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THE GI/G/1 QUEUE WITH UNIFORMLY LIMITED VIRTUAL WAITING TIMES. --ETC(U)
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THE GI/G/1 QUEUE WITH UNIFORMLY LIMITED VIRTUAL WAITING TIMES; THE FINITE DAM

by

DO LE MINH*

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*On leave from The New South Wales Institute of Technology,
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THE GI/G/1 QUEUE WITH UNIFORMLY LIMITED VIRTUAL WAITING TIMES;

THE FINITE DAM.

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ABSTRACT: This paper studies the GI/G/1 queueing system in which no customer can stay longer than a fixed interval D. This is also a model for the dam with finite capacity, instantaneous water supply and constant release rule. Using analytical method together with the property that the queueing process 'starts anew' probabilistically whenever an arriving customer initiates a busy period, ~~we obtain~~ various transient and stationary results for the system is obtained.

Key words: GI/G/1 queue. Limited virtual waiting times. Finite dam. Regenerative processes. Actual waiting times. Lost service times. Virtual waiting times. Busy cycle. Transient results. Stationary results.

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1. Introduction.

In the single-server queueing model with uniformly bounded virtual waiting times studied here, the server never permits the amount of work still to be handled by him at any moment to exceed a constant D. This means that the total time spent by each customer in the system is limited by D. If this time exceeds D, then the customer has to cut short his assigned service time.

This model is also the model for the dam with finite capacity, instantaneous water supply and constant release rule. The content of the dam is equivalent to the virtual waiting time of the queueing system. If at some input moment, the supply is so large that the content exceeds the capacity, then overflow occurs.

The $E_j/D/1$ model has been investigated by Ghosal [9] and Prabhu [15]; the M/G/1 model by Kovalenko [11], Takács [23] and Cohen [5]. Cohen [4] also obtained several results for the model in which the inter-arrival times and service times have rational Laplace-Stieljes transforms. Daley [7,8] studied the distribution of the stationary waiting time of the GI/G/1 model. (See also [1] and [2]). In this paper, we shall obtain a fairly complete solution for the GI/G/1 model. Explicit results will be obtained for the M/G/1 model.

The key to our analysis of the system is that many of its processes are <u>regenerative</u>; that is, they restart probabilistically whenever a customer initiates a busy period. Regenerative processes in this sense were introduced by Smith [17,18] and have been used

by many authors to study the stationary behaviour of many queueing systems. (See, for example, [3], [5], [6], [12], [13], [16] and [19]). By using the regenerative properties along with analytical methods in this paper, we shall show that not only the stationary behaviour of the system can be studied but its transient characteristics can also be obtained. These methods also give us insight into the probabilistic structure of the system. In Section 3, the mathematical description of the transient behaviour of the system is obtained from its behaviour within a busy cycle. In Section 4, the mathematical description of its stationary behaviour is also obtained from its behaviour within a busy cycle. The behaviour of the system within a busy cycle and the stochastic laws for the busy cycle are then studied in Section 5. In Section 6, due to the special structure of the M/G/1 queue, we shall obtain explicit results for the behaviour within a busy cycle and the stochastic laws for the busy cycle of this system.

2. <u>The formal model and notation</u>.

We are given

(D.1) a real, positive number D, $D < \infty$;

(D.2) an integer-valued, non-negative random variable $\underline{m}_0$, $E\{\underline{m}_0\}<\infty$;

(D.3) a real, non-negative random variable $\underline{w}_0$, $P\{\underline{w}_0 \leqslant D\} = 1$;

(D.4) Two independent sequences of independent and identically distributed, real, positive random variables $\{\underline{t}_k, k \geqslant 1\}$ and $\{\underline{s}_k, k \geqslant 1\}$. We assume that each of $\underline{s}_1$ and $\underline{t}_1$ has a finite first moment.

$\underline{m}_0$ is the number of customers in the system at time t=0- and $\underline{w}_0$ is the virtual waiting time at time t=0-. Let customers $(\underline{m}_0+1)$, $(\underline{m}_0+2)$, ..., k ,.... arrive at the time epochs $\underline{\tau}_{\underline{m}_0+1}$, $\underline{\tau}_{\underline{m}_0+2}$, ..., $\underline{\tau}_k$, ... where $0 = \underline{\tau}_{\underline{m}_0+1} < \underline{\tau}_{\underline{m}_0+2} < \ldots < \underline{\tau}_k < \ldots$ Let $\underline{\tau}_{k+1} - \underline{\tau}_k = \underline{t}_k$, for all $k > \underline{m}_0$.

Let the <u>assigned</u> service time of the k^{th} customer be $\underline{s}_k$. We assume that no customer can stay in the system longer than the time interval of length D. If the total actual waiting time and service time of a customer exceeds D, then he shall cut short his assigned service time.

We write

(D.5) $\Psi(z) = E\{\exp(-z\underline{s}_1)\}$ for $Re(z) \geqslant 0$;

(D.6) $\Omega(z) = E\{\exp(-z\underline{t}_1)\}$ for $Re(z) \geqslant 0$.

The customers are served in order of their arrivals and there is no limit on the size of the waiting room.

Besides $\underline{l}_k$, we want to study the following random variables:

(D.7) $\underline{w}_k$ = the actual waiting time of the k^{th} customer ($k>\underline{m}_0$);

(D.8) $\underline{w}_\infty$ = limit in distribution of $\underline{w}_k$ when $k\to\infty$, if this exists;

(D.9) $\underline{l}_k$ = the lost service time of the k^{th} customer ($k>\underline{m}_0$),

i.e. $\underline{l}_k = \max(\underline{w}_k + \underline{s}_k, D) - D$;

(D.10) $\underline{l}_\infty$ = limit in distribution of $\underline{l}_k$ when $k\to\infty$, if this exists;

(D.11) $\underline{p}_1$ = the duration of the initial busy period;

(D.12) $\underline{p}_v$ = the duration of the v^{th} busy period, $v\geq 2$;

(D.13) $\underline{i}_1$ = the duration of the first idle period;

(D.14) $\underline{i}_v$ = the duration of the v^{th} idle period, $v\geq 2$;

(D.15) $\underline{c}_1 = \underline{p}_1 + \underline{i}_1$ = the duration of the initial busy cycle;

(D.16) $\underline{c}_v = \underline{p}_v + \underline{i}_v$ = the duration of the v^{th} busy cycle, $v\geq 2$;

(D.17) $\underline{n}_1$ = the number of customers served during the initial busy period, including the $\underline{m}_0$ customers in the system at time $t=0-$;

(D.18) $\underline{n}_v$ = the number of customers served during the v^{th} busy period, $v\geq 2$;

(D.19) $\underline{v}(t)$ = the virtual waiting time at time t, $t\geq 0$;

(D.20) $\underline{v}(\infty)$ = limit in distribution of $\underline{v}(t)$ when $t\to\infty$, if this exists;

(D.21) $\underline{\nu}(t)$ = the total number of customers arriving during the time interval $[0,t]$, including the $\underline{m}_0$ customers in the system at time $t=0-$;

(D.22) $\underline{a}(t)$ = the time difference between t and the time of the first arrival during the interval (t,∞).

The results will be expressed in the following forms:

(D.23) $W_{m,w}(x,\xi,z) = \sum_{k=m+1}^{\infty} x^k E\{\exp(-\xi\underline{\tau}_k - z\underline{w}_k) \mid \underline{m}_0=m, \underline{w}_0=w\}$

for $0<|x|<1$, $Re(\xi)\geq 0$, $|z|<\infty$, $m\geq 0$, $w\geq 0$;

(D.24) $L_{m,w}(x,\xi,z) = \sum_{k=m+1}^{\infty} x^k E\{\exp(-\xi\underline{\tau}_k - z\underline{\ell}_k) \mid \underline{m}_0=m, \underline{w}_0=w\}$

for $0<|x|<1$, $Re(\xi)\geq 0$, $Re(z)\geq 0$, $m\geq 0$, $w\geq 0$;

(D.25) $V_{m,w}(x,\xi,z,s) = \int_0^{\infty} \exp(-\xi t) E\{x^{\underline{\nu}(t)} \exp(-z\underline{v}(t) - s\underline{a}(t)) \mid \underline{m}_0=m, \underline{w}_0=w\} dt$

for $0<|x|<1$, $Re(\xi)\geq 0$, $|z|<\infty$, $Re(s)\geq 0$, $m\geq 0$, $w\geq 0$;

(D.26) $C_{m,w}(x,\xi,-z) = E\{x^{\underline{n}_1} \exp(-\xi\underline{c}_1 + z\underline{i}_1) \mid \underline{m}_0=m, \underline{w}_0=w\}$

for $0<|x|\leq 1$, $Re(\xi)\geq 0$, $Re(z)\leq 0$, $m\geq 0$, $w\geq 0$;

(D.27) $W_{\infty}(z) = E\{\exp(-z\underline{w}_{\infty})\}$ for $|z|<\infty$;

(D.28) $L_{\infty}(z) = E\{\exp(-z\underline{\ell}_{\infty})\}$ for $Re(z)\geq 0$;

(D.29) $V_{\infty}(z) = E\{\exp(-z\underline{v}(\infty)\}$ for $|z|<\infty$.

We shall need the following intermediate Laplace-Stieljes transforms:

(D.30) $\hat{W}_{m,w}(x,\xi,z) = E\left\{\sum_{k=m+1}^{\underline{n}_1} x^k \exp(-\xi\underline{\tau}_k - z\underline{w}_k) \mid \underline{m}_0=m, \underline{w}_0=w\right\}$

for $0<|x|\leq 1$, $Re(\xi)\geq 0$, $|z|<\infty$, $m\geq 0$, $w\geq 0$;

(D.31) $\hat{L}_{m,w}(x,\xi,z) = E\left\{\sum_{k=m+1}^{\underline{n}_1} x^k \exp(-\xi\underline{\tau}_k - z\underline{\ell}_k) \mid \underline{m}_0=m, \underline{w}_0=w\right\}$

for $0<|x|\leq 1$, $Re(\xi)\geq 0$, $Re(z)\geq 0$, $m\geq 0$, $w\geq 0$;

(D.32) $\hat{V}_{m,w}(x,\xi,z,s) = E\left\{\int_0^{\underline{c}_1} x^{\underline{\nu}(t)} \exp(-\xi t - z\underline{v}(t) - s\underline{a}(t)) dt \mid \underline{m}_0=m, \underline{w}_0=w\right\}$

for $0<|x|\leq 1$, $Re(\xi)\geq 0$, $|z|<\infty$, $Re(s)\geq 0$, $m\geq 0$, $w\geq 0$;

(D.33) $P_{m,w}(x,\xi,s) = E\left\{x^{\underline{n}_1} \exp(-\xi\underline{p}_1 - s\underline{i}_1) \mid \underline{m}_0=m, \underline{w}_0=w\right\}$

for $0<|x|\leq 1$, $Re(\xi)\geq 0$, $Re(s)\geq 0$, $m\geq 0$, $w\geq 0$.

It is important to point out here that there are two types of busy periods which Cohen [4,p.284] calls strong and weak busy periods. While two consecutive strong busy periods are separated with probability one by an idle period of non-zero duration; a weak busy period may be followed by an idle period of zero duration. In other words, if the $(k+1)^{th}$ customer arrives at the instant the k^{th} customer departs, the strong busy period continues while the weak busy period terminates and a new weak busy period starts. We consider both types of busy periods in this paper. If the result is applicable for both, no notational distinctions are made. If a result is applicable to the strong busy period only, then a superscript "s" is added to the notation. If a result is applicable to the weak busy period only, then the superscript "w" is added.

Remarks:

(R.1) From (D.30) and (D.31), we have

$$E\{\underline{n}_1 | \underline{m}_0=m,\underline{w}_0=w\} = \hat{W}_{m,w}(1,0,0)+m = \hat{L}_{m,w}(1,0,0)+m \; ; \tag{2.1}$$

$$E\{\underline{n}_2\} = \hat{W}_{0,0}(1,0,0) = \hat{L}_{0,0}(1,0,0) \; . \tag{2.2}$$

3. Regenerative results for the transient behaviour.

In this section, we shall show that the study of the actual waiting time of each customer, the lost service time of each customer and the virtual waiting time at each epoch can be reduced to the study of the lost servive times within one busy cycle alone; that is, for $0<|x|<1$, $\mathrm{Re}(\xi)\geq$ $\mathrm{Re}(s)\geq 0$, $m\geq 0$, $w\geq 0$, $W_{m,w}(x,\xi,z)$ $(|z|<\infty)$, $L_{m,w}(x,\xi,z)$ $(\mathrm{Re}(z)\geq 0)$ and $V_{m,w}(x,\xi,z,s)$ $(|z|<\infty)$ can be obtained from $\hat{L}_{m,w}(x,\xi,z)$ $(\mathrm{Re}(z)\geqslant 0)$ and $C_{m,w}(x,\xi,0)$. The arguments are based mainly on the regenerative property that the continuations of many processes in this system beyond the end of a busy cycle are the probabilistic replicas of these processes commencing at the beginning of that busy cycle.

THEOREM 1 For $0<|x|<1$, $\mathrm{Re}(\xi)\geqslant 0$, $|z|<\infty$, $m>0$, $w>0$,

$$W_{0,0}(x,\xi,z) = \hat{W}_{0,0}(x,\xi,z) \,/\, [1-C_{0,0}(x,\xi,0)] \quad ; \tag{3.1}$$

$$W_{m,w}(x,\xi,z) = \hat{W}_{m,w}(x,\xi,z) + C_{m,w}(x,\xi,0)W_{0,0}(x,\xi,z) \quad . \tag{3.2}$$

PROOF: We have , from (D.23) and (D.30), for $m\geqslant 0$, $w\geqslant 0$,

$$W_{m,w}(x,\xi,z) = \hat{W}_{m,w}(x,\xi,z) + E\left\{\sum_{k=1}^{\infty} x^{\underline{n}_1+k} \exp(-\xi\underline{\tau}_{\underline{n}_1+k} - z\underline{w}_{\underline{n}_1+k}) \,\middle|\, \underline{m}_0=m, \underline{w}_0=w\right\} . \tag{3.3}$$

Now since the queueing process starts again probabilistically when the $(\underline{n}_1+1)^{th}$ customer initiates the second busy period at time $\underline{\tau}_{\underline{n}_1+1} = \underline{c}_1$, $\underline{c}_1$ and $\underline{n}_1$ are independent of $\underline{w}_{\underline{n}_1+k}$ for all $k\geq 1$. Also, for all $k\geq 1$, $\underline{w}_{\underline{n}_1+k}$ are independent of $\underline{m}_0$ and $\underline{w}_0$. Thus we can write

$$(3.4)\quad W_{m,w}(x,\xi,z) = \hat{W}_{m,w}(x,\xi,z)$$

$$+ C_{m,w}(x,\xi,0)\,E\left\{\sum_{k=1}^{\infty} x^k \exp(-\xi \underline{\tau}_k - z\underline{w}_{\underline{n}_1+k}) \mid \underline{m}_0=0, \underline{w}_0=0\right\} .$$

Furthermore, if $\underline{m}_0=0$ and $\underline{w}_0=0$, then $\underline{w}_{\underline{n}_1+k}$ will have the same distribution as $\underline{w}_k$. Thus (3.2) is proved. As (3.2) is also applicable when $\underline{m}_0=0$, $\underline{w}_0=0$, (3.1) follows.□

<u>THEOREM 2:</u> For $0<|x|<1$, $\mathrm{Re}(\xi)\geqslant 0$, $\mathrm{Re}(z)\geqslant 0$, $m>0$, $w>0$,

$$(3.5)\quad L_{0,0}(x,\xi,z) = \hat{L}_{0,0}(x,\xi,z) \Big/ \left[1-C_{0,0}(x,\xi,0)\right] \quad ;$$

$$(3.6)\quad L_{m,w}(x,\xi,z) = \hat{L}_{m,w}(x,\xi,z) + C_{m,w}(x,\xi,0)\,L_{0,0}(x,\xi,z) \quad .$$

<u>PROOF:</u> The proof is omitted because it is analogous to that of Theorem 1.□

Theorems 1 and 2 relate $W_{m,w}(x,\xi,z)$ and $L_{m,w}(x,\xi,z)$ to $\hat{W}_{m,w}(x,\xi,z)$, $\hat{L}_{m,w}(x,\xi,z)$ and $C_{m,w}(x,\xi,0)$. We shall now show that $V_{m,w}(x,\xi,z,s)$ can be obtained from $W_{m,w}(x,\xi,z)$ and $L_{m,w}(x,\xi,z)$. This is an important relation which is of interest by itself because it enables us to find the mathematical description of the behaviour of a queue in continuous time if we know its behaviour at a certain set of discrete-time epochs.

<u>LEMMA 1:</u> For $0<|x|<1$, $\mathrm{Re}(\xi)\geqslant 0$, $|z|<\infty$, $m>0$, $w>0$,

$$(3.7)\quad V_{0,0}(x,\xi,z,s) = \hat{V}_{0,0}(x,\xi,z,s) \Big/ \left[1-C_{0,0}(x,\xi,0)\right]$$

$$(3.8)\quad V_{m,w}(x,\xi,z,s) = \hat{V}_{m,w}(x,\xi,z,s)$$

$$+ C_{m,w}(x,\xi,0)\,V_{0,0}(x,\xi,z,s)$$

<u>PROOF:</u> The proof of this lemma is omitted because it is analogous

to that of Theorem 1. Here, we use the property that if $\underline{m}_0 = 0$, then for all $t \geq 0$, $\underline{u}(t+\underline{\tau}_1)$, $\underline{v}(t+\underline{\tau}_1)$ and $\underline{a}(t+\underline{\tau}_1)$ have the same distributions as $\underline{u}(t)+\underline{n}_1$, $\underline{v}(t)$ and $\underline{a}(t)$ respectively. □

<u>LEMMA 2:</u> For $0<|x|\leq 1$, $\mathrm{Re}(\xi) \geq \mathrm{Re}(s) \geq 0$, $|z|<\infty$, $m\geq 0$, $w\geq 0$,

$$
\begin{aligned}
(3.9)\quad (z+s-\xi)\hat{V}_{m,w}(x,\xi,z,s)
&= -x^m \exp(-zw) - zx^m \exp((s-\xi)w)/(s-\xi) \\
&\quad + \left[1-x\Psi(z)\Omega(s)\right]\hat{W}_{m,w}(x,\xi,z)/x \\
&\quad + z\left[1-x\Psi(\xi-s)\Omega(s)\right]\hat{W}_{m,w}(x,\xi,\xi-s)/x(s-\xi) \\
&\quad + \exp(-zD)\Omega(s)\left[\hat{L}_{m,w}(x,\xi,z)-\hat{L}_{m,w}(x,\xi,0)\right] \\
&\quad + z\exp((s-\xi)D)\Omega(s)\left[\hat{L}_{m,w}(x,\xi,\xi-s)-\hat{L}_{m,w}(x,\xi,0)\right]/(s-\xi) \\
&\quad + (z+s-\xi)C_{m,w}(x,\xi,0)/(s-\xi) \quad .
\end{aligned}
$$

<u>PROOF:</u> For $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(s)\geq 0$, $|z|<\infty$, $0\leq m<(\underline{n}_1-1)$, $w\geq 0$,

$$
\begin{aligned}
(3.10)\quad \hat{V}_{m,w}(x,\xi,z,s) &= \\
& E\left\{\sum_{k=m+1}^{\underline{n}_1-1} x^k \int_{\underline{\tau}_k}^{\underline{\tau}_{k+1}} \exp(-\xi t - z(\underline{\tau}_k+\min(\underline{w}_k+\underline{s}_k,D)-t)-s(\underline{\tau}_{k+1}-t))\,dt \,\middle|\, \underline{m}_0=m,\underline{w}_0=w\right\} \\
&+ E\left\{x^{\underline{n}_1}\int_{\underline{\tau}_{\underline{n}_1}}^{\underline{p}_1} \exp(-\xi t - z(\underline{\tau}_{\underline{n}_1}+\min(\underline{w}_{\underline{n}_1}+\underline{s}_{\underline{n}_1},D)-t)-s(\underline{\tau}_{\underline{n}_1+1}-t))\,dt \,\middle|\, \underline{m}_0=m,\underline{w}_0=w\right\} \\
&+ E\left\{x^{\underline{n}_1}\int_{\underline{p}_1}^{\underline{c}_1} \exp(-\xi t - s(\underline{\tau}_{\underline{n}_1+1}-t))\,dt \,\middle|\, \underline{m}_0=m,\underline{w}_0=w\right\} \quad .
\end{aligned}
$$

As $\underline{w}_{k+1} = \underline{\tau}_k + \min(\underline{w}_k+\underline{s}_k\ D) - \underline{\tau}_{k+1}$ for $k<\underline{n}_1$ and $\underline{p}_1 = \underline{\tau}_{\underline{n}_1} + \min(\underline{w}_{\underline{n}_1}+\underline{s}_{\underline{n}_1},D)$, (3.10) becomes

(3.11) $\hat{V}_{m,w}(x,\xi,z,s) =$

$$\frac{1}{z+s-\xi} E\left\{\sum_{k=m+1}^{\underline{n}_1-1} x^k\left[\exp(-\xi\underline{\tau}_{k+1}-z\underline{w}_{k+1}) - \exp(-\xi\underline{\tau}_k-z(\min(\underline{w}_k+\underline{s}_k,D))-s\underline{t}_k)\right]\middle|\underline{m}_0=m,\underline{w}_0=w\right\}$$

$$+\frac{1}{z+s-\xi} E\left\{x^{\underline{n}_1}\left[\exp(-\xi\underline{p}_1-s(\underline{c}_1-\underline{p}_1)) - \exp(-\xi\underline{\tau}_{\underline{n}_1}-z(\min(\underline{w}_{\underline{n}_1}+\underline{s}_{\underline{n}_1},D))-s\underline{t}_{\underline{n}_1})\right]\middle|\underline{m}_0=m,\underline{w}_0=w\right\}$$

$$+\frac{1}{s-\xi} E\left\{x^{\underline{n}_1}\left[\exp(-\xi\underline{c}_1) - \exp(-\xi\underline{p}_1-s(\underline{c}_1-\underline{p}_1))\right]\middle|\underline{m}_0=m,\underline{w}_0=w\right\}$$

Now observe that for $k\le\underline{n}_1$, $\underline{\ell}_k = \max(\underline{w}_k+\underline{s}_k,D)-D$ and hence

(3.12) $$\exp(-\xi\underline{\tau}_k-z(\min(\underline{w}_k+\underline{s}_k,D))-s\underline{t}_k)$$

$$= \exp(-\xi\underline{\tau}_k-z(\underline{w}_k+\underline{s}_k)-s\underline{t}_k) + \exp(-\xi\underline{\tau}_k-zD-s\underline{t}_k) - \exp(-\xi\underline{\tau}_k-z(\max(\underline{w}_k+\underline{s}_k,D))-s\underline{t}_k)$$

$$= \exp(-z\underline{s}_k)\exp(-s\underline{t}_k)\exp(-\xi\underline{\tau}_k-z\underline{w}_k) - \exp(-zD)\exp(-s\underline{t}_k)\left[\exp(-\xi\underline{\tau}_k-z\underline{\ell}_k)-\exp(-\xi\underline{\tau}_k)\right].$$

Thus (3.11) can now be written as

(3.13) $$\hat{V}_{m,w}(x,\xi,z,s)=-x^m\exp(-zw)/(z+s-\xi)$$

$$+\left[1-x\Psi(z)\Omega(s)\right]\hat{W}_{m,w}(x,\xi,z)\ /\ x(z+s-\xi)$$

$$+\exp(-zD)\Omega(s)\left[\hat{L}_{m,w}(x,\xi,z) - \hat{L}_{m,w}(x,\xi,0)\right]\ /\ (z+s-\xi)$$

$$+\ P_{m,w}(x,\xi,s)\ /\ (z+s-\xi)$$

$$+\left[C_{m,w}(x,\xi,0) - P_{m,w}(x,\xi,s)\right]/\ (s-\xi)$$

It is easy to prove that (3.13) is also applicable when $n_1 = m+1$. Now since $\hat{V}_{m,w}(x,\xi,z,s)$, $P_{m,w}(x,\xi,s)$ and $C_{m,w}(x,\xi,0)$ are analytic for $0<|x|\leq 1$, $Re(\xi) \geq Re(s)\geq 0$, $|z|<\infty$, letting $z=\xi-s$ in (3.13) yields

$$(3.14)\quad P_{m,w}(x,\xi,s) = x^m \exp((s-\xi)w) - \left[1-x\Psi(\xi-s)\Omega(s)\right]\hat{W}_{m,w}(x,\xi,\xi-s)/x - \exp((s-\xi)D)\Omega(s)\left[\hat{L}_{m,w}(x,\xi,\xi-s)-\hat{L}_{m,w}(x,\xi,0)\right].$$

If we substitute this equation back into (3.13), we obtain (3.9).□

<u>THEOREM 3:</u> For $0<|x|<1$, $Re(\xi)\geq Re(s)\geq 0$, $|z|<\infty$, $m\geq 0$, $w\geq 0$,

$$\begin{aligned}(3.15)\quad (z+s-\xi)V_{m,w}(x,\xi,z,s) &= -x^m\exp(-zw) - zx^m\exp((s-\xi)w)/(s-\xi)\\ &+ \left[1-x\Psi(z)\Omega(s)\right]W_{m,w}(x,\xi,z)/x\\ &+ z\left[1-x\Psi(\xi-s)\Omega(s)\right]W_{m,w}(x,\xi,\xi-s)/x(s-\xi)\\ &+ \exp(-zD)\Omega(s)\left[L_{m,w}(x,\xi,z)-L_{m,w}(x,\xi,0)\right]\\ &+ z\exp((s-\xi)D)\Omega(s)\left[L_{m,w}(x,\xi,\xi-s)-L_{m,w}(x,\xi,0)\right]/(s-\xi)\end{aligned}$$

<u>PROOF:</u> The proof is straightforward from Theorems 1,2 and Lemmas 1,2.□

It remains to show that $\hat{W}_{m,w}(x,\xi,z)$ can be obtained from $\hat{L}_{m,w}(x,\xi,z)$ and $C_{m,w}(x,\xi,-z)$.

<u>THEOREM 4:</u> For $0<|x|\leq 1$, $Re(\xi)\geq Re(z)\geq 0$, $m\geq 0$, $w\geq 0$,

$$\begin{aligned}(3.16)\quad &[1-x\Psi(z)\Omega(\xi-z)]\hat{W}_{m,w}(x,\xi,z)\\ &= x^{m+1}\exp(-zw) - xC_{m,w}(x,\xi,-z)\\ &\quad - x\exp(-zD)\Omega(\xi-z)[\hat{L}_{m,w}(x,\xi,z)-\hat{L}_{m,w}(x,\xi,0)]\end{aligned}$$

PROOF: Since $\hat{V}_{m,w}(x,\xi,z,s)$, $C_{m,w}(x,\xi,0)$ and $P_{m,w}(x,\xi,s)$ are analytic for $0 \le x \le 1$, $Re(\xi) \ge Re(z) \ge 0$, $m \ge 0$, $w \ge 0$, (3.16) is obtained by putting $s = \xi - z$ in (3.13). $\square$

Remarks:

(R.2) (3.1) and (3.7) are the generalizations of (3.3) and (4.6) in [5,pps. 6,13] respectively.

(R.3) When $D \to \infty$, then Theorem 3 becomes Theorem 2 in [21]. While Takács derived the latter directly, the former is obtained via Lemma 2, which will also be useful in the derivation of Theorem 6 later.

(R.4) If we let $x \to 1$, $z \to 0$, $s \to 0$ in (3.14) and (3.16) and then use l'Hospital's Rule to obtain the limit when $\xi \to 0$, we shall obtain the following Generalized Wald's Lemma:

(3.17) $$E\{\underline{p}_1\} = E\{\underline{w}_0\} + [E\{\underline{n}_1\} - E\{\underline{m}_0\}]E\{\underline{s}_1\} - E\left\{\sum_{k=m+1}^{\underline{n}_1} \underline{\lambda}_k\right\};$$

(3.18) $$E\{\underline{c}_1\} = [E\{\underline{n}_1\} - E\{\underline{m}_0\}]E\{\underline{t}_1\}.$$

4. Regenerative results for the stationary behaviour.

For the queueing system studied in this paper, it has been proved that, unless $\underline{s}_1 = \underline{t}_1 = \text{constant} < D$, each of the processes $\{\underline{w}_k, k>\underline{m}_0\}$, $\{\underline{\ell}_k, k>\underline{m}_0\}$ and $\{\underline{v}(t), t\geq 0\}$ has a unique stationary distribution which is independent of the initial conditions (See [7]). In this section, we shall show that the study of the stationary behaviour of the system can also be reduced to the study of the lost service times within one busy cycle alone; that is, the expressions for $W_\infty(z)$ $(|z|<\infty)$, $L_\infty(z)$ $(\text{Re}(z)\geq 0)$ and $V_\infty(z)$ $(|z|<\infty)$ can be obtained from $\hat{L}_{0,0}(1,0,z)$. Here, we shall use a general theorem in the literature stating that the stationary distribution of a regenerative process, if it exists, is the 'time average' or 'customer average' of the process over a regenerative cycle. (See [5])

This allows us to state the following theorem without proof:

THEOREM 5:

$$(4.1)\qquad W_\infty(z) = \hat{W}_{0,0}(1,0,z)/E\{\underline{n}_2\} \quad \text{for } |z|<\infty \quad ;$$

$$(4.2)\qquad L_\infty(z) = \hat{L}_{0,0}(1,0,z)/E\{\underline{n}_2\} \quad \text{for } \text{Re}(z)\geq 0 \quad .$$

Together with (3.16), (2.2), the assertion for $W_\infty(z)$ and $L_\infty(z)$ is now true. The next theorem will allow us to find the distribution function of the stationary virtual waiting time in terms of the distribution functions of the stationary actual waiting time and the stationary lost service time.

THEOREM 6: For $z < \infty$,

$$(4.3) \quad V_\infty(z) = 1 - E\{\underline{p}_2\}/E\{\underline{c}_2\} + \{1-?(z)\}W_\infty(z)/zE\{\underline{t}_1\}$$

$$+ \exp(-zD)[L_\infty(z)-1]/zE\{\underline{t}_1\} \ .$$

PROOF: This is because $V_\infty(z) = \hat{V}_{0,0}(1,0,z,0)/E\{\underline{c}_2\}$. Upon applying l'Hospital rule to (3.13), we prove the theorem. □

Remarks:

(R.5) When $D\to\infty$, then (4.3) becomes a well-known result due to Takács [20] for the classical GI/G/1 queue.

5. The stochastic laws for the busy cycles.

Let

(D.34) M_z = the set of all those functions of z which are analytic in the domain $Re(z)>0$ and continuous, free from zeros, uniformly bounded in $Re(z)\geq 0$;

(D.35) N_z = the set of all those functions of z which are analytic in the domain $Re(z)<0$ and continuous, free from zeros, uniformly bounded in $Re(z)\leq 0$;

(D.36) R_z = the set of all those functions $\Phi(z)$ which are defined for $Re(z)=0$ on the complex plane and can be represented in the form

(5.1) $$\Phi(z) = E\{\underline{\zeta}\exp(-z\underline{\eta})\} ,$$

where $\underline{\zeta}$ is a complex (or real) random variable with $E\{|\underline{\zeta}|\}<\infty$ and $\underline{\eta}$ is a real random variable.

Let us define the following transformations on R_z:

(D.37) $$T_z\{\Phi(z)\} = E\{\underline{\zeta}\exp(-z\underline{\eta}^+)\} ;$$

(D.38) $$T_z^*\{\Phi(z)\} = \Phi(z) - T_z\{\Phi(z)\} ;$$

(D.39) $$U_z\{\Phi(z)\} = E\{\underline{\zeta}\delta(\underline{\eta}\geq 0)\exp(-z\underline{\eta})\} ;$$

(D.40) $$U_z^*\{\Phi(z)\} = \Phi(z) - U_z\{\Phi(z)\} = E\{\underline{\zeta}\delta(\underline{\eta}<0)\exp(-z\underline{\eta})\} ;$$

(D.41) $$V_z\{\Phi(z)\} = E\{\underline{\zeta}\delta(\underline{\eta}>0)\exp(-z\underline{\eta})\} ;$$

(D.42) $$V_z^*\{\Phi(z)\} = \Phi(z) - V_z\{\Phi(z)\} = E\{\underline{\zeta}\delta(\underline{\eta}\leq 0)\exp(-z\underline{\eta})\} ,$$

where $\underline{\eta}^+ = \max(0,\underline{\eta})$ and $\delta(A)$ is the indicator function of any event A; that is, $\delta(A)=1$ if A occurs and $\delta(A)=0$ if A does not occur.

Clearly, $T_z\{\Phi(z)\}$, $U_z\{\Phi(z)\}$ and $V_z\{\Phi(z)\}$ belong to M_z and $T_z^*\{\Phi(z)\}$, $U_z^*\{\Phi(z)\}$ and $V_z^*\{\Phi(z)\}$ belong to N_z. Also, it is easy to show that

$$(5.2) \quad U_z\{\Phi(z)\} = T_z\{\Phi(z)\} + \lim_{z\to-\infty} T_z^*\{\Phi(z)\} \quad ;$$

$$(5.3) \quad U_z^*\{\Phi(z)\} = T_z^*\{\Phi(z)\} - \lim_{z\to-\infty} T_z^*\{\Phi(z)\} \quad ;$$

$$(5.4) \quad V_z\{\Phi(z)\} = T_z\{\Phi(z)\} - \lim_{z\to\infty} T_z\{\Phi(z)\} \quad ;$$

$$(5.5) \quad V_z^*\{\Phi(z)\} = T_z^*\{\Phi(z)\} + \lim_{z\to\infty} T_z\{\Phi(z)\} \quad ;$$

$$(5.6) \quad T_z\{\Phi(z)\} = U_z\{\Phi(z)\} + \lim_{z\to 0} U_z^*\{\Phi(z)\} \quad ;$$

$$(5.7) \quad T_z\{\Phi(z)\} = V_z\{\Phi(z)\} + \lim_{z\to 0} V_z^*\{\Phi(z)\} \quad .$$

This means that the closed form expressions for these transformations can be obtained if that for $T_z\{\Phi(z)\}$ is known. The following lemma, which is due to Takács [22], will enable us to obtain $T_z\{\Phi(z)\}$ explicitly:

<u>LEMMA 3:</u> If $\Phi(z)\in R_z$, then for $Re(z)>0$, we have

$$(5.8) \quad T_z\{\Phi(z)\} = \tfrac{1}{2}\Phi(0) + \lim_{\varepsilon\to 0} \frac{z}{2\pi i} \int_{L_\varepsilon} \frac{\Phi(s)}{s(z-s)} ds \quad ,$$

where the path of integration L_ε $(\varepsilon>0)$ consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$ and again from $z=i\varepsilon$ to $z=i\infty$.

<u>PROOF:</u> See Theorem 2 in [22].□

In this paper, we have shown in Sections 3 and 4 that both the transient and stationary behaviours of the system can be studied in term of the lost service times within one busy cycle alone.

In this section, analytic methods are given for finding the integral equations that would theoretically allow us to obtain results for the lost service times within a busy cycle $\hat{L}_{m,w}(x,\xi,z)$ and the stochastic laws of the busy cycle $C_{m,w}(x,\xi,-z)$ simultaneously. These equations will be expressed in terms of the transformations defined in (D.37)-(D.42).

Basically, this method simply involves the re-arrangement of (3.16) into identities whose left hand sides belong to M_z and right hand sides belong to N_z. By Liouville's Theorem, they are functions independent of z. The integral equations will be obvious when these functions are known.

First, for the sake of simplicity, let us write

(D.43) $$\hat{Q}_{m,w}(x,\xi,z) = \hat{L}_{m,w}(x,\xi,z) - \hat{L}_{m,w}(x,\xi,0)$$

for $0<|x|\leqslant 1$, $\mathrm{Re}(\xi)\geqslant 0$, $\mathrm{Re}(z)\geqslant 0$, $m\geqslant 0$, $w\geqslant 0$.

From (3.16), we have

(5.9) $$\hat{L}_{m,w}(x,\xi,0) = \hat{W}_{m,w}(x,\xi,0) = [x^{m+1}-xC_{m,w}(x,\xi,0)]/[1-x\Omega(\xi)] .$$

This means that $\hat{L}_{m,w}(x,\xi,z)$ will be known if $\hat{Q}_{m,w}(x,\xi,z)$ and $C_{m,w}(x,\xi,-z)$ are known.

For $0<|x|\leqslant 1$, $0\leqslant\mathrm{Re}(z)\leqslant\mathrm{Re}(\xi)$, we now assert that $[1-\Psi(z)\Omega(\xi-z)]$ can be factorized into the form

(5.10) $$[1-x\Psi(z)\Omega(\xi-z)] = g^{+}(x,\xi,z)/g^{-}(x,\xi,z) ,$$

where $g^{+}(x,\xi,z)\in M_z$ and $g^{-}(x,\xi,z)\in N_z$.

Such factorization always exists as we can write

(5.11) $$g^{+}(x,\xi,z) = \exp\left\{T_z\{\ln(1-x\Psi(z)\Omega(\xi-z))\}\right\} ;$$

(5.12) $$g^{-}(x,\xi,z) = \exp\left\{-T_z^{*}\{\ln(1-x\Psi(z)\Omega(\xi-z))\}\right\} .$$

In fact, $g^{+}(x,\xi,z)$ and $g^{-}(x,\xi,z)$ are determined up to a multiplicative function of x and ξ. For if we also have $[1-x\Psi(z)\Omega(\xi-z)] = h^{+}(x,\xi,z)/h^{-}(x,\xi,z)$ where $h^{+}(x,\xi,z) \in M_z$ and $h^{-}(x,\xi,z) \in N_z$ then by Liouville's Theorem $g^{+}(x,\xi,z)/h^{+}(x,\xi,z) = g^{-}(x,\xi,z)/h^{-}(x,\xi,z) = F(x,\xi)$. If $\Psi(z)$ or $\Omega(z)$ is a rational function of z, then the more useful expressions of $g^{+}(x,\xi,z)$ and $g^{-}(x,\xi,z)$ have been obtained in [21] (equations 43, 44, 50 and 51).

(3.16) can now be re-arranged as

$$(5.13)\quad g^{+}(x,\xi,z)\hat{W}_{m,w}(x,\xi,z) - x^{m+1}T_z\{g^{-}(x,\xi,z)\exp(-zw)\} + xT_z\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}_{m,w}(x,\xi,z)\}$$
$$= x^{m+1}T_z^{*}\{g^{-}(x,\xi,z)\exp(-zw)\} - xT_z^{*}\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}_{m,w}(x,\xi,z)\} - xg^{-}(x,\xi,z)C_{m,w}(x,\xi,-z)$$

for $0<|x|\le 1$, $\mathrm{Re}(\xi)\ge \mathrm{Re}(z)\ge 0$, $m\ge 0$, $w\ge 0$. As the left hand side of this equation belongs to M_z and its right hand side belongs to N_z, applications of Liouville's Theorem and analytic continuation yields

$$(5.14)\quad g^{+}(x,\xi,z)\hat{W}_{m,w}(x,\xi,z) - x^{m+1}T_z\{g^{-}(x,\xi,z)\exp(-zw)\} + xT_z\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}_{m,w}(x,\xi,z)\} = R(x,\xi)$$

for $0<|x|\leqslant 1$, $\mathrm{Re}(\xi)\geqslant 0$, $\mathrm{Re}(z)\geqslant 0$, $m\geqslant 0$, $w\geqslant 0$; and

$$(5.15)\quad x^{m+1}T_z^{*}\{g^{-}(x,\xi,z)\exp(-zw)\} - xT_z^{*}\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}_{m,w}(x,\xi,z)\} - xg^{-}(x,\xi,z)C_{m,w}(x,\xi,-z) = R(x,\xi)$$

for $0<|x|\le 1$, $\mathrm{Re}(\xi)\ge 0$, $\mathrm{Re}(z)\leqslant 0$, $m\geqslant 0$, $w\geqslant 0$.

Also, observing that from (3.12),

$$\hat{W}_{m,w}(x,\xi,z)\exp(zD)\Psi(z) - \hat{Q}_{m,w}(x,\xi,z)$$

$$= E\left\{\sum_{k=m+1}^{\underline{n}_1} x^k \exp(-\xi\underline{\tau}_k - z(\min(\underline{w}_k+\underline{s}_k-D,0)))\,\middle|\,\underline{m}_0=m,\underline{w}_0=w\right\} \in N_- ,$$

we re-arrange (3.16) differently and then apply Liouville's theorem and analytic continuation to obtain

$$(5.16)\quad \frac{\hat{Q}_{m,w}(x,\xi,z)}{g^+(x,\xi,z)} - x^{m+1}T_z\left\{\frac{\exp(z(D-w))\Psi(z)}{g^+(x,\xi,z)}\right\} + xT_z\left\{\frac{C_{m,w}(x,\xi,-z)\exp(zD)\Psi(z)}{g^+(x,\xi,z)}\right\} = S(x,\xi)$$

for $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(z)\geq 0$, $m\geq 0$, $w\geq 0$; and

$$(5.17)\quad -\frac{\hat{W}_{m,w}(x,\xi,z)\exp(zD)\Psi(z) - \hat{Q}_{m,w}(x,\xi,z)}{g^-(x,\xi,z)} + x^{m+1}T_z^*\left\{\frac{\exp(z(D-w))\Psi(z)}{g^+(x,\xi,z)}\right\} - xT_z^*\left\{\frac{C_{m,w}(x,\xi,-z)\exp(zD)\Psi(z)}{g^+(x,\xi,z)}\right\} = S(x,\xi)$$

for $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(z)\leq 0$, $m\geq 0$, $w\geq 0$.

The expressions of $R(x,\xi)$ and $S(x,\xi)$, which are dependent on the type of busy cycle we are interested in, will enable us to find the expressions of $\hat{Q}_{m,w}(x,\xi,z)$ and $C_{m,w}(x,\xi,-z)$ as in the following theorems:

<u>THEOREM 7 :</u> $C^s_{m,w}(x,\xi,-z)$ and $\hat{Q}^s_{m,w}(x,\xi,z)$ satisfy the following simultaneous integral equations:

$$(5.18)\qquad g^-(x,\xi,z)C^s_{m,w}(x,\xi,-z) = x^m U^*_z\{g^-(x,\xi,z)\exp(-zw)\}$$

$$- U^*_z\{g^-(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}^s_{m,w}(x,\xi,z)\}$$

$$(0< x\leqslant 1,\ Re(\xi)\geqslant 0,\ Re(z)\leqslant 0,\ m\geqslant 0,\ w\geqslant 0);$$

$$(5.19)\qquad \frac{\hat{Q}^s_{m,w}(x,\xi,z)}{g^+(x,\xi,z)} = \frac{-x^{m-1}+xC^s_{m,w}(x,\xi,0)}{g^+(x,\xi,0)}$$

$$+ x^{m+1}T_z\left\{\frac{\exp(z(D-w))\Psi(z)}{g^+(x,\xi,z)}\right\} - xT_z\left\{\frac{C^s_{m,w}(x,\xi,-z)\exp(zD)\Psi(z)}{g^+(x,\xi,z)}\right\}$$

$$(0< x\leqslant 1,\ Re(\xi)\geqslant 0,\ Re(z)\geqslant 0,\ m\geqslant 0,\ w\geqslant 0).$$

<u>PROOF:</u> We have $p\{\underline{i}^s_v = 0\} = 0$ for all $v\geqslant 1$. Hence $\lim_{z\to-\infty}C^s_{m,w}(x,\xi,-z)=0$. Thus if we let $z\to-\infty$ in (5.15), we shall obtain

$$(5.20)\qquad R^s(x,\xi) = x^{m+1}\lim_{z\to-\infty} T^*_z\{g^-(x,\xi,z)\exp(-zw)\}$$

$$- x\lim_{z\to-\infty} T^*_z\{ g^-(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}^s_{m,w}(x,\xi,z)\} .$$

Upon substituting this back into (5.15), on behalf of (5.3), we obtain (5.18). Also, we have $\lim_{z\to 0}g^+(x,\xi,z)\neq 0$ and from (D.43), $\lim_{z\to 0}\hat{Q}_{m,w}(x,\xi,z) = 0$. Thus if we let $z\to 0$ in (5.16), remembering that $\lim_{z\to 0} T_z\{\Phi(z)\} = \lim_{z\to 0}\Phi(z)$, we obtain

$$(5.21)\qquad S(x,\xi) = \frac{-x^{m+1}+xC_{m,w}(x,\xi,0)}{g^+(x,\xi,0)} .$$

Upon substituting this back into (5.16), we obtain (5.19). $\square$

<u>THEOREM 8:</u> $C^{w}_{m,w}(x,\xi,-z)$ and $\hat{Q}^{w}_{m,w}(x,\xi,z)$ satisfy the following simultaneous integral equations:

$$(5.22)\quad g^{-}(x,\xi,z)C^{w}_{m,w}(x,\xi,-z) = x^{m}V^{\star}_{z}\{g^{-}(x,\xi,z)\exp(-zw)\}$$
$$-\ V^{\star}_{z}\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}^{w}_{m,w}(x,\xi,z)\}$$
$$(\ 0<|x|\leqslant 1,\ \mathrm{Re}(\xi)\geqslant 0,\ \mathrm{Re}(z)\leqslant 0,\ m>0,\ w>0)$$

$$(5.23)\quad g^{-}(x,\xi,z)C^{w}_{0,0}(x,\xi,-z) = g^{-}(x,\xi,z) - \lim_{z\to\infty} g^{+}(x,\xi,z)$$
$$-\ V^{\star}_{z}\{g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}^{w}_{0,0}(x,\xi,z)\}$$
$$(0<|x|\leqslant 1,\ \mathrm{Re}(\xi)\geqslant 0,\ \mathrm{Re}(z)\leqslant 0)$$

$$(5.24)\quad \frac{\hat{Q}^{w}_{m,w}(x,\xi,z)}{g^{+}(x,\xi,z)} = \frac{-\ x^{m+1}\ +\ xC^{w}_{m,w}(x,\xi,0)}{g^{+}(x,\xi,0)}$$
$$+\ x^{m+1}T_{z}\left\{\frac{\exp(z(D-w))\Psi(z)}{g^{+}(x,\xi,z)}\right\} - xT_{z}\left\{\frac{C^{w}_{m,w}(x,\xi,-z)\exp(zD)\Psi(z)}{g^{+}(x,\xi,z)}\right\}$$
$$(0<|x|\leqslant 1,\ \mathrm{Re}(\xi)\geqslant 0,\ \mathrm{Re}(z)\geqslant 0,\ m\geqslant 0,\ w\geqslant 0),$$

<u>PROOF:</u> We have $p\{\underline{w}^{w}_{k}=0\} = 0$ for $0<\underline{m}_{0}<k\leqslant\underline{n}_{1}$. Hence $\lim_{z\to\infty}\hat{W}^{w}_{m,w}(x,\xi,z) = x\delta_{m,0}$ where $\delta_{i,j}$ is the Kronecker delta. Thus, for $m>0$, $w>0$, letting $z\to\infty$ in (5.14), we obtain

$$(5.25)\quad R^{w}(x,\xi) = -x^{m+1}\lim_{z\to\infty}T_{z}\{g^{-}(x,\xi,z)\exp(-zw)\}$$
$$+x\lim_{z\to\infty}T_{z}\{\ g^{-}(x,\xi,z)\exp(-zD)\Omega(\xi-z)\hat{Q}^{w}_{m,w}(x,\xi,z)\}\quad .$$

Upon substituting this back into (5.15), on behalf of (5.5), we obtain (5.22). When $\underline{m}_{0}=0$ and $\underline{w}_{0}=0$, we first modify (5.14) to have the term $\left[\hat{W}^{w}_{0,0}(x,\xi,z)-x\right]$ included. (5.23) can now be derived by the same method as that for (5.22). For (5.24), the proof is similar to that for (5.19). $\square$

Remarks:

(R.6) When $D \to \infty$, (5.18) becomes

(5.26) $$C^{s}_{m,w}(x,\xi,-z) = x^{m} U^{*}_{z}\{g^{-}(x,\xi,z)\exp(-zw)\} / g^{-}(x,\xi,z)$$

which is equation (198) in [24]. (3.16) now can be written as

(5.27) $$\hat{W}^{s}_{m,w}(x,\xi,z) = x^{m+1} U_{z}\{g^{-}(x,\xi,z)\exp(-zw)\}/g^{+}(x,\xi,z) .$$

This equation, together with (3.1), (3.2), (5.6), (5.26) yields

(5.28) $$W_{m,w}(x,\xi,z) = x^{m+1} T_{z}\{g^{-}(x,\xi,z)\exp(-zw)\}/g^{+}(x,\xi,z),$$

a well-known result due to Pollaczek [14], Kingman [10] and Takács [21].

6. The strong M/G/1 queue.

In this section, we shall concentrate on the queueing system in which the arrival process is a Poisson process; that is, $P\{t_1 \leq t\} = 1-\exp(-\lambda t)$ for $t \geq 0$. We shall obtain explicit expression for $C^s_{m,w}(x,\xi,-z)$. The argument is based on the property that in this system, the idle periods are exponential distributed and independent of the busy periods.

THEOREM 9:

(a) For $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(z)\leq 0$, $m\geq 0$, $w>0$,

(6.1) $$C^s_{m,w}(x,\xi,-z) = \lambda P^s_{m,w}(x,\xi,0)/(\lambda+\xi-z)$$

(b) For $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $m\geq 0$, $w\geq 0$,

(6.2) $$P^s_{m,w}(x,\xi,0) =$$

$$\frac{x^{m+1}e^{-\theta w}\Psi(\theta) + x^{m+1}\theta\, e^{-\theta D} g^+(x,\xi,\theta)\left[\dfrac{1}{2(\lambda+\xi-\lambda x)} - \lim_{\varepsilon\to 0}\dfrac{1}{2\pi i}\displaystyle\int_{L_\varepsilon}\dfrac{\exp(s(D-w))\Psi(s)\,ds}{[\lambda+\xi-s-\lambda x\Psi(s)]s}\right]}{1 + \lambda x\theta e^{-\theta D} g^+(x,\xi,\theta)\left[\dfrac{1}{2(\lambda+\xi)(\lambda+\xi-\lambda x)} - \lim_{\varepsilon\to 0}\dfrac{1}{2\pi i}\displaystyle\int_{L_\varepsilon}\dfrac{\exp(sD)\,\Psi(s)\,ds}{[\lambda+\xi-s][\lambda+\xi-s-\lambda x\Psi(s)]s}\right]}$$

where

(D.44) $\theta \equiv \theta(x,\xi)$ is the root of the equation

(6.3) $$\lambda+\xi-z-\lambda x\Psi(z) = 0$$

in the domain $\mathrm{Re}(z)\geq 0$ and

(6.4) $$g^+(x,\xi,z) = [\lambda+\xi-z-\lambda x\Psi(z)]/[\theta-z] .$$

<u>PROOF:</u> If $P\{\underline{t}_1 \leq t\} = 1-\exp(-\lambda t)$ for $t \geq 0$, then for all $v \geq 1$, $\underline{i}_v$ is independent of both $\underline{p}_v$ and $\underline{n}_v$ and $E\{\exp(-z\underline{i}_v)\} = \lambda/(\lambda+z)$ for $\mathrm{Re}(z) \geq 0$. Thus we obtain (6.1). (3.16) can now be written as

$$(6.5) \quad \hat{W}^s_{m,w}(x,\xi,z) = \frac{x^{m+1}(\lambda+\xi-z)\exp(-zw) - \lambda x P^s_{m,w}(x,\xi,0) - \lambda x\exp(-zD)\hat{Q}^s_{m,w}(x,\xi,z)}{\lambda+\xi-z-\lambda x\Psi(z)}$$

for $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(z)\geq 0$, $m\geq 0$, $w\geq 0$. Now, since $\hat{W}^s_{m,w}(x,\xi,z)$ is analytic in the domain $\mathrm{Re}(z)\geq 0$, letting $z=\theta$ as defined in (D.44) yields

$$(6.6) \quad P^s_{m,w}(x,\xi,0) = x^{m+1}e^{-\theta w}\Psi(\theta) - e^{-\theta D}\hat{Q}^s_{m,w}(x,\xi,\theta)$$

for $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $m\geq 0$, $w\geq 0$. Also, from (5.19), we can write

$$(6.7) \quad \frac{\hat{Q}^s_{m,w}(x,\xi,z)}{g^+(x,\xi,z)} = -\frac{x^{m+1}}{g^+(x,\xi,0)} + x^{m+1}T_z\left\{\frac{\exp(z(D-w))\Psi(z)}{g^+(x,\xi,z)}\right\}$$
$$+ \lambda x P^s_{m,w}(x,\xi,0)\left[\frac{1}{g^+(x,\xi,0)(\lambda+\xi)} - T_z\left\{\frac{\exp(zD)\Psi(z)}{g^+(x,\xi,z)(\lambda+\xi-z)}\right\}\right]$$

for $0<|x|\leq 1$, $\mathrm{Re}(\xi)\geq 0$, $\mathrm{Re}(z)\geq 0$, $m\geq 0$, $w\geq 0$. As $g^+(x,\xi,z)$ takes the form of (6.4), we let $z\to\theta$ in (6.7) and then eliminate $\hat{Q}^s_{m,w}(x,\xi,\theta)$ from the resulting equation and (6.6) to prove (6.2). □

<u>Remark:</u>

(R.6) When $D\to\infty$, (6.2) and (6.6) will become a well-known result for the residual busy period of the M/G/1 queue :

$$(6.8) \quad P^s_{m,w}(x,\xi,0) = x^{m+1}e^{-\theta w}\Psi(\theta) \quad (0<|x|\leq 1,\ \mathrm{Re}(\xi)\geq 0,\ m\geq 0,\ w\geq 0).$$

REFERENCES:

[1] Afanas'eva L.G. (1965) 'Existence of a limit distribution in queueing systems with bounded sojourn time' Theory Prob. Applications , 10, 515-522.

[2] Afanas'eva L.G. and Martynov A.V. (1967) 'On ergodic properties of queues with constraints' Theory Prob. Applications , 12, 104-109.

[3] Brown, M and Ross, S.M. (1972) 'Asymptotic properties of cummulative processes', SIAM J. Appl. Math. , 22, 93-105.

[4] Cohen, J.W. (1969) 'The Single Server Queue' , North-Holland Publishing Company - Amsterdam, London.

[5] Cohen, J.W. (1976) 'On Regenerative Processes in Queueing Theory' , Lecture Notes in Economics and Mathematical System No. 121, Springer-Verlag, Berlin-New York.

[6] Crane, M.A. and Iglehart, D.L. (1974) 'Simulating stable stochastic systems, I: General multi-server queues' J. Assoc. Comput. Mach. , 21, 103-113.

[7] Daley, D.J. (1964) 'Single-server queueing systems with uniformly limited queueing time', J. Aust. Math. Soc, 4 , 489-505.

[8] Daley, D.J. (1965), 'General customer impatience in the queue GI/G/1', J. Appl. Prob. , 2, 186-205.

[9] Ghosal, A (1960) 'Emptiness in the finite dam', Ann. Math. Statist, 31 , 803-808.

[10] Kingman, J.F.C. (1962) ' On the algebra of queues', J. Appl. Prob. , 3, 285-326.

[11] Kovalenko, I.N. (1961) 'Some queuing problems with restrictions' Theory Prob. Applications, 6 , 234-238.

[12] Lemoine, A.J. (1974) 'Limit theorems for generalized single server queues', Adv. Appl. Prob. , 6, 159-174.

[13] Minh, D.L. (1979) 'Analysis of the exceptional queueing system by the use of regenerative processes and analytical methods' Math. of Operat. Res. , 24,

[14] Pollaczek F. (1952) 'Fonctions caracteristiques de certaines répartitions definies au moyen de la notion d'ordre. Application à la theorie des attentes' , C.R. Acad. Sci. Paris., 234, 2334-2336

[15] Prabhu, N.U. (1965) 'Queues -and Inventories' John Wiley and Sons, New York.

[16] Ross, S.M. (1970) 'Applied Probability Models with Optimization Applications' , Holden Day, San Francisco.

[17] Smith, W.L. (1953) 'On the distribution of queueing times' Proc. Cambridge Phil. Soc. A, 49, 449-461

[18] Smith, W.L. (1955) 'Regenerative stochastic processes' Proc. Roy. Soc. A. , 232, 6-31.

[19] Stidham, S. Jr. (1972) 'Regenerative processes in the theory of queues, with applications to the alternating-priority queue' Adv. Appl. Prob., 4, 542-577.

[20] Takács, L. (1963) 'The limiting distribution of the virtual waiting time and the queue size for a single-server queue with recurrent input and general service times, Sankhyā A, 25, 91-100.

[21] Takacs, L. (1970) 'A fundamental identity in the theory of queues' Ann. Inst. Statist. Math. , 22, 339-348.

[22] Takacs, L. (1972) 'On a linear transformation in the theory of probability' Acta. Sci. Math. (Szeged) 33, 15-24.

[23] Takacs, L. (1974) 'A single-server queue with limited virtual waiting time', J. Appl. Prob. , 11, 612-617.

[24] Takacs, L. (1976) 'On fluctuation problems in the theory of queues' Adv. Appl. Prob., 8, 548-583.

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This paper studies the GI/G/1 queueing system in which no customer can stay longer than a fixed interval D. This is also a model for the dam with finite capacity, instantaneous water supply and constant release rule. Using analytical method together with the property that the queueing process "starts anew" probabilistically whenever an arriving customer initiates a busy period, we obtain various transient and stationary results for the system.

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