 Kent published four reports concerning the effectiveness of fragmentation shell against aircraft[9-12]. In these reports he considered costs and interior and exterior ballistic factors as well as terminal effectiveness in comparing proposed anti-aircraft artillery shells of 90mm, 105mm, 4.7 inch and 6 inch. He readily admitted that the weakest point in his analysis was the nature of the assumptions he was forced to make about the effectiveness of fragment hits on the target. In general, he pointed out, small fragments could be lethal to personnel and fuel cells, although large fragments would be required to damage engines or other less vulnerable structural components. As a plausible compromise measure of effectiveness he initially used the average of two numbers for given shells, namely, their fragment number ratio (as determined by pit tests) and their weight ratio (since larger shells should have more larger fragments). However, he soon substituted another criterion, namely, the product of the number of fragments emitted by a shell and their average effective range. He argued that this would be a reasonable measure of shell effectiveness by considering the target to be composed of a large number, n, of small (differential) areas of mean size, a, such that the total presented area of the aircraft could be written as $A_p = na$. If N_a is the fractional number of hits on area a such that $N_a = (N_\Omega/R^2) a < < 1$, analogous to Equation (11) above, then the probability of not hitting the area a is, by the binomial theorem

$$\left(1-N_a\right)^n = 1 - nN_a + \frac{n(n-1)}{2!} N_a^2 \cdots \approx e^{-nN_a} = e^{-\left(N_\Omega/R^2\right)A_p} \qquad (12)$$

since $n > > 1$, $N_a < < 1$ and $A_p = na$. Then to a good approximation, provided the target is in the fragment spray, the chance of at least one lethal hit is

$$p = 1 - e^{-\left(N_\Omega/R^2\right)A_p} \approx \left(N_\Omega/R^2\right)A_p \qquad (13)$$

so that Equation (13) reduces to Equation (3) for $A_p N_\Omega < < R^2$.

[9]R. H. Kent, "A Comparison of Antiaircraft Guns of Various Calibers", BRL R125, 1 Dec 38. (AD#492691L)

[10]R. H. Kent, "The Probability of Hitting an Airplane as Dependent upon Errors in the Height Finder and Director", BRL R127, 14 Dec 38. (AD#701865)

[11]R. H. Kent, "The Probability of Hitting Various Parts of an Airplane as Dependent on the Fragmentation Characteristics of the Projectile", BRL R132, 6 Mar 39. (AD#491787)

[12]R. H. Kent, "The Probability of Disabling an Airplane", BRL R155, 5 Sep 39. (AD#491799)

This argument of Kent's certainly captures the spirit of the argument generally presented today for the Poisson distribution as an approximation to the binomial distribution. If we consider a (space or time) interval A divided into n parts of size a = A/n with N_A equal to the constant average rate of occurrence of a random event in the interval A, then the probability of one event occurring in the sub-interval A/n is $N_A A/n = p$, and the probability of no events occurring is $1 - N_A A/n$. If these events are randomly distributed, the probability of finding exactly s of them in n intervals (or trials) is given by the binomial distribution

$$g(s; n) = \frac{n!}{s!(n-s)!} (N_A A/n)^s (1 - N_A A/n)^{n-s} \tag{14}$$

for s = 0, 1, 2... n and g = 0 for all other s. In the limit as $n \to \infty$ and $p \to 0$ but $np = N_A A \ll n$ remains finite, we have

$$\lim_{n\to\infty} g = \frac{(NA)^s}{s!} \left[\lim_{n\to\infty} \frac{n!}{(n-s)!\, n^s} \right] \left[\lim_{n\to\infty} (1 - N_A A/n)^n \right] \left[\lim_{n\to\infty} (1 - N_A A/n)^s \right]$$

$$= \frac{(N_A A)^s}{s!} e^{-N_A A} = P(S) \tag{15}$$

which is the Poisson distribution. The third limit in Equation (15) is obviously unity while the first can be shown to be unity as follows. Divide the numerator and denominator by (n-s)! and obtain

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-s+1)}{n^s} = \lim_{n\to\infty} 1(1 - \frac{1}{n})(1 - \frac{2}{n})\dots\left[1 - \frac{(s-1)}{n}\right] = 1. \tag{16}$$

If we let $z = -N_A A/n$, the second limit in Equation (15) becomes the $(-N_A A)$ power of the limit which defines the base of the natural logarithm, namely,

$$\lim_{z\to 0} \left[(1+z)^{1/z} \right]^{(-N_A A)} = e^{-N_A A} \tag{17}$$

Finally, the probability that one or more events will occur in the interval is

$$\sum_{s=1}^{\infty} P(S) = \sum_{s=0}^{\infty} P(S) - P(0) = 1 - e^{-N_A A} \quad . \tag{18}$$

since

$$\sum_{s=0}^{\infty} P(S) = \sum_{s=0}^{\infty} \frac{(N_A A)^S}{s!} e^{-N_A A} = e^{+N_A A} e^{-N_A A} = 1 \quad . \tag{19}$$

Equation (18) is the same as Equation (13) if we identify $N_A = N_\Omega/R^2$ and $A = A_p$, and gives the probability of at least one lethal hit if N_A is the number of effective or lethal fragments per unit area of fragment front, such that $N_L = N_A A = N_\Omega A_p/R^2$ is the expected number of lethal hits and is much smaller than the number of trials (total fragment number).

Kent went on to let $GR^2 dR$ represent the probability that the aircraft lies in an elementary volume of the fragment spray. Here G is a function expressing the aiming errors of a particular gun versus aircraft encounter. Then the probability that a shell burst will be effective can be written as

$$P = \int_0^{R_2} (1 - e^{-(N_\Omega/R^2)A_p}) GR^2 \, dR \tag{20}$$

where R_2 is the effective range of a fragment (a single value if all fragments are alike or an average value in the case of natural fragmentation). When $A_p N_\Omega << R^2$, then Equation (20) becomes

$$P = G A_p (N_\Omega R_2) \tag{21}$$

provided $G A_p N_\Omega$ is independent of R. For comparable aiming systems against the same target, $G A_p$ will be the same, so that $(N_\Omega R_2)$ or even $N_\Omega \Omega R_2 = (NR_2)$ can be used to compare the relative effectiveness of different size shells. Kent expanded the exponential in Equation (20) and retained the first few terms in order to estimate P for the shells being considered. He also suggested that G have the form of an error function, but did not carry out any calculations employing this suggestion. Finally, he estimated that the probability of a direct

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hit on the aircraft by a shell was as great as the probability of killing an unprotected essential component like the pilot. In order to do this he replaced the aircraft by two intersecting cylinders, one representing the fuselage and the other representing the wings. He concluded that prompt point detonating fuzes should be considered in addition to time fuzes, especially if aiming techniques were improved.

In the fourth report of this series, Kent analyzed the results of a Navy report[13] which gave estimates of two types of disability or "kill" based on experimental firings. Type A was expected to cause mission abandonment and an immediate forced landing, while type B should cause an eventual forced landing short of base without mission abort. These two types of kill were given for hits on the engine, lubricating system, fuel system, engine controls, engine bearers, pilot and surface controls. Kent suggested that these disability estimates be treated as conditional probabilities of kill (type A or B) given a hit on the ℓth component, namely, $(p_{K/H})_\ell$. If each component had a presented area, $(A_p)_\ell$, and p_{H1} is the probability of hitting a unit area of the aircraft, then $(A_p)_\ell \, p_{H1}$ is the probability of hitting the ℓth component. The product of this probability with $(p_{K/H})_\ell$ then gives the probability of disabling the airplane by a hit on a given essential part. Since all these probabilities are small, an approximation for the net probability of killing an aircraft hit by a single burst is

$$P_{K/H} = \left[\sum_\ell (p_{K/H})_\ell \, (A_p)_\ell \right] p_{H1} = \left[\sum_\ell A_{v\ell} \right] p_{H1} = A_v \, p_{H1} \qquad (22)$$

Here we have used the symbols $A_{v\ell}$ and A_v to represent the component and total vulnerable areas of the target. Kent did not use the symbols or the phrase vulnerable area, but he computed the sums for both kill types using the Navy data. The Navy report gave the number of fragments hitting each part and the number of estimated A and B kills. As an estimate of $(p_{K/H})_\ell$ one might use the ratio number of kills N_K divided by number of hits N_H. Instead, Kent considered it better practice to apply Bayes' rule and take $p_{K/H} = (N_K + 1)/(N_H + 2)$.

Since the Navy airplane had one engine and one pilot, Kent could not directly transfer these results to multi-engined Army bombers with pilot and bombardier. He recommended that the best method of obtaining estimates of kill probabilities would be to subject such an airplane to fragment attack. Because the Navy tests did not involve a large number of hits on vital parts, Kent estimated that the $p_{k/H}$ values he calculated could easily be in error by 50% or

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[13]*Navy Report 15, "A. A. Gunnery, Damage Effect as Determined by tests against XF 7B-1 Airplane Navy".*

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more. Since no tests had been made on a multi-engined bomber, $P_{K/H}$ estimates based on the Navy report would be even less reliable. Of course the $(p_{K/H})_\ell$ values used by Kent were for an average over all fragments emitted by a particular shell at a particular location relative to the target.

About the same time in England Cunningham and coworkers[14] considered a similar problem in which they analyzed the effectiveness of machine gun fire by an RAF defender attacking a twin-engined bomber. In their report they gave an admirable summary description of the goal they were trying to achieve:

"The volume of data required to describe with complete fidelity all the changing conditions which occur in any one actual combat is clearly too large to admit either of specification, or of useful analysis. This difficulty seems, at first sight, to preclude the possibility of a mathematical theory of combat for which much realism can be claimed. It has nevertheless been proved that the statistical results of a large population of similar combats, on which strategical interest is mainly centered, must depend only on mean values of the varying influences..."

In this report reference is made to a previously published "Mathematical Theory of Air Combat" by L. B. C. Cunningham without date or report number. This report is now unavailable, but modifications to it appear in Appendix I of the report we are now considering. This appendix speaks of "the whole potentially vulnerable area", S, of a bomber, and the set of fractions, θ_ℓ, contributed by the ℓth vital component to this area as determined by firing trials such that the actual total vulnerable area, A_V, is

$$A_V = \left(\sum_\ell \theta_\ell \right) S \tag{23}$$

The pilot, gunner and fuel tanks were considered "singly - vulnerable" components, while each engine consisted of a "duplicated pair, whose contribution to the bomber's vulnerability may be analyzed by the standard method". Such components are now called multiply - vulnerable. Unfortunately, Cunningham's other reports are now unavailable so we are unable to explain how he used Equation (23). However, he seems to have influenced a number of later writers as we shall see.

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[14]*L. B. C. Cunningham, E. C. Cornford, W. Rudoe and J. Knox, "An Analysis of the Performance of a Fixed-gun Fighter Armed with Guns of Different Calibres in Single Home-defence Combat with a twin-engined Bomber", Feb 40.*

B. THE VARIABLE TIME PROXIMITY FUSE

In 1942, the U. S. Office of Scientific Research and Development issued a significant report, OSRD 738, authored by a group consisting of Professor Garrett Birkhoff, Dr. Ward F. Davidson, Dr. David R. Inglis, Professor Marston Morse, Professor J. Von Neumann and Dr. Warren Weaver[15]. In Appendix I of the present report brief summaries are given of the background of some of these men and some others who were involved in this type of work in England and the United States. The OSRD report we are now considering compared the effectiveness of time and proximity fuzing for shells fired by guns on ships at sea defending themselves against high level bomber attack. It is worth quoting part of the introduction to this report since what is said is still true today in many cases, although sometimes not so clearly recognized:

"It was well understood that many items of experimental knowledge are now lacking; and that it would therefore be necessary, at various points in the discussion, to use estimates based not on established fact, but based rather on judgment or even guesswork. A serious attempt has been made to call attention to all such soft spots in the argument; bracketing figures are frequently used to indicate the possible range from 'average' estimates down to 'low' estimates and up to 'high' estimates; and in the cases of major importance, there is some indication of the sensitivity with which the final results respond to changes in estimated quantities".

Proximity fuzes were relatively new and the aim was to find the ratio of kill probabilities for the same shell with two different fuzes. Even large absolute errors should not seriously influence estimates of relative effectiveness. As these authors remarked, the advantage of the proximity fuze was found not to be affected importantly by assumptions made concerning target vulnerable area, initial fragment velocity or the nature of velocity loss due to air drag. Their general method was to multiply the probability that a burst occur at a certain point relative to the target by the conditional probability that a burst at that point would inflict specified damage and to sum these products for all possible locations of the burst to obtain the total probability that a given round inflict the specified damage. The ratio of these probabilities for proximity and time fuzes was called the advantage ratio.

A cartesian coordinate system was located with origin at the effectively stationary target which is being attacked in its forward hemisphere while in level flight. The z axis was chosen anti parallel to the shell trajectory at burst and the x-axis was chosen horizontal. A Gaussian distribution with circular transverse (x, y) error σ and

[15] G. Birkhoff, W. F. Davidson, Dr. R. Inglis, M. Morse, J. Von Neumann and W. Weaver, "The Probability of Damage to Aircraft through Anti-aircraft Fire. A Comparison of Fuzes When Used against High Level Bombers Attacking a Concentrated Target", OSRD 738, 16 Jul 42.

range error σ_z was used to describe the probability that a burst occur in a volume near the target, that is, for a time fuze

$$P_B = \frac{1}{(2\pi)^{3/2}\sigma^2\sigma_z} \iiint \exp\left[-\frac{1}{2}\left\{\frac{(x^2+y^2)}{\sigma^2} + \frac{z^2}{\sigma_z^2}\right\}\right]dxdy\,dz \qquad (24)$$

or simply $(2\pi\sigma^2)^{-1} \exp\left[-(x^2+y^2)/(2\sigma^2)\right]$ dxdy for a proximity fuze which possesses a burst surface described by $z = z(x, y)$. If $R = (x^2 + y^2 + z^2)^{1/2}$ and ψ is the latitude measured from the x, y plane toward the z-axis ($\psi = \pi/2 - \theta$ where θ is the co-latitude), then the expected number of lethal hits on the target will be a function $N_L(R, \psi)$ because of symmetry about the z-axis. These authors used the poisson distribution to represent the conditional probability that at least one hit be lethal to the target, that is $D = (1-e^{-N_L})$, as in Equation (18) above, following the practice of Pearson and Welch in a British report now unavailable. They also used the usual air-drag law

$$m\, dv/dt = m\, v\, dv/dR = -C_D\, \rho_a\, \overline{A}\, v^2 \qquad (25)$$

where m is the fragment mass, C_D is the drag coefficient, ρ_a is the air density and $\overline{A}$ is the average area presented by the fragment in flight. Following Pearson and Bishop[16] they represented this area for a randomly rotating fragment by

$$\overline{A} = k\, m^{2/3} \qquad (26)$$

where k is a constant shape factor. If we integrate Equation (25) and use Equation (26), we obtain for straight trajectories

$$v = v_0\, e^{-C_D\, \rho_a\, \overline{A}R/m} = v_0 e^{-C_D\, \rho_a\, kR/m^{1/3}} = v_0 e^{-\alpha\, R/m^{1/3}} \qquad (27)$$

We discuss the derivation of Equation (26) in Appendix II. For the number of lethal hits they wrote

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[16] *E. S. Pearson and D. J. Bishop, "The Derivation of a Retardation Law for Shell Fragments", A. C. 2478, 5 Aug 42.*

$$N_L = \frac{N}{2\pi \ R^2} \ K(\psi) \ A_v(R) = \frac{NA_v(0)}{2\pi \ R^2} \ K(\psi) \ \frac{A_v(R)}{A_v(0)} \qquad (28)$$

which is of the form $N_L = \dfrac{N_\Omega A}{R^2}$ used in Equations (18), (13) and (3)

above with $N_\Omega = N \dfrac{K(\psi)}{2\pi}$ and $A = A_v(R)$ rather than A_p. Here $K(\psi)$ is the
ratio of the number of fragments per unit solid angle to the average
number per unit solid angle over the hemisphere forward of the shell's
equatorial plane. The function $K(\psi)$ was determined for 2.5° intervals from
experimental shell firings and depends on altitude because of decreasing
remaining velocity of the shell. Here N is the total number of
fragments larger than some minimum size (.04 oz) and $A_v(R)$ is "a sort

of vulnerable area" which the target presents to the burst and is defined
by Equation (28). In order to estimate this area, its value was assumed
to be divided into three component areas, $A_c = A_v/2$, representing the

crew (pilot and bombardier), and $A_M = A_F = A_v/4$, representing the

motors and fuel cells respectively. As they remarked, there was "some
evidence that two fragments hitting a fuel tank at the same point (or
near together) at short intervals of time are especially effective in
starting fires". Threshold lethality criteria were chosen for the
three component areas, namely, 120 foot - pounds of fragment energy
for A_c, the ability to perforate 3/16 inch of mild steel for A_M and
the ability to perforate 1/8 inch of mild steel for A_F. The fragments
were divided into fairly homogeneous groups with the i-th group
containing n_i fragments of average mass m_i, residual speed $v (m_i, R)$
from Equation (27) and vulnerable area $A_i [m_i, v(m_i, R)]$. Here

$A_i = A_{vi} = A_{ci} + A_{Mi} + A_{Fi} = \displaystyle\sum_\ell A_{\ell i}$ for all groups on the assumption

that they all have some sufficiently high launch speed and mass. However,
as R increases, one or more of the component vulnerable areas will
drop out of the sum for $(A_i = A_{vi} (R)$ because of loss in velocity, so

that a plot of each A_i versus R will be a step function approaching

zero. Consequently, the total vulnerable area

$$A_v(R) = (\sum_i \ n_i \ A_i) \ / \ (\sum_i \ n_i) = \frac{1}{N} \sum_i n_i \sum_\ell A_{\ell i} \qquad (29)$$

will also be a step function. Figure 1 reproduces their figure for
effectiveness versus R for a Navy 5 inch shell with initial fragment
speed $v_o = 3500$ ft/sec. Near the target for example the contribution
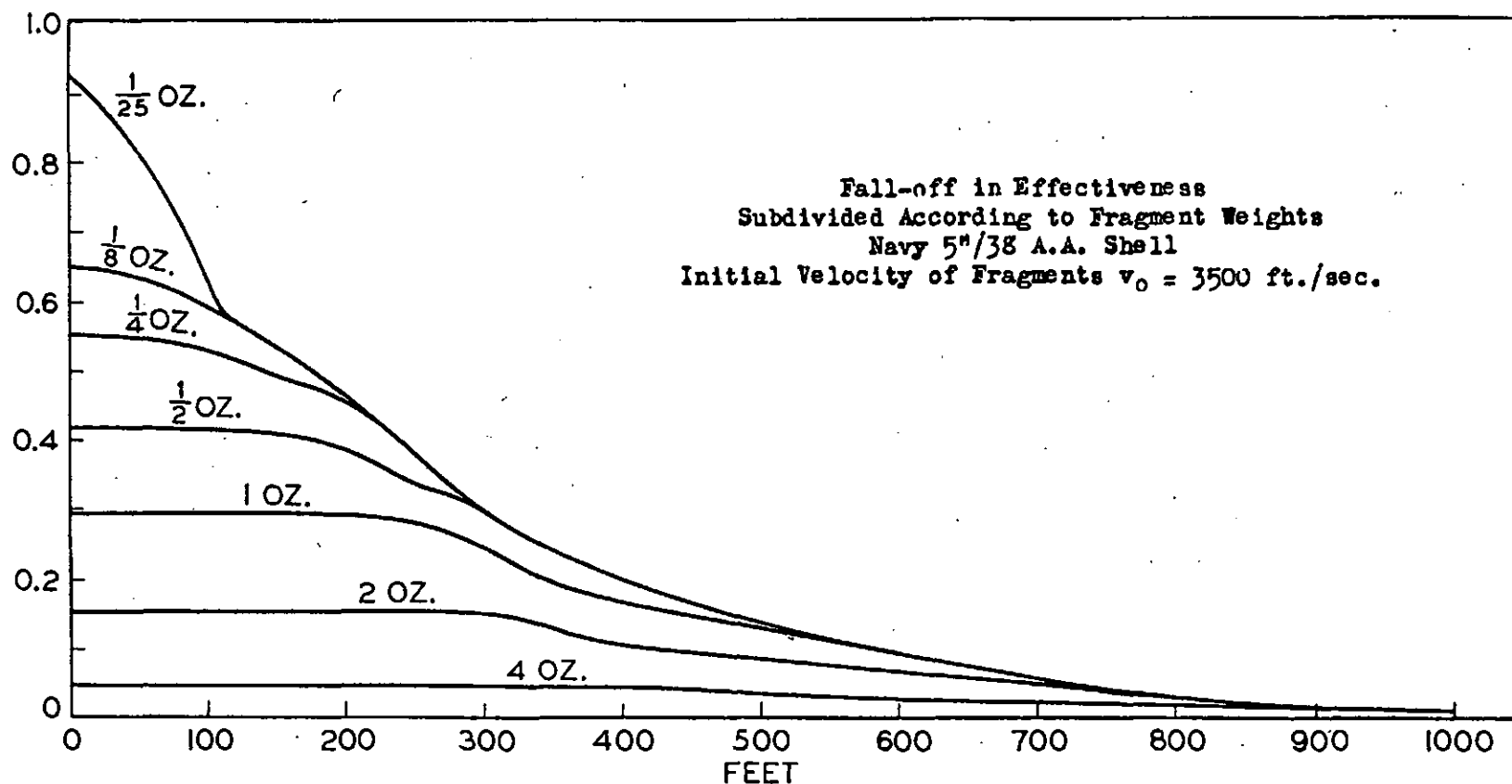from 1/2 oz. fragments is given by a vertical line segment between

Figure 1. Fall-off in Effectiveness Subdivided According to Fragment Weights
(from OSRD 738)

the 1/2 oz. and the 1 oz. curves. These 1/2 oz. fragments show losses in effectiveness near 150 feet and 300 feet. Their vertical scale is arbitrary, but if they had plotted $A_v(R) / A_v(0)$ the curve would approach unity as $R \to 0$. One also sees why they could approximate the cumulative effectiveness curve by an exponential, namely,

$$A_v(R) = A_v(0) \, e^{-CR} \tag{30}$$

where C depends on ρ_a, C_D and k as well as fragment mass. The ratio $A_v(R) / A_v(0)$ was introduced into Equation (28) for convenience, since it could be computed for any $A_v(0)$, assuming the same division into component areas. Then the value of $A_v(0)$ could be chosen later to fit a particular size target of the same type. The above scheme for estimating $A_v(R)$ and $N_L(R,\psi)$ is straightforward and similar schemes have been widely used since. It is interesting to note the authors' remarks on this scheme since they are often true even today:

"It must be confessed that the computation of $N_L(R,\psi)$ is a somewhat shaky affair. But it is comforting to note that the final results of the whole study are rather surprisingly resistant to alterations in the method of computing N_L".[17]

If we recall that their study was of relative calculated kill probabilities rather than of absolute values, then this result is not too surprising. However, they also noted close agreement between the kill probabilities, p_1, for time fuzes which they calculated and those calculated by the British:

"Taking into account the variety and complexity of the different assumptions and the different methods of calculation, the above comparisons of p_1 seem almost miraculously close; while the essential coincidence of the ranges of value for the advantage ratios, as just stated, is also reassuring".[18]

If we put Equation (30) into Equation (28), we find

$$N_L = \frac{NA_v(0)}{2\pi} K(\psi) \, e^{-CR}/R^2 = K(\psi)e^{-CR}/R^2 \tag{31}$$

[17]Reference 15, p. B-7.

[18]Reference 15, p. 46.

which for large R is approximately equal to the damage function, that is, $D = 1 - e^{-N_L} \approx N_L$. It is clear that $R^2 D \approx R^2 N_L \to 0$ as $R \to 0$ and that $R^2 D \to 0$ as $R \to 0$. For a proximity fuze we can express $D(R, \psi)$ in cylindrical coordinates, namely, $D(r, z)$ where $r = (x^2 + y^2)^{1/2}$. Their calculations showed that $D(r, z)$ has its principal maximum for $z = z(r)$, a surface of revolution which is the optimum burst surface for a proximity fuzed shell. From their tables it is apparent that this surface is approximately a cone with the target near the apex. Such a burst surface would place the target in the center of the side spray. In general they concluded that proximity fuzes were anywhere from twice as effective to an order of magnitude more effective than time fuzes, depending on circumstances. The advantage ratios were found by evaluating numerically or by power series integrals over the product probability distribution of D and the Gaussian error function, namely,

$$p_1 = \frac{1}{(2\pi)^{3/2}\sigma^2\sigma_z} \iiint e^{-\frac{1}{2}\left[\frac{(x^2+y^2)}{\sigma^2} + \frac{z^2}{\sigma_z^2}\right]} D(x, y, z)\, dx\, dy\, dz \qquad (32)$$

for the time fuze and

$$p(\sigma) = \frac{1}{2\pi\sigma^2} \iint e^{-\frac{(x^2+y^2)}{2\sigma^2}} D[x, y, z(x, y)]\, dx\, dy \qquad (33)$$

for the proximity fuze. Here p_1 or $p(\sigma)$ are the expected values of the probability function D when x, y and z are random variables with Gaussian probability distributions.

It is interesting to note that the concept of vulnerable area presented in Equation (29) is more general than Kent's given by Equation (22) where Kent's $A_{v\ell} = A_\ell$ is explicity equal to $\frac{1}{N} \sum_i n_i$ $A_{\ell i}$, that is, equal to the average effect of all the fragments on the ℓth component when launched from a particular location. Kent estimated component $(P_{K/H})_\ell$ from experiments and used $A_{p\ell}$ to estimate hit probability. The association of $A_{p\ell}$ with $(P_{K/H})_\ell$ to form $A_{v\ell}$ does not eliminate the need to estimate $(P_{K/H})_\ell$ experimentally and certainly does not imply that the conditional kill probability of the ℓth component $(P_{K/H})_\ell = A_{v\ell}/A_{p\ell}$ can be reduced merely by increasing $A_{p\ell}$, that is, by simply placing the sensitive components in a bigger box! As is clear

from the discussions above, $A_{v\ell}$ or $A_{v\ell i}$ or A_{vi} are "sort of areas".
They are perhaps better described as effectiveness parameters which are
defined to have the dimensions of an area by the role they play in
Equation (28). We can just as well speak of total or component $P_{K/H}$
as estimated from experimental firings (or in some other way) using
a threshold or some more sophisticated lethality criterion. This has
the advantage of avoiding confusion between "vulnerable areas" and
physical (spatial) areas. A similar confusion can arise concerning
so-called "lethal areas" as we shall see.

Shortly after, a British report by Kendall[19] gave a similar comparison
of time and proximity fuze effectiveness versus aircraft, but for
rockets instead of gun projectiles. Since launch stresses on rocket
missiles are not as great as those on gun-launched shells, pre-formed
fragments can be used instead of relying on natural fragmentation. This
forces the designer to choose one or more fragment sizes and shapes.
Kendall also included systematic bias in the aiming error and introduced
an important approximation for the chance of at least one lethal hit.
He adopted the coordinate systems of Equations (32) and (33) above
and assumed that a proximity fuze would burst approximately on the cone
specified by the co-latitude $\theta = \theta_o$ ($z = R \cos \theta$). He included a
transverse bias error α by rotating the system about the z axis until
the projection of the burst point on the x, y plane lay on the x-axis
with mean position $(-\alpha, 0)$ instead of $(0,0)$ as for no bias. Since
$x = R \sin \theta_o \cos \phi$ and $y = R \sin \theta_o \sin \phi$, then Equation (33) becomes

$$p(\sigma,\alpha) = \frac{1}{2\pi\sigma^2}\int_0^R \int_0^{2\pi} \exp\left[-\frac{1}{2\sigma^2}\left\{R^2\sin^2\theta_o + \alpha^2 - 2\alpha \sin\theta_o R \cos\phi\right\}\right]$$
$$D(R,\theta_o) \sin^2\theta_o R\, dR\, d\phi = \frac{\sin^2\theta_o}{\sigma^2} e^{-\alpha^2/2\sigma^2}\int_0^R e^{\left[\frac{\sin^2\theta_o}{2\sigma^2}\cdot R^2\right]}$$

$$D(R,\theta_o)\, I_o\,(bR)\, R\, dR \qquad (34)$$

where $I_o (bR) = \frac{1}{2\pi}\int_0^{2\pi} e^{bR \cos\phi}\, d\phi$ is a Bessel function with
$b = \alpha \sin\theta_o/\sigma^2$. For D in Equation (33) he proposed the approximation

$$D = 1 - e^{-Ke^{-cr}/r^2} \approx e^{-r^2/(2R_2^2)} = e^{-R^2 \sin^2\theta_o/(2R_2^2)}. \qquad (35)$$

Kendall called the adjustable constant R_2 the probability radius and
pointed out that for $N_L = K e^{-cr/r^2} = 1$

[19]D. G. Kendall, "The Chances of Damage to Aircraft from A. A. Rockets
Fitted with Time or Proximity Fuzes", A. C. 2435, 28 Jul 42.

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$$e^{-r^2/(2R_2^2)} = 1-e^{-1} = .632 \tag{36}$$

so

$$R_2 = 1.044 \ r \approx r \tag{37}$$

Thus R_2 can be thought of as defining a circle in the x, y plane centered on the target within which we expect at least one lethal hit. This is comparable to Kent's radius of overhitting in Equation (1) above, so that πR_2^2 is a 100% "lethal area" in a probability sense. If we use Equation (35) in Equation (34) with no bias error ($\alpha = 0$), we find

$$p(\sigma) = \frac{1}{\sigma^2} \int_0^{r_0} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{R_2^2}\right)r^2} r\,dr = \left(\frac{R_2^2}{\sigma^2 + R_2^2}\right)\left[1 - e^{-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{R_2^2}\right)r_0^2}\right] \tag{38}$$

since $r^2 = R^2 \sin^2\theta_0$ gives $r\,dr = \sin^2\theta_0\,R\,dR$. Equation (34) with Equation (35) can also be integrated in closed form for $\alpha \neq 0$, provided we neglect the integral from r_0 to ∞. In this case we find

$$p(\alpha,\sigma) = \frac{1}{\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{R_2^2}\right)r^2} I_0\left(\frac{\alpha r}{\sigma^2}\right) r\,dr$$

$$= \frac{R_2^2}{\sigma^2 + R_2^2} e^{-\frac{\alpha^2}{2(\sigma^2 + R_2^2)}} \tag{38a}$$

Kendall did not choose R_2 on the basis of how closely Equation (35) could match the original function $1 - \exp(-Ke^{-cR/R^2})$ for given K and c. Rather he chose R_2 on the basis of how well Equation (38) for given σ and r_0 could be matched to $p(\sigma)$ computed numerically using the original function. Kendall also considered the problem of optimizing σ in Equation (38a) to compensate for an unavoidable α, as well as the probability of damage from a salvo of rockets equipped with proximity fuzes. His treatment of the subject was later repeated by Carlton[20] and his approximation for D in Equation (35) above came to be called the Carlton approximation.

For a time – fuzed warhead without bias errors but with equal range and transverse random errors ($\sigma_r = \sigma$), Kendall used symmetry to integrate over ϕ and assumed that taking $\theta = \theta_0$ would describe the side spray well enough, so Equation (32) above becomes

[20]A. G. Carlton, "The VT Fuze in Anti-Aircraft Gunfire", TM122, Aug 45, the Johns Hopkins Applied Physics Laboratory.

$$P_1 = \sqrt{\frac{\pi}{2}} \frac{\sin \theta_o}{\sigma^3} \int_0^\infty e^{-\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{R_3^2}\right)R^2} R^2 dR \approx \frac{\pi}{2} \sin \theta_o \frac{R_3^3}{(\sigma^2 + R_3^2)^{3/2}} \qquad (39)$$

neglecting the integral from R_o to ∞ and using the approximation

$$D = e^{-R^2/(2R_3^2)}. \qquad (40)$$

analogous to Equation (35). The factor $\frac{\pi}{2}$ $\sin \theta_o$ can be buried in the probability radius R_3 which defines a kind of "lethal volume". Kendall also briefly considered the cases in which $\sigma_z \neq \sigma$ and in which non-zero bias occur. In a later report[21] he applied these ideas to the prediction of an optimum size and shape fragment in a pre-formed fragment warhead attacking a high altitude bomber. He concluded that the fragment should be square on the two faces which intially form part of the shell wall and should weigh from 1/8 oz to 3/8 oz, assuming no change in fragment launch speed from shells in service. He and his co-workers pointed out that this conclusion was quite sensitive to the threshold criterion used to define the perforating power of a lethal fragment and recommended that much more experimental information was needed to relate perforating power to aircraft damage before much confidence could be placed in these conclusions.

A little after Kendall's paper appeared in 1942 Professor N. F. Mott published a report on this subject[22] in which he discussed methods of choosing the optimum size fragment to be used in a controlled fragmentation warhead attacking aircraft. For example, against dive bombers, damage to controls or one engine may stop the attack even though the aircraft can return to base. If attrition of a high altitude bomber fleet is desired, then damage to all engines or hits on the fuel may cause a crash short of base. To achieve these different objectives, different numbers and sizes of fragments might be desired. Since adequate estimates of the vulnerability of vital components to fragment attack were required but not available, Mott contented himself with an outline of a method while pointing out the weakness of the experimental base on which it rested. The proof that available vulnerability estimates based on controlled firings against aircraft were inadequate he took to be the fact that predictions based on these component vulnerability estimates did not agree with the results of combat data:

[21]A. G. Walters, D.G. Kendall and L. Rosenhead, "Controlled Fragmentation of the 3" (Parallel Sided) Rocket Shell, Fitted with a Time Fuze, in the H. A. A. Role. The Optimum Mass and Shape of the Fragments when they are all of Equal Size", A. C. 4467, 26 Jul 43.

[22]N. F. Mott, "Damage to Aircraft by H. A. A. Fire", A. C. 2480, Aug 42.

'Estimates of the vulnerability of two engined bombers obtained by adding these components do not agree with results obtained in combat. These can be deduced from the results of H. A. A. shooting in this country, where casualties can be compared with the estimated accuracy, and from a comparison between the number of hits on Bomber Command aircraft which return and the estimated number shot down by (German) H. A. A. In both cases the numbers shot down are at least three times as great as they should be on the assumption that damage to controls, pilot or both engines is normally necessary to cause a crash".

Mott devoted half of his report to a discussion of the shortcomings of estimated component vulnerable areas in the light of combat data. Of course he was aware of the limitations of combat data, but he also knew the limitations of predictive models based on inadequate experiments. It is not surprising that a man of Mott's ability (see Appendix I) showed a healthy skepticism toward his own computational estimates and measured the degree of their predictive ability by comparisons with real-world combat data.

In section six of his report, Mott assumed that a shell burst projected a uniform random distribution of N identical fragments each travelling the same straight line distance R_E before it became ineffective. Each aircraft in a fleet of T aircraft was looked upon as a sphere which presents a vulnerable area A and the poisson probability $(1-e^{-N}L)$ with $N_L = N_\Omega A/R^2$ was used as the probability of at least one lethal hit. This estimate for N_L is more like Kent's Equation (3) above since it does not include the angular dependence of the fragment spray as described in Equation (28). The use of the poisson approximation he attributed to Cunningham without citing a reference. The number of casualties or aircraft which receive at least one lethal hit from a burst somewhere in the fleet is

$$C = T_v \int_0^{R_E} \left(1 - e^{N_L}\right) 4\pi R^2 dR = 4\pi T_v \left[R_E^3/3 - \int_0^{R_E} e^{-N_\Omega A/R^2} R^2 dR \right] \quad (41)$$

where $T_v = T/\left(\frac{4}{3} \pi R_E^3\right)$ is the number of targets per unit volume of air space. Equation (41) is a three dimensional analog of Equation (1) above, but differs from it in that the use of the poisson expression for at least one lethal hit avoids the necessity of introducing the idea of an overkill radius.

For vulnerable components Mott used the symbol A^1 with $\left(1 - e^{-N_L^1}\right) \approx N_L^1 = N_\Omega A^1/R^2 \ll 1$, so that the fraction of target components of a given type (say an engine) which receive at least one lethal hit becomes

$$\frac{C}{T} = 1/\left(\frac{4}{3}\pi \, R_E^3\right)\int_0^{R_E} N_\Omega A^1 \, 4\pi dR = 3 \, N_\Omega \, A^1/R_E^2 \tag{42}$$

since N_Ω and A^1 were not taken to be functions of R. In the case of redundant components as in a two-engine bomber, Mott recommended using $\left(1-e^{-N_L^1}\right)^2 \approx \left(N_L^1\right)^2$.

In section seven of his report Mott discussed the question of optimum fragment size. He adopted Equation (27) above and solved it for R, obtaining,

$$R = (C_D \, \rho_a \, k)^{-1} \, m^{1/3} \, \ln \, (v_o/v) \, . \tag{43}$$

Next he adopted a threshold lethality criterion based on the momentum needed to perforate a given thickness, T, of wood

$$mv = GT\overline{A} = GTk \, m^{2/3} \tag{44}$$

where G is a constant and the average presented area was adopted from Pearson as in Equation (26) above. Mott used Equation (44) to define a minimum mass, m_m,

$$m^{1/3}v = m_m^{1/3} \, v_o = GTk \tag{45}$$

which depends on the launch speed, v_o, as well as the target thickness, T, and the shape factor, k. He then used Equation (45) to eliminate v_o/v in Equation (43) to obtain an expression for the effective range

$$R_E = (C_D \, \rho_a \, k)^{-1} \, m^{1/3} \, \ln \, (m/m_m)^{1/3} \tag{46}$$

a form he attributed to Welch without reference. Finally he noted that if M is the mass of the shell, then the number of fragments will be N = M/m, while the chance of a lethal hit on a target component will be proportional to the product NR_E as in Equation (21) above,

$$NR_E = \frac{1}{3} \, M \, (C_D \, \rho_a \, k)^{-1} \, m^{-2/3} \, \ln \, (m/m_m) \tag{47}$$

from Equation (46). This function of m has a maximum for

$$m/m_m = e^{3/2} = 4.48 \qquad (48)$$

so that all fragments should have this mass. If there are two vulnerable components, one of area A_1 vulnerable to fragments of minimum mass m_1 and a second of area A_2, vulnerable to a minimum mass $m_2 > m_1$, then the chance of a lethal hit will be proportional to

$$m^{-2/3} \left[A_1 \ln (m/m_1) + A_2 \ln (m/m_2) \right] \qquad (49)$$

by analogy with Equation (47) above. This expression has a maximum with respect to m when

$$m = \left[m_1^{A_1} \, m_2^{A_2} \, e^{3/2 \, (A_1 + A_2)} \right]^{1/A_1 + A_2)} \, . \qquad (50)$$

Finally, Mott returned to the evaluation of Equation (41) in the case where $N_\Omega A/R^2$ is not negligible compared to unity and plotted $C/(T_V A)$ versus (m/m_m) since the upper limit of the integral in Equation (41) depends on this ratio by Equation (46). He evaluated the integrals numerically both for a singly vulnerable component using $\left(1 - e^{-Nl_L} \right)$ and for a multiply vulnerable component, using $\left(1 - e^{-Nl_L} \right)^2$ as explained above. As expected, these measures of effectiveness exhibited maxima as a function of (m/m_m) with a broader maximum near 10 to 15 for singly vulnerable component and with a narrower lower maximum near 4 to 6 for two redundant components.

In an appendix Mott briefly considered the case of natural shell fragmentation. Adopting a suggestion of Welch which he described in more detail in a later report[23], Mott wrote for the number of fragments with masses between m and m + dm

$$dN = N(m) \, dm = N_o \, e^{-(m/m_o)^{1/3}} \, d \left[(m/m_o)^{1/3} \right] \qquad (51)$$

where N_o and m_o are chosen to fit experimental mass distributions. Instead of using the value R_E in Equation (41), he used an average value, $\bar{R}_E$, computed by weighting R_E according to the distribution

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[23]N. F. Mott and E. H. Linfoot, "A Theory of Fragmentation", A. C. 3348, 11 Jan 43.

given in Equation (51), namely,

$$\bar{R}_E = \int R_E \, d(N/N_o) = \frac{1}{3} \left(C_D \, \rho_a \, k/m_m^{1/3} \right)^{-1} \int_y^\infty y(\ln y) \, e^{-y} dy \qquad (52)$$

where $y = (m/m_o)^{1/3} = (m/m_m)^{1/3}$, if we design our naturally fragmenting shell so that $m_o = m_m$. The infinite upper limit of Equation (52) is an adequate approximation when almost all of the fragments are very small compared to the original shell mass. Since $\bar{R}_E$ is also a function of (m/m_m), Mott could have plotted $C/(T_v A)$ versus $(\bar{m}/m_m)$ for naturally fragmenting shell examples also, but he did not.

In a later report[24] Mott compared the effectiveness of natural and controlled fragmentation and concluded that "the optimum average size with natural fragmentation is so small that if all fragments had this size they would scarcely be large enough to damage the target". In addition he remarked, "the chance of damage is very insensitive to mean fragment size, and cannot be improved by more than 20 per cent by finer uncontrolled fragmentation". In his calculations he assumed an aircraft to have a vulnerable area of ten square feet with resistance to damage equivalent to the resistance to perforation offered by two inches of wood. As before he used an average over all aspects of the target. As we can see from Equation (41) above, the vulnerable area was taken to be a fixed number for a given target and shell which entered into the estimation of the number of lethal hits $N_L = N_\Omega \, A/R^2$.
Of course $N_L = N_L(R)$ while fragment size and speed entered through R_E in Equation (46) or $\bar{R}_E$ in Equation (52). Again his treatment of vulnerable area is more like Kent's than that given in OSRD 738.

Meanwhile, at the Ordnance Office in Washington, D.C. similar studies were being carried out by Prof. Marston Morse and his associates. Prof. Morse had already collaborated in OSRD 738 and continued his interest as revealed by a series of reports from 1943 to 1945. For damage to materiel targets Morse preferred an energy threshold criterion

$$mv^2 = K^2 \, T \, A = \text{constant} \qquad (53)$$

instead of Equation (44) above. Here again T is the target thickness (wood or steel plate equivalence). However, Morse did not simply use the average area presented by a fragment in flight as the striking

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[24] N. F. Mott, "Optimum Fragment Size for A. A. Shell", A. C. 3366, 15 Jan 43.

area. Instead he preferred the maximum area or some quantity related to it, at least when deliberately trying to overestimate the protection needed by an aircraft from its own bomb-fragments in very low-level runs[25]. If we solve Equation (53) for v (given m) then we have an expression for the minimum striking speed needed to perforate the target thickness, T, namely, the limit speed

$$v_L = K \ (TA)^{1/2} \ m^{-1/2} \tag{54}$$

which, as Morse points out, is a particular case of the De Marre formula

$$v_L = K \ (TA)^{\alpha_1} \ m^{-1/2} \tag{55}$$

This later served as the basis for the formula

$$v_L = 10^{C_1} \ (TA)^{\alpha_1} \ m^{\beta_1} \ (\sec \theta)^{\gamma_1} \tag{56}$$

used in Project Thor[26] where C_1, α_1, β_1 and γ_1 are empirical constants. Morse proposed checking the British hypothesis of random rotation in flight leading to $\bar{A}$ in Equation (26) above by using photography. He also discussed a method of determining the mass distribution of a naturally fragmenting bomb or shell from observations of the frequency distribution of fragment hole areas in witness panels.

Since proximity fuzes had become available by 1943 a number of reports appeared which discussed the optimum height of burst for a shell or bomb used against personnel on the ground. Birkhoff and Lewy[27] compared the results of three such reports with experimental data and found satisfactory agreement on the preferred height. The first report they cited was by Mott[28] for the British 3.7 inch shell, but it is now unavailable. The second was by Chandrasekhar[29] and treated a 105mm shell. Chandrasekhar used a coordinate system with

[25]M. Morse, R. Baldwin and E. Kolchin, "Report on the Uniform Orientation and Related Hypotheses for Bomb Fragments, with Applications to Retardation and Penetration Problems", T. D. B. S. 3, 30 Jan 43.

[26]Project Thor, TR47, "The Resistance of Various Metallic Materials to Perforation by Steel Fragments; Empirical Relationships for Fragment Residual Velocity and Residual Weight", Ballistic Analysis Laboratory, The Johns Hopkins University, Apr 61. (AD#322781)

[27]G. Birkhoff and H. Lewy, "Optimum Height of Burst for 105mm Shell", BRL MR 178, 24 Jun 43. (AD#89856)

[28]N. F. Mott, Army Operations Research Group Memo No. 18, 18 Feb 43.

[29]S. Chandrasekhar, "Optimum Height for the Bursting of a 105mm Shell", BRL MR 139, 15 Apr 43. (AD#492801)

the x, y plane defining the ground plane and z = h representing the height of burst. For a horizontal shell with its axis parallel to the x axis he took the equatorial side spray of fragments to be symmetric about the y, z plane with an angular width of 2Δ ($\Delta \approx 7.5°$). For prone ("poker chip") men of area A the area presented normal to the slant range R is Ah/R so the expected number of hits is

$N_L = N_\Omega \, Ah/R^3$. Chandrasekhar used Equation (51), which he also attributed to Welch, for the mass distribution of fragments, but he implicitly assumed a uniform spatial distribution within the narrow side spray so that N_Ω is an angular step function which

depends on average effective range as determined by the mass distribution, average shape, air drag and lethality threshold parameters expressed in Equation (52) above. His expression for the number of casualties was similar to Kent's Equation (1)

$$C = {}^{T}A \iint_{I} dxdy + {}^{T}A \iint_{I} dxdy \left[\frac{1}{4\Delta} \int_{m_{min}}^{\infty} N(m)dm \right] Ah/R^3 \qquad (57)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad II$

where region I was a rectangular area of overhitting instead of Kent's circular area and T_A is the number of targets per unit area of ground.

One side of the target area was defined by the extent of the side spray while the other was defined by the average effective range. The mass integral in Equation (57) was divided by the angular width $2\Delta \approx 2 \sin \Delta$ as well as by a factor of two because only half the fragments were projected at the ground. Chandrasekhar smoothed and extrapolated experimental fragment mass data in order to tabulate $\int_m^{\infty} N(m)dm$ versus m. He then used Equation (45) as his lethality criterion and tabulated $\overline{R}_E$

in Equation (52) versus m. Next he tabulated the integrand in region II of Equation (57) for various values of h (50 ft < h < 120 ft) and $\sqrt{x^2+y^2}$, the radial distance on the ground measured from directly below the burst point. Since A was taken to be a constant it did not enter into the tabulation of the integral, so that $C/(AT_A)$ could be found as a function of h by summing over values tabulated for various distances $\sqrt{x^2+y^2}$ since $R = \sqrt{h^2+x^2+y^2}$. He concluded that the optimum value for

h was about 75 feet with little sensitivity to A = 2.5 ft^2 or 5 ft^2 or Δ = 7.5° or 15°. At this height he estimated that each shell burst. would cover an area about 20 ft by 150 ft, indicating that such bursts should be placed in a grid pattern with these dimensions for optimum effect. Obviously Equation (57) predicts zero casualties for h = 0 since prone "poker chip" men are safe from a burst at ground level in this model, so this formula does not reduce to Kent's Equation (1) which envisions standing targets. It does however include a mass distribution of fragments as well as air drag effects which vary with mass and an

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average shape, although, like Kent's formula, it is limited to a particular shell orientation.

Soon after, Morse and co-workers[30] discussed the same problem but included foxhole protection. In this model a man is considered barely safe if the slant height $R = h/\sin \zeta$ where ζ is measured in the R, h plane between R and the ground plane. If $\zeta = 30°$ we have a so-called "30 degree" foxhole with the maximum effective distance on the ground measured from a point directly below the burst $r_2 = \sqrt{3}\, h$. Like Mott, Morse used the poisson expression for the probability of at least one lethal hit. For the 500 pound general purpose bomb he constructed detailed contour plots of this probability on the ground plane for two angles of fall (45° and 60°), four burst heights (0, 30, 50 and 100 feet) and two types of target. For personnel targets he used the energy threshold criterion of 58 foot pounds, while for truck targets his threshold criterion for lethal fragments was their ability to perforate 1/8 inch of mild steel plate. For personnel he introduced a terrain cover function $(1-r/r_2)$ multiplied by the presented area. He computed the expected number of casualties by placing men or trucks at grid points in a square target area, for example, 400 men ten feet apart, evaluating the probability of at least one lethal hit, 1- exp $\left[-N_L(x,y)\right]$, at each grid point and summing over all grid points. This eliminated the need to evaluate integrals. He concluded that a burst height of about 50 feet would be optimum against shielded personnel or trucks (shielding each other).

Morse soon followed this report with another on the effectiveness of the 105mm shell[31]. He used the same methodology and concluded that the optimum burst height varied from 30 feet to 65 feet as shielding increased from natural terrain only to terrain plus 45° foxholes. Morse compared his results with those of Chandrasekhar mentioned above and enumerated the following differences between the two models: (1) Chandrasekhar assigned a vulnerable area to a prone man (in the ground plane) while Morse represented a man by a sphere which always presents the same area to the radius vector $\vec{R}$. Morse also added a representation of terrain shielding; (2) Chandrasekhar smoothed and extended experimental pit results for the mass distribution, while Morse used the results without modification; (3) Morse used the experimental angular distribution of fragments[32] while Chandrasekhar used a uniform distribution over a narrow equatorial spray; (4) Morse used

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[30]*M. Morse, W. Transue and R. Kuebler, "The Optimum Point of Burst of a 500 lb. G. P. Bomb Equipped with a Proximity Fuze", T. D. B. S. 7, 22 Apr 43.*

[31]*M. Morse, W. Transue and R. Kuebler, "On the Optimum Height of Burst of a 105mm H. E. Shell, M1, Used against Personnel", T. D. B. S. 11, 1 Jun 43.*

[32]*N. A. Tolch, "Number, Angular Distribution, and Velocity of Fragments from 105mm H. E. Shell M1", BRL R386, 28 Jul 43. (AD#491869)*

an energy threshold of 58 foot pounds for his lethality criterion while Chandrasekhar used $m^{1/3} v = 210$ (pound)$^{1/3}$ ft/sec; and (5) Morse used a summation technique over a grid while Chandrasekhar integrated with an assumed constant target density, T_A. In addition it might be mentioned that Morse included non-zero angles of fall.

As mentioned above, Birkhoff and Lewy pointed out that the conclusions of Morse and Chandrasekhar concerning optimum burst height were in reasonable agreement with each other and with the results of experiments performed for the Field Artillery Board at Fort Bragg, all employing the 105mm shell. In addition, Mott's conclusions concerning the British 3.7 inch shell were also about the same.

"All studies agree that a variation of ± 20% from optimum in the height of burst will decrease the effectiveness by at most 10%. Within these limits, the results of the four studies are in agreement with each other, in spite of numerous differences in the mathematical formulations used"[27]

The underlining of the words quoted above by Birkhoff and Lewy seems to indicate their pleasant surprise that a variety of simple models could lead to the prediction of the same useful result, confirmed by experiment. The experimental result was about 45 feet for optimum burst height (Appendix to Chandrasekhar's report) while, model estimates ranged above and below this by as much as 78%. However, the particular point of agreement noted above was of a relative rather than of an absolute nature, namely, that effectiveness for h values on either side of the optimum height of burst did not change very much from the maximum value of effectiveness. Encouraged by this result, Lewy later published a report on the optimum burst height for demolition bombs which employ a blast rather than fragment lethal mechanism[33]. Lewy also made suggestions on how to handle terrain shielding[34]. In addition, he published a report on the effectiveness of a 300 pound bomb against personnel, following the methods of Kent and Chandrasekhar[35]. Soon after, Huntoon[36] remarked that a dispersion in height of burst which is inevitable for a proximity fuze, could actually be an advantage in view of the variety of target conditions actually encountered.

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[33]H. Lewy, "On the Optimum Height of Explosion of Demolition Bombs", BRL MR185, 29 Jun 43. (AD#493725)

[34]H. Lewy, "On the Shielding Effect of Small Elevations of Terrain from Explosions of Small but Variable Height", BRL MR195, 14 Jul 43. (AD#493415)

[35]H. Lewy, "Determination of the Number of Casualties Caused by the Explosion of a 300 lb. Bomb", BRL R403, 11 Sep 43. (AD#491874)

[36]R. D. Huntoon, "Effects on Bomb Damage of Dispersion in Height of Burst", O. S. R. D. 1867, 27 Sep 43.

C. Further Development.

During the next year or so a number of reports appeared which provided additional background for the further development of our subject. Here we will mention some examples. Thomas[37] pointed out that Equation (51) above, or its analog involving $(m)^{1/2}$, can be obtained by simple general arguments and need not be based on fracture models involving cubes, planes or lines. This implies that the ability to fit data with a formula of this type is not an argument in favor of a particular breakup mechanism. For example, there is no need to assume simultaneous fracturing in any regular manner, since any kind of random sequential breakup leading to a decrease in average size and an increase in total fragment number will do quite as well as any other.

Gurney[38] used an energy balance to derive expressions for the launch speed of fragments from cylindrical or spherical shells filled with high explosive. When the metal casing breaks it has expanded about 50% or so to a radius $r = a$ and allowed the detonation products to drop in density to a value ρ. If we assume that the radial speed of expansion of these products is a linear function of r, namely, $v = (r/a)v_o$, with the products in contact with the case having the launch speed of the case fragments, $\dot{v}_o$, then, if we neglect all other energy sinks such as metal or gas heating, we can partition the detonation energy into two parts, shared by the case and the detonation product gases as follows

$$E_c C = \frac{1}{2} M v_o^2 + \int_0^a \left[\frac{1}{2} \rho_c \left(r v_o / a \right)^2 \right] \left(4\pi \; r^2 \; dr \right) \qquad (58)$$

where E_c is the energy per unit mass released by a spherical explosive charge of mass $C = \rho_c \; (\frac{4}{3} \pi a^3)$, detonated at its center. Here M is the mass of the spherical shell encasing the charge. If we carry out the integration in Equation (58) and solve for v_o we obtain

$$v_o = \sqrt{(2E) \; / \; [(M/C) + .6]} \qquad (59)$$

If we carry out the same sort of analysis for an infinitely long cylindrical charge of mass per unit length $C = \rho_c \; (\pi a^2)$ detonated along its axis with M equal to the mass per unit length of the cylindrical metal shell, using volume element $(2\pi r \; dr)$ instead of $(4\pi \; r^2 \; dr)$ in Equation (58), we obtain Equation (59) with .5 instead of .6 in the

[37]L. H. Thomas, "Comments on Mott's Theory of the Fragmentation of Shells and Bombs", BRL R398, 4 Sep 43. (AD#36152)

[38]R. W. Gurney, "The Initial Velocities of Fragments from Bombs, Shells, Grenades", BRL R405, 14 Sep 43. (AD#36218)

brackets. In this case v_o approaches $\sqrt{4E_c}$ for a bare cylindrical charge (M/C → o) or slightly less than this for a bare spherical charge. If this approximation were correct in this limit, we would expect to be able to use measured values for the heat of detonation for E_c (cal/gm).

Gurney noted that for TNT filled shells with .18 < M/C < 16.67 an appropriate value for E_c was 715 cal/gm, leading to $\sqrt{2E_c}$ = 8,000 ft/sec, which is considerably less than the measured heat of detonation. Similarly he noted that we cannot expect such an approximate formula to agree with observation in the limit as M/C becomes arbitrarily large. Experimentally v_o → 0 well before M/C → ∞ since a heavy walled vessel will completely contain the detonation of a small amount of explosive inside it. Still, over the range of military interest Equation (59) and its analog for a cylinder can serve as a useful approximation. Gurney also noted G. I. Taylor's discussion of the expansion velocity of a thin walled cylindrical bomb detonated at one end rather than axially, but did not compare the two models[39].

Thomas[40] applied the cylindrical analog of Equation (59) to a 90mm shell divided for computational purposes into ten cross sectional parts perpendicular to its axis. In this way the actual variations of the M/C ratio along the length of the shell could be approximated by using Gurney's formula with a different M/C for each section. Thomas also attempted to account for the lack of parallelism between the axis and individual sections of the wall by simply doubling the actual angles, since he assumed that the shell would expand to about twice its diameter before it fragmented. However, in order to obtain agreement with panel data, he was forced to assume a distribution about each angle (normal distribution with standard deviation ± 7 degrees). In addition, he fitted Equation (51) to panel and pit data, using the half power form. He was obviously not satisfied with this angular analysis, since about six months later he published a report showing the relation between the Taylor and Gurney models of a cylindrical H. E. filled shell[41]. By using an asymptotic solution method to take account of the transverse velocity of the detonation products in Taylor's theory he was able to derive Gurney's formula in the final expanded state. Taylor's prediction of the angle of departure of the fragments in this limit remained the same, so that this angle could be added to

[39]*G. I. Taylor, R. C. 193. Also see "Analysis of the Explosion of a Long Cylindrical Bomb Detonated at One End", Paper No. 30 (1941) in The Scientific Papers of Sir Geoffrey Ingram Taylor, Vol. III, Aerodynamics and the Mechanics of Projectiles and Explosions, Ed. by G. K. Batchelor, Cambridge University Press, 1963.*

[40]*L. H. Thomas, "Analysis of the Distribution in Mass, in Speed and in Direction of Motion of the Fragments of the M71 (90mm) A. A. Shell, when Filled with TNT, and when Filled with Ednatol", BRL R434, 3 Dec 43.*

[41]*L. H. Thomas, "Theory of the Explosion of Cased Charges of Simple Shape", BRL R475, 18 Jul 44. (AD#491945)*

the original angle of the shell's double ogive shape to obtain a better estimate of the net angle of departure for each section. This type of approximation is still in use today.

A more exact theory which retains the radial component of casing velocity and leads to results similar to Taylor's has been published by Allison and Schriempf[42].

Late in 1943, Kent and co-workers[43] used the methods so far developed to compare the standard 90mm A. A. shell effectiveness with that of several proposed modifications involving time plus contact fuzing as well as filling with shapnel balls or eight ounce cubes. The advantage of using balls or cubes dispersed by spin and air blast rather than by a large explosive charge would be their filling of the region forward of the shell where the target was likely to be in contrast to the expanding annular pattern characteristic of standard shell design. The disadvantage was the smaller number and slower speed of the fragments. They concluded that none of the proposed designs would rival the potential of a proximity fuzed shell. This report is also noteworthy for its nine appendices in which the model assumptions then current were reviewed. In their Appendix VII they summarized the available vulnerable area estimates for aircraft and estimated that their ratios for various fragments and aircraft were correct to about ± 50%, while they might be absolutely correct to within a factor of two. In their final appendix they repeated Mott's derivation of an optimum size fragment mass given in Equations (47) and (48) above, apparently without being aware of Mott's work over a year before their own. Shortly after, Thomas and Birkhoff[44] employed similar methods to conclude that a 10% to 30% advantage could be gained by filling the 90mm A. A. shell with ednatol explosive instead of TNT.

The personnel area target problem continued to receive attention in the years from 1943 to 1945. Morse and co-workers[45] compared the effectiveness of the 500 pound bomb loaded with explosive in ten different ways and subjected to pit, panel and velocity tests at Aberdeen Proving Ground. In order to do this they employed two indices, a space index, I, describing effectiveness against a collection of targets surrounding a burst in mid air (A. A. Shell versus bomber

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[42]F. E. Allison and J. T. Schriempf, "Explosively Loaded Metallic Cylinders. II", J. Appl. Phys. 31 (5), 846 (1960).

[43]G. Burkhoff, R. H. Kent and D. R. Inglis, "Comparative Effectiveness of Different 90mm Shell Against Aircraft", BRL R310, 27 Oct 43.

[44]L. H. Thomas and G. Burkhoff, "Use of Ednatol or TNT in A. A. Shell", BRL R441, 23 Dec 43.

[45]M. Morse, W. Transue and R. Kuebler, "Fragment Damage Indices. A Comparison of Fragment Effects of Different Loadings of the 500 lb. G. P. Bomb, AN-M64, and the 20 lb. Fragmentation Bomb, AM-M41", T. D. B. S. 16, 27 Sep 43.

squadron), and a plane index J, describing effectiveness against targets distributed in a plane (bomb or artillery shell versus troops on the ground). Both indices were computed by summing the trajectory lengths over which each fragment is considered to be effective according to various threshold criteria. For the index J each element in the sum was weighted as r rather than r^2 to account for the slower increase in number of targets in a plane than in a sphere. Both indices were evaluated for effective trajectory lengths greater than some radius of overhitting and a more complete index, K, was obtained by adding the number of targets inside the circle or sphere of overhitting[46,47]. Morse's ideas of using fragmentation data, postulating threshold criteria for effectiveness, accounting for air drag losses, and so forth were little different from those of his contemporaries. He differed from them mostly in the quantities he chose to sum together. Later in 1945, Morse's method of making his conclusions about the superiority of one or another type of shell was criticized by Scheffe[48]. Scheffe admitted that he was in the uncomfortable position of raising difficulties without having anything better to offer. Still his criticisms are worth considering since some of them apply to all writers on this subject, including those working in the field today. He pointed out that assuming that the possession of a certain energy or power to perforate a given thickness of steel or wood implies the ability to inflict a certain type of damage on a man or materiel target requires a leap in our chain of reasoning. How are we to justify, for example, our assumption that ability to perforate 1/8 inch of mild steel correlates with the complex phenomenon of damaging a plane or truck in a certain way? To attempt to do this by calculation seems not only very complicated but also highly uncertain. To accomplish it experimentally would seem to involve prohibitive expenditures of time and materiel. Even apart from these uncertainties Scheffe pointed out that uncertainties in the experimental fragmentation data upon which all such calculations are based carry over into any conclusions which are drawn. He advanced some general considerations which indicated that a difference in the Morse indices should not be regarded as statistically significant unless it exceeded 30 per cent. Since the differences calculated by Morse were almost always smaller

[46]*M. Morse, W. Transue and R. Kuebler, "The Choice of Bombs According to the Type of Fragment Damage Required. New Methods for Comparison of Different Missiles as Well as the Same Missile with Different Loadings", T. D. B. S. 18, 23 Dec 43.*

[47]*M. Morse, W.Transue and M. Heins, "The Expected Number of Effective Fragment Hits from Ground Bursts of Bombs Taking Account of the Angle of Fall and Remaining Velocity of the Bomb, and the Average Height of Targets", T. D. B. S. 49, 5 Jan 45.*

[48]*H. Scheffe, "Some Comments on the Morse Theory of Fragment Damage", OSRD 6426, 3 Dec 45.*

than this, the implication was that the conclusions drawn concerning the superiority of one shell type over another were not meaningful. For example, the most widely fluctuating measurement which entered into the calculations was the number of fragments observed in the pit experiments which led to a standard deviation of 13%. If at least two standard deviations were required to distinguish the effectiveness of one explosive fill compared to another, we see that the calculated differences were not usually statistically significant.

Sheffe﹐ pointed out how experimental uncertainties, namely, a lack of reproducibility in a certain measurement technique, lead to uncertainties in calculated effectiveness parameters. He also noted that the sand pit technique itself might contain a built in bias because of secondary break-up and referenced a report by Gurney and Lewy[49]. These authors remarked that the fragmentation of bombs had been re-investigated using a set up in which the sand-pit was covered with four inches of celotex or with fourteen inches of sawdust since it was feared that in earlier experiments many of the high velocity fragments had been further broken up by impact on sand.. The results indicated that for the 500 pound bomb the fragments were more than twice as heavy and less than half as numerous as in the earlier sand pit experiments, while for the smaller 100 pound bomb the fragments were about twice as heavy and half as numerous as before. Again the question of secondary breakup in celotex or sawdust could be raised, but these authors did not do so. Instead they commented that the observed changes in fragmentation did not seem to affect calculations of effectiveness very much since the reduction in the number of fragments is compensated by an increase in their range. For example, for the 100 lb bomb, separate calculations showed that doubling the average mass and halving the number of fragments produced a change of fifteen per cent in the effectiveness index. The question of breakup in softer recovery media was studied later by Bentz[50] who concluded that pit data gives a number of fragments at least several times the true value.

The British also continued an active interest in estimating the the effectiveness of anti-personnel weapons. In 1944, Pearson and co-workers[51] introduced the name "lethal area" and pioneered the use

[49] R. W. Gurney and H. Lewy, "On the Comparison of the Efficiency of the 20 lb. Fragmentation Bomb, the 23 lb. Parachute Bomb, the 100 lb. G. P. Bomb, and the 500 lb. G. P. Bomb Singly and in Clusters for Attack by Ground Bursts-on Aircraft Dispersed on Open Ground", BRL MR289, 24 Apr 44. (AD#492802)

[50] W. H. Bentz, "Secondary Fragment Break-up in Cane Fiber, Cardboard, and Sand," BRL R704, 3 Aug 49. (AD#492317)

[51] E. S. Pearson, N. L. Johnson and D. F. Mills, "The Calculations of Lethal Areas for Anti-personnel Weapons, with Special Reference to the 25-pr. Shell", A. C. 5885, 9 Mar 44.

of a lethality criterion which was not a simple step function like
possession of 58 ft-lbs of energy or ability to perforate a one-inch
thick board. The lethal area, A_L, they interpreted as follows. If
T targets are distributed on average uniformly over an area $S >> A_L$,
such that $T_A = T/S$ is the target density, then on average it is to be
expected that $A_L T_A = A_L T/S$ targets will be incapacitated. The
lethal area was computed by integrating the damage function $D(x, y)$
over all possible ground positions. Thus

$$A_L = \iint D(x, y) \, dx \, dy = \iint \left[1 - e^{-N_L(x,y)} \right] dx \, dy \qquad (60)$$

using the poisson distribution as explained above. We see that this
notion is similar to Kent's Equation (1) above if we rewrite
Equation (1) in the form $C = T_A A_L$. The major conceptual difference
is the use of $(1 - e^{-N_L})$ for the damage function instead of adding an
area integral to a circle of 100% effectiveness as in Equation (1).
We have seen that Kent himself eventually adopted the poisson form
(see Equation (13) above). From Equation (60) and the assigned
interpretation $C = A_L T_A$, it is clear that a "lethal area" may have

the dimensions of an area, but it is not a spatial region on the
ground which can be paced off or staked out. Rather it is an
effectiveness parameter or casualty index which when multiplied by
the number of targets per unit area gives the number of casualties to
be expected on average.

Pearson also suggested using the "Zuckerman man" instead of the
"1.0-inch wood man", by which he meant a vulnerable area or kill
probability which varied continuously from a minimum to a maximum
value. However, he proposed no functional form for this variation,
remarking that it should be determined experimentally. In practice he
used the "1.0-inch wood man".

Carlton[20], in the report already mentioned above, introduced a
different meaning for "lethal area", considering instead an attack on
a single (aircraft) target by a proximity fuzed shell. As mentioned
before, the burst occurs on the cone $\theta = \theta_o$ at a distance $z = R \cos \theta_o$
in front of the aircraft dictated by the sensitivity of the fuze.
On the assumption of symmetry about the z axis, Carlton discussed the
conditional damage probability $D(r)$ for fuzes with radial miss r
found by averaging $D(r, z)$ over the values of z which occur in the
burst pattern. He then defined the "lethal area" or "effective size"
of the target as the integral of $D(r)$ in the x, y plane, namely,

$$A_{LC} = 2\pi \int_0^\infty D(r) \, r \, dr \, . \tag{61}$$

In addition, he discussed the probability of damaging the target, given functioning fuzes, as

$$P(\sigma) = \frac{1}{\sigma^2} \int_0^\infty D(r) \, e^{-\frac{r^2}{2\sigma^2}} \, r \, dr \tag{62}$$

which becomes Equation (38) above if we use Equation (35) for $D(r)$. Here Equation (61) is Equation (62) when all miss distances are equally probable instead of following a Gaussian distribution so that in Equation (61) we are assuming a uniform distribution of burst point projections on the x, y plane which contains a single target at the origin, whereas in Equation (60) above we are assuming a uniform distribution of targets in the x, y plane with a single burst point projected onto the origin. Carlton's type of "lethal area" has been labeled A_{LC} to distinguish it from A_L in Equation (60). The difference is conceptual rather then computational. In either case we are dealing with a "casualty index" or "effectiveness parameter" which has a meaning only in a probabilistic sense. Because of this, the results of a single, real burst may differ widely from the estimate computed, even if the encounter is modeled in great detail. Such estimates as we are dealing with have significance only for the strategist who considers repeated encounters of the same type. This was pointed out by Cunningham in 1940 in the quotation we cited above[14]. Obviously we should not use Equation (61) when all miss distances are not equally probable as usually occurs when a single target is being aimed at. Only if σ is much larger than some finite effectiveness limit r_2 will Equation (61) be a valid approximation, provided the upper limit is r_2 instead of infinity. The Carlton type of lethal area calculation for a single target is still widely used, sometimes without realizing that it assumes either a single burst near an infinite plane of uniformly distributed targets or a single target near an infinite plane of uniformly distributed bursts. At least it gives a lower bound for an estimate of relative effectiveness.

In 1945 Lewy and Gurney[52] generalized Kent's formula, Equation (1), retaining the idea of a radius of overhitting, but considering a burst height, h, above the ground. In addition, they approximated the angular distribution of fragments from a shell by an equatorial side spray of angular width 2Δ superimposed on a spherical spray, each spray with its own constant number of fragments per steradian, N_Ω. The effectiveness of the spherical spray is, of course, independent of the

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[52]H. Lewy and R. W. Gurney, "Rapid Methods for the Prediction of Damage by Fragments in its Dependence on the Variables of the Tactical Situation", BRL R530, 14 Feb 45. (AD#PB165304)

attitude of the projectile at burst. However, the effectiveness of the side spray depends not only on its width and the height of burst but also on the angle of fall ω as defined in Equation (8) above. A symmetric side spray is bounded by the two nappes of a right circular cone, each of half angle $(\pi/2-\Delta)$, which is the co-latitude measured from the axis of the shell (or tangent to its trajectory at burst). The solid angle occupied by the side spray is 4π minus twice the area of spherical cap intercepted by one nappe of the cone. From Equation (2) this is $\Omega = 4\pi - 4\pi [1-\cos (\pi/2-\Delta)] = 4\pi \sin \Delta$. If we use a result obtained by Menaechmus and Appolonius in the fourth and third centuries B. C., we note that the intersection of the cone with the ground plane forms a conic section which is an ellipse of $\omega > \pi/2 - \Delta$, a parabola if $\omega = \pi/2 - \Delta$ and a hyperbola if $\omega < \pi/2 - \Delta$. Since the hyperbola is a common case it is shown in Figure 2. Also shown in Figure 2 is a circle of radius r_2 which is the maximum radius of lethality on the ground. Lewy and Gurney considered "30° foxhole" shielding so that $r_2 = \sqrt{3h}$. In Figure 2 this circle is shown intersecting both branches of the hyperbola and limiting the potentially lethal area of the side spray to the region without grid lines. Of course this circle might intersect only one branch of the hyperbola or not intersect it at all.

Let us consider a right-handed coordinate system in Figure 2 with origin at the burst point, z^1-axis pointing along the projectile velocity vector at the moment of burst and y^1-axis pointing up from the Figure. Then the equation of the cone is

$$(x^1)^2 + (y^1)^2 = (z^1)^2 \tan^2 (\theta^1) = (z^1)^2 / \tan^2\Delta \qquad (63)$$

since the co-latitude $\theta^1 = \pi/2-\Delta$. If we add $(z^1)^2$ to both sides of Equation (63) the cone equation becomes

$$(x^1)^2 + (y^1)^2 + (z^1)^2 = (z^1)^2 / \sin^2 \Delta . \qquad (64)$$

We may express this equation in another coordinate system obtained by a counterclockwise rotation about the y^1-axis through an angle $\omega+\pi/2$, namely

$$x^1 = - x \sin \omega + z \cos \omega \qquad (65)$$

$$y^1 = y$$

$$z^1 = - x \cos \omega - z \sin \omega .$$

Now the equation of the cone is

$$\alpha = \pi/2 - \omega$$
$$\beta = \Delta$$

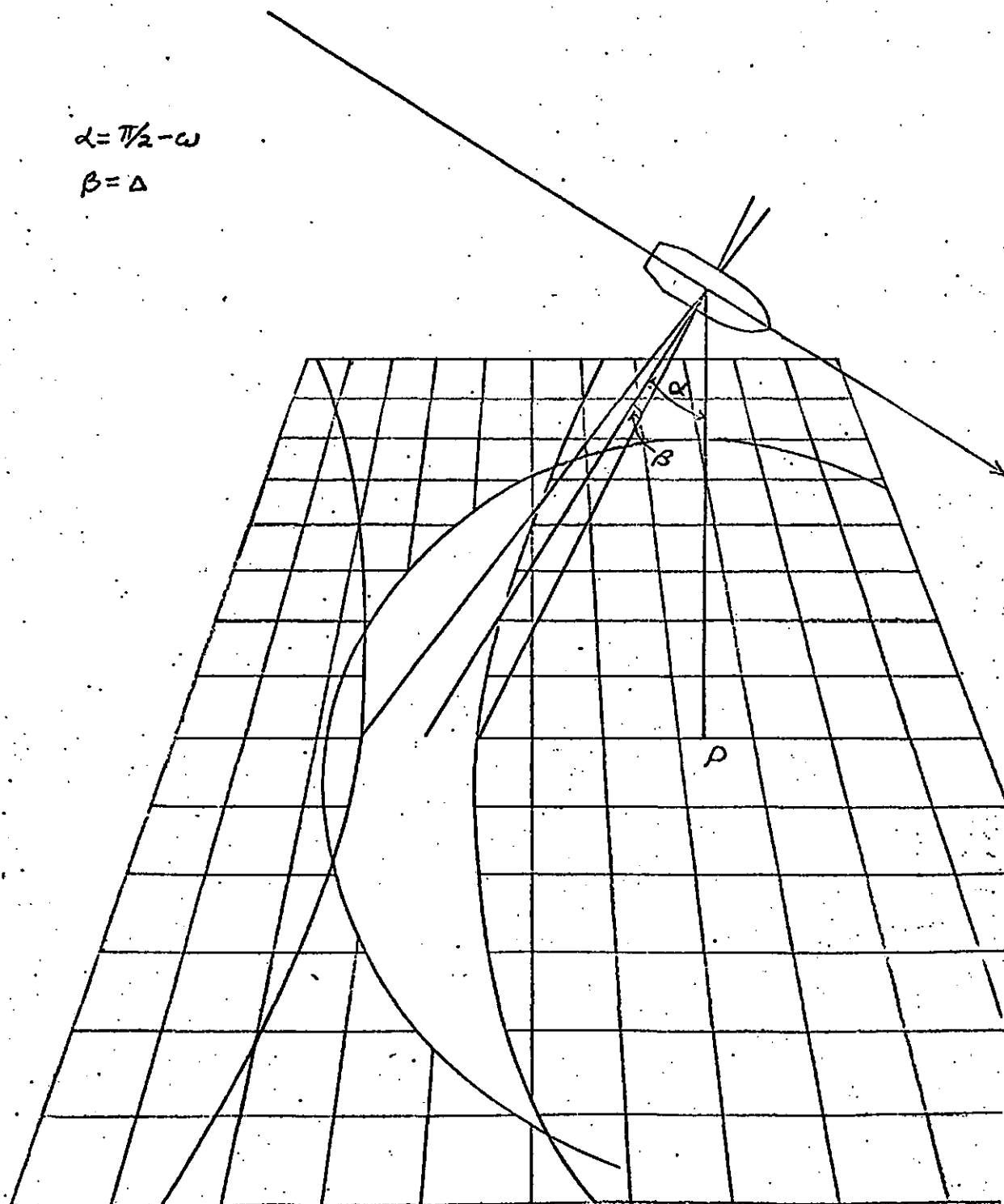Figure 2. Geometry of a Burst over the Ground Plane (from BRL R530)

$$(x^2+y^2 + z^2) \sin^2 \Delta = (-x \cos \omega - z \sin \omega)^2 . \tag{66}$$

In this coordinate system the equation of the ground plane is $z = -h$ and the equation of the conic section mentioned above is obtained by letting $z = -h$ in Equation (66). If we do this, divide by $(\cos^2 \omega - \sin^2 \Delta)$ and add

$$x_o^2 = \left[h \frac{\sin \omega \cos \omega}{(\cos^2\omega-\sin^2\Delta)} \right]^2 \tag{67}$$

to both sides, the equation of the conic section is

$$(x-x_o)^2 - y^2 \frac{\sin^2 \Delta}{(\cos^2\omega-\sin^2\Delta)} = \left[h \frac{\sin \Delta \cos \Delta}{(\cos^2\omega-\sin^2\Delta)} \right]^2 = a^2. \tag{68}$$

Equation (66) is the same as Equation (9) above since $\theta = \pi/2 - \Delta$ so $\cos \theta = \sin \Delta$. The conic section of Equation (68) is centered at the point $(x_o,0,-h)$. If we divide Equation (68) by a^2, we put it into standard form, namely

$$\frac{1}{a^2} (x-x_o)^2 - \frac{1}{b^2} y^2 = 1 \tag{69}$$

with
$$b^2 = h^2 \cos^2\Delta / (\cos^2 \omega - \sin^2 \Delta) \tag{70}$$

Here a, defined in Equation (68), is the semi-transverse axis and b is the semi-conjugate axis. The eccentricity is

$$\epsilon = \sqrt{a^2+b^2} / a = \cos \omega / \sin \Delta \tag{71}$$

and its relation to the discriminant δ of Equation (10) above is $\delta = 4(\epsilon^2-1)$. As stated before, for $\omega < (\pi/2-\Delta)$ we have a hyperbola ($\epsilon>1$), for $\omega = (\pi/2-\Delta)$ we have a parabola ($\epsilon=1$), and for $\omega > (\pi/2-\Delta)$ we have an ellipse ($\epsilon<1$) which becomes a circle if $\omega = \pi/2$ ($\epsilon=0$), the case of a vertical fall.

For a vertical fall ($\omega=\pi/2$), $x_o = 0$ in Equation (65), and Equation (68) is

$$x^2 + y^2 = (h/\tan \Delta)^2 \tag{72}$$

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so the bounding conic section is a circle of radius h/tan Δ centered directly below the burst point. No side spray fragments fall inside this circle and no casualties are possible due to this spray unless h/tan Δ < r_2. If h → o this circle of exclusion shrinks to a point and we return to the case considered by Kent. The circle of exclusion also shrinks to a point as Δ → $\pi/2$ since this represents a spherical fragment distribution. In this case it is meaningless to distinguish between an equatorial side spray and a uniform spherical background.

For a horizontal fall (an unlikely event), ω. = o and the eccentricity ϵ = 1/sin Δ > 1 for Δ < $\pi/2$, so we have a hyperbola with x_0 = 0 from Equation (67). The side spray is concentrated directly below the burst point. We can re-write Equation (68) as

$$x^2 - y^2 \left(\frac{1}{\epsilon^2 - 1} \right) = \left(\frac{h^2}{\epsilon^2 - 1} \right) \left[\frac{1}{\tan^2 \Delta (\csc^2 \Delta - 1)} \right] = \frac{h^2}{\epsilon^2 - 1} \tag{73}$$

when ω = o. Now if h → 0, the boundary degenerates into two straight lines given by

$$y = \pm \sqrt{\epsilon^2 - 1} \ x \tag{74}$$

If Δ → $\pi/2$ (uniform spherical spray) for ω = o, the branches of the hyperbola move far apart since a → ∞ in Equation (68), so the circle of lethality of radius r_2 fixed by shielding becomes the only limiting factor.

Lewy and Gurney introduced the approximation

$$N_\Omega(R) = \exp(-b - cR) \tag{75}$$

for the number of effective fragments per steradian where the constants a and b were evaluated by fits to experimental data over the range h < R < 2 h as discussed by Thomas[53]. For prone men they used the "poker chip" approximation (A_p/R^2) (h/R) for the solid angle subtended by each target. For unshielded standing men they used $(A_p/R^2)(r/R)$. In the case of a ground burst r = $\sqrt{R^2 - h^2}$ → R and Kent's integral in Equation (1) above was found to involve a tabulated function, if we use Equation (75) for N_Ω, namely, with t = cr,

[53]L. H. Thomas, "Computing the Effect of Distance on Damage by Fragments", BRL R468, 18 May 44.(AD#492590)

$$2\pi\ A_p\ e^{b}\int_{cr_1}^{\infty}(e^{-t}/t)\ dt\ =\ 2\pi\ A_p\ e^{b}E_1\ (cr_1) \qquad (76)$$

where E_1 is an exponential integral. Similarly for air bursts against unshielded prone men, we have

$$2\pi\ h\ A_p c\ e^{b}\int_{cR_{min}}^{\infty}(e^{-t}/t^2)\ dt\ =\left[2\pi\ h\ A_p\ e^{b}/R_{min}\right]E_2\ (cR_{min}) \qquad (77)$$

using $t = cR$. Again, E_2 is an exponential integral. For shielded prone men Equation (77) has a finite upper limit and numerical integration must be used. Clearly Equation (77) or the more general case including shielding vanishes as $h \rightarrow 0$ implying no casualties if all targets are prone or in foxholes when a ground burst occurs. For given ω and Δ, the number of casualties increases as h increases, passes through a maximum and decreases as h continues to increase since more fragments become ineffective due to air drag and more fragments fall outside the circle of radius $R_2 = \sqrt{3}h$ determined by shielding. Lewy and Gurney illustrated their modified version of Equation (1) for a 500 lb bomb with remaining velocity of 1000 ft/sec bursting with angle of fall $\omega = 55°$ over men in 30° foxholes and found an optimum burst height near $h = 60$ feet. They plotted the number of casualties divided by the target density, namely, C/T_A, versus h, a quantity

which they did not name, but which we have seen had already been dubbed a "lethal area" by Pearson (see Equation (60) above). It is a casualty index which applies to a single burst over a plane containing a uniform distribution of targets.

We will close this section of our review with a brief account of a report by Whitcomb[54] who modified Kent's original formulation expressed in Equation (4) above in order to broaden the discussion of optimization. Since Equation (3) set equal to unity defines the "radius of overhitting", R_1, we have

$$R_1^2\ =\ N_\Omega\ A_p\ =\ \frac{N}{\Omega}\ A_p\ =\ \frac{M/m}{\Omega}\ A_p \qquad (78)$$

where the total number of fragments N is equal to the mass of the shell case, M, divided by the mass of a fragment, m, when all fragments are alike. From Equation (43) we have

$$R\ =\ \alpha\ m_m^{1/3}\left(\frac{m}{m_m}\right)^{1/3}\ \ln\left[\left(\frac{v_o}{v_m}\right)\left(\frac{v_m}{v}\right)\right]\ =\ \alpha\ m_m^{1/3}y\ \ln\left[\chi^{1/2}y^{3/4}\right] \qquad (79)$$

[54]S. E. Whitcomb, "Fragmentation Efficiency of Bombs", BRL R532, 5 Mar 45.(AD#492432)

where $y = \left(m/m_m\right)^{1/3}$ with the minimum mass, m_m, able to perforate and damage a target is given by a threshold criterion like Equation (56) with $\alpha_1 = 3/4$, $\beta_1 = -\frac{1}{4}$, $\gamma_1 = 0$ and $\gamma^{-3/4} = 10^{C_1} A^{\alpha_1}$, so the target steel equivalent thickness is

$$T = \gamma \, m^{1/3} \, v_L^{4/3} = \gamma \, m_m^{1/3} \, v_m^{4/3} \qquad (80)$$

defining m_m and the corresponding maximum striking speed, v_m. The parameter χ is defined from Gurney's cylinder formula, the analog of Equation (59) above, namely, with .5 instead of .6.

$$\chi = \left(\frac{v_o}{v_m}\right)^2 = \frac{v_o^2}{2E} = \frac{C}{C+2M} \qquad (81)$$

so $\frac{1-\chi}{1+\chi} = \frac{M}{M+C} = \frac{M}{W}$ where W is the warhead mass the sum of the explosive charge mass, C, and the case mass, M. The number of fragments can now be written

$$N = \frac{M}{m} = \frac{M}{m} \left(\frac{W}{M+C}\right)^{\frac{m_m}{m}} = \frac{W}{m_m} \frac{1}{y^3 (1+\chi)/(1-\chi)} = \frac{W}{m_m w} \qquad (82)$$

so $\frac{W}{m_m} = M\, w/m$. If we divide the square of Equation (79) by (78) and use Equation (82) for $\frac{M}{m}$ we find

$$\left(\frac{R}{R_1}\right)^2 = \left[\alpha^2 \, m_m^{2/3} \Big/ \left\{\frac{A_p}{\Omega} \frac{W}{m_m w}\right\}\right] \left[y \ln\left(\chi^{1/2} y^{3/4}\right)\right]^2 = \frac{wz}{P} \qquad (83)$$

where $z = \left[y \ln\left(\chi^{1/2} y^{3/4}\right)\right]^2$ and $P = W A_p \Big/ \left[\Omega m_m^{5/3} \alpha^2\right]$. Whitcomb used as his measure of efficiency Equation (4) above divided by $T_A A_p$ (W/m_m) which gives an efficiency

$$C/\left[T_A A_p (W/m_m)\right] = \frac{\pi R_1^2}{A_p(W/m_m)} \left[1 + \ln \left(\frac{R}{R_1}\right)^2\right]$$

$$= \frac{\pi (M/m) A_p/\Omega}{A_p M w/m} \left[1 + \ln \left(\frac{wz}{P}\right)\right] = \frac{\pi}{\Omega w} \left[1 + \ln \left(\frac{wz}{P}\right)\right] \qquad (84)$$

In this treatment he is considering ground bursts only so r = R (since h = o) as well as vertical angle of fall only. However, by varying the parameters w, z and P in Equation (84) he tried to discuss the effect of size and design changes. Since P is proportional to weight he could use it to compare the efficiency of two bombs of the same design but different size. However, both w and z depend on both the C/M ratio (through χ) and the fragment size (through y), so his formulation does not lend itself to a clear discussion of the effect of changing these two design parameters even for vertical fall and ground burst. By plotting a number of curves with only one parameter varied at a time, he did however conclude that for an optimum bomb weight relative to a given type of target, there should exist both an optimum C/M ratio and an optimum fragment size.

III. EFFECTIVENESS STUDIES AFTER WORLD WAR TWO

A. The Refinement of Analytical Methods.

Soon after World War Two a large number of aircraft became available for experimentation and hundreds were expended for this purpose. Assessments were made of various levels of damage done during controlled test firings and these judgments were expressed as component vulnerable areas[55-60]. Much of this work was directed toward recommending families of antiaircraft and artillery weapons[61-62].

[55]*H. K. Weiss and A. Stein, "Airplane Vulnerability and Overall Armament Effectiveness," BRL MR462, May 47.*(AD#9093)

[56]*J. M. Garmousakis, "The Effect of Blast on Aircraft," BRL R645, 47.* (AD#376940)

[57]*H. K. Weiss and A. Stein, "Vulnerability of Aircraft to 75mm Airburst Shell," BRL MR 482, Sep 48.* (AD#41431)

[58]*A. Stein and H. Kostiak, "Damage by Controlled Fragments to Aircraft and Aircraft Components," BRL MR487, Feb 49.* (AD#53127)

[59]*H. K. Weiss, J. Christian and L. M. Peters, "Vulnerability of Aircraft to 105mm and 75mm HE Shell," BRL R687, Mar 49.* (AD#54768)

[60]*A. Stein and H. Kostiak, "Methods for Obtaining the Terminal Ballistic Vulnerability of Aircraft to Impacting Projectiles with Application to Generic Jet Fighter, Generic Jet Bomber, F-47 Piston Fighter and B-50 Piston Bomber," BRL R768, Jun 51.*(AD#130730)

[61]*J. H. Frazer, H. P. Hitchcock, R. H. Kent, F. G. King, J. R. Lane, T. E. Stern and H. K. Weiss, "A Study of a Family of Antiaircraft Weapons," BRL TN119, Sep 49.*(AD#378732)

[62]*F. E. Grubbs, R. H. Kent, J. R. Lane and H. K. Weiss, "A Family of Field Artillery," BRL R771, Jul 51.* (AD#377190)

The vulnerability of fuel and ammunition was given particular attention[63,64]. Since similar work was being done elsewhere, a conference on aircraft vulnerability was held in 1949.[65]

In 1946, Fano[66] had concluded that reliable effectiveness estimates could not be made "because of insufficient knowledge of the vulnerability characteristics of targets". By 1949 methodology studies began to appear once more. Weiss[67] discussed the Lewy and Gurney approximation in Equation (73) above. He used the 1/2 power form of Equation (51) to obtain for the number of fragments with mass greater than m

$$N = N_o \int_m^\infty e^{-(m/m_o)^{1/2}} d\left[(m/m_o)^{1/2}\right] = N_o \ e^{-(m/m_o)^{1/2}} . \tag{85}$$

He also assumed a threshold lethality criterion

$$m \ v^\gamma = m_m \ v_o^\gamma = L_1 \tag{86}$$

where γ and L_1 are constants and m_m is the minimum mass. If $\gamma = 1$ we have a momentum threshold like Mott's in Equation (44) above, or if $\gamma = 2$ we have Morse's energy threshold as in Equation (53). In addition he made the usual assumption as in Equation (27) above, that

$$v = v_o \ e^{-\alpha R m^{-1/3}} \tag{87}$$

If we let $m = m_m$ in Equation (85), we have $N_m = N_o \ e^{-(m_m/m_o)^{1/2}}$ and dividing Equation (85) by N_m, we obtain the ratio

[63]A. Stein and M. G. Torsch, "Effectiveness of Incendiary Ammunition Against Aircraft Fuel Tanks," BRL MR484, Oct 48. (AD#42383)

[64]K. S. Jones, "A Comparative Study of the Vulnerability of Warheads Loaded with TNT and with Composition B," BRL MR486, Jan 49. (AD#802306)

[65]"Report on First Working Conference on Aircraft Vulnerability," BRL MR488, Mar 49. (AD#377164)

[66]V. Fano, "Discussion of the Optimum Characteristics of Weapons for Most Efficient Fragmentation," BRL R594, Jan 46. (AD#492600)

[67]H. K. Weiss, "Justification of an Exponential Fall-off Law for Number of Effective Fragments," BRL R697, Feb 49. (AD#801777)

$$N/N_m = e^{-\left[\left(\frac{m}{m_o}\right)^{1/2} - \left(\frac{m_m}{m_o}\right)^{1/2}\right]} = e^{-\left(\frac{m_m}{m_o}\right)^{1/2}\left[\left(\frac{m}{m_m}\right)^{1/2} - 1\right]} \qquad (88)$$

If we put Equation (87) into Equation (86), take the square root then the logarithm, then multiply by $\left(\frac{m}{m_m}\right)^{1/3}$, we find

$$.5 \ \alpha\gamma R \ m_m^{-1/3} = \left(\frac{m}{m_m}\right)^{1/3} \ln\left(\frac{m}{m_m}\right)^{1/2} \approx 1.11\left[\left(\frac{m}{m_m}\right)^{1/2} - 1\right] \qquad (89)$$

where the second equality is approximately true for $1 \le \left(\frac{m}{m_m}\right) \le 15$ to within 10%. From Equation (89) we find $\left[\left(\frac{m}{m_m}\right)^{1/2} - 1\right] \approx .45 \ \alpha\gamma R \ m_m^{-1/3}$

which we may substitute in Equation (88) to obtain

$$N = N_m \ e^{-\left[\left(\frac{m_m}{m_o}\right)^{1/2}\left(.45 \ \alpha\gamma R \ m_m^{-1/3}\right)\right]} = N_m \ e^{-CR} \qquad (90)$$

where $C = .45 \ \alpha\gamma \ m_m^{1/6} \ m_o^{-1/2}$. If we divide Equation (90) by Ω to obtain $N_\Omega = N/\Omega$, we see that Lewy and Gurney's approximation in Equation (73) above, based on particular experimental data, should hold approximately for any shell to which Equations (85), (86) and (87) can be applied, at least in the range of (m/m_m) specified. Thus Weiss was able to generalize Equation (73), relating e^b and c to the parameters of the problem. The same could be done using the 1/3 power law of Equation (51) or any other power. It is interesting to note the

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expected number of lethal hits on vulnerable area $A_v(0)$ according to Equation (90) or (73), namely

$$N_L = N_\Omega A_v(0)/R^2 = \frac{N_m}{\Omega} \frac{A_v(0) e^{-cR}}{R^2} = K_1 e^{-cR}/R^2 \tag{91}$$

which has the same form as Equation (31) above if we neglect the angular dependence of $K(\psi)$ and take K_1 to be constant. As we recall, the authors of OSRD 738 arrived at this form by using Equation (30) to approximate the behavior of A_v in Figure 1 while Weiss, Lewy and Gurney obtained it as an approximation to N or N_Ω. In either case the dependence on R is clear. The factor e^{-cR} involves loss of effectiveness because of air drag ($\alpha = C_D \rho_a k$) as well as a damage threshold criterion (m_m) and a fragment mass distribution (N_o and m_o). The factor $\frac{1}{R^2}$ enters because of the assumption that the fragments emanate from a point in space, the burst point. The solid angle Ω may be 4π for full spherical symmetry, or 2π for a hemisphere as in Equation (31) or some lesser amount for a more restricted fragment spray.

Later in the same year Weiss published another report[68] detailing modifications to the methods already developed in World War Two. For a uniform side spray of half angle Δ thrown forward to co-latitude θ_0 by the remaining velocity of the shell Weiss used $\Omega = 4\pi \sin \Delta \sin \theta_o$. For very high altitude bursts against aircraft, Weiss pointed out that $c \approx 0$ in Equation (91) so that Equation (32) becomes

$$
\begin{aligned}
P_1 &= \frac{\Delta \sin \theta_o}{(2\pi)^{1/2} \sigma^2 \sigma_z} \int_o^\infty \left[1 - e^{-K_1/R^2} \right] e^{-K_2 R^2} R^2 dR \\
&= \frac{\Delta \sin \theta_o}{\sqrt{2\pi} \sigma^2 \sigma_z K_2^{3/2}} \int_o^\infty \left(1 - e^{-b^2/y^2} \right) y^2 e^{-y^2} dy \\
&= \frac{\Delta \sin \theta_o}{\sqrt{2\pi} \sigma^2 \sigma_z K_2^{3/2}} \frac{\sqrt{\pi}}{4} \left[1 - e^{-2\sqrt{K_1 K_2}} (1 + 2\sqrt{K_1 K_2}) \right]
\end{aligned} \tag{92}
$$

[68]H. K. Weiss, "A Method for Computing the Probability of Killing a Multiply Vulnerable Aircraft Target with 'N' Rounds of Fragmenting Shell," BRL MR495, 12 Sep 49. (AD#24574)

for fixed fuze angle θ_o, neglecting the integral from R_o to ∞. The final result in Equation (92) can be found in integral tables[69]. Here

$$K_2 = \frac{1}{2} \left[\left(\frac{\sin \theta_o}{\sigma} \right)^2 + \left(\frac{\cos \theta_o}{\sigma_2} \right)^2 \right] , \quad y^2 = K_2 R^2 \text{ and } b^2 = K_1 K_2.$$

Air drag was included by using a series expansion of e^{-cR} and integrating term by term.

Weiss also considered various other questions. For instance, how are we to combine component kill probabilities in the case of duplicated or multiply vulnerable components? A difficulty arises for example when we consider two pilots sitting close together so that a round bursting close to one is also near the other. Because of this lack of position independence, the probability of killing pilots computed by averaging over all burst positions will not be the same as that obtained by computing the probability of killing each one independently and then combining the probabilities. In an appendix he concluded that the approximation of treating duplicated components as if they were independent is good enough for practical purposes. Weiss also considered the combined mechanical time and point detonating fuze for which the probability of a direct hit should be evaluated and then combined with the probability of fragment kill. Finally, he discussed the correlation between successive burst positions when systematic errors exceed random errors, especially in the case of a maneuvering target.

In 1951, Taylor and Kravitz[70] modified the work of Lewy and Gurney by changing their integration variables to two angles instead of an angle and a distance. In addition, they approximated the damage probability function as

$$D = 1 - e^{-K_1 e^{-cR/R^2}} \approx \frac{K_1}{R^2} e^{-cR} \left(1 - e^{-\alpha R} - \alpha R e^{-\beta R} \right) \tag{93}$$

instead of Equation (35) above. Here α and β are purely empirical constants obtained by fitting over an appropriate range for R. This enabled them to integrate over one angle in terms of Bessel functions, but numerical methods were required to integrate over the second angle.

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[69] I. M. Ryshik and I. S. Gradstein, Tables of Series, Products and Integrals, Moscow 63.

[70] W. C. Taylor and S. Kravitz, "A Method for Computing an Estimate of the Antipersonnel Effect of an Airburst Shell," BRL MR555, 13 Aug 51. (AD#802074)

About the same time, Trauring[71] in an appendix elaborated upon Lewy and Gurney's numerical method of evaluating their distance integral.

The question of shaping the nose of a fragmenting bomb or shell received some attention also at this time. Kent[72] reasoned that if personnel targets could be considered to be uniformly distributed on the ground, then it should be advantageous to shape the warhead so that a uniform distribution of fragments on the ground would result. Kent considered the case of a bomb falling vertically with negligible remaining velocity and bursting at height h. He assumed that the fragments are projected normal to the metal case and wrote $2\pi \sigma$ xds for the number of fragments in a ring of width ds on the bomb surface located a distance x from the axis, where σ is the constant number of fragments per unit surface area. If θ is the co-latitude measured from the bomb axis to the radius from the bomb center to a point on the ground, then r = h tan θ is the distance on the ground to this point from directly below the burst point. The fragments from the ring of width ds on the bomb fall on a ground ring of area $2\pi rdr = 2\pi$ (h tan θ) (h sec^2 θ d θ), so the number of fragments per unit area of ground is the ratio

$$\frac{2\pi \sigma \text{ x ds}}{2\pi \text{ h}^2 \tan \theta \sec^2 \theta \text{ d } \theta} = \Sigma \tag{94}$$

where we wish Σ to be constant. Since ds = $\sqrt{1+(dy/dx)^2}$ dx and

tan θ = dy/dx with d(tan θ) = sec^2 θ d θ, Equation (87) becomes

$$x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \left(h^2 \frac{\Sigma}{\sigma}\right) \left(\frac{dy}{dx}\right) d\left(\frac{dy}{dx}\right) \tag{95}$$

The first integral is

$$\left(\frac{dy}{dx}\right) = \left(\frac{x}{\ell^2}\right) \sqrt{\ell^2 + (x^2)/4} \tag{96}$$

where $\ell^2 = h^2 \frac{\Sigma}{\sigma}$ and $\left(\frac{dy}{dx}\right) = 0$ for x = 0. The second integral gives a curve through the origin

[71]M. Trauring, "The Effectiveness of Various Weapons Used in Air Attack on Ground Troops," BRL754, 17 May 51.(AD#377188)

[72]R. H. Kent, "The Shape of a Fragmentation Bomb to Produce Uniform Fragment Densities on the Ground," BRL R762, 15 Jun 51. (AD#377098)

$$y = \left(\frac{4}{3\ell^2}\right)\left[\left(\ell^2 + \frac{x^2}{4}\right)^{3/2} - \ell^3\right] \tag{97}$$

a solution which Kent attributed to a Professor Barnett. Since x and y increase without limit in Equation (97), there obviously must be a change in shape, perhaps to a cylinder, at some finite x value. In other words, Equation (97) describes only a nose shape. It is somewhat of a curiosity since it does not consider such problems as non-vertical angles of fall, the presence of nose fuzes or the question of how to initiate the explosive charge in order to produce the assumed normal launch direction for the fragments. Since aerodynamic considerations impose some requirements on the design of a bomb or shell nose shape, one might also inquire about the flight characteristics provided by Equation (97) as well as the effect of remaining velocity. Still, the whole idea of shaping warheads for optimum effect embodied in Equation (97) stimulated a number of people to pursue the topic. Using the same assumptions, Weiss[73] considered more general methods of determining optimum angular distributions. His conclusion was that the number of fragments per unit solid angle should increase with angle measured from the nose, reach a maximum and then vanish as the equator is approached. Of course, a bomb or shell with a main equatorial side spray thrown forward by a large remaining velocity will have this type of dynamic fragment distribution. Other workers published reports considering the same problem[74,75].

In 1952 Weiss[76] published a fairly comprehensive discussion of the methodology in use at the time. He included an explicit discussion of the addition of the static fragment launch velocity $\vec{v}_o$, and the remaining velocity of the shell, $\vec{v}_r$, to obtain the resultant launch velocity $\vec{v}$, with components

$$v \sin\theta = v_o \sin\theta_o$$

$$v \cos\theta = v_o \cos\theta_o + v_r \tag{98}$$

[73]H. K. Weiss, "Optimum Angular Fragment Distributions for Air-Ground Warheads," BRL R829, Sep 52. (AD#377182)

[74]"A Theoretical Basis for the Optimization of Munition Shape and Munition Distribution Having a Generalized poisson Formulation," Aug 62, Honeywell Ordnance Division, Hopkins, Minnesota.

[75]L. F. Nichols and W. R. Benson, "Optimum Fragment Distributions Using the Calculus of Variations," Picatinny Arsenal TM1213, Jun 63.

[76]H. K. Weiss, "Methods for Computing the Effectiveness of Fragmentation Weapons Against Targets on the Ground," BRL R800, Jan 52. (AD#397181)

from which we can obtain the dynamic launch speed

$$v = \left[v_o^2 + v_r^2 + 2\, v_r\, v_o \cos \theta_o \right]^{1/2} \tag{99}$$

and launch angle

$$\theta = \arc \tan \left[\left(v_o \sin \theta_o \right) / \left(v_o \cos \theta_o + v_r \right) \right] \tag{100}$$

where θ and θ_o are measured from the shell axis which is taken to be parallel to $\vec{v}_r$.

Since effectiveness depends on both fragment number and speed, Weiss reasoned that for a given total weight, $W = M + C$, there ought to be an optimum value of C/M for a given target damage criterion since larger C favors larger v, while larger M favors larger N. With Equation (86) as damage criterion determining m_m, the number greater than this minimum mass, N_m, can be written from Equation (85). If we divide this number by the case mass, we obtain the number of potentially lethal fragments per unit case mass, namely,

$$N_m/M = (N_o/M)\ e^{-\left(\dfrac{m_m}{m_o}\right)^{1/2}} = \frac{1}{2}\, m_o\ e^{-\left(\dfrac{m_m}{m_o}\right)^{1/2}} \tag{101}$$

since the average mass can be calculated as

$$\frac{M}{N_o} = \bar{m} = \int_0^\infty m\ e^{-\left(\dfrac{m}{m_o}\right)^{1/2}}\ d\left[\left(\dfrac{m}{m_o}\right)^{1/2} \right] = m_o \int_0^\infty x^2 e^{-x}\, dx = m_o (2!). \tag{102}$$

Equation (101) is a function of m_o which has a maximum for

$$\bar{m} = (2\, m_o) = m_m/2 \tag{103}$$

so choosing a design which results in natural fragmentation characterized by $m_o = m_m/4$ in Equation (85) will maximize the number of lethal fragments obtainable per unit case mass. Choosing an average mass equal to half the minimum mass as in Equation (103) for natural fragmentation

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is in sharp contrast to Mott's choice for the mass of a controlled or preformed fragment equal to 4.48 m_m as expressed in Equation (48) above. If we rely on natural fragmentation, the best we can do is make the average mass ineffectively small according to Equation (103). If we put Equation (103) in Equation (101) we find

$$(N_m/M)_{Max} = (2/m_m)e^{-2} = .27/m_m = .27 \frac{v_o^{\gamma}}{L_1} \qquad (104)$$

In the case of zero air drag we should choose a controlled fragment size of m_m, so that $(N_m/N) = 1/m_m$, illustrating that natural fragmentation can only be 27% as efficient by Equation (104). Of course, if we took air drag into account as Mott did, we would choose a fragment size larger still.

For zero air drag, Weiss went on to assume that fragment size optimization had been carried out by adopting Equation (103), and discussed the maximum number of potentially lethal fragments as a function of the ratio M/W. In other words, he first determined an optimum fragment size and then an optimum launch speed, v_o.

He realized that the problem should strictly speaking not be treated in a stepwise fashion since all parameters should be optimized simultaneously. However, he hoped to gain some insight by this simplified procedure. From Gurney's formula for a cylinder we have $\left(v_o^2/4E\right) = 1/\left(1+2 \text{ M/C}\right)$ and

$$\frac{M}{W} = \frac{M}{C+M} = \frac{M/C}{1+M/C} = \frac{1-(v_o^2/4E)}{1+(v_o^2/4E)} = \frac{1}{\frac{C}{M} + 1} \qquad (105)$$

as in Equation (81) above. If we multiply Equations (104) and (105)

$$(N_m/M)_{Max} (M/W) = \left(.27 \ v_o^{\gamma}/L_1\right)\left[1 - \left(v_o^2/4E\right)\right]/\left[1+\left(v_o^2/4E\right)\right]. \qquad (106)$$

For the dynamic problem we should use Equation (99) above for the launch speed instead of the static launch speed v_o. If we neglect v_r and maximize Equation (106) with respect to v_o we find this occurs for

$$\left(v_o^2/4E\right) = \sqrt{1 + (2/\dot{\gamma})^2} - (2/\dot{\gamma}) \qquad (107)$$

which is 0.414 for $\dot{\gamma} = 2$ giving $\frac{C}{M} = 1.414$ in Equation (105). Similarly, $\dot{\gamma} = 3/2$ gives $\frac{C}{M} = 1.000$ and $\dot{\gamma} = 1$ gives $\frac{C}{M} = .618$. Strictly speaking, this does not give us $(N_m/W)_{Max}$ because of the stepwise procedure. Corrections for air drag would tend to raise the optimum value of average mass and lower the optimum value of launch velocity.

Weiss also discussed the desirability of using a function other than the simple step function implied by Equation (86) to express the conditional kill probability of a target which has been hit, $P_{K/H}$. Equation (86) implies that for $L = m\,v^{\dot{\gamma}} < L_1$, $P_{K/H} = 0$ and for $L > L_1$, $P_{K/H} = 1$. It would seem more reasonable for $P_{K/H}$ to increase from zero to unity over a range of values for L. Weiss considered a linear relation

$$P_{K/H} = (L-L_1) \,/\, (L_2-L_1) \qquad (108)$$

for $L_1 < L < L_2$, with $P_{K/H} = 0$ for $L \leq L_1$ and $P_{K/H} = 1$ for $L \geq L_2$. If we use $N_m = N_o e^{-(m_m/m_o)^{1/2}}$ and $m_m = L_1 v_o^{-\gamma}$ in Equation (90), we obtain

$$N = N_o \exp\left\{ - m_o^{-1/2} \left[L_1^{1/2} v_o^{-\gamma/2} + \left(.45\,\alpha\dot{\gamma}\, L_1^{1/6} v_o^{-\gamma/6} \right) R \right] \right\} \qquad (109)$$

for the number of potentially effective fragments as a function of R. Weiss used L instead of L_1 in Equation (109) and argued qualitatively that the appearance of L to fractional powers in Equation (109), especially the 1/6 power in the coefficient of R, would not change the R dependence of N or N_T very much even if a relation like Equation (108) were used. Consequently, he convinced himself that it would not be worthwhile to consider this additional complication in spite of the fact that it seems reasonable to do so. However, he gave no quantitative arguments to justify this position.

It seems that Weiss found himself in this difficulty because of his method of arriving at Equation (90) and so Equation (91). If he had adopted the viewpoint of OSRD 738 as expressed in Equation (28) above in which decrease in effectiveness due to air drag is incorporated into A_v rather than N, then he would not have found himself in this position. We can rewrite Equation (28) as

$$N_L = \frac{NA_v(0)}{2\pi R^2} \; K(\psi) \; P_{K/H}(R) \tag{110}$$

where the target kill probability, given a hit, is $P_{K/H}(R) = A_v(R) \; /$ $A_v(0)$ with $A_v(0) \leq A_p$. Now Equation (108) or some other functional form dependent on $m \, v^\gamma(R)$ can be used in Equation (110) to estimate N_L and the damage function $D = 1 - e^{-N_L}$. We shall see an example of another functional form for $P_{K/H}(R)$ in Equation (113) below.

In a later part of his report, Weiss suggested another approximation for the damage function, namely,

$$D = 1 - e^{-N_L} \approx N_L = K \, e^{-cR}/R^2 \tag{111}$$

and $D = 1$ for $R \leq R_1$. Here R_1 is Kent's radius of overhitting in a three-dimensional context and with air drag and target vulnerability contained in the argument of the exponential in the coefficient c. This approximation not only unifies the idea of a radius of overhitting with the poisson expression for the probability of at least one lethal hit, but also simplifies many integrals involving the product of D with other factors like aiming error functions or ground target cover functions. Weiss evaluated a number of such integrals for special cases in closed form, but required numerical integration for general cases. He used this approximation to estimate ground lethal areas in later reports[77-78]. Weiss advocated the use of integrable formulations whenever possible since these more easily contribute to our general understanding of a problem. In particular, he pointed out that aim error could be neglected when calculating the effectiveness of an area weapon since it was smaller than the extent of the target. However, he was careful to include it when its standard deviation, σ, was comparable to or larger than either the extent of the target or the extent of the weapon's lethal mechanism. His approach was in sharp contrast to some of the purely empirical formulations in use at the time[79]. The Weiss

[77]H. K. Weiss, "Description of a Lethal Area Computation Problem," BRL MR723, Sep 53. (AD#21133)

[78]H. K. Weiss, "Methods for Computing the Effectiveness of Area Weapons," BRL R879, Sep 53. (AD#24478)

[79]L. N. Enequist, "Rapid Estimation of Lethal Areas for Mortar and Similar Thin Walled H. E. Projectiles," BRL MR635, Dec 52. (AD#3766)

approximation in Equations (110) and (111) can be compared with the Kendall-Carlton approximation in Equation (35) and the Taylor-Kravitz approximation in Equation (93).

For many years various authors had mentioned the desirability of improving the threshold lethality criterion for disabling personnel, usually taken to be $\frac{1}{2} m v^2 = 58$ ft-pounds. In 1956 Allen and Sperrazza[80] reviewed previous criteria and proposed a new criterion based on extensive firing tests. They adopted a purely empirical formula

$$P_{K/H} = 1 - \exp\left[-a\left(mv^{\dot{\gamma}} - b\right)^n\right] \tag{113}$$

and found that $\dot{\gamma} = 3/2$ seemed to fit the data best. The other three positive parameters, a, b and n, assumed various values (with n not far from 0.5) when this formula was used to describe probability of incapacitation in various military stress situations. Incapacitation within five minutes for a defender was assigned a different probability than for an attacker, given a random hit. If this formula is used instead of Equation (108) or the simpler threshold criterion, then it is clear that simply integrable expressions for lethal area become much more difficult. Since Equation (113) is a purely empirical formula, it may be desirable to find another formula to fit the data which allows integrations to be carried out in closed form.

In the next decade lethal area methodology underwent various refinements, but no basic changes. In 1963 Myers[81] codified and updated the basic methods discussed by Weiss to include more complicated cover functions[82] and the effects of blast. He also used Equation (113) for his casualty criterion, treating each mass group separately and summing over all groups to obtain the net $P_{K/H}$ for the distribution centered at a particular polar angle θ (or latitude ψ). In addition, he treated fragment angular dependence by using average values of $K(\psi)$ at five degree intervals (instead of 2.5° intervals as in OSRD738). The effects of air drag were estimated through $v(R)$ in Equation (113) for which he used the usual formula. Because of these complications he always had to use numerical integration schemes. This 1963 report of Myers has become the standard for the industry[83] and is still used

[80]F. Allen and J. Sperrazza, "New Casualty Criteria for Wounding by Fragments," BRL R996, Oct 56. (AD#137681)

[81]K. A. Myers, "Lethal Area Description," BRL TN1510, Jul 63. (AD#612041)

[82]B. W. Harris and K. A. Myers, "Cover Functions for Prone and Standing Men Targets on Various Types of Terrain," BRL MR1203, Mar 59. (AD#309048)

[83]Reference 2 in Joint Munitions Effectivenss Manual, "Lethal Areas of Selected U. S. Army, U. S. Navy, and U. S. Marine Corps Surface to Surface Weapons Against Personnel and Materiel Targets," Jul 73.

today with only minor modifications. It has the virtue of including in some way or other all of the factors which are felt to be important. However, like most complicated codifications it has the disadvantage of obscuring the experimental and theoretical approximations which are buried in the code and lull the unwary user into believing that the answers are very accurate or that minor differences in the answers are significant. Since complexity impedes understanding, it is not helpful as a design tool either, and in fact is generally used merely to evaluate designs already arrived at by other means. Myers himself seemed to realize the approximate nature of his description and discussed in particular the use of the poisson approximation for the damage function instead of the Binomial which some people felt to be more correct. As he pointed out, "the Binomial itself must be considered only as an approximation since fragments projected from the same shell cannot really be considered as 'independent events'. He added that:

"To the author's knowledge, no extensive comparisons of lethal areas computed using the Binomial and poisson forms for the probability of incapacitation have been made. However, in some isolated cases where such comparisons have been made, the difference did not appear to be significant, particularly in light of other assumptions inherent in the lethal areas concept".[81]

B. The Adoption of Monte Carlo Methods for Air Defense Problems.

In order to treat the complexities of the air defense problem, a different (Monte Carlo) technique has been adopted instead of the analytical techniques which were used until the early 1950's. In 1950, F. G. King[84] described his method. Although he mentions a previous partial formulation of the problem by Cunningham in England, his report is clearly the ancestor of the end-game analyses which are still in use today. Although today's analyses are very complicated and detailed, King's original reason for using this method was simplicity:

"The lotto method proposed in the present report seems very simple when compared with the formulation of the problem in mathematical symbolism. The basic vulnerability data is known for the most part only to order of magnitude and at best only to ten or twenty per cent, so that a simple method is in order. If the basic data were known within one per cent, the lotto method would still be justified as a short cut to an approximate answer but would be inefficient for getting the answer to one per cent accuracy."

King's lotto or monte carlo method is a mathematical experiment in which the events of an anti aircraft attack are acted out. First, for a given engagement range a burst is positioned with respect to the target by drawing a card at random from a box which represents the distribution of aiming errors at the range in question. King used

[84]F. G. King, "Lotto Method of Computing Kill Probability of Large Warheads," BRL MR530, Dec 50. (AD#802148)

one thousand cards as a discrete representation of a trivariate normal distribution about the aim point. Second, the distance and angle from the burst point to each vulnerable component was measured, using a model airplane on a target stand and a model burst on a warhead stand to a scale of 1:72. A component was taken to be shielded if at least its center of mass was shielded. Third, using approximate treatments of missile and target velocity, air drag at a given altitude, warhead fragmentation characteristics and component vulnerability data, the kill probability of each vital component was estimated as a function of miss distance (and angle if angular data was available). Unfortunately King gives no detailed description of how this crucial step was actually carried out, but it is safe to assume that it involved the techniques we have discussed previously. Fourth, the kill probability of each component computed in step three for a particular miss distance was turned into a zero or a one by reading down a list of random numbers between zero and one. If the next random number read was less than the computed kill probability the component was considered killed, otherwise not. Sometimes this step was repeated several times and the results averaged before going on. Fifth, for each burst it was decided whether or not the aircraft was killed, subject to the requirement that a certain number of engines or pilots must be killed or that fuel be hit or that the burst occur within the blast kill envelope. Sixth, steps one through five were repeated several hundred times. Seventh, the single shot kill probability at the given engagement range was found by averaging the single burst kill probabilities obtained in each step six. A two-shot kill probability could be determined by taking an average over a number of pairs of bursts, considering cumulative damage, and so on. Eighth, since engagements between an aircraft and a rapid fire gun or battery of guns consist of single or multiple shots at successively different ranges, a new box of cards was introduced in step one and steps one through seven were repeated for a number of representative ranges. This eighth step can also be repeated an appropriate number of-times to improve the accuracy of the prediction. However, there is no point to repeating it too many times since overall accuracy is already limited by the accuracy of the experimental fragmentation and vulnerability data. An application of part of this method to a particular system was later made by Sacks and King[85].

In 1953 Juncosa and Young[86] computerized King's method after replacing parts of it with mathematical idealizations. In particular, they adopted a suggestion which they attributed to J. Von Neumann that the airplane and its blast kill envelope be replaced by ellipsoids resembling the

[85] *S. Sacks and F. G. King, "Single-shot Probability of an A-Kill on the B-29 Type Bomber by Nike I," BRL MR 553, Jul 51.* (AD#377282)

[86] *M. L. Juncosa and D. M. Young, "A Mathematical Formulation for Ordvac Computation of the Probability of Kill of an Airplane by a Missile," BRL R867, May 53.* (AD#17267)

the fuselage, wings, empennage and engines. In addition, they replaced all vulnerable components by points to simplify shielding by the aircraft parts represented by ellipsoids. Third, they restricted their considerations to a particular angle of attack, and finally, they evaluated three dimensional integrals over the product of aim and damage functions by a monte carlo method or by numerical quadratures instead of averaging over the results of several hundred randomly chosen burst points.

In 1960, Stiegler[87] modified the method of Juncosa and Young. He allowed random (monte carlo) selection of straight line approaches by the missile toward the target for elevation angles other than zero degrees and azimuths other than 45 degrees. He also computed a missile homing point which is the center of the target "glitter points" (radar reflecting points) rather than the center of the aircraft. He also computed the intersection of "glitter points" with the fuze cone and took direct hits into account. Stiegler's method was later modified by Monks[88-90] and various versions of this technique are still in use today.

As can be seen, the development of the monte carlo method since 1950 has followed the same trend as the development of lethal area methodology and similar remarks can be made. More and more detail has been included so that no feature be left out of the model. Unfortunately, the resulting complicated codes and the precision to which the final answer is expressed tend to make the unwary user place more credence in the answers than is justified. By and large the methodology cannot be used as a design tool, and is only used to evaluate weapon designs arrived at in other ways.

Although monte carlo methods have dominated the evaluation of air defense effectiveness, there have been some reports on the development of analytical methods in this area. For example, in 1966, Banash[91] proposed using the Kendall-Carlton approximation in Equation (40) above with $R_3^2 = 1.25 \text{ K}$ so

[87]A. D. Stiegler, "A Mathematical Formulation for ORDVAC Computation of the Single-Shot Kill Probabilities of a General Missile Versus a General Aircraft," BRL MR1306, Nov 60. (AD#249957)

[88]B. G. Monks, "A Mathematical Simulation of the SA-2 Missile for Evaluation of its Lethality," BRL-MR1690, Aug 65. (AD#368987L)

[89]B. G. Monks, "Aberdeen Blast-Fragmentation, Fixed Angle Fuze Missile End Game Simulation (AMEGS)," AMSAA TR66, Apr 73.

[90]B. G. Monks, "A Focused Blast-Fragmentation Warhead Evaluation Model," AMSAA TR70, Feb 73.

[91]R. C. Banash, "An Analytic Procedure for Evaluating Blast-Fragmentation Warheads in Air Defense," BRL R1317, Apr 66. (AD#485264)

$$D = 1 - e^{-K/R^2} \approx e^{-\dfrac{R^2}{2R_3^2}} = e^{-\dfrac{R^2}{2.5K}} \qquad (114)$$

where air drag is being neglected. Equation (114) is a fair approximation for miss distances in the range $0 < R < \sqrt{5K}$ and eventually improves again as $R \to \infty$ and $D \to 0$. He compared kill probabilities calculated by his analytical method with those obtained by the monte carlo method in four examples involving three hypothetical warheads versus missile and aircraft targets. In most cases the agreement was within 6%.

C. Current Vulnerability Methodology. Warhead Optimization.

Whatever the method used to estimate effectiveness, there is a continuing need for vulnerable area information or kill probability, given a hit, since targets and weapons are constantly changing. For aircraft the current state of the art has been summarized in a recent report[92]. In Section II above we quoted professor Mott[22] who remarked on the wide discrepancy between combat data and the effectiveness estimates of his day. He would be encouraged to hear that matters have improved somewhat. Combat damage data has been used to develop vulnerable areas for at least three helicopters.

"In all three cases, the average vulnerable areas obtained from the combat damage data and generated in vulnerability studies are consistent."[93]

While this agreement is not for overall effectiveness, still it is encouraging in that combat data can be used to provide guidance wherever possible. As these authors remark, "Data on aircraft damage which occurs in combat are very valuable since they originate in the 'real world' conditions of military conflict.... thus the body of aircraft combat damage data provided by the collections from Southeast Asia serves as the main source of information on aircraft reactions to the effects of component damage while airborne." They also remark on use of such data for vulnerability reduction. Still in many cases not only is combat damage data not available, but not even controlled experimental information on specific threat performance or target vulnerability is available.

[92]D. W. Mowrer, R. E. Walther, R. D. Mayerhofer and R. N. Schumacher, "Aircraft Vulnerability Assessment Methodology. Volume I - General," BRL R1796, Jul 75. See also Volumes II to XII for particular subsystems. Volume I also appeared as JTCG/AS-76-V-004, Jul 77.(AD#B005878L)

[93]Reference 92, p 72.

"The availability and validity of these data greatly influence the reliability of the results of any study based on these data. It is generally difficult to quantify this influence in terms of confidence limits, but some general comments are germane. A completely sufficient body of experimental data is seldom, if ever available for use in a study.... In fact, in most studies these considerations require that the analyst make a number of assumptions to fill the information gaps. Thus, certain errors may be introduced into the study by the necessity of making these assumptions. Furthermore, all of the analysis procedures require certain simplifying assumptions and approximations, which introduce additional errors...."

"The sensitivity of the complex models used in vulnerability studies (particularly in computer programs) to the assumptions, approximations and various data inputs is neither apparent nor readily determined."[94]

Still, some few sensitivity studies have been carried out in related areas and according to these authors the final answers are highly sensitive to all of the parameters examined, namely, fragment residual weight, speed and shape, shotline location and grid size, and component detail and $P_{K/H}$ values. Other factors obviously introducing error whose magnitude has not yet been examined were also discussed. Some of these are neglect of all but the largest residual fragment pieces as well as target pieces in subsequent perforations, neglect of ricochets, the need to use single plate results repetitively to describe multiplate perforations, the need to estimate equivalent thicknesses of non-plate components and the necessity of considering only a small number of target aspects.

This report does not give details on how component $P_{K/H}$ values are estimated. Often such information is kept classified. However, some unclassified reports discuss the matter in a general way. For example, Johnson[95] points out that for a given fragment mass striking a particular component at a certain angle the kill probability is represented by a multi-step function of $P_{K/H}$ versus striking speed, v. There is a threshold speed below which $P_{K/H}$ = 0 and the various steps reflect the cumulative damage of sub components. However, $P_{K/H}$ does not approach unity no matter how high the speed in Figure 3. This seems to reflect the definition used, namely, $P_{K/H} = A_v/A_p$, the vulnerable portion of the area divided by the measured presented area,

[94]Reference 92, pp 80-81.

[95]W. P. Johnson, "An Application of the Weibull-Gnedenko Distribution Function for Generalizing Conditional Kill Probabilities of Single Fragment Impacts on Target Components," BRL R1789, May 75. (AD#B004591L)
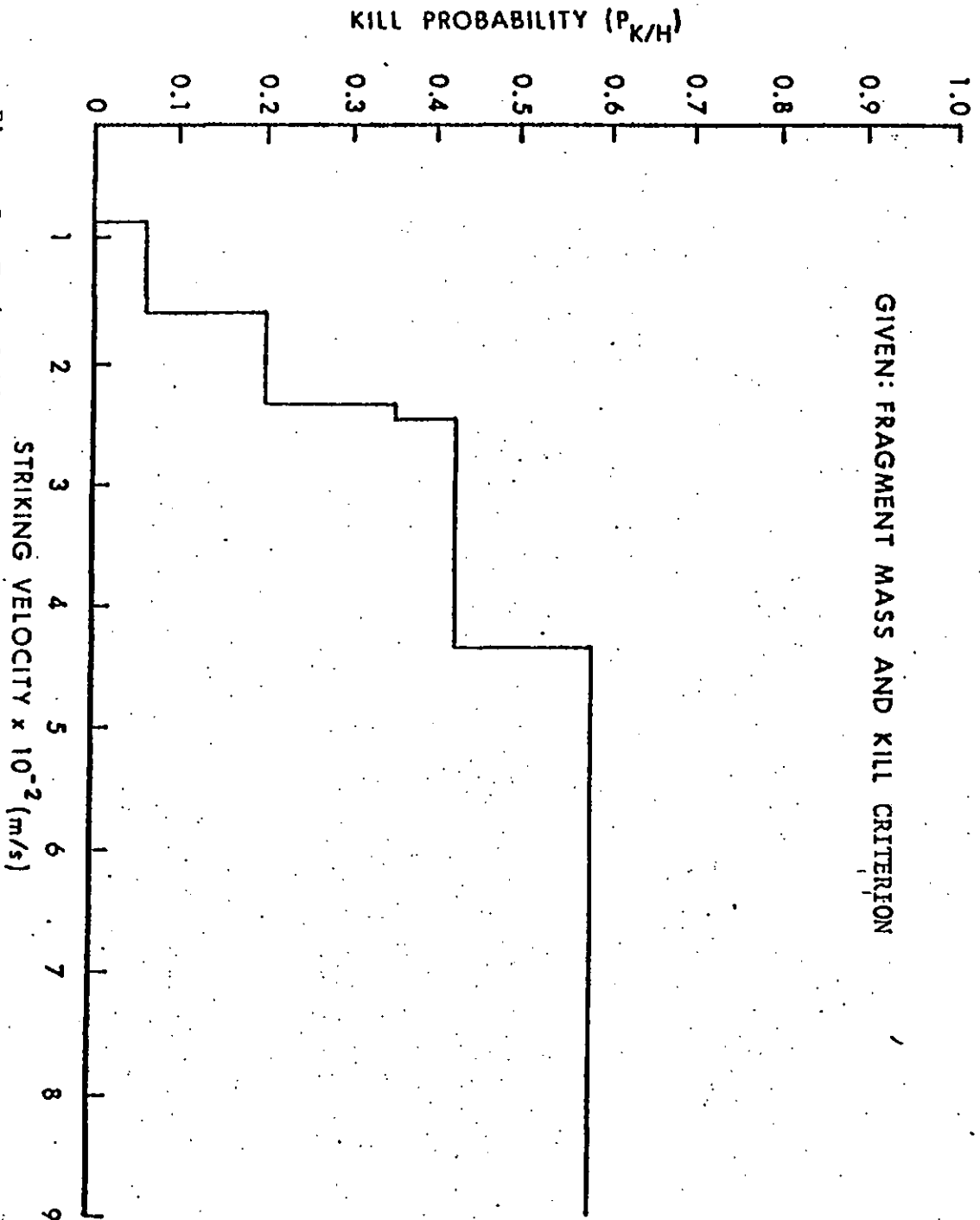
Figure 3. Typical Kill Probability versus Striking Speed Curve for a Given Fragment Mass and Kill Criterion (from BRL R1789)

with $A_v < A_p$ because parts of the component are non-critical dead space. This literal interpretation of A_v as an area whose maximum value is the sum of a number of critical areas less than A_p can lead to unnecessary semantic difficulties. Since $P_{K/H}$ is not allowed to approach unity in this interpretation, it can be objected that it should not be called a probability. This is a valid objection since one requirement for a probability distribution is that the sum of the probablities of all its subsets be equal to unity. The objection can be avoided by calling $P_{K/H}$ a damage estimator or something similar rather than a probability. A simple way of avoiding the problem is to deal only with A_v as in the first form of Equation (28) above. We can of course multiply A_v by any form of unity we desire without changing the result. We could multiply by A_p/A_p with $(A_v)_{Max} < A_p$, but the ratio A_v/A_p could not be called a probability. Instead we could meet the objection by multiplying by some A/A such that $A = (A_v)_{Max}$ to obtain a probability A_v/A. In Equation (28) A was chosen to be $A_v(0)$ so that $A_v(R)/A_v(0)$ in Equation (29) would be unity for $R = 0$. As we recall, the assumption was that only fragments capable of damaging the target at short range ($R = 0$) would be considered. Since $R = 0$ corresponds to $v = v_o$ by Equation (27), we can write for the target conditional kill probability

$$P_{K/H} = A_v(R)/A_v(0) = A_v(v)/A_v(v_o) \tag{115}$$

with subscripts added if components are being considered. Another alternative would be to let A be equal to the sum of the presented areas of the vulnerable subcomponents (in the case of a component) or the sum of the presented areas of all such subcomponents (for the whole target). In this way parts of the target which are not even potentially vulnerable to the weapon in question are excluded from consideration. Thus the difficulty introduced by using an A_p which includes both potentially vulnerable and invulnerable parts can be eliminated.

Johnson suggests using a function of the form

$$P_{K/H} = 1 - \exp \; [-a \; (mv - b)^n] \tag{117}$$

although he multiplies it by a factor $(P_{K/H})_{Max} < 1$ since he accepts Figure 3 as a probability. Here we have let $(P_{K/H})_{Max} = 1$ so $P_{K/H}$ can

be a probability. Equation (117) as it stands is the well-known Weibull-Gnedenko distribution and would be the same as Equation (113) above if the variable were mv^γ instead of mv. There are analytical advantages to using a form like Equations (117) or (113) instead of a multitude of graphs like Figure 3. In addition, it can help to unify our description of both personnel and materiel vulnerability.

Another functional form has been proposed by Capillo[96], namely,

$$A_V/A = \left(1 + B_p^{cv}\right)^{-1} \tag{118}$$

where A, B, p and c are constants. In this formulation a separate set of constants is required for each mass (and angle) so it is not as succinct as Equation (117). In addition, it does not resemble Equation (113). Later in his report Capillo gives a rather succinct verbal summary of the state of the art:

"Vulnerability assessment, as it now stands, is an empirically based quasi-science drawing knowledge and interpretation from historical data, experimental tests, limited analytics, and engineering experience and judgment."

In some of the reports we have considered so far we have seen that one or another parameter like burst height or fragment mass has been selected and varied in order to optimize effectiveness. This tradition has continued, although, when numerical methods are used, the range of the variable selected is usually rather narrow and only a few curves over this range are compared. Occasionally a more general treatment using analytical methods has appeared. In 1950 Dougherty[97] discussed optimizing the fragment angular distribution, $K(\psi)$, and concluded that a cone pattern was desirable for a proximity fuze. He also gave a derivation of the damage function which started with certain assumptions and employed summation techniques instead of limiting processes as in Equations (12) to (18) above. He let

$$N_L = \overline{N}p \tag{119}$$

where $p = A_V(R)/A_V(0) = P_{K/H}$ and $\overline{N} = NK(\psi) A_V(0)/(2\pi R^2)$ is the expected number of hits so that N_L is the expected number of lethal

[96] G. C. Capillo, "Fighter Aircraft Vulnerable Area Data Patterns: Modeling, Data Reduction, and Interpolation Schemes," BRL MR2648, Aug 76. (AD#B013348L)
[97] C. B. Dougherty, "Optimization of Warhead and Fusing Parameters," Rand Corporation, RM349, Mar 50.

hits. He then assumed that the chance of a hit follows a poisson distribution with expected value $\overline{N}$, while the chance of a kill obeys a binomial distribution

$$G(y; n) = \binom{n}{y} p^y (1-p)^{n-y} \tag{120}$$

which is the chance of exactly y fragments killing the target out of n hits (or trials). The chance of at least one fragment killing the target is then

$$\sum_{y=1}^{n} G(y;n) = \sum_{y=0}^{n} G(y;n) - G(0;n) = 1 - (1-p)^n \tag{121}$$

since $\sum_{y=0}^{n} G(y;n) = 1$. If the hit probability is assumed to follow a poisson distribution, then the joint probability of at least one fragment hitting the target and at least one of these killing it is

$$D = \sum_{n=1}^{\infty} \left\{ \left(e^{-\overline{N}} \frac{\overline{N}^n}{n!} \right) \left[1 - (1-p)^n \right] \right\}$$

$$= \sum_{n=1}^{\infty} \left(e^{-\overline{N}} \frac{\overline{N}^n}{n!} \right) - e^{-\overline{N}} \sum_{n=1}^{\infty} \left[\frac{[\overline{N}(1-p)]^n}{n!} \right]$$

$$= \left[\sum_{n=0}^{\infty} \left(e^{-\overline{N}} \frac{\overline{N}^n}{n!} \right) - e^{-\overline{N}} \right] - e^{-\overline{N}} \left[\sum_{n=0}^{\infty} \frac{[\overline{N}(1-p)]^n}{n!} - 1 \right]$$

$$= \left[1 - e^{-\overline{N}} \right] - e^{-\overline{N}} \left[e^{\overline{N}(1-p)} - 1 \right]$$

$$= 1 - e^{-\overline{N}p} = 1 - e^{-N_L} \tag{122}$$

which is the usual damage function, but arrived at in a way which brings out better its nature as a joint probability.

It may be objected that the hit probability also would be described better by a binomial distribution especially if the number of fragments, N, is not very large. In this case we can modify

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Dougherty's derivation by letting the probability that one fragment hit the target be

$$P_H = K(\psi) \, A_v(0) \, / \, (2\pi R^2) \tag{123}$$

such that $\overline{N} = N P_H$ as before. Now the joint probability is

$$D = \sum_{n=1}^{N} \binom{N}{n} P_H^n \left(1 - P_H\right)^{N-n} \left[1 - \left(1-P\right)^n\right]$$

$$= 1 - \left(1-P_H\right)^N - \sum_{n=1}^{N} \binom{N}{n} P_H^n \left(1-P_H\right)^{N-n} \left(1-P\right)^n$$

$$= 1 - \left(1-P_H\right)^N - \left(1-P_H\right)^N \left[\sum_{n=0}^{N} \binom{N}{n} \left\{\frac{P_H(1-P)}{(1-P_H)}\right\}^n - 1 \right]$$

$$= 1 - \left(1-P_H\right)^N \left\{1 + \frac{P_H(1-P)}{(1-P_H)}\right\}^N$$

$$= 1 - \left(1-P P_H\right)^N = 1 - \left(1-P_{K/H} \, P_H\right)^N \tag{124}$$

where we have used the identity $(1+X)^N = \sum_{n=0}^{N} \binom{N}{n} X^n$ in the next to last line, then rearranged the factor in brackets amd cancelled $(1-P_H)^N$. An approximation to Equation (124) is

$$D = 1 - \left(1-P P_H\right)^N = 1 - e^{N \ln (1-P P_H)}$$

$$\approx 1 - \exp\left[-N P_H P\right] = 1 - \exp\left[-\overline{N} P\right] = 1 - e^{-N L} \tag{125}$$

which is the usual damage function. The second form in Equation (125) is merely an identity while the third form is an approximation for $P_H P \ll 1$. As we see, the result is equivalent to using the poisson

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approximation to the binomial for the hit probability in Equation (122). (Recall Equation (15) above also). Now however, we see that

$$P_H P = P_H P_{K/H} = \frac{K(\psi) A_v(o)}{2\pi R^2} \frac{A_v(R)}{A_v(o)} \ll 1 \tag{126}$$

can be satisfied by having P_H small (even if $P_{K/H}$ is close to unity) or by having $P_{K/H}$ small (even if P_H is close to unity) or by having both small. Thus even burst close to a tough target can be reasonably represented by the approximation in Equation (125). A burst close to a weaker target might require Equation (124). We must remember, of course, that any of these representations are model idealizations and presume event independence. They also presume a model geometry which treats the burst as a point as in Equation(2) above. Thus even if $A_v(R) = A_p = 4\pi \sin^2 (\theta/2) R^2$, we see that its largest value is $A_v(R) = 2\pi R^2$, since $\theta_{Max} = \pi/2$. That is, the burst is exterior to the target. More often $A_v(R) < A_p$.

In 1964 Sewell[98] presented an interesting discussion of optimizing the charge-to-metal ratio, $x = C/M$, subject to weight or volume constraints. He did not attempt to maximize kill probabilities, lethal areas or other measures of effectiveness. Instead, he looked only at the warhead, not at the target or engagement conditions and maximized the launch function, $E = M v_o^\gamma$, for $\gamma = 1$, 3/2 and 2. In other words he postulated that a first cut in warhead design might be made by considering the maximum total momentum, energy (or something in between) which could be imparted to the warhead case. Since $M = M(x)$, then

$$\frac{dE}{dx} = \frac{\partial E}{\partial M} \frac{dM}{dx} + \frac{\partial E}{\partial v_o} \frac{dv_o}{dx} = 0 \tag{127}$$

will give us a condition for a maximum of E with respect to $x = C/M$. For example, for a fixed weight $W = C + M = M(1+x)$, so $M = W/(1+x)$ and

$$\frac{dM}{dx} = -W / (1+x)^2 \tag{128}$$

We may rewrite Gurney's cylinder formula, the analog of Equation (59) above, as $v_o = (2E)^{1/2} (1+.5x)^{-1/2} x^{1/2}$, so

[98]R. G. S. Sewell, "Fixed-Weight and Fixed-Volume Constraints on Optimum Charge-to-Metal Ratios in Warhead Design," NOTS TP 3430, Mar 64.

$$\frac{dv_o}{dx} = .5 \ (2E)^{1/2} \ x^{-1/2} \ (1+.5x)^{-3/2} \tag{129}$$

and for $E = M v_o^\gamma$ we have $\frac{\partial E}{\partial M} = v_o^\gamma$, while $\frac{\partial E}{\partial v_o} = \gamma M v_o^{\gamma-1}$. If we put these relations in Equation (127) we find

$$x = C/M = \sqrt{(\gamma/2)^2+1} \ + \ (\gamma/2 - 1) \tag{130}$$

as the optimum value. Thus, if $\gamma = 2$, $x = 1.414$ and if $\gamma = 3/2$, $x = 1.000$ while if $\gamma = 1$, $x = .618$ exactly as in Equations (105) to (107) above. It is not hard to show that putting Equation (107) into Equation (105) and solving for C/M yields Equation (130), so the equivalence holds for every γ. This is not surprising since optimization of E with respect to M/W (or C/M) is theoretically independent of fragment size in this view.

Sewell repeated the same procedure for a sphere of fixed weight, finding somewhat smaller values of C/M to be optimum. In addition, he discussed both geometries subject to a fixed volume constraint, namely, constant $V = C/\rho_E + M/\rho_m$, for which the optimum C/M values will depend on the ratio of explosive and metal densities, ρ_E/ρ_m. In 1976 Zulkoski[99] modified Sewell's treatment by postulating various factors to represent the experimental fact that cylinders of finite length do not obey the Gurney formula except at one cross section, say at the middle. Fragments launched from sections near the ends will have different (lower) speeds because of differences in explosive gas product confinement. His approach otherwise was that of Sewell.

Perhaps the most comprehensive attempt to optimize warhead characteristics, at least in the case of area targets on the ground, is a 1967 report by Scherich and Kitchen[100]. The parameters they considered were warhead height of burst, angle of fall and remaining velocity as well as upper and lower limits of the side spray and up to ten fragment classes. They used a numerical integration scheme to evaluate the lethal area integral and the well-known method of steepest ascent to seek out at least a local maximum. To avoid missing larger values of the lethal area they also made provision for searches over wider ranges of the parameters. However, because of the complexity of the method

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[99]T. Zulkowski, "Development of Optimum Theoretical Warhead Design Criteria," NWC TP5892, Aug 76.

[100]E. L. Scherich and T. Kitchen, "Weapon Optimization Techniques," Martin Marietta, AFATL-TR-67-128, Oct 67.

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used, it appears to be difficult to gain insight into the optimization process.

IV. CONCLUSION

In this report we have chosen to view our subject from an historical perspective because of a conviction that much can be learned in this way. Hopefully, the repetition or continuation of mistakes and dead ends can be avoided, while hints or suggestions for new directions can be gathered for the future. In addition, the meaning of what we are now doing might be clarified by seeing how it arose from the unification of a number of different viewpoints. Unfortunately, we have only seen the early British and the American Army (BRL) views of the subject. More could be learned from a broader perspective, but this report is already long enough. A certain amount of new material has also been woven into the text to assist in clarifying our ideas and to lay the groundwork for future advances. As was mentioned in the introduction, a later report in this series will consider methods of simplifying descriptions of effectiveness so that analytical tools for optimizing warhead designs will be more accessible. Still another report will be devoted to the origins and possible improvement of the methodology which has evolved to describe the effectiveness of anti-armor weapons. Since direct hits by single lethal mechanisms (massive penetrators) are generally required, the methodology should be simpler in some ways. However, since direct hits are required, we must always include aiming errors in our overall effectiveness analysis. Lethal areas will not do. In any case, there are enough similarities to include such a discussion in the present report series, but enough differences to warrant a separate report.

APPENDIX I. SOME EARLY WORKERS IN THE FIELD

Warren Weaver. Professor of Mathematics, Throop College, 1917-18; California Institute of Technology, 1919-20; Chairman of Mathematics Department, University of Wisconsin, 1920-32; Director of the Division of Natural Science, General Education Board, 1932-37; Rockefeller Foundation, 1932-55 (V. P. 1955-59); Sloan - Kettering Trustee, 1954-67; National Science Foundation, 1956-60; Courant Institute, 1962-72.

John Von Neumann. Research Professor of Mathematics, Institute for Advanced Study, Princeton, 1933 on.

David R. Inglis. Professor, Johns Hopkins University, 1938 on.

Ward F. Davidson. Professor of Electrical Engineering,University of Michigan, 1916-22; Director of Research, Brooklyn and Consolidated Edison Company, 1922-42.

Sir Nevill Francis Mott. Lecturer, University of Manchester, 1929-30; Fellow and Lecturer, Gonville and Caius College, Cambridge, 1930-33; Professor of Theoretical Physics and Laboratory Director, University of Bristol, 1933-54; Cavendish Professor of Physics, Cambridge University, 1954-71; Senior Research Fellow, Imperial College London, 1971-3; Nobel Laureate in Physics, 1977.

Egon S. Pearson. University College, London, 1921-60.

Marston Morse. Professor of Mathematics, Harvard University, 1926-35; Institute for Advanced Study, Princeton, 1935-62.

Martin Schwarzschild. Professor, Columbia University, 1940-47; Professor, Princeton University, 1947 on.

Robert G. Sachs. Instructor, Purdue University 1941-43; Ballistic Research Laboratory, 1943-45; Director of Theoretical Physics Division, Argonne National Laboratory, 1945-47; Professor of Physics, University of Wisconsin, 1947-64; Assoc. Director, Argonne National Laboratory, 1964-68; Professor of Physics, University of Chicago, 1964-68; Director, Enrico Fermi Institute, 1968-73; Director, Argonne National Laboratory, 1973 on.

Hans Lewy. Lecturer, Brown University, 1933-35; Professor, University of California at Berkeley, 1935 on.

Subrahmanyan Chandrasekhar. Fellow of Trinity College, Cambridge, 1933-37; University of Chicago, Professor, 1937-46; Distinguished Service Professor of Theoretical Astrophysics, 1947 on.

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Llewellyn H. Thomas. Professor of Physics, Ohio State, 1919-43; Ballistics Research Laboratory, 1943-45; Senior Staff, Watson Computing Laboratory and Professor of Physics, Columbia, 1946-68; Professor, N. Carolina State, Raleigh, 1968-76.

Garrett Birkhoff. Professor of Mathematics, Harvard, 1936 on. National Academy of Science, V. P. American Mathematical Society, 1958-9; President, Society of Industrial and Applied Mathematics, 1967-68.

Ronald Wilfred Gurney. Palmer Physical Laboratory, Princeton, 1928-29; University of Bristol, 1935-39; Ballistic Research Laboratory, 1940-47; Johns Hopkins University, 1948-49; University of Maryland, Institute for Fluid Dynamics and Applied Mathematics, 1950-52.

Edgar Bright Wilson, Jr. Professor, Harvard, 1936 on.

Lewis Rosenhead. Professor of Mathematics, Liverpool 1933-73.

David George Kendall. Lecturer, Oxford, 1946-62; Professor of Mathematical Statistics, Cambridge, 1962 on; President, London Mathematical Society, 1972-74.

Henry Schefe'. Instructor, University of Wisconsin, 1935-37; Oregon State, 1947-41, Princeton, 1941-44; Professor, Syracuse, 1944-46; Professor, University of California at Los Angeles, 1946-48; Columbia, 1948-55; University of California, Berkeley, 1953-58; Princeton, 1959-60.

APPENDIX II. FRAGMENT PRESENTATION AREA AND SHAPE FACTOR

In their paper[16] mentioned above, Pearson and Bishop discussed the results of gun firings of various sizes and shapes of simulated fragments (square plates ranging from .02 oz to .50 oz). They observed that the loss in speed with distance was approximately a linear function of speed, namely,

$$dv/dR = -\beta v \qquad \text{(A-1)}$$

so that

$$dv/dt = -\beta v^2 \qquad \text{(A-2)}$$

where $\beta = -C_D \rho_a \bar{A}/m$, if Equation (25) above holds. However, the experimental data indicated that β was proportional to $m^{-1/3}$ instead of m^{-1} as might be expected if $\bar{A}$ were independent of mass, m. For this to be true, $\bar{A}$ would have to be proportional to $m^{2/3}$ as in Equation (26) above. This is clearly true in the case of a sphere for which $m = \rho \, (\frac{\pi}{6}) \, d^3$ where ρ is its density and d is its diameter. The presented area of a sphere is, of course, $\bar{A} = \frac{\pi}{4}d^2 = \frac{\pi}{4} \, (\frac{6}{\pi\rho})^{2/3} \, m^{2/3}$ which is proportional to $m^{2/3}$. For a cube in face-on flight, $m = \rho d^3$ and $\bar{A} = d^2 = (\frac{1}{\rho})^{2/3} \, m^{2/3}$, while for edge-on flight, $\bar{A} = \sqrt{2} \, (\frac{1}{\rho})^{2/3} \, m^{2/3}$. In 1940 Taylor[101] remarked that rectangular blocks of dimensions a, b and c, when fired from guns, would probably rotate so that all aspects of presented area relative to the flight path would occur with about equal frequency. If such a rectangular parallelepiped tumbled randomly in flight, the average presentation area would be

$$\bar{A} = \frac{1}{2} \, (ab + bc + ca) = \frac{(ab + bc + ca)}{2\rho^{2/3} \, (abc)^{2/3}} \, m^{2/3} = k \, m^{2/3} \qquad \text{(A-3)}$$

when put in the form of Equation (26). Pearson and Bishop then postulated that arbitrary shapes such as occur for fragments of bombs or shells would also rotate randomly with $\bar{A} = k \, m^{2/3}$. By approximating real fragments from a 3.7 inch shell with rectangular parallelepipeds, they estimated an average value for the shape factor, k, in the case

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[101]*G. I. Taylor, A. C. 79 (1940).*

of steel fragments. This basic approach was repeated in a number of later British reports which employed a greater variety of fragment sizes, shapes and initial speeds.

In this country, Schwarzschild and Sachs[102] used high speed cinematography of bomb detonations to determine the launch speeds and retardation of fragments in air. However, fragment orientations in flight were not determined because of the great distance between the camera and the burst point. They mentioned that spark photography experiments on real fragments launched from a gun by Charters seemed to confirm the random rotation hypothesis. However, a report published soon after by Charters and co-workers[103] was more modest in its conclusions. They presented their data in terms of drag coefficients based on maximum and minimum area as well as average area calculated by approximating fragment shapes with ellipsoids, as suggested by Schwarzschild and Sachs. Although rotation was observed, its exact nature (random orientation or not) was difficult to observe. Schwarzschild and Sachs also proposed using the 1.8 power of the speed instead of the 2.0 power on the right side of Equation (A-2) above for subsonic speeds, in which case

$$v = \left(v_o^{1/5} - \beta R/5 \right)^5 \qquad \text{(A-4)}$$

The question of whether or not fragments from real warheads actually follow the random rotation hypothesis at least over distances of tens or hundreds of meters is difficult to answer satisfactorily in an experiment. If one is dealing with a large number of fragments of various sizes and shapes, we might also ask whether or not it is worth knowing a characteristic shape factor for each fragment, since we are interested in the average effects of the whole group of fragments. For this purpose an average shape factor characteristic of the group should serve just as well. This has the further advantage of providing an easy way to estimate the striking area of a fragment as it hits the target, a factor which is needed if we are to use penetration equations based on relations like Equation (56) above. The random orientation hypothesis also seems to have appealed to a number of people because of its simplicity, apart from the question of describing a large collection of objects statistically. At the suggestion of Morse and co-workers[104] various investigators began to measure average

[102]M. Schwarzschild and R. G. Sachs, "Properties of Bomb Fragments," BRL R347, 7 Apr 43.

[103]W. F. Braun, L. H. Thomas and A. C. Charters, "Retardation of Fragments," BRL R425, 15 Nov 43. (AD#PB22113)

[104]M. Morse, W. Transue and M. Heins, "The Theory of the Presentation Areas of Fragments and the Icosahedron Area Gage," T.D.B.S. 44, 20 Sep 44.

presentation areas, accepting random rotation. For example, Simpson and Bushkovitch[105] adopted Morse's suggestion and built an apparatus to measure twenty uniformly distributed orientations by projecting fragment shadows in directions corresponding to the twenty faces of an icosahedron. While this was done only for a limited number of faces and relatively small number of selected fragments, it was a systematic way to determine experimentally an average shape factor, k. These authors mentioned parallel efforts in England. This technique was refined in succeeding years[106-109] and the proportionality between $\bar{A}$ and $m^{2/3}$ has been well established experimentally for fragments recovered from a variety of weapons with steel cases. Presumably for other metals we need only employ a ratio of densities to the 2/3 power as indicated by Equation (A-3) to calculate a form factor for drag or penetration calculations, provided we have similar geometrical shapes.

It is interesting to note that Equation (A-3) states that the average presentation area of a rectangular block is equal to one-fourth of its surface area, namely, one-fourth of 2(ab + bc + ca). This is also clearly true for a sphere and other simple shapes. The result can be generalized to any irregularly shaped convex body by a simple argument. Choose cartesian and spherical coordinate systems with a common origin such that the radius vector, $\vec{R}$, is parallel to the normal vector, $\vec{dA_s}$, which represents an element of surface area. The magnitude of the projection of $\vec{dA_s}$ on the z axis is $dA_s \cos \theta$ which is also the size of the shadow this element of area makes on the x-y plane. If we vary the angle ϕ, holding θ constant, the size of this shadow stays the same. If we vary θ, holding ϕ constant, the shadow size varies from dA_s to zero as θ varies from 0 to $\pi/2$. For $\theta > \pi/2$, the element of area is in the half-space below the x-y plane, is obscured by the rest of the body, and its shadow size is taken to be zero. Consequently, the average shadow size or presentation area, giving equal weight to all aspects is

[105]M. H. Simpson and A. V. Bushkovitch, "Fragment Contour Projector and the Presentation Areas of Bomb and Shell Fragments," BRL R501, 8 Nov 44. (AD#715954)

[106]A. V. Bushkovitch, "Presentation Areas of Shell Fragments," BRL R536, Apr 45.

[107]J. E. Shaw, "A Measurement of the Drag Coefficient of High Velocity Fragments," BRL R744, Oct 50. (AD#801550)

[108]W. R. Porter, J. L. Machamer and W. O. Ewing, "Electro-Optic Icosahedron Gage," BRL R877, Sep 53. (AD#21197)

[109]D. J. Dunn, Jr., and W. R. Porter, "Air Drag Measurements of Fragments," BRL MR915, Aug 55. (AD#77240)

$$d\bar{A} = \frac{\int_0^{2\pi}\int_0^{\pi} dA_s \cos\theta \sin\theta d\theta d\phi}{\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi} = \frac{2\pi dA_s\int_0^{\pi/2} \cos\theta \sin\theta d\theta}{4\pi} = \frac{dA_s}{4} \quad (A-5)$$

since the integral over θ from $\pi/2$ to π is zero because dA_s is obscured by the rest of the body as already mentioned. This result may be repeated for each element of surface area, although generally speaking, a translation of the origin will be required to make the radius vector parallel to the element of surface area in question. Since the total surface area is simply the sum of all its elements, we have the desired result,

$$\bar{A} = 1/4 \, A_s \qquad (A-6).$$

This result is sometimes attributed to Cauchy although no reference has been found. We can use this result to derive Equation (A-3) by noting that

$$A_s = KV^{2/3} = K(\frac{m}{\rho})^{2/3} \qquad (A-7)$$

where K is a dimensionless geometric ratio and V is the volume of a body of mass, m, and density, ρ. If we put Equation (A-7) into Equation (A-6), we obtain Equation A-3) in the form

$$\bar{A} = (\frac{K}{4\rho^{2/3}}) \, m^{2/3} = k \, m^{2/3}. \qquad (A-8)$$

Here the shape factor, k, depends only on two quantities. One is the K-number defined by Equation (A-7) as the ratio of suitable powers of two extensive, geometric properties of the body, its surface area and its volume. The other is the negative two-thirds power of an intensive, physical property of the body, its density. In particular, the shape factor, k, is independent of mass. However, this does not eliminate the possibility that certain k values may occur more frequently for certain mass groups because of metal processing or some other factor.

While the meaning of K in Equation (A-7) is intuitively evident and its numerical value is easily found for simple geometrical shapes, it is worth noting that a K-number can be defined for any convex body no matter how complicated its shape. It may be very difficult to calculate this number for an oddly-shaped body, but we only need to define it in order to derive Equation (A-8). The use of an

icosahedron gage or some other technique will enable us to estimate its value to the accuracy we desire. These assertions can be clarified by the following whittling experiment. If we start with a rectangular parallelepiped with edges A, B and C we may whittle out any convex shape we want by a series of slices each of which produces a small pyramid which is discarded, leaving behind new vertices where three edges meet in addition to the old ones. A curved surface may be produced as accurately as we please by making a large enough number of small enough cuts (an infinite number in the geometrical limit). Suppose, for example, we start by cutting off a corner with slant heights a, b and c corresponding to the edges A, B and C. If we take edge a as the pyramid height and the triangle with b and c as sides for the base, we find the volume $\frac{1}{3}$ ($\frac{1}{2}$ ab) c $= \frac{1}{6}$ abc. The surface area of the pyramid is the sum of the triangular areas of its four faces, namely, $\frac{1}{2}$ ab, $\frac{1}{2}$ bc, $\frac{1}{2}$ ca and $\sqrt{s(s-a)(s-b)s-c)}$ where s $= \frac{1}{2}$ (a+b+c). This sum divided by $(\frac{1}{6}$ abc$)^{2/3}$ is the K-number of the pyramid. The surface area of the remaining mass is the original surface area minus that part of the pyramid surface area which was exposed before the cut plus the newly exposed surface area. The volume of the remaining mass is simply the original volume minus the pyramid volume. Thus, the K-number of the remaining mass is

$$K = \frac{A_s}{V^{2/3}} = \frac{2[AB+BC+CA] - \frac{1}{2}(ab+bc+ca) + \sqrt{s(s-a)(s-b)(s-c)}}{[ABC - \frac{1}{6} abc]^{2/3}} \quad ; \quad (A-9)$$

If we slice off a new pyramid which contains one of the newly produced vertices, the calculation will be complicated by the fact that, generally speaking, no right angles will be present. Still, the calculation of a K-number for the remaining mass is analytically possible. The same is true if we had started with a polyhedron other than a rectangular parallelepiped. As the number of cuts increases the analytical formulas for the K-number of the remaining mass become increasingly cumbersome and only become simple again as some simple geometric shape is reached. We will not pursue this matter here since our only purpose is to illustrate the fact that a K-number can be defined for arbitrary convex shapes like those which most often occur in natural fragmentation.

Finally, it is interesting to note the dependence of v on the fragment mass, m, in Equation (27) above, namely, in

$$v = v_o e^{-\alpha Rm^{-1/3}} \quad (A-10)$$

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where $\alpha = C_D \rho a.k$ varies from near zero for very high altitudes $(\rho a \approx 0)$ to about $.6(.001 \text{ gm/cm}^3)$ $(.38 \text{ cm}^2/\text{gm}^{2/3}) \approx .2 \times 10^{-3} \text{ gm}^{1/3}/\text{cm} = .02 \text{ gm}^{1/3}/\text{meter}$ near sea level. Here we have used $k = .38 \text{ cm}^2/\text{gm}^{2/3}$ for a cubical steel fragment, using $a=b=c$ in Equation (A-3). If the c dimension were 0.1 or 10 times $a=b$, then k would only be about twice as large. For natural fragmentation c is usually less than five times $a=b$, giving a k variation of only about 25% from the cubical value for a mass variation of two orders of magnitude. Because of this it is common to choose a single representative value of k to describe a large collection of fragments of different sizes and shapes. Let us do this so that α is the same for all fragments in Equation (A-10), neglecting the even smaller variation of α with altitude for a given burst. In addition, let us assume that v_0 is the same for all fragments, an approximation which is nearly true if the charge to mass ratio is nearly constant over the length of a shell whose lethal fragments are mostly found in a zone near the equatorial plane. Then the ratio of the speeds of two masses, m_1 and m_2, at distance R is

$$R^1 = \exp[-\alpha R (m_1^{-1/3} - m_2^{-1/3})]. \qquad (A-11)$$

If $m_2 = 1$ gm then R^1 can be found as a function of m_1 and R. Table I gives representative values of R^1 for $m_2 = 1$ gm and $\alpha = .02 \text{ gm}^{1/3}$ meter versus m_1 (gm) and R (meters). If we compare various masses with $m_2 = 0.1$ or 10.0 gm we obtain somewhat different values of R^1. Still Table I gives us a feeling for the variations to be expected over the variable ranges of interest and for the size of the corrections we might expect if we were to employ different values of k or α for various mass groups. Generally speaking, such corrections are not justified unless the other errors in our model are much less than they usually are.

TABLE I. Speed ratio of various masses, m (gm), at various distances from the burst point, R (meters), compared to a 1 gm mass, using $\alpha = .02 \text{ gm}^{1/3}/\text{meter}$.

m_2 \ R	0	10	20	40
.01	1	.48	.23	.05
.1	1	.79	.63	.40
1	1	1	1	1
10	1	1.11	1.23	1.54
100	1	1.18	1.37	1.89

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