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QUATERNION UTILIZATION IN A
MISSILE MODEL.

THESIS

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(10) William John Thome
Capt USAF

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QUATERNION UTILIZATION IN A MISSILE MODEL

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

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Capt USAF

Graduate Strategic and Tactical Science

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Preface

The primary purpose of this paper was to modify an air-to-air missile model used by the Cruise Missile Independent Survivability Team (CMIST) of the Aeronautical Systems Division (ASD). The model did not have incorporated within it, data pertinent to the cruise missile. The modification, integrated the required data into the model so that the model could be used to evaluate an air-to-air missile's performance against the cruise missile.

The model itself utilizes a hypercomplex number algebra which avoids usage of direction cosine matrices and enables real time of flight output. Prior to the completion of this thesis, no adequate documentation existed that explained the algebra used and its translation into computer code. Bits and pieces were gathered together and combined into the appendices of this work, which should prove helpful in understanding the model.

I wish to thank Captain Aaron DeWispelare for his encouragement and help in completing this work. I also wish to thank Molly Bustard, my favorite librarian, who consistently provided a bright spot during many a dreary day these last eighteen months. I must also acknowledge Bob and Diane Turelli, who provided an escape from the academic grind when it was needed, and finally my mother, sister and her family who gave me the moral help I needed to complete my studies.

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List of Symbols

A	Complex number
$\bar{A}$	Conjugate of A
$\lvert A \rvert$	Absolute value of A
B	Transformation matrix
B^T	Transpose of B
B^{-1}	Inverse of B
b	Bisector of $x_1 + x_1'$
d	$2/N(q)$
i	Unit vector along the x-axis
ImA	Imaginary portion of A
j	Unit vector along the y-axis
k	Unit vector along the z-axis
$N(q)$	Norm of quaternion (q)
p,q,r,s,t	Quaternions
q^*	Conjugate of q
$\lvert q \rvert$	Magnitude of q
ReA	Real portion of A
T	Complex portion of the tilt quaternion
x,u,v,y	Vectors
$\lvert v \rvert$	Magnitude of v
Z	Complex portion of a factored vector

List of Abbreviations

arg	Argument
ASD	Aeronautical Systems Division
AZ	Azimuth
CMIST	Cruise Missile Independent Survivability Team
DCM	Direction Cosine Matrix
EL	Elevation
FAAC	First Ann Arbor Corporation
LHS	Left Hand Side
LOS	Line of Sight
RHS	Right Hand Side

Abstract

The air-to-air missile model used by the Cruise Missile Independent Survivability Team did not contain data useful to the cruise missile. The objective of this study was to modify the subroutine SIGNAL in the model to incorporate the cruise missile data. The modification required an understanding of quaternion algebra utilized within the model to represent three-dimensional motion. These quaternions allow real time outputs from the model for use by tactical ranges. The study contains a discussion of quaternions and their algebra.

I Introduction

Problem Statement

The Cruise Missile Independent Survivability Team (CMIST) obtained the computer model of the air-to-air missile performance package utilized in the ACEVAL/AIMVAL program. The model involves a new technique of representing the mechanics of the air-to-air missile's flight by using quaternions.

The hypercomplex quaternions were not used regularly in a computer at the Aeronautical Systems Division. Major Ken Madsen (CMIST) expressed a desire for an analysis of how quaternions are utilized in the model to be able to make future changes if necessary. In addition, he requested a modification to the infrared signature subroutine (SIGNAL) which would incorporate infrared data generated for the cruise missile and not currently in the model.

Background

In 1843 Sir William Rowan Hamilton developed a new algebra of quadruples of numbers which he named quaternions. His concept was intended to represent motion in three-space, predating the familiar vector methods of J. Willard Gibbs. Hamilton's quaternion theory, although the foundation of vector algebra, was from the viewpoint of the student of mathematical physics, a confusing mixture of scalars and vectors. The vector which represented three-dimensional motion seemed to play a servile role as part of the quaternion. The real power of the quaternion, however, came from the vector aspects of the algebra which was later extracted by J. Willard Gibbs. Gibbs simplified the quaternion methods of Hamilton and originated the vector analysis techniques that we are familiar with today (Ref 3).

Because the concept of a three-dimensional vector was more acceptable than the hypercomplex quaternion (Ref 9), vector algebra, a subset of quaternion algebra, was adopted by the mathematical community to describe three-dimensional motion. With the advent of recent computer missile simulations, however, different computational methods were required to reduce the computation times to the real times of flight. To accomplish this, the concept of quaternion algebra was resurrected and found capable of not only reducing computation time but also storage space normally required by the nine element direction cosine matrices.

II Subroutine SIGNAL

The missile model obtained by CMIST was designed to work with actual launch and target aircraft in mock combat on a controlled range. While the aircraft maneuver, data is relayed to a range computer by telemetry pods. The computer tracks and stores their flight profiles and also does real time missile launch calculations using the missile model to determine if a hit or miss is achieved.

The missile model acquired by CMIST was to do some effectiveness studies of several air-to-air missiles against the cruise missile. The model, however, did not contain the required data to calculate the infrared signal generated by the cruise missile and could not be used until the data was incorporated into the model.

The infrared signal is calculated within the model in a subroutine called SIGNAL. The subroutine then uses the approximation assumption of the basic model that the signal strength is directly proportional to the maximum line of sight tracking rate. Utilizing this assumption, the subroutine compares the calculated signal to several threshold values to determine if the strength of the signal is adequate for the gimbaled seeker head to track the target aircraft at the calculated line of sight rate.

The modification proposed not only involved the incorporation of the infrared data, but also required a redefinition of the line of sight reference. This involved determining the line of sight in the target coordinate system by means discussed in the Methodology.

III Methodology

Two major problems were encountered in the modification of the CMIST missile model. First, the model did not utilize standard direction cosine coordinate transformations, an accepted technique to represent spatial relationships among several coordinate systems, in SIGNAL. Secondly, the method employed by SIGNAL involved hypercomplex quaternions which are foreign to many engineering students. (The interested reader will find a discussion of quaternion development and their properties in Appendix A.) This section discusses the relative merits of direction cosines and quaternions, with a justification for the methodology selected.

Direction Cosines

Direction cosines have been the accepted engineering technique to represent spatial relationships among several coordinate systems. They are relatively easy to use because direction cosines follow the rules of conventional vector algebra and can be stored in a computer as elements of an array. A vector can then be defined in another coordinate system by utilizing the array as a directional cosine matrix and using matrix multiplication to transform the vector coordinates. For example, the vector $\vec{x}$ will be redefined in some primed coordinate system by multiplying it by the Direction Cosine Matrix (DCM) which links the unprimed coordinate system to the primed system

$$\vec{x}(DCM) = \vec{x}' \quad .$$

Direction cosines, although familiar and relatively easy to use on a computer also have several disadvantages. These consist of

1) Each matrix requires a nine element storage allocation.

2) Trigonometric functions are required to form the elements.

3) Many multiplications of the elements must be performed in a proper order.

The direction cosine method would result in a major rewrite of the subroutine SIGNAL but would enable a familiar engineering method to be employed.

Quaternion Method

The quaternion method employed by the model derives a quaternion to relate one coordinate system to another. This methodology is discussed in Appendix B. The quaternion q then operates on a vector $\vec{x}$ converting it to $\vec{x}'$ by

$$q^{-1} \vec{x} q$$

where q^{-1} is the inverse of q. (The interested reader is again referred to Appendix A.)

Quaternions consist of only four elements and thus are advantageous over direction cosines since they require fewer arithmetic operations and less storage space. The disadvantage of quaternions is the requirement of understanding their unique properties and operating rules.

Selection

To avoid a major revision of the computer code, quaternions were selected to modify SIGNAL. First, quaternions did not require a conversion of the vectors from the quaternion factored format into the conventional format used by direction cosines. Secondly, after the operations were completed the resulting vectors would have to be reconverted to the factored format used by the rest of the program. Finally, the construction of direction cosine matrices could be avoided along with a dimensioning of their storage space. Quaternions were perceived to be the way to go because their methods would have to be understood in either case.

IV Modification of SIGNAL

Data Incorporation

The modification of the subroutine was to incorporate the infrared data supplied by CMIST. The data used was generated by a computer program into a set of data points at various azimuths and elevations about the cruise missile. This data was then incorporated into several arrays which served as a tabulated reference, the idea being to enter an array with an appropriate range and angle off from the target and be able to select the relative infrared signal strength.

Coding

The principal problem in obtaining the infrared signal was to convert the missile seeker line of sight to the target coordinate system so the tabulated data could be used. The range vector is defined in the universal coordinate system. This same vector in the quaternion factored format is:

$$\vec{r} = (r_1 + kZ)i \quad .$$

A quaternion q, which relates the universal coordinate system to that of the target coordinate system is used to redefine $\vec{r}$ in the target system by

$$q^{-1}\,\vec{r}\,q \quad .$$

The interested reader is referred to the methods of Appendix B.

With the range vector defined in target coordinates, the relative position of the seeker line of sight in azimuth and elevation can be determined. With this relative position, a data point can be computed from the arrayed information. Using this data point, the received signal strength is reduced by a factor of one over the range squared. Upon determination of the strength of the received signal, the program reverts to the original line of sight rate tolerance calculations and the simulation continues.

Validation

The modification to SIGNAL was validated by comparing the infrared signature output of the old model with that of the modified routine. To accomplish this, the function which formerly computed the infrared signal based upon aspect angle and range was extracted from the original model. Using this extracted function, a series of data points were generated to fill the azimuth and elevation data arrays used by the modified routine. Once the arrays were filled, infrared signal strengths were generated using both models for various aspects and ranges of three, six and nine thousand feet.

Old Model

The original infrared data generating function used in the old model, utilized data generated for a particular aircraft by the Navy at China Lake. This data was plotted into several cardioid shaped curves (Figure 1) which varied in size as a function of range and altitude. A function was

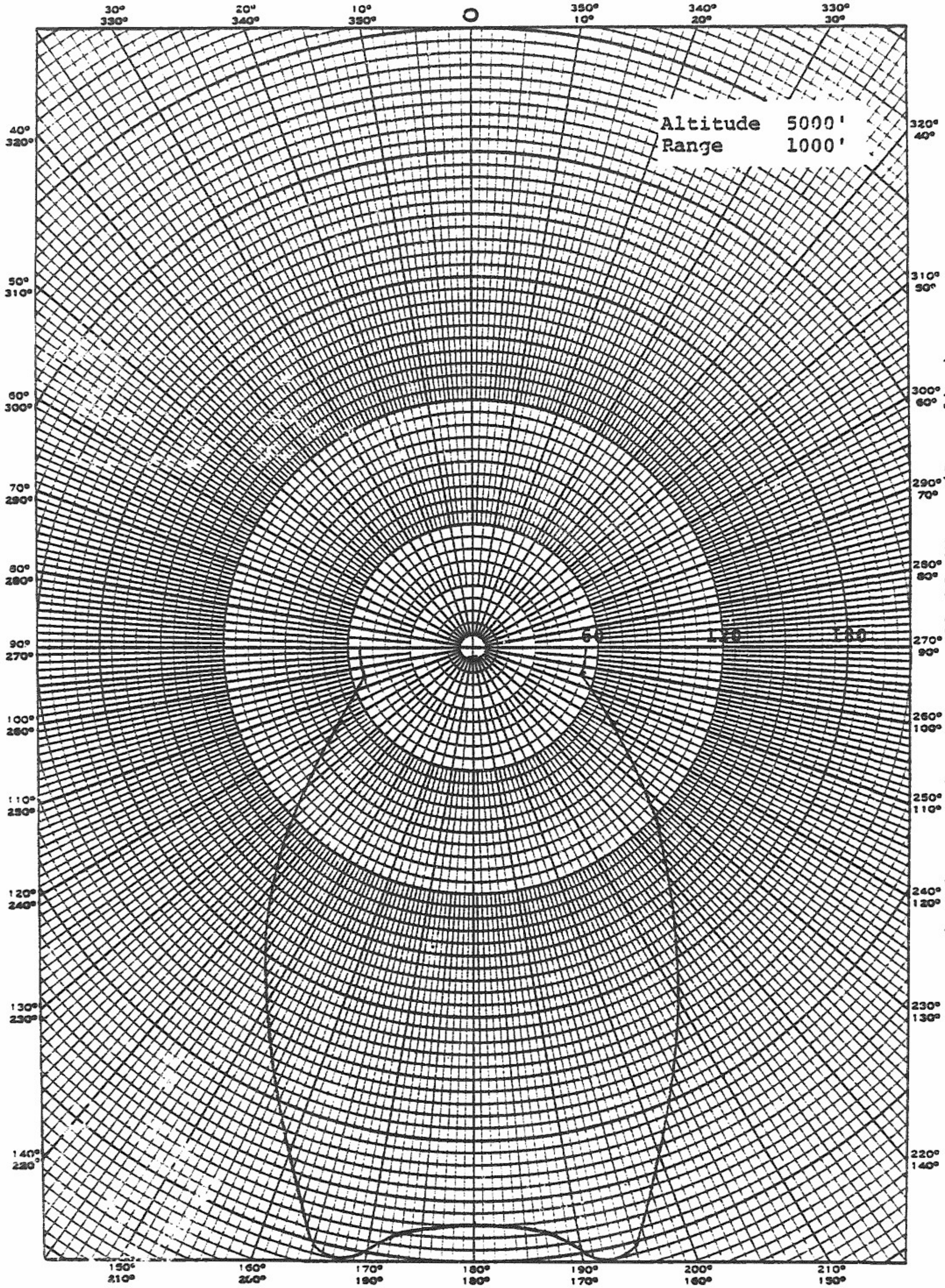

Fig. 1. IR Signature Cardioid (Watts/Steradian)

fit to these curves for different aspects with parameters of atmospheric attenuation, range and the cosine of the aspect angle. The old model used this functional format to determine the infrared signal. A unique feature of this function was that the output was in the form of decibels which are called Phasey dB's. The data supplied by China Lake was in decibels formed by taking the natural logarithm of the signal and multiplying it by a factor of ten rather than the standard method of ten times the logarithm (base 10) of the signal. The model uses these Phasey dB's in all of its signal comparisons, something to be aware of when trying to follow the logic.

Comparing Results

Once the old infrared function was understood, data points were generated for 1000, 5000 and 13,000 feet ranges at various aspects and elevations. These data points (in watts/steradian) were next inserted into the azimuth and elevation arrays utilized by the modified routine. (The interested reader is referred to Appendix D.)

Once both infrared routines were set up using information based upon the same aircraft, a comparison set of data was generated by both methods to determine the accuracy of the new infrared routine. The results are as shown in Table I.

Some typical launch ranges for an IR missile at various aspects in the rear hemisphere of the target (180° being a direct tail shot) were selected. A comparison of the old and new IR signature routines shows that the new model varies at most by less than two-and-one-half percent from the old and that at an extreme aspect angle.

TABLE I

Comparison of Old and New Models

Range	Aspect	IR Signal (Phasey dB) Old	New	% Difference
3,000	180	102.8	103.4	0.6
6,000	180	87.9	88.0	0.1
9,000	180	79.1	79.4	0.3
3,000	170	104.2	103.6	0.6
6,000	170	89.0	87.7	1.5
9,000	170	80.0	78.9	1.4
3,000	165	102.7	103.7	1.0
6,000	165	87.3	87.5	0.2
9,000	165	78.3	78.7	0.5
3,000	150	98.0	99.2	1.2
6,000	150	82.2	82.5	0.4
9,000	150	73.0	73.5	0.7
3,000	140	94.6	95.8	1.3
6,000	140	78.7	78.7	0.0
9,000	140	69.3	69.6	0.4
3,000	135	92.9	94.3	1.5
6,000	135	76.8	77.1	0.4
9,000	135	67.3	67.9	0.9
3,000	120	87.2	88.8	1.8
6,000	120	70.7	71.1	0.6
9,000	120	61.0	61.6	1.0
3,000	105	83.1	84.9	2.2
6,000	105	66.3	66.7	0.6
9,000	105	56.4	57.1	1.2
3,000	90	82.6	84.6	2.4
6,000	90	65.5	65.9	0.6
9,000	90	55.4	56.1	1.3

V Summary and Conclusions

Quaternions, although initially appearing to be a combination of a vector and a scalar, possess several desirable properties when compared to conventional vector methods. These properties have been applied in a missile model which has been acquired by the CMIST branch at ASD. The model was obtained to evaluate several air-to-air missiles against a cruise missile target.

To use the model, data which represent the cruise missile characteristics had to be incorporated. This required a working knowledge of some of the internal mechanics of the model and how the quaternions were utilized. A working knowledge of the model's architecture and coding was also required to modify the subroutine SIGNAL which would take infrared data supplied by CMIST and integrate it with the existing model. This research accomplished the incorporation of the required data and compared the results with the infrared data generating function which the model has originally incorporated.

The most significant result of this research was the modification to the subroutine SIGNAL and a greater detailed description of how quaternions are used in the model than available in the documentation. The modification of SIGNAL saved the Air Force from contracting the work out and is currently capable of generating the required data to complete an evaluation of the cruise missile. Also, utilizing the detailed description of the computer code along with the description of how it applies to the quaternions, further modifications should be possible if necessary in the future.

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Appendix A

Quaternions

Initial Development

Quaternions are specified using four real parameters, a scalar and three other units i, j and k. The four parameters specify the properties that a quaternion possesses when it acts as an operator. These properties allow for the rotation of a line vector through a given angle about an axis through its origin and a stretching of the vector by a given ratio. Of the four parameters; two are required to specify the axis of rotation, one to specify the angle of rotation and one to specify the ratio of stretch (Ref 8).

A quaternion q can be expressed in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k \ .$$

Thus it consists of two parts, a scalar q_0 and a vector

$$\vec{q} = q_1 i + q_2 j + q_3 k$$

where q_1, q_2 and q_3 are the rectangular Cartesian coordinates of a point P and i, j, k represent unit vectors in the positive direction of the x, y and z axes, respectively. The vector defines a line-vector from the origin O to the point P (Ref 8).

Two quaternions q and q' are defined to be equal if (Ref 2)

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$q' = q_0' + q_1' i + q_2' j + q_3' k$$

and $\quad q_0 = q_0'$, $q_1 = q_1'$, $q_2 = q_2'$ and $q_3 = q_3'$.

There also exists a zero quaternion (Ref 2)

$$0 + 0i + 0j + 0k \quad \text{or simply} \quad (0,0,0,0)$$

such that

$$q + (0,0,0,0) = q_0 + q_1 i + q_2 j + q_3 k = q .$$

Two quaternions are added by adding their scalar parts and the corresponding coefficients of their vector parts (Ref 2)

$$q + q' = (q_0+q_0') + (q_1+q_1')i + (q_2+q_2')j + (q_3+q_3')k .$$

Thus the sum of two quaternions is also a quaternion.

For any scalar λ a quaternion can be defined (Ref 2)

$$\begin{aligned} \lambda q &= \lambda(q_0 + q_1 i + q_2 j + q_3 k) \\ &= \lambda q_0 + \lambda q_1 i + \lambda q_2 j + \lambda q_3 k . \end{aligned}$$

Also for $\lambda = -1$

$$(-1)q = -q_0 - q_1 i - q_2 j - q_3 k$$

and

$$q + (-1)q' = (q_0-q_0') + (q_1-q_1')i + (q_2-q_2')j + (q_3-q_3')k .$$

So, quaternions obey the algebraic laws of addition and scalar multiplication (Ref 2).

It can also be shown that for any quaternions p, q and r and any scalars μ and λ (Ref 2)

$$p + q = q + p$$

$$(p+q) + r = p + (q+r)$$

$$\lambda q = q\lambda \ , \ (\lambda\mu)q = \lambda(\mu q)$$

$$(\lambda+\mu)q = \lambda q + \mu q$$

and

$$\lambda(p+q) = \lambda p + \lambda q \ .$$

To examine quaternion multiplication, the vector elements of a quaternion must be examined. In a Cartesian coordinate system, three mutually perpendicular axes exist: x, y and z. For a right-handed system, the positive directions are as shown in Figure 2. Rotation is defined as positive when viewed from the origin (0) as a clockwise rotation of the coordinate axis. Thus rotation from x to y about z, y to z about x, and z to x about y are all positive rotations. Counterclockwise rotation is defined as negative (Ref 7).

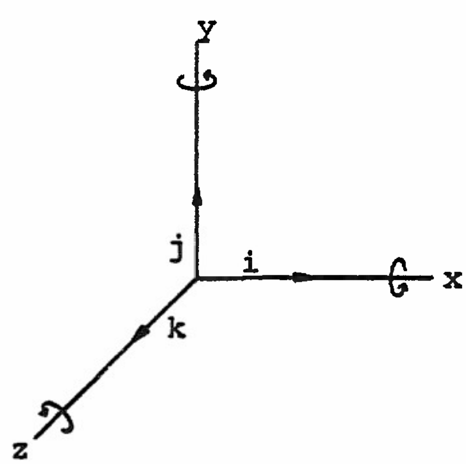

Fig. 2. Cartesian Coordinate System

The vectors i, j and k are defined as unit vectors oriented along the x, y and z axes, with each specifying a direction along its respective axis and possessing a magnitude of one. The rotation of the axis can be defined by the multiplication of two of the unit vectors, i.e. the multiplication of i into j or ij is defined to be the turning of j by +90° in the plane normal to i (Ref 6). Therefore

$$ij = k .$$

Similarly,

$$jk = i , \; ki = j$$

and $$ji = -k , \; kj = -i , \; ik = -j .$$

Furthermore, since

$$ij = k$$

and $$ik = -j$$

then $$-j = ik = i(ij) = i^2 j$$

or $$i^2 j = -j$$

therefore $$i^2 = -1 .$$

Similarly $$j^2 = k^2 = ijk = -1 .$$

All the familiar algebraic rules of multiplication except for the commutative law are valid for quaternions.

This relates to the fact that the order in forming the products of i, j and k determines positive and negative rotations and, therefore, the order must be preserved (Ref 6).

From the previous discussion a quaternion product can be defined (Ref 2).

$$\begin{aligned} qr &= (q_0 + q_1 i + q_2 j + q_3 k)(r_0 + r_1 i + r_2 j + r_3 k) \\ &= q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 + q_0 (r_1 i + r_2 j + r_3 k) \\ &\quad + r_0 (q_1 i + q_2 j + q_3 k) + (q_2 r_3 - q_3 r_2) i \\ &\quad + (q_3 r_1 - q_1 r_3) j + (q_1 r_2 - q_2 r_1) k \\ &= q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 + q_0 (r_1 i + r_2 j + r_3 k) \\ &\quad + r_0 (q_1 i + q_2 j + q_3 k) + \begin{vmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} . \end{aligned}$$

Now consider a vector $\vec{v}$, which is a special case of a quaternion whose scalar portion is zero

$$\vec{v} = 0 + ai + bj + ck \quad .$$

The quaternion product of two vectors $\vec{v}$ and $\vec{v}'$ would be

$$\begin{aligned} \vec{v}\vec{v}' &= 0 - aa' - bb' - cc' + 0(a'i + b'j + c'k) \\ &\quad + 0(ai + bj + ck) + \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix} \end{aligned}$$

or

$$\vec{v}\vec{v}' = (aa' + bb' + cc')(-1) + \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix} .$$

There exists a unit sphere with origin O and an orthagonal coordinate system as shown in Figure 3. The vector $\vec{v}$ is defined as oriented along i with vector $\vec{v}'$ located in the i, j plane, both with origins at O and $\vec{v}'$ forming an angle θ with $\vec{v}$. ($\theta < 90°$)

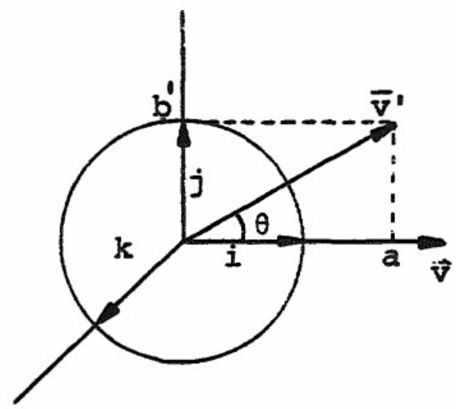

Fig. 3. Unit Sphere

The vectors are now defined

$$\vec{v} = ai$$

and $$\vec{v}' = a'i + b'j \quad . \quad a' = a \quad .$$

By definition

$$|\vec{v}| = a \; , \; |\vec{v}'| = [(a')^2 + (b')^2]^{1/2}$$

$$\text{Sin}\theta = \frac{b'}{[(a')^2 + (b')^2]^{1/2}} = \frac{b'}{|\vec{v}'|}$$

$$\text{Cos}\theta = \frac{a'}{[(a')^2 + (b')^2]^{1/2}} = \frac{a'}{|\vec{v}'|}$$

$$\therefore \quad \vec{v} = |\vec{v}|i \quad \text{and} \quad \vec{v}' = |\vec{v}'|\cos\theta i + |\vec{v}'|\sin\theta j$$

A vector product can now be defined as

$$\vec{v}\vec{v}' = |\vec{v}|i\ [|\vec{v}'|(i\cos\theta + j\sin\theta)]$$

$$= |\vec{v}||\vec{v}'|(-\cos\theta + k\sin\theta)$$

or

$$\vec{v}\vec{v}' = -|\vec{v}||\vec{v}'|\cos\theta + |\vec{v}||\vec{v}'|k\sin\theta \quad .$$

Substituting

$$\vec{v}\vec{v}' = -a\left(\sqrt{(a')^2+(b')^2}\right)\cos\theta + a\left(\sqrt{(a')^2+(b')^2}\right)k\sin\theta$$

but

$$\sqrt{(a')^2+(b')^2}\ \cos\theta = a'$$

and

$$\sqrt{(a')^2+(b')^2}\ \sin\theta = b'$$

so

$$\vec{v}\vec{v}' = -aa' + ab'k \quad . \qquad (1)$$

Now consider the quaternion vector product

$$\vec{v}\vec{v}' = -(aa' + bb' + cc') + \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix}$$

but

$$b = c = c' = 0$$

so

$$\vec{v}\vec{v}' = -aa' + \begin{vmatrix} i & j & k \\ a & 0 & 0 \\ a' & b' & 0 \end{vmatrix}$$

or

$$\vec{v}\vec{v}' = -aa' + ab'k \qquad (2)$$

which is the same as (1). The product of two vectors can be defined as a quaternion whose scalar part is

$$S(\vec{v}\vec{v}') = -aa'$$

or equivalently $-|\vec{v}||\vec{v}'|\cos\theta$, and vector part is

$$V(\vec{v}\vec{v}') = ab'k$$

or equivalently $|\vec{v}||\vec{v}'|k\sin\theta$. These correspond with the scalar and vector products as defined in the vector algebra of J. Willard Gibbs (Ref 2). Gibbs defined the scalar or dot product of two vectors $\vec{u}$ and $\vec{v}$ as

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

and the vector or cross product as

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta\vec{e}$$

where $\vec{e}$ is a unit vector perpendicular to $\vec{u}$ and $\vec{v}$ and θ is the angle between u and v. Thus a quaternion product may be defined as

$$\vec{q}\vec{q}' = \vec{q} \times \vec{q}' - \vec{q} \cdot \vec{q}' \quad .$$

Properties of Quaternions

The conjugate of a quaternion is similar to the conjugate of a complex number (Ref 2). If

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

then the conjugate of q or q*

$$q^* = q_0 - q_1 i - q_2 j - q_3 k \quad .$$

The product of a quaternion q and its conjugate q* is defined as the Norm of q or N(q)

$$\begin{aligned} N(q) = qq^* &= (q_0+q_1 i+q_2 j+q_3 k)(q_0-q_1 i-q_2 j-q_3 k) \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

and further

$$q^*q = qq^* \qquad \text{(Ref 2)}$$

The Norm of a quaternion is a scalar and $N(q) = 0$ implies that $q_0 = q_1 = q_2 = q_3 = 0$. If $N(q) = 1$, q is called a unit quaternion (Ref 2).

From the quaternion product of two vectors

$$\vec{v}\vec{v}' = -(aa' + bb' + cc') + \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix} ,$$

the conjugate of the quaternion formed by the product is

$$(\vec{v}\vec{v}')^* = -(aa' + bb' + cc') - \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix}$$

or equivalently

$$(\vec{v}\vec{v}')^* = \vec{v}'\vec{v} = -(aa' + bb' + cc') + \begin{vmatrix} i & j & k \\ a' & b' & c' \\ a & b & c \end{vmatrix} .$$

For any two quaternions

$$q = (q_0 + \vec{q}) , \; q' = (q_0' + \vec{q}')$$

whose product is

$$qq' = (q_0 + \vec{q})(q_0' + \vec{q}') = q_0q_0' + q_0\vec{q}' + q_0'\vec{q} + \vec{q}\vec{q}'$$

the conjugate of their product will be

$$(qq')^* = q_0q_0' - q_0\vec{q}' - q_0'\vec{q} + (\vec{q}\vec{q}')^*$$

$$= q_0q_0' - q_0\vec{q}' - q_0'\vec{q} + \vec{q}'\vec{q} \; ,$$

or

$$(qq')^* = (q_0' - \vec{q}')(q_0 - \vec{q}) \quad .$$

Therefore, the conjugate of the product of two quaternions is equal to the product of the individual conjugates taken in reverse order (Ref 2).

$$(qq')^* = (q')^*(q)^*$$

From this property, the Norm of the product of two quaternions can be determined.

$$N(qq') = (qq')(qq')^*$$

$$= q(q')(q')^*q^*$$

$$= qN(q')q^*$$

Since N(q) is a scalar

$$qN(q')q^* = qq^*N(q') = N(q)N(q') \quad .$$

If q is not the zero quaternion then $N(q) \neq 0$. Using this fact the inverse of q can now be defined (Ref 6).

$$N(q) = qq^*$$

$$1 = q \frac{q^*}{N(q)}$$

or

$$q^{-1} = \frac{q^*}{N(q)}$$

and

$$qq^{-1} = q \frac{q^*}{N(q)} = \frac{N(q)}{N(q)} = 1 \quad .$$

It can also be shown that

$$N(q^{-1}) = \frac{1}{N(q)} \quad .$$

Using the concept of the inverse of a quaternion, the property of division can now be addressed. For any three quaternions p, q and r where

$$qr = p \quad \text{or} \quad rq = p$$

solutions for r may be obtained by multiplying both sides of the equations by the inverse of q. In this way two solutions are apparent

$$q^{-1}qr = q^{-1}p \; ; \quad rqq^{-1} = pq^{-1}$$

or

$$r_1 = q^{-1}p \; ; \qquad r_2 = pq^{-1} \; .$$

where r_1 is called the right hand quotient of p by q and r_2 is the left hand quotient of p by q. These solutions in general are different and demonstrate again that order must be preserved in quaternion multiplication (Ref 2).

Finally, from the Norm of q

$$\begin{aligned} N(q) &= qq^* \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= q_0^2 + |\vec{q}|^2 \end{aligned}$$

and the magnitude of q is defined

$$|q| = [N(q)]^{1/2} = [q_0^2 + q_1^2 + q_2^2 + q_3^2]^{1/2}$$

$$|q| = [q_0^2 + |\vec{q}|^2]^{1/2} \quad . \quad \text{(Ref 2)}$$

Product of a Quaternion and a Vector

To review, the quaternion product of two vectors $\vec{u}$ and $\vec{v}$

$$\vec{u}\vec{v} = \vec{u} \times \vec{v} - \vec{u} \cdot \vec{v}$$

and the product of two quaternions q and r

$$qr = (q_0+\vec{q})(r_0+\vec{r}) = q_0 r_0 + q_0\vec{r} + r_0\vec{q} + \vec{q} \times \vec{r} - \vec{q}\cdot\vec{r} \quad .$$

The conjugate of q is

$$q^* = q_0 - \vec{q}$$

and the Norm of q is

$$N(q) = qq^* = q_0^2 + |\vec{q}|^2$$

where

$$\vec{q} = q_1 i + q_2 j + q_3 k$$

and

$$|\vec{q}| = (q_1^2 + q_2^2 + q_3^2)^{1/2} \quad .$$

The magnitude of the quaternion q is

$$|q| = \Big(N(q)\Big)^{1/2} = (q_0^2 + |\vec{q}|^2)^{1/2} \quad .$$

For a vector $\vec{x}$ and a quaternion q, where

$$q = q_0 + \vec{q} \; , \; q^{-1} = \frac{q^*}{N(q)} = \frac{q_0 - \vec{q}}{N(q)}$$

the quaternion product of q and $\vec{x}$ is

$$q\vec{x} = q_0\vec{x} + (\vec{q}X\vec{x}) - (\vec{q}\cdot\vec{x})$$

$$\vec{x}q = q_0\vec{x} + (\vec{x}X\vec{q}) - (\vec{x}\cdot\vec{q}) \quad .$$

The right hand quotient of q^{-1} and $\vec{x}$ is

$$q^{-1}\vec{x} = \frac{q^*}{N(q)}\vec{x} = \frac{(q_0 - \vec{q})\vec{x}}{N(q)}$$

$$= \frac{q_0\vec{x} - (\vec{q}X\vec{x}) + (\vec{q}\cdot\vec{x})}{N(q)}$$

or

$$q^{-1}\vec{x} = \frac{q_0\vec{x} + (\vec{x}X\vec{q}) + (\vec{q}\cdot\vec{x})}{N(q)} \quad .$$

The left hand quotient of $\vec{x}$ and q^{-1} is

$$\vec{x}q^{-1} = \frac{\vec{x}(q_0 - \vec{q})}{N(q)}$$

$$= \frac{q_0\vec{x} - (\vec{x}X\vec{q}) + (\vec{x}\cdot\vec{q})}{N(q)}$$

or

$$\vec{x}q^{-1} = \frac{q_0\vec{x} + (\vec{q}X\vec{x}) + (\vec{x}\cdot\vec{q})}{N(q)} \quad .$$

In general, the product of a vector and a quaternion yields a quaternion whose scalar part is the dot product $(\vec{x}\cdot\vec{q})$ and whose vector part is the scalar of the quaternion times the vector and the vector cross product $(q_0\vec{x} + (\vec{q}X\vec{x}))$. However, in the case where $\vec{q}$ and $\vec{x}$ are perpendicular $(\vec{q}\cdot\vec{x} = 0)$, and the vector $q_0\vec{x} + \vec{q}X\vec{x}$ is obtained. In this case both $q_0\vec{x}$ and $\vec{q}X\vec{x}$ are normal to each other and the resultant vector

$q_0\vec{x} + \vec{q}X\vec{x}$ is normal to $\vec{q}$. Figure 4 shows the geometrical relationships of $\vec{x}q$ and $\vec{q}$ oriented into the page (Ref 5).

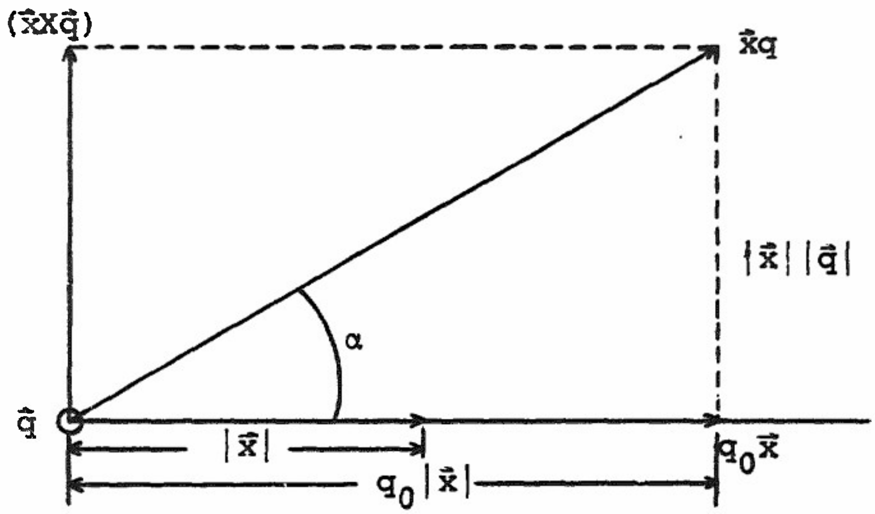

Fig. 4. Vector Times Quaternion

It can be seen that

$$\tan\alpha = \frac{|\vec{x}|\,|\vec{q}|}{q_0|\vec{x}|} = \frac{|\vec{q}|}{q_0}$$

also

$$|\vec{x}q| = [(|\vec{x}|\,|\vec{q}|)^2 + (q_0|\vec{x}|)^2]^{1/2}$$

$$= [|\vec{x}|^2|\vec{q}|^2 + q_0^2|\vec{x}|^2]^{1/2}$$

$$= [|\vec{x}|^2(q_0^2 + |\vec{q}|^2)]^{1/2}$$

$$= [|\vec{x}|^2|q|^2]^{1/2}$$

$$|\vec{x}q| = |\vec{x}|\,|q| \quad .$$

Figure 4 shows that α is a negative rotation around $\vec{q}$, and in summary multiplying a vector $\vec{x}$ on the right by a quaternion q (where $\vec{q}$ is the vector part of q) rotates the vector $\vec{x}$ negatively by an angle $\tan^{-1}\left(\frac{|\vec{q}|}{q_0}\right)$ and changes its length by a $|q|$ factor (Ref 5).

Now consider the right hand quotient of q^{-1} and $\vec{x}$,

$$q^{-1}\vec{x} = \frac{q_0\vec{x} + (\vec{x}X\vec{q}) + (\vec{q}\cdot\vec{x})}{N(q)} \quad .$$

Since $\vec{x}$ and $\vec{q}$ are perpendicular,

$$\vec{q}\cdot\vec{x} = 0$$

and the results are similar to the previous $\vec{x}q$ except the magnitude is reduced.

$$|q^{-1}\vec{x}| = \left[\left(\frac{|\vec{x}|\,|\vec{q}|}{N(q)}\right)^2 + \left(\frac{q_0|\vec{x}|}{N(q)}\right)^2\right]^{1/2}$$

$$= \left[\left(\frac{|\vec{x}|}{N(q)}\right)^2 |q|^2\right]^{1/2}$$

$$= \frac{|\vec{x}|\,|q|}{N(q)}$$

Since $|q| = \left(N(q)\right)^{1/2}$ then

$$N(q) = |q|^2$$

Therefore,

$$|q^{-1}\vec{x}| = \frac{|\vec{x}|}{|q|}$$

and

$$\tan\alpha = \frac{\dfrac{|\vec{x}|\,|\vec{q}|}{N(q)}}{\dfrac{q_0|\vec{x}|}{N(q)}} = \frac{|\vec{x}|\,|\vec{q}|}{q_0|\vec{x}|} = \frac{|\vec{q}|}{q_0} \quad .$$

Because $\vec{x}$ is perpendicular to $\vec{q}$, $\vec{x}q$ is also perpendicular to $\vec{q}$ and if q^{-1} is applied to $\vec{x}q$ to form a vector $\vec{y}$

$$\vec{y} = q^{-1}(\vec{x}q)$$

$$\vec{y} = q^{-1}\vec{x}q \quad .$$

The vector $\vec{y}$ is rotated from $\vec{x}$ negatively by an angle of $\theta = 2\alpha$. Now consider a special case where the vector $\vec{x}$ lies along $\vec{q}$ and has a magnitude of $\lambda|\vec{q}|$. Then

$$\vec{x} = \lambda\vec{q}$$

where λ is a real number. If the same operation is carried out

$$\vec{y} = q^{-1}\vec{x}q = q^{-1}(\lambda\vec{q})q$$

$$\begin{aligned}
\vec{y} &= q^{-1}[\lambda(q - q_0)]q \\
&= \lambda[q^{-1}q - q^{-1}q_0]q \\
&= \lambda[1 - q_0q^{-1}]q \\
&= \lambda[q - q_0q^{-1}q] \\
&= \lambda[q - q_0] \\
&= \lambda\vec{q} = \vec{x} \quad .
\end{aligned}$$

So, a vector that lies along $\vec{q}$ and operated upon by $q^{-1}\vec{x}q$ is not changed in magnitude or direction (Ref 5).

Finally, consider a general vector $\vec{x}$ and a quaternion $q = q_0 + \vec{q}$. The vector $\vec{x}$ can be resolved into components (Ref 5)

$$\vec{x} = \vec{x}_1 + \vec{x}_2$$

along q and perpendicular to q such that

$$\vec{x}_1 = \frac{\vec{q}.\vec{x}}{|\vec{q}|^2}\vec{q}$$

where

$$\frac{\vec{q}\cdot\vec{x}}{|\vec{q}|} = |\vec{x}| \quad \text{along } \vec{q}$$

and

$$\vec{x}_2 = \frac{(\vec{q}X\vec{x}) \text{ X } \vec{q}}{|\vec{q}|^2}\vec{q}$$

where

$$\frac{\vec{q}X\vec{x}}{|\vec{q}|} = |\vec{x}| \quad \text{perpendicular to } \vec{q} \text{ .}$$

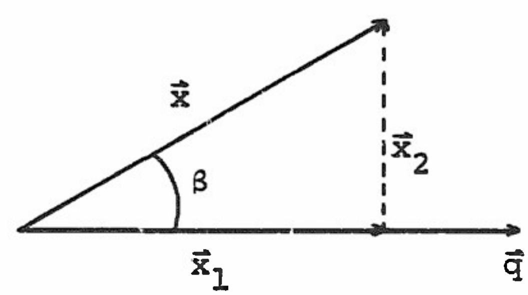

Fig. 5. Components of $\vec{x}$

The dot product of $\vec{q}$ and $\vec{x}$ (oriented as in Figure 5) with the angle β between them is

$$\vec{q}\cdot\vec{x} = |\vec{q}|\,|\vec{x}|\cos\beta$$

or

$$\cos\beta = \frac{\vec{q}\cdot\vec{x}}{|\vec{q}|\,|\vec{x}|} .$$

The cross product of $\vec{q}$ and $\vec{x}$ is

$$\vec{q}X\vec{x} = |\vec{q}|\,|\vec{x}|\sin\beta$$

or

$$\sin\beta = \frac{\vec{q}X\vec{x}}{|\vec{q}|\,|\vec{x}|} .$$

Therefore,

$$|\vec{x}_1| = |\vec{x}|\cos\beta = |\vec{x}| \frac{\vec{q}\cdot\vec{x}}{|\vec{q}|\,|\vec{x}|}$$

or

$$|\vec{x}_1| = \frac{\vec{q}\cdot\vec{x}}{|\vec{q}|}$$

and

$$|\vec{x}_2| = |\vec{x}|\sin\beta = |\vec{x}| \frac{\vec{q}X\vec{x}}{|\vec{q}|\,|\vec{x}|}$$

or

$$|\vec{x}_2| = \frac{\vec{q}X\vec{x}}{|\vec{q}|}$$

Now a vector $\vec{y}$ exists such that

$$\begin{aligned}
\vec{y} &= q^{-1}\vec{x}q \\
&= q^{-1}(\vec{x}_1 + \vec{x}_2)q \\
&= q^{-1}\vec{x}_1 q + q^{-1}\vec{x}_2 q \\
&= \vec{x}_1 + q^{-1}\vec{x}_2 q = \vec{y}_1 + \vec{y}_2
\end{aligned}$$

where

$$\vec{y}_1 = \vec{x}_1 \text{ along } \vec{q}$$

and

$$\vec{y}_2 = q^{-1}\vec{x}_2 q \text{ perpendicular to } \vec{q} \quad .$$

The vector $\vec{y}_2$ is normal to $\vec{q}$ and is a negative rotation about $\vec{q}$ through an angle

$$\theta = 2 \tan^{-1} \frac{|\vec{q}|}{q_0} \quad .$$

When $\vec{q}$ is oriented along the Cartesian x-axis, the rotation from $\vec{x}_2$ to $\vec{y}_2$ is a negative rotation, but this negative space rotation is equivalent to a positive rotation of the coordinate system, with $q^{-1}\vec{x}q$ representing a positive rotation of coordinates and $q\vec{x}q^{-1}$ a negative rotation (Ref 5).

Tilt and Roll Quaternions

To utilize quaternions in computer programs rotating coordinate systems in three dimensions, they must be factored into a convenient format (Ref 1). This format factors the quaternion into two quaternions which divide the rotation into two successive transformations, tilt and roll. Because of this, the factored quaternions are called the tilt quaternion and the roll quaternion (Ref 5).

A quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ can be written as $q = q_0 + q_1 i + k(q_3 + iq_2)$, where $k(q_3 + iq_2) = jq_2 + kq_3$. For any complex number a + bi,

$$k(a + bi) = (a - bi)k ,$$

or k times a complex number is equal to its conjugate times k (Ref 1). Now q can be written

$$q = Q_{01} + kQ_{32}$$

where

$$Q_{01} = q_0 + iq_1 \quad \text{and} \quad Q_{32} = q_3 + iq_2 \quad .$$

The conjugate and Norm can be written:

$$q^* = \overline{Q}_{01} - kQ_{32} \text{ , where } \overline{Q}_{01} = q_0 - iq_1$$

$$N(q) = |Q_{01}|^2 + |Q_{32}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

Similarly, a vector $\vec{x} = ix_1 + jx_2 + kx_3$, may be factored into

$$\vec{x} = ix_1 + kZ \quad \text{where } Z = x_3 + ix_2$$

or

$$\vec{x} = (x_1 + kZ)i \quad \text{where } Z = x_2 - ix_3 \quad .$$

For any two quaternions q and r, their product may now be written

$$\begin{aligned} qr &= (Q_{01} + kQ_{32})(R_{01} + kR_{32}) \\ &= Q_{01}R_{01} + Q_{01}kR_{32} + kQ_{32}R_{01} + kQ_{32}kR_{32} \\ &= Q_{01}R_{01} + k\overline{Q}_{01}R_{32} + kQ_{32}R_{01} + k^2\overline{Q}_{32}R_{32} \\ &= (Q_{01}R_{01} - \overline{Q}_{32}R_{32}) + k(\overline{Q}_{01}R_{32} + Q_{32}R_{01}) \end{aligned}$$

Consider a quaternion $q = Q_{01} + kQ_{32}$ with $q_0 \neq 0$. The quaternion can be factored in two parts

$$q = (1 + k\frac{Q_{32}}{Q_{01}})Q_{01}$$

$$= (1 + kT)Q_{01} = tr \quad \text{where } T = \frac{Q_{32}}{Q_{01}}$$

$$\text{or} \quad T = t_3 + it_2 \quad .$$

The tilt quaternion t is therefore

$$t = 1 + kT$$

and the roll quaternion is

$$r = Q_{01} = q_0 + iq_1 \quad .$$

Tilt

The vector portion of the tilt quaternion kT is located in the plane normal to the x-axis. It transforms the x-axis by $t^{-1}\vec{x}t$ a new axis x' representing a tilt from x results (Figure 6).

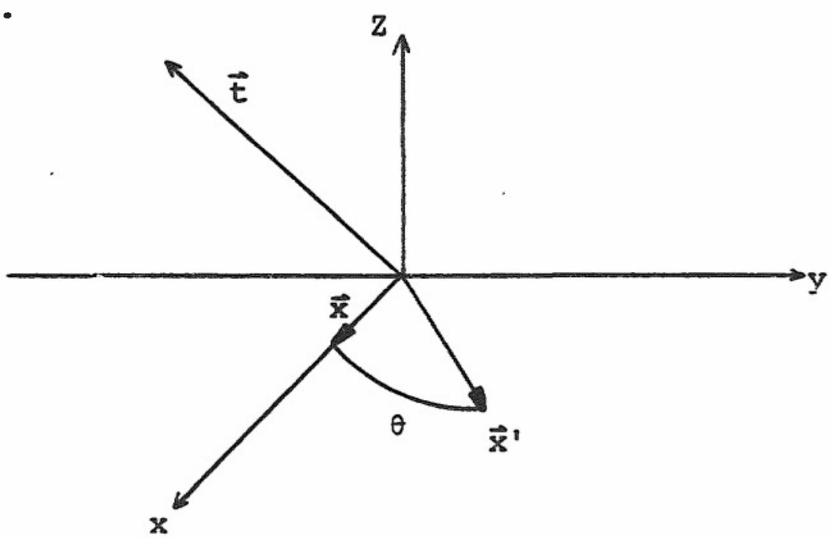

Fig. 6. Tilt

The vector $\vec{t}$ remains normal to $\vec{x}$ and $\vec{x}'$ and represents the axis of rotation of $\vec{x}$ to $\vec{x}'$ through an angle θ. The tangent of θ being

$$\tan\frac{\theta}{2} = \frac{|\vec{t}|}{t_0} = |\vec{t}| = |T| = (t_2^2 + t_3^2)^{1/2} \quad .$$

The y and z axis are also tilted to y' and z' axis but this is not shown in Figure 6 (Ref 5).

<u>Roll</u>

The roll quaternion r has the form $r = (1 + ir_1)$. The inverse is

$$r^{-1} = \frac{r^*}{N(r)} = \frac{r^*}{rr^*} \quad .$$

So for a given vector $\vec{x} = (x_1 + kZ)i$ and an $\vec{x}' = (x_1' + kZ')i$ where $Z = x_2 - ix_3$ and $Z' = x_2' - ix_3'$,

$$\vec{x}' = r^{-1}\vec{x}r \quad . \quad \text{(Ref 5)}$$

r represents a roll about the x-axis through an angle ϕ. This is accomplished by

$$(x_1' + kZ')i = r^{-1}(x_1 + kZ)ir$$

or

$$(x_1' + kZ')i = (x_1 + kZ\frac{r}{r^*})i$$

Thus

$$x_1' = x_1$$

and

$$Z' = Z\frac{r}{r^*} = Z\frac{r \cdot r}{rr^*} = Z\frac{r^2}{N(r)} = Z\frac{r^2}{|r|^2} .$$

The angle ϕ is defined by

$$\tan\frac{\phi}{2} = \frac{|\vec{r}|}{r_0} = |\vec{r}| = r_1 . \quad \text{(Ref 5)}$$

Therefore

$$r = 1 + i \tan\frac{\phi}{2}$$

$$|r|^2 = rr^* = 1 + \tan^2\frac{\phi}{2}$$

$$r^2 = (1 + i \tan\frac{\phi}{2})(1 + i \tan\frac{\phi}{2})$$

$$= 1 - \tan^2(\frac{\phi}{2}) + i\, 2 \tan\frac{\phi}{2}$$

and

$$\frac{r^2}{|r|^2} = \cos\phi + i \sin\phi .$$

For a positive roll

$$Z' = Z(\cos\phi + i \sin\phi)$$

or

$$x_2' + ix_3' = (x_2 + ix_3)(\cos\phi + i \sin\phi)$$

and

$$x_2' = x_2\cos\phi - x_3\sin\phi$$
$$x_3' = x_2\sin\phi + x_3\cos\phi .$$

Therefore $$Z' = Ze^{i\phi} . \quad \text{(Ref 5)}$$

Appendix B

Convenient Computer Coding Relations

Complex Factoring

The tilt quaternion t represents a transfer of coordinates from an unprimed system to a primed system (Ref 5). A general vector $\vec{x}$ defined:

$$\vec{x} = (x_1 + kZ)i \quad \text{with} \quad Z = x_2 - ix_3$$

will become

$$\vec{x}' = (x_1' + kZ')i \quad \text{with} \quad Z' = x_2' - ix_3' \quad .$$

This is accomplished by:

$$\vec{x}' = t^{-1}\,\vec{x}\,t$$

where

$$t = 1 + kT \quad \text{and} \quad T = t_3 + it_2 \quad .$$

By expanding

$$\vec{x}' = \frac{(1 - kT)}{N(t)}\,\vec{x}\,(1 + kT)$$

$$= \frac{(1 - kT)}{1 + |T|^2}\,\vec{x}\,(1 + kT)$$

Let

$$d = \frac{2}{1 + |T|^2} \qquad (3)$$

then

$$\vec{x}' = \frac{d}{2}(1 - kT)\,\vec{x}\,(1 + kT)$$

$$= \frac{d}{2}(1 - kT)(x_1 + kZ)i\,(1 + kT)$$

$$= \frac{d}{2}(1 - kT)(x_1 + kZ)(1 - kT)i \quad .$$

Expanding

$$(x_1' + kZ')i = \frac{d}{2}[(x_1 + kZ - kTx_1 - kTkZ)(1 - kT)i]$$

$$= \frac{d}{2}[x_1 - x_1kT + kZ - kZkT - kTx_1 + kTx_1kT - kTkZ + kTkZkT]i$$

$$= \frac{d}{2}[x_1 - kx_1T + kZ + \bar{Z}T - kTx_1 - x_1\bar{T}T + \bar{T}Z - kT^2\bar{Z}]i$$

$$= \frac{d}{2}[x_1 + \bar{Z}T - x_1\bar{T}T + \bar{T}Z + k(Z - 2Tx_1 - T^2\bar{Z})]i$$

$$(x_1' + kZ')i = \frac{d}{2}[x_1 - x_1|T|^2 + \bar{Z}T + \bar{T}Z + k(Z - 2Tx_1 - T^2\bar{Z})]i$$

$$x_1' + kZ' = \frac{d}{2}[x_1 - x_1|T|^2 + \bar{Z}T + \bar{T}Z] + \frac{d}{2}k[Z - 2Tx_1 - T^2\bar{Z}] \quad .$$

Therefore

$$x_1' = \frac{d}{2}[x_1(1 - |T|^2) + \bar{Z}T + \bar{T}Z]$$

and

$$Z' = \frac{d}{2}[Z - 2Tx_1 - T^2\bar{Z}] \quad . \quad \text{(Ref 5)} \qquad (4)$$

Let

$$b = x_1 + x_1'$$

then

$$b = x_1 + \frac{d}{2}\,[x_1(1 - |T|^2) + \bar{Z}T + \bar{T}Z]$$

but

$$\bar{Z}T = (x_2 + ix_3)(t_3 + it_2) = x_2t_3 - x_3t_2 + i(x_2t_2 + x_3t_3)$$

and

$$\bar{T}Z = (t_3 - it_2)(x_2 - ix_3) = x_2t_3 - x_3t_2 - i(x_3t_3 + x_2t_2) \ .$$

So

$$\bar{Z}T + \bar{T}Z = 2x_2t_3 - 2x_3t_2$$

and

$$b = \frac{d}{2}\,[\frac{2}{d}\,x_1 + x_1(1 - |T|^2) + 2\mathrm{Re}(\bar{T}Z)]$$

where

$$\mathrm{Re}(\bar{T}Z) = x_2t_3 - x_3t_2 \ .$$

Substituting

$$\frac{2}{d} = 1 \div |T|^2$$

$$b = \frac{d}{2}\,[x_1(1 + |T|^2) + x_1(1 - |T|^2) + 2\mathrm{Re}(\bar{T}Z)]$$

$$b = \frac{d}{2}\,[2x_1 + 2\mathrm{Re}(\bar{T}Z)]$$

$$b = d\,[x_1 + \mathrm{Re}(\bar{T}Z)] \qquad (5)$$

$$b = d\,[x_1 + t_3x_2 - t_2x_3] \ . \qquad \text{(Ref 5)} \qquad (6)$$

From (4) we have

$$Z' = \frac{d}{2}[Z - 2Tx_1 - T^2\bar{Z}] \tag{7}$$

Subtracting Z from both sides

$$\begin{aligned} Z' - Z &= \frac{d}{2}[Z - 2Tx_1 - T^2\bar{Z}] - Z \\ &= \frac{d}{2} Z - Z - \frac{d}{2}[2Tx_1 + T^2\bar{Z}] \\ &= Z(\frac{d}{2} - 1) - \frac{d}{2}[2Tx_1 + T^2\bar{Z}] \end{aligned}$$

but

$$\frac{d}{2} - 1 = \frac{1}{1 + |T|^2} - 1 = \frac{-|T|^2}{1 + |T|^2} = \frac{d}{2}(-T\bar{T})$$

Substituting

$$\begin{aligned} Z' - Z &= -\frac{d}{2} ZT\bar{T} - \frac{d}{2}[2Tx_1 + T^2\bar{Z}] \\ &= -\frac{d}{2} T[2x_1 + Z\bar{T} + T\bar{Z}] \\ &= -\frac{d}{2} T[2x_1 + 2\mathrm{Re}(\bar{T}Z)] \\ Z - Z' &= -bT \end{aligned}$$

So

$$Z' = Z - bT \qquad \text{(Ref 5)} \tag{8}$$

A reversal of the tilt transformation

$$\vec{x}' = t^{-1} \vec{x} t$$

yields

$$t \vec{x}' t^{-1} = tt^{-1} \vec{x} tt^{-1} = \vec{x} \ .$$

This leads to the equations of the inverse transformation

$$d = \frac{2}{1 + |T|^2} \tag{9}$$

$$b = d(x_1' - t_3 x_2' + t_2 x_3') \tag{10}$$

$$x_1 = b - x_1' \tag{11}$$

$$Z = Z' + bT \quad . \qquad \text{(Ref 5)} \tag{12}$$

<u>Forming the Tilt for Range</u>

For a given vector r in an unprimed coordinate system there exists a quaternion t which will tilt the vector $\vec{r}$ to the x'-axis (Ref 5). Let $\vec{r}$ be defined by:

$$\vec{r} = (r_1 + kZ)i \quad \text{where} \quad Z = r_2 - ir_3$$

and $|\vec{r}| = R$. Now to determine t which will tilt the x'-axis to $\vec{r}$ by

$$\vec{r}' = t^{-1}\, \vec{r}\, t$$

or

$$(r_1' + kZ')i = t^{-1}\, \vec{r}\, t \quad .$$

But $r_1' = R$ and $Z' = 0$ since $\vec{r}$ is to be oriented along the x'-axis. Thus utilizing

$$b = r_1 + r_1' = r_1 + R$$

and

$$Z' = Z - bT = 0$$

the tilt quaternion $t = 1 + kT$ can be found.

$$bT = Z$$

or

$$T = \frac{Z}{b} = \frac{r_2 - ir_3}{R + r_1}$$

If on the other hand, a quaternion t is desired which will tilt the x'-axis to $\vec{r}$, the following equations are used ($|\vec{r}| = R$, $Z' = 0$, $r_1' = R$).

$$b = d[r_1' - \mathrm{Re}(\bar{T}Z')] = dR$$

where

$$d = \frac{2}{1 + |T|^2}$$

$$r_1 = b - R = R(d - 1)$$

$$Z = bT = dRT$$

where

$$d - 1 = \frac{1 - |T|^2}{1 + |T|^2} \ ; \quad |T| = \tan\frac{\theta}{2}$$

$$d - 1 = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \cos^2(\tfrac{\theta}{2})$$

$$d - 1 = \cos\theta$$

and

$$r_1 = R(d - 1) = R\cos\theta$$

Roll Representation Due to Tilt

Because missile seeker heads are gimbaled to move in azimuth and elevation Euler angles, a tilt of the x-axis will be accomplished by movements constrained by the fixed axes (Ref 5). The tilt quaternion

$$t = 1 + kT$$

where

$$T = \frac{r_2 + ir_3}{R + r_1} = t_3 + it_2$$

transforms from i, j, k to i', j', k' coordinates by tilting i through an angle τ. The case illustrated in Figure 7 shows a change in azimuth followed by a change in elevation (Ref 5). To model this motion, the desired roll angle ϕ is assumed known and through the inverse of this roll transform the coordinates to the tilt position defined by t. Then by using t^{-1} the original position is obtained (Ref 5).

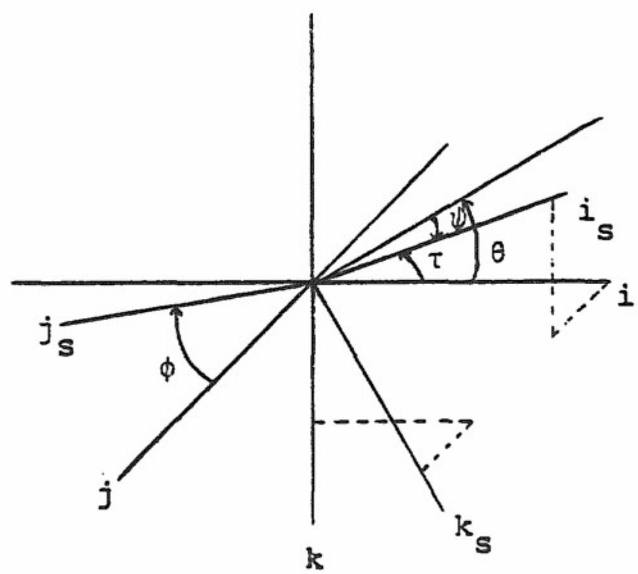

Fig. 7. Gimbaled Tilt (τ)

The j coordinate in the final position following the azimuth and elevation change can be defined in factored form

$$j_s = (j_{s1} + k\ J_s)i$$

where

$$j_s = 0i + j_{s2}j + 0k \quad . \qquad \text{(Ref 5)}$$

Since $j_{s1} = 0$ and $ki = j$

$$J_s = 1$$

For an inverse or negative roll

$$Z' = Ze^{-i\phi}$$

or

$$J'_s = J_s e^{-i\phi} = e^{-i\phi}$$

$$j_{s1} = j'_{s1} = 0$$

and for an inverse tilt

$$t = (1 + kT)$$

$$T = (t_3 + it_2)$$

where

$$d = \frac{2}{1 + |T|^2}$$

and

$$b = d[j'_{s1} - \mathrm{Re}(\overline{T}J'_s)] = -d\mathrm{Re}(\overline{T}e^{-i\phi}) \quad . \quad \text{(Ref 5)}$$

Therefore

$$J''_s = J'_s + bT = e^{-i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})T$$

but

$$J''_s = (j''_{s2} - ij''_{s3}) \quad \text{where} \quad j''_{s3} = 0$$

So

$$-\mathrm{Im}J''_s = 0$$

and

$$\mathrm{Im}[e^{-i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})]T = 0 \quad . \quad \text{(Ref 5)}$$

Since for any complex number A

$$\frac{A - \overline{A}}{2} = \mathrm{Im}(A) \quad \text{and} \quad \frac{A + A}{2} = \mathrm{Re}(A)$$

$$\frac{i}{2}[(e^{-i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})T) - (e^{i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})\overline{T})] = 0$$

Collecting terms

$$e^{-i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})T = e^{i\phi} - d\mathrm{Re}(\overline{T}e^{-i\phi})\overline{T}$$

$$e^{-i\phi} - e^{i\phi} = d\mathrm{Re}(\overline{T}e^{-i\phi})(T - \overline{T})$$

$$e^{-i\phi} - e^{i\phi} = 2id\mathrm{Re}(\overline{T}e^{-i\phi})\mathrm{Im}(T)$$

$$e^{-i\phi} - e^{i\phi} = id\mathrm{Im}(T)(\overline{T}e^{-i\phi} + Te^{i\phi})$$

Multiplying both sides by $e^{i\phi}$

$$1 - e^{2i\phi} = id\mathrm{Im}(T)(\bar{T} + Te^{2i\phi})$$

Collecting terms,

$$1 - id\mathrm{Im}(T)\bar{T} = e^{2i\phi}[1 + id\mathrm{Im}(T)T] \quad . \qquad \text{(Ref 5)}$$

Let

$$A = 1 - id\mathrm{Im}(T)\bar{T}$$

then

$$\bar{A} = 1 + id\mathrm{Im}(T)T$$

and

$$e^{2i\phi} = \frac{1 - id\mathrm{Im}(T)\bar{T}}{1 + id\mathrm{Im}(T)\bar{T}} = \frac{A}{\bar{A}}$$

$$e^{2i\phi} = \frac{A}{\bar{A}} = \frac{A^2}{\bar{A}A} = \frac{A^2}{|A|^2} \quad . \qquad \text{(Ref 5)}$$

Since

$$r = e^{i\phi} = \left(\frac{A^2}{|A|^2}\right)^{1/2} = \frac{A}{|A|} \qquad (13)$$

The roll quaternion r acts through the angle ϕ to restore the tilted coordinates to the zero roll position where the tilt quaternion transforms them to their original position.

Forming a Roll for Range

The roll transfer relation is formed from the relations

$$\frac{r}{r^*} = \frac{A}{|A|} \tag{14}$$

This is accomplished by utilizing the relationships developed previously for a tilt for range where

$$t = 1 + kT \qquad T = \frac{r_2 - ir_3}{R + r_1}$$

and

$$b = dR$$

$$d = \frac{b}{R} = \frac{R + r_1}{R} \quad .$$

Then

$$A = 1 - id\mathrm{Im}(T)\overline{T} = 1 - i\,\frac{R + r_1}{R}\left(\frac{-r_3}{R + r_1}\right)\left(\frac{r_2 + ir_3}{R + r_1}\right)$$

$$A = 1 - i\,\frac{-r_2r_3 - ir_3^2}{R(R + r_1)} = 1 - \frac{r_3^2 - r_2r_3i}{R(R + r_1)}$$

and finally

$$A = 1 - \frac{r_3^2}{R(R + r_1)} + \frac{r_2r_3}{R(R + r_1)}\,i$$

Rotating Coordinate Systems

Let two rotating coordinate systems be related by a tilt

$$t = 1 + kT$$

that is

$$\vec{x}' = t^{-1} \vec{x} t$$

with angular velocities

$$\vec{\omega} = i\omega_1 + j\omega_2 + k\omega_3$$

and

$$\vec{\omega}' = i\omega_1' + j\omega_2' + k\omega_3'$$

at a later time $t_o + \Delta t$ the systems will relate through an angle $|\vec{\omega}|\Delta t$ about the $\vec{\omega}$ axis. Therefore, the $\vec{\omega}$ rotation is similar to a quaternion (Ref 5)

$$q_{\Delta t} = 1 + \vec{\omega}_q$$

or

$$q_{\Delta t} = 1 + \frac{\vec{\omega}}{|\vec{\omega}|} \tan \frac{|\vec{\omega}|\Delta t}{2} \; .$$

Since for small angles $\tan\alpha = \alpha$

$$q_{\Delta t} = 1 + \frac{\vec{\omega}\Delta t}{2} \quad \text{(Ref 5)}$$

Let $\frac{\vec{\omega}\Delta t}{2}$ equal a vector $\vec{x}$, and $\vec{x} = ix_1 + kX$. The components of x can be defined

$$x_1 = \frac{\omega_1 \Delta t}{2} \quad x_2 = \frac{\omega_2 \Delta t}{2} \quad \text{and} \quad x_3 = \frac{\omega_3 \Delta t}{2}$$

and let

$$X = \frac{(\omega_3 + i\omega_2)\Delta t}{2} = \frac{O\Delta t}{2} \; . \quad O = \omega_3 + i\omega_2 \text{ (Ref 5)}$$

There also exists a $q'_{\Delta t}$ such that

$$q'_{\Delta t} = 1 + ix'_1 + kX' = 1 + \frac{\vec{\omega}'\Delta t}{2} .$$

Let s be a quaternion that relates the unprimed coordinate system to the primed system at time $(t_o + \Delta t)$. Assuming s is a tilt as was t then

$$s = 1 + kS = 1 + k(T + \Delta T) .$$

Now the quaternions $q_{\Delta t}s$ and $tq'_{\Delta t}$ both transform from unprimed to primed coordinates at time t_o (Ref 5). They also transform from unprimed to primed coordinates at a $t_o + \Delta t$ and are therefore equivalent (Ref 5)

$$q_{\Delta t}s = atq'_{\Delta t}$$

where a is a real number. Let a = 1 + h so that when $\Delta t = 0$, h = 0 and $q_{\Delta t}s = tq'_{\Delta t}$. Substituting for $q_{\Delta t}$, s, t and $q'_{\Delta t}$ and neglecting second order terms

$$\text{LHS} = (1 + ix_1 - \bar{X}T) + k(X + T + \Delta T - ix_1T) + \text{ord}(\Delta t^2)$$

$$\text{RHS} = (1 + ix'_1 - \bar{T}X' + h) + k(X' + T + ix'_1T + hT) + \text{ord}(\Delta t)^2.$$

where $\text{ord}(\Delta t)^2$ is on the order of Δt^2. Collecting corresponding parts:

$$ix_1 - XT = ix'_1 - \bar{T}X' + h \qquad (15)$$

$$X + \Delta T - ix_1T = X' + hT + ix'_1T \quad . \text{(Ref 5)} \qquad (16)$$

Equating the real and imaginary parts of (15):

$$h = \text{Re}(\bar{T}X' - \bar{X}T) \qquad (17)$$

$$x_1 = \mathrm{Im}(\bar{X}T) = x_1 + \mathrm{Im}(\bar{T}X)$$

$$= x_1' - \mathrm{Im}(\bar{T}X') \qquad \text{(Ref 5)} \qquad (18)$$

If a quaternion p is now defined as

$$p = (x_1' - \mathrm{Im}(\bar{T}X')\ \frac{\Delta t}{2}$$

then $p = \omega_1' - \mathrm{Im}(\bar{T}O') = \omega_1 + \mathrm{Im}(\bar{T}O)$ which constrains $\vec{\omega}$ and $\vec{\omega}'$ so that p will remain a tilt from unprimed to primed coordinates (Ref 5). From (17)

$$h = \mathrm{Re}(\bar{T}X' - \bar{X}T)$$

$$= \mathrm{Re}(\bar{T}X') - \mathrm{Re}(\bar{X}T) = \mathrm{Re}(\bar{T}X') - \mathrm{Re}(\bar{T}X)$$

thus

$$h = \mathrm{Re}[\bar{T}(X' - X)] \quad . \qquad \text{(Ref 5)}$$

Now from (16)

$$\Delta T = X' - X + T[h + i(x_1 + x_1')]$$

$$= (X' - X) + T[\mathrm{Re}(\bar{T}(X' - X)) + i(x_1 + x_1')]$$

So

$$\Delta T = (1 + |T|^2)(X' - X) + T[\mathrm{Re}(\bar{T}(X' - X)) - \bar{T}(X' - X)$$
$$+ i(x_1 + x_1')]$$

$$= \frac{2}{d}\ (X' - X) + iT[x_1' - \mathrm{Im}(\bar{T}X') + x_1 + \mathrm{Im}(\bar{T}X)]$$

and

$$\Delta T = \frac{2}{d} [X' - X + 2iT(x_1' - \text{Im}\bar{T}X')] \quad \text{(Ref 5)}$$

Dividing both sides by Δt and taking the limit as $\Delta t \to 0$

$$\dot{T} = \frac{1}{d} (O' - O) + iTp \quad \text{(Ref 5)}$$

where

$$O = (\omega_3 + i\omega_2) \quad \text{and} \quad \omega_2 = \frac{dx_2}{dt}, \ \omega_3 = \frac{dx_3}{dt}$$

As long as $p = \omega_1 + \text{Im}(\bar{T}O) = \omega_1' - \text{Im}(\bar{T}O')$ the unprimed and primed coordinates will remain related by a tilt. (Ref 5)

$$1 + kT$$

and T can be determined by

$$T = \int \dot{T}\, dt \ . \quad \text{(Ref 5)}$$

If one of the coordinate systems is fixed then $\omega = 0$, $O = 0$ and $p = 0$ giving $\omega_1' = \text{Im}(\bar{T}O')$ and

$$\dot{T} = \frac{O'}{d} \ . \quad \text{(Ref 5)} \qquad (19)$$

In both cases $\dot{T}$ is numerically integrated to find the tilt t (Ref 5). To relate a fixed axis system to a non-rolling system with rolling components in the j' k' plane only, an intermediate rolling coordinate system R with angular velocity

$$\vec{\omega}_R = \omega_{R_1} + kO'$$

with $\omega_{R_1} = \text{Im}(\bar{T}O')$ so that

$$\dot{T} = \frac{O'}{d} .$$

Therefore, the tilt relating R to the fixed (F) system can be found by numerical integration. If the quaternion q(t) relating R to F is

$$q(t) = 1 + kT(t)$$

where $T(t) = T_o + \int \dot{T}\, dt$ and a roll quaternion r(t) relating the non-rolling system to the rolling system:

$$r(t) = e^{i\phi(t)/2} ,$$

the quaternion s(t) relating the non-rolling to the fixed system is:

$$s(t) = q(t)\, r(t) = [1 + kT(t)]e^{i\phi(t)/2} . \text{ (Ref 5)}$$

Therefore, a $\phi(t)$ must be found such that ω_1 of the non-rolling system equals zero (Ref 5).

The quaternion transforms s(t) from F to non-rolling (NR) axes at time t. For a small time interval Δt and a time $(t - \Delta t)$, the quaternion $[s(t - \Delta t)]^{-1}$ transforms NR to F and if

$$p = [s(t - \Delta t)]^{-1}\, s(t) = P_{01} + kP_{32}$$

transforms NR at $(t - \Delta t)$ to NR at time (t). Since $\omega_{NR_1} = 0$

$$(\omega_{NR_1})(\Delta t) = \mathrm{Im}(P_{01}) = 0$$

substituting into p

$$p = \frac{[e^{-i\phi(t - \Delta t)/2}][1 - kT(t - \Delta t)][1 + kT(t)]e^{i\phi(t)/2}}{N[s(t - \Delta t)]}$$

and

$$P_{01} = \frac{[e^{-i\phi(t - \Delta t)/2}][1 + \overline{T(t - \Delta t)}\ T(t)]e^{i\phi(t)/2}}{1 + |T(t - \Delta t)|^2} \quad \text{(Ref 5)}$$

Let

$$\Delta\phi = \phi(t) - \phi(t - \Delta t)$$

then

$$P_{01} = [1 + \overline{T(t - \Delta t)}\ T(t)]e^{i\Delta\phi/2} \ . \quad \text{(Ref 5)}$$

If $Im(P_{01}) = 0$ then

$$\arg[1 + \overline{T(t - \Delta t)}\ T(t) + \arg(e^{i\Delta\phi/2})] = 0 \quad \text{(Ref 5)}$$

If A is a complex number, then

$$A = a + bi \ .$$

When expressed in polar form

$$A = (a^2 + b^2)^{1/2}\left[\frac{a}{(a^2 + b^2)^{1/2}} + i\left(\frac{b}{(a^2 + b^2)^{1/2}}\right)\right]$$

or

$$A = r(\cos\phi + i \sin\phi) = re^{i\phi}$$

where $r = (a^2 + b^2)^{1/2}$ and $\cos\phi = \frac{a}{r}$, $\sin\phi = \frac{b}{r}$ (Ref 9).

The argument (arg) of A is:

$$\arg A = \phi \ . \quad \text{(Ref 9)}$$

Therefore if b(ImA) is equal to zero then $\cos\phi = 1$ and $\sin\phi = 0$. Which means that $\phi = 0$.

Since P_{01} consists of two complex factors

$$[1 + \overline{T(t - \Delta t)}\, T(t)] \quad \text{and} \quad e^{i\Delta\phi/2}$$

their arguments must sum to zero, otherwise $\phi \neq 0$. Then

$$\arg[1 + \overline{T(t - \Delta t)}\, T(t)] = -\arg(e^{i\Delta\phi/2})$$

but

$$\arg e^{i\Delta\phi/2} = \frac{\Delta\phi}{2}$$

and

$$\frac{\Delta\phi}{2} = -\arg[1 + \overline{T(t - \Delta t)}\, T(t)]$$

$$\Delta\phi = -2\arg[1 + \overline{T(t - \Delta t)}\, T(t)]$$

or

$$\Delta\phi = -2\arg[1 + T(t - \Delta t)\, \overline{T(t)}] \tag{20}$$

The model numerically integrates (20) using the present and past values of T which is numerically integrated from equation (19) (Ref 5).

Appendix C

Comparison to Conventional Transformation

Consider a coordinate system as in Figure 8. The vector kT, where T = $t_3 + it_2$ in the complex x_2, x_3 plane, defines a new coordinate system which is rotated about the x_1 axis through an angle γ with major axis y_1, y_2, y_3.

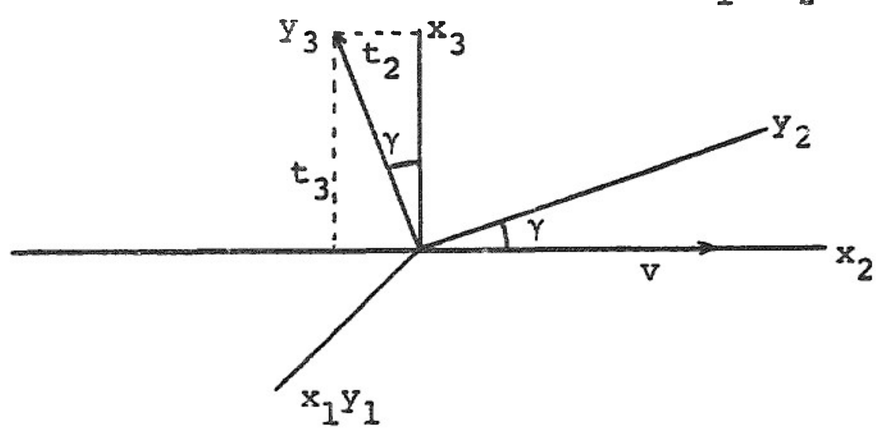

Fig. 8. x_1 Rotation

Now consider a rotation about the y_3 axis as depicted in Figure 9. These two rotations represent a tilt of the x_1 axis to the y_1' position in space.

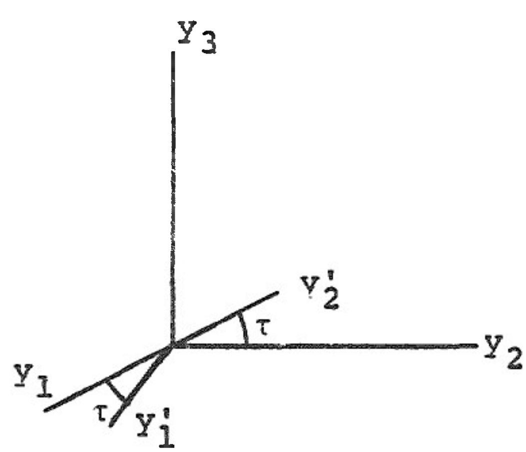

Fig. 9. y_3 Rotation

To accomplish the tilt from x_1 to x_1' through the angle θ, using direction cosines, a two step process must be utilized. Let the rotation about x_1 be represented by direction cosine matrix A and the rotation about y_3 be represented by direction cosine matrix B (Ref 4). Then

$$\vec{y} = A\vec{x}$$

rotates $\vec{x}$ to $\vec{y}$ in x coordinates,

$$\vec{y}' = B\vec{y}$$

rotates $\vec{y}$ to $\vec{y}'$ in y coordinates. So the original vector $\vec{x}$ rotated by A then B is

$$\vec{x}' = A^T B\vec{y}$$

in x coordinates. (For an orthogonal matrix $A^T = A^{-1}$) The completed tilt can be represented by

$$\vec{x}' = C\vec{x}$$

where

$$C = A^T BA \quad .$$

From Figure 8

$$\sin\gamma = -\frac{t_2}{|\vec{t}|}, \text{ and } \cos\gamma = \frac{t_3}{|\vec{t}|} \quad .$$

From the quaternion definition of τ,

$$\tan\frac{\tau}{2} = |\vec{t}| = |T|$$

then

$$\tau = 2\tan^{-1}|T| \quad .$$

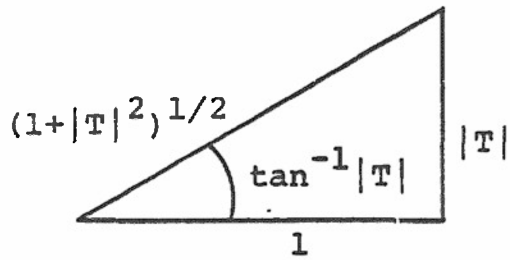

Fig. 10. $\tan^{-1}|T|$

From Figure 10

$$\cos(\tan^{-1}|T|) = \frac{1}{(1 + |T|^2)^{1/2}}$$

$$\sin(\tan^{-1}|T|) = \frac{|T|}{(1 + |T|^2)^{1/2}}$$

$$\cos\tau = \cos 2(\tan^{-1}|T|)$$

$$\cos\tau = 2\cos^2(\tan^{-1}|T|) - 1$$

$$= \frac{2}{1 + |T|^2} - \frac{1 + |T|^2}{1 + |T|^2} = \frac{1 - |T|^2}{1 + |T|^2}$$

and

$$\sin\tau = \sin 2(\tan^{-1}|T|)$$

$$= 2\sin(\tan^{-1}|T|)\cos(\tan^{-1}|T|)$$

$$= 2\left[\frac{|T|}{(1 + |T|^2)^{1/2}}\right]\left[\frac{1}{(1 + |T|^2)^{1/2}}\right]$$

$$= \frac{2|T|}{1 + |T|^2} .$$

Now that the functions $\sin\gamma$, $\cos\gamma$, $\sin\tau$ and $\cos\tau$ are defined, the transformation matrices can be formed:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$C = A^T B A$$

$$C = \begin{bmatrix} \cos\tau & \cos\gamma\sin\tau & \sin\gamma\sin\tau \\ -\sin\tau\cos\gamma & \cos^2\gamma\cos\tau+\sin^2\gamma & \sin\gamma\cos\gamma\cos\tau-\sin\gamma\cos\gamma \\ -\sin\gamma\sin\tau & \sin\gamma\cos\gamma\cos\tau-\sin\gamma\cos\gamma & \sin^2\gamma\cos\tau+\cos^2\gamma \end{bmatrix}$$

Substituting and carrying out the vector multiplication

$$x_1' = \frac{1 - |T|^2}{1 + |T|^2} x_1 + \frac{2t_3}{1 + |T|^2} x_2 + \frac{2t_2}{1 + |T|^2} x_3$$

$$x_2' = -\frac{2t_3}{1 + |T|^2} x_1 + \frac{1 + |T|^2 - 2t_3^2}{1 + |T|^2} x_2 + \frac{2t_2t_3}{1 + |T|^2} x_3$$

$$x_3' = \frac{2t_2}{1 + |T|^2} x_1 + \frac{2t_2t_3}{1 + |T|^2} x_2 + \frac{1 + |T|^2 - 2t_2^2}{1 + |T|^2} x_3 \quad .$$

From the tilt derivation for a positive rotation of coordinates

$$x_1' = b - x_1 \quad \text{and} \quad Z' = Z - bT$$

or

$$x_2' - ix_3' = (x_2 - ix_3) - b(t_3 + it_2)$$

therefore

$$x_2' = x_2 - bt_3$$

and

$$x_3' = x_3 + bt_2$$

where

$$b = d(x_1 + t_3x_2 - t_2x_3) \quad \text{and} \quad d = \frac{2}{1 + |T|^2} .$$

Substituting

$$x_1' = \frac{1 - |T|^2}{1 + |T|^2} x_1 + \frac{2t_3}{1 + |T|^2} x_2 - \frac{2t_2}{1 + |T|^2} x_3$$

$$x_2' = - \frac{2t_3}{1 + |T|^2} x_1 + \frac{1 + |T|^2 - 2t_3^2}{1 + |T|^2} x_2 + \frac{2t_2t_3}{1 + |T|^2} x_3$$

$$x_3' = \frac{2t_2}{1 + |T|^2} x_1 + \frac{2t_2t_3}{1 + |T|^2} x_2 + \frac{1 + |T|^2 - 2t_2^2}{1 + |T|^2} x_3 .$$

Therefore, the tilt quaternion gives the same result as the three matrix multiplications.

Similarly for a negative roll but a positive coordinate transformation

$$\vec{x}' = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\emptyset & \sin\emptyset \\ 0 & -\sin\emptyset & \cos\emptyset \end{bmatrix}$$

and

$$x_1' = x_1$$

$$x_2' = x_2\cos\phi + x_3\sin\phi$$

$$x_3' = -x_2\sin\phi + x_3\cos\phi$$

From roll quaternion development

$$r = 1 - i\tan\frac{\phi}{2}$$

$$|r|^2 = 1 + \tan^2\frac{\phi}{2}$$

$$r^2 = 1 - \tan^2\frac{\phi}{2} - i\,2\tan\frac{\phi}{2}$$

$$\frac{r^2}{|r|^2} = \cos\phi - i\sin\phi$$

Then

$$Z' = (\cos\phi - i\sin\phi)(x_2 + ix_3)$$

$$x_2' + ix_3' = x_2\cos\phi + x_3\sin\phi - ix_2\sin\phi + ix_3\cos\phi$$

or

$$x_2' = x_2\cos\phi + x_3\sin\phi$$

$$x_3' = -x_2\sin\phi + x_3\cos\phi$$

Rolling Coordinate Systems

From the discussion on rolling coordinate systems in Appendix B, a p quaternion is formed by:

$$p = q_{\Delta t}\, s$$

where p relates a rolling and non-rolling set of coordinate systems at any time t. From the development of p, equation (19) is derived which implies the parameter ω_1' of the rotational velocity vector ω is

$$\omega_1' = \mathrm{Im}(\bar{T}O') \quad .$$

This is equivalent to:

$$\omega_1' = \mathrm{Im}(t_3 - it_2)(\omega_3' + i\omega_2')$$

$$= t_3\omega_2' - t_2\omega_3' \qquad (21)$$

To arrive at (21) utilizing conventional direction cosines, it must first be understood that the product of two quaternions will result in a negative coordinate rotation. This is a reversal of the direction discussed previously in this appendix and result in the following changes:

$$\sin\gamma = \frac{t_2}{|T|} \quad , \quad \cos\gamma = -\frac{t_3}{|T|}$$

$$\cos\tau = \frac{1 - |T|^2}{1 + |T|^2} \quad , \quad \sin\tau = \frac{2|T|}{1 + |T|^2} \quad .$$

In addition, the cosine matrices A and B are now

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{vmatrix}$$

and

$$B = \begin{vmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad .$$

To describe the vector $\vec{\omega}$ which relates the non-rolling coordinate system to the rolling coordinate system, the following relation is used:

$$\vec{\omega}'C(t) = \dot{C}(t)$$

where

$$C = ABA^T \quad .$$

The vector $\vec{\omega}$ can be found:

$$\vec{\omega}' = \dot{C}(t)C^T(t)$$

where

$$\dot{C}(t) = \frac{d}{dt}(ABA^T)$$

or

$$\vec{\omega}' = \dot{A}A^T + A\dot{B}B^TA^T + AB(\dot{A}^TA)(AB)^T$$

Substituting for A, $\dot{A}$, A^T, $\dot{A}^T$, B, $\dot{B}$ and B^T

$$\vec{\omega}' = \begin{vmatrix} 0 & \dot{\gamma}\sin\gamma\sin\tau-\tau\cos\gamma & -\dot{\gamma}\cos\gamma\sin\tau-\dot{\tau}\sin\gamma \\ -\dot{\gamma}\sin\gamma\sin\tau+\dot{\tau}\cos\gamma & 0 & -\dot{\gamma}+\dot{\gamma}\cos\tau \\ \dot{\gamma}\cos\gamma\sin\tau+\dot{\tau}\sin\gamma & \dot{\gamma}-\dot{\gamma}\cos\tau & 0 \end{vmatrix}$$

which implies

$$\omega_1' = \dot{\gamma}(1 - \cos\tau)$$

$$\omega_2' = -\dot{\gamma}\cos\gamma\sin\tau - \dot{\tau}\sin\gamma$$

and $$\omega_3' = -\dot{\gamma}\sin\gamma\sin\tau + \dot{\tau}\cos\gamma \quad .$$

Utilizing the equations for ω_2' and ω_3', $\dot{\gamma}$ can be solved for:

$$\dot{\gamma} = -\frac{\omega_2'\cos\gamma + \omega_3'\sin\gamma}{\sin\tau} \quad .$$

Substituting into the equation for ω_1

$$\omega_1' = (\omega_2'\cos\gamma + \omega_3'\sin\gamma)\left(\frac{\cos\tau - 1}{\sin\tau}\right) \quad .$$

Substituting for $\cos\gamma$, $\sin\gamma$, $\cos\tau$ and $\sin\tau$:

$$\omega_1 = \left(-\frac{\omega_2't_3}{|T|} + \frac{\omega_3't_2}{|T|}\right)\left(-\frac{2|T|^2}{1 + |T|^2}\right)\left(\frac{1 + |T|^2}{2|T|}\right)$$

or

$$\omega_1 = \omega_2't_3 - \omega_3't_2 \tag{22}$$

Equations (21) and (22) are identical and the same results are obtained.

Appendix D

Computer Listing and Guide

Introduction

The subroutine SIGNAL was modified so that a table of signal strengths for various aspects and ranges could serve as a data base for calculations of maximum seeker line of sight (LOS) rates for target tracking. To accomplish this, azimuth and elevation data arrays (Figure 11) were inserted along with the proper code (Figure 12) to evaluate the geometry and extract the proper values from the data base to interpolate a proper signal strength.

The missile model uses many common variables so that quaternions need to be defined once and then through the common variables be available to various subroutines as necessary. The program DRIVER manages the overall model by reading the input data, initializing the variables and setting up quaternion relationships among the various coordinate systems. Where appropriate, the coding from DRIVER has been extracted to help define the work variables which form the tilt and roll quaternions utilized in SIGNAL.

The following sections describe the code used to modify SIGNAL. The reader is encouraged to read Appendices A and B prior to reading the Azimuth and Elevation section. The final section is an example problem to demonstrate how the interpolation technique works.

```
C
C
C
C
 2300 DATA AZARAY /100.0,80.0,40.0,0.0,
     2           110.0,85.0,45.0,10.0,
     3           120.0,90.0,50.0,20.0,
     4           130.0,95.0,55.0,25.0,
     5           135.0,100.0,60.0,30.0,
     6           140.0,110.0,65.0,35.0,
     7           150.0,120.0,70.0,40.0,
     8           170.0,130.0,80.0,45.0,
     9           200.0,140.0,85.0,50.0,
     A           220.0,150.0,90.0,55.0,
     B           240.0,165.0,100.0,60.0,
     C           265.0,180.0,110.0,70.0,
     D           280.0,195.0,120.0,80.0/
      DATA (FLARAY(IEL),IEL=1,52)/
     1        100.0,80.0,40.0,0.0,
     2        115.0,85.0,43.0,10.0,
     3        122.0,91.0,47.0,14.0,
     4        131.0,98.0,53.0,18.0,
     5        143.0,101.0,60.0,22.0,
     6        157.0,110.0,68.0,29.0,
     7        166.0,119.0,74.0,33.0,
     8        177.0,127.0,85.0,43.0,
     9        191.0,134.0,90.0,50.0,
     A        200.0,144.0,96.0,57.0,
     B        230.0,157.0,101.0,70.0,
     C        268.0,173.0,110.0,75.0,
     D        280.0,195.0,120.0,80.0/
       DATA (ELARAY(IEL),IEL=53,104)/
     1        100.0,80.0,40.0,0.0,
     2        108.0,83.0,44.0,9.0,
     3        115.0,87.0,48.0,12.0,
     4        120.0,92.0,51.0,16.0,
     5        127.0,99.0,56.0,18.0,
     6        137.0,103.0,63.0,25.0,
     7        150.0,111.0,70.0,31.0,
     8        168.0,121.0,78.0,41.0,
     9        192.0,135.0,85.0,50.0,
     A        230.0,143.0,98.0,58.0,
     B        250.0,157.0,105.0,68.0,
     C        260.0,168.0,109.0,76.0,
     D        280.0,195.0,120.0,80.0/
```

Fig. 11. Azimuth and Elevation Data Arrays

```
      DATA RANDSQ /0.0,1.0E6,25.0E6,169.0E6/,ELAND/0.0,15.0,30.0,45.0,
     1            60.0,75.0,90.0,-75.0,-60.0,-45.0,-30.0,-15.0,0.0/,
     2            PHASEY/1.0783914E9/
      DATA ANDAZ/0.0,15.0,30.0,45.0,60.0,75.0,90.0,105.0,120.0,135.0,
     2          150.0,165.0,180.0,180.0/
      ITEMPA=R*0.001+1
      IRNG=ITEMPA
      IF (ITEMPA .GE. 2) IRNG=2
      IF (ITEMPA .GE. 6) IRNG=3
C     TRANSFORM LOS FROM FIXED (F) TO U COORDINATES
      CALL RESOLV(DLRF1,DLRFR,-DLRF1,FTOU,TEMPA,ZWORKA)
C     TRANSFORM LOS FROM U TO TARGET (T) COORDINATES
      TEMPB=DUT*(TEMPA+WORKAR*TUTR+WORKAI*TUTI)
      TEMPC=TEMPB-TEMPA
      WORKBR=WORKAR-TUTR*TEMPB
      WORKBI=WORKAI-TUTI*TEMPB
      ZWORKB=ZWORKB*ZRUT
C     FIND ELEVATION AND AZIMUTH ANGLES
      IF(ABS(TEMPC) .LT. 0.0001) TEMPC=SIGN(0.0001,TEMPC)
      AZ= ATAN2(WORKBR,TEMPC)
      EL= ASIN(WORKBI*RRECIP)
C     INTERPOLATE IN AZIMUTH
      TEMPA=ABS(AZ)*RADDEG
      ITEMPA=(TEMPA/15.)+1
      IN=1
      IF (ABS(AZ) .GE. PI) IN=0
      TEMPB=(TEMPA-ANDAZ(ITEMPA))*12.*DEGRAD
      TEMPA=COS(TEMPB)*HALF
      TEMPC=AZARAY(IRNG,ITEMPA)
      TEMPD=AZARAY(IRNG,ITEMPA+IN)
      ANDAL=(TEMPC+TEMPD)*HALF+(TEMPC-TEMPD)*TEMPA
      TEMPC=AZARAY(IRNG+1,ITEMPA)
      TEMPD=AZARAY(IRNG+1,ITEMPA+IN)
      ANDAU=(TEMPC+TEMPD)*HALF+(TEMPC-TEMPD)*TEMPA
C     INTERPOLATE IN ELEVATION
      IF (EL .LT. 0.) ITEMPA=ITEMPA+17
      TEMPC=ELARAY(IRNG,ITEMPA)
      TEMPD=ELARAY(IRNG,ITEMPA+IN)
      ANDEL=(TEMPC+TEMPD)*HALF+(TEMPC-TEMPD)*TEMPA
      TEMPC=ELARAY(IRNG+1,ITEMPA)
      TEMPD=ELARAY(IRNG+1,ITEMPA+IN)
      ANDEU=(TEMPC+TEMPD)*HALF+(TEMPC-TEMPD)*TEMPA
C     INTERPOLATE BETWEEN AZIMUTH AND ELEVATION
      TEMPA=ABS(EL)*RADDEG
      ITEMPA=(TEMPA/15.)+1
```

Fig. 12a. Code

```
      TEMPB=(TEMPA-FLOAT(ITEMPA))*12.*DEGRAD
      TEMPA=COS(TEMPB)*HALF
      BNDL=(BNDAL+BNDEL)*HALF+(BNDAL-BNDEL)*TEMPA
      BNDU=(BNDAU+BNDEU)*HALF+(BNDAU-BNDEU)*TEMPA
C     COMPUTE IRRADIANCE AS FUNCTION OF 1/R**2
      TEMPD=R**2
      TEMPC=RBNDSQ(IRNG)
      TEMPB=(BNDL*TEMPC-BNDU*RBNDSQ(IRNG+1))/(BNDU-BNDL)
      TEMPA=BNDL*(TEMPC+TEMPB)
      RADANC=TEMPA/(TEMPD*TEMPB)
      TENLNH=10.0*ALOG(RADANC*PHASEY/TEMPD)
```

Fig. 12b. Code

Range

The first computation

$$ITEMPA = R * 0.001 + 1$$

takes the actual magnitude of the range vector and divides it by 1000. This indexes the range into a range bin with respect to the data. The data arrays are constructed such that the row value corresponds to the range bin and the column value corresponds to the LOS azimuth/elevation measured from the target back to the missile seeker. There are four range indexes and 13 angle indexes.

The value IRNG is next assigned by determining if ITEMPA is between zero or one (IRNG = 1), between two and six (IRNG = 2), or greater than six (IRNG = 3). These values correspond to ranges

$$\begin{array}{lll} IRNG = 1 & & 0 \le range < 1000 \\ IRNG = 2 & & 1000 \le range < 5000 \\ IRNG = 3 & & 5000 \le range < 13000 \end{array}$$

Azimuth and Elevation

The first step in setting up the geometry of the missile-target relationship is to transform the line of sight (LOS) from the fixed (launch aircraft) coordinates to U (range) coordinates. This is done by calling the subroutine RESOLV which utilizes a direction cosine matrix (FTOU). The calling sequence is CALL RESOLV(DLRF1, DLRFR, -DLRFI, FTOU, TEMPA, ZWORKA). DLRF1 is the x component, DLRFR is the y component and DLRFI is the z component of the range vector in fixed (F) coordinates. (DLRFI is negative to aid later computations.) FTOU specifies the direction cosine matrix to be used (F coordinates to U coordinates). Finally TEMPA and ZWORKA are dummy arguments which return the range vector in U coordinates in factored format

$$\vec{r} = (r_1 + kZ)i$$

where r_1 is the TEMPA value returned and Z = ZWORKA = $(r_2 - ir_3)$. The components r_2 and r_3 are the y and z values of $\vec{r}$ in the U system.

Now that the LOS is defined in the U system which contains both the missile and target, the LOS is redefined in the target system to determine the azimuth and elevation of the seeker from the target reference.

Because the seeker is a gimbaled system, both a tilt and roll are required to match the LOS from the target to the seeker reference system. The required transformation utilizes the tilt quaternion t = 1 + k(ZTUT) and the roll transformation ZRUT.

The tilt quaternion is constructed first through the work variables TEMPA, TEMPB, WORKAR, and WORKAI:

$$\text{TEMPB} = \text{DUT}*(\text{TEMPA} + \text{WORKAR} * \text{TUTR} + \text{WORKAI} * \text{TUTI}) .$$

DUT is defined in the DRIVER as the factor d of the tilt quaternion (ZTUT) from U to T coordinates where

$$\text{DUT} = \frac{2}{1 + |T|^2} . \qquad \text{(d from U to T)}$$

TUTR is the real part and TUTI is the imaginary part of ZTUT which is also computed in DRIVER. The sequence from DRIVER is as follows:

```
      TEMPA  =  VT + VTU1
      TUTR   =  VTUR/TEMPA
      TUTI   =  VTUI/TEMPA
      TEMPA  =  TUTI**2
DUT  =  TWO/(ONE + TUTR**2 + TEMPA)   .
```

The variables used here are VT, the magnitude of target velocity; VTU1, the x-coordinate of velocity; VTUR, the y-coordinate of velocity; and VTUI, the z-coordinate of velocity. The target coordinate system's orientation is defined by VTU1. TEMPA in this case is

$$\text{TEMPA} = VT + v_1 \qquad \text{VTU1} = v_1$$

$$\text{TUTR} = \frac{v_2}{VT + v_1} \qquad \text{VTUR} = v_2$$

$$\text{TUTI} = \frac{v_3}{VT + v_1} \qquad \text{VTUI} = v_3 \quad .$$

TEMPA is then redefined

$$\text{TEMPA} = \frac{v_3^2}{(VT + v_1)^2} \quad .$$

Since the tilt

$$t = (1 + kT)$$

and $T = t_3 + it_2$ where $t_2 = \frac{v_3}{VT + v_1}$ and $t_3 = \frac{v_2}{VT + v_1}$ then the magnitude of T:

$$|T| = (t_2^2 + t_3^2)^{1/2}$$

DUT therefore is

$$d = DUT = \frac{2}{1 + \left(\frac{v_2}{VT + v_1}\right)^2 + \left(\frac{v_3}{VT + v_1}\right)^2}$$

$$= \frac{2}{1 + t_2^2 + t_3^2} = \frac{2}{1 + |T|^2}$$

Returning to the SIGNAL sequence, TEMPB is:

$$TEMPB = b = d[r_1 + t_3 r_2 - t_2 r_3] ,$$

and

r_1 = TEMPA

r_2 = WORKAR

$-r_3$ = WORKAI (From the RESOLV calling parameter)

t_3 = TUTR

t_2 = TUTI .

With these parameters the new range vector $\vec{r}'$ can be constructed.

$$TEMPC = TEMPB - TEMPA$$

is equivalent to

$$r_1' = b - r_1 \quad \text{(from App. B (11))}$$

$$\text{WORKBR} = \text{WORKAR} - \text{TUTR*TEMPB}$$

$$= r_2 - t_3b$$

and

$$\text{WORKAI} = \text{WORKAI} - \text{TUTI*TEMPB}$$

$$= -r_3 - t_2b$$

ZWORKB is defined by

$$\text{ZWORKB} = \text{WORKAR} + i\text{WORKBI}$$

or equivalently

$$Z' = r_2' - ir_3' \quad . \quad \text{(from App. B (8))}$$

From the discussion of a quaternion times a vector

$$Z' = Z - bT \quad ,$$

where

$$Z = r_2 - ir_3 = \text{WORKAR} + i(\text{WORKAI}) .$$

This can further be broken down into

$$Z' = r_2 - bt_3 - i(r_3 + bt_2)$$

or

$$\text{WORKBR} = r_2 - bt_3$$

$$\text{WORKBI} = -r_3 - bt_2 \quad .$$

The roll due to tilt is next considered using the roll transformation from U to T coordinates (ZRUT) which is defined in DRIVER by the sequence:

```
WORKAR  =  ONE - DUT*TEMPA
WORKAI  =  -DUT*TUTI*TUTR
TEMPA   =  CABS(ZWORKA)
RUTR    =  -WORKAR/TEMPA
RUTI    =  -WORKAI/TEMPA
```

This sequence is identical to the development of the roll due to tilt discussion where the roll quaternion $r = (1 + ir_1)$ and the roll transformation is

$$e^{i\phi} = \frac{r}{r^*} = \frac{A}{|A|} .$$

The complex number is:

$$A = 1 - id\mathrm{Im}(T)\bar{T}$$

where $T = t_3 + it_2$, $\bar{T} = t_3 - it_2$ and d = DUT. ZRUT which is equivalent to $\frac{r}{r^*}$ is found by

$$\text{WORKAR} = \mathrm{Re}(A) = 1 - dt_2^2$$

$$\text{WORKAI} = \mathrm{Im}(A) = -dt_2t_3$$

$$A = \text{ZWORKA} = \text{WORKAR} + i\text{WORKAI}$$

therefore

$$A = 1 - d(t_2^2 + it_2t_3)$$

where

$$iIm(T)\bar{T} = it_2(t_3 - it_2) = t_2^2 + it_2t_3 \quad .$$

Since

$$TEMPA = CABS(ZWORKA) = |A| \quad ,$$

and

$$RUTR = -\frac{Re(A)}{|A|} \quad ,$$

$$RUTI = -\frac{Im(A)}{|A|} \quad \text{then} \quad ZRUT = \frac{A}{|A|} \quad .$$

The roll takes place about the x-axis of the coordinate system and ZRUT is negative to transform the coordinates of Z to their zero tilt origin. This is done by

$$Z_1' = Z_1 \frac{r}{r^*} = Ze^{-i\phi}$$

or

$$ZWORKB = ZWORKB*ZRUT \quad .$$

With the line of sight defined in target coordinates, the azimuth and elevation can be computed as

$$AZ = ATAN2(WORKBR,TEMPC)$$

$$EL = ASIN(WORKBI*RRECIP)$$

where WORKBR is the y coordinate, TEMPC the x coordinate, WORKBI the z coordinate and RRECIP is the reciprocal of the range magnitude.

Signal Strength Interpolation

The infrared signal strength is obtained by interpolating the values obtained from the data matrices AZARAY and ELARAY. The interpolation is done by a comparison of values based on range, azimuth and elevation from the target. Once a representative signal is calculated it is compared to minimum standard values characteristic of the missile seeker head to determine the maximum line-of-sight tracking rate.

Azimuth

The program initially converts azimuth (AZ) to degrees, a more convenient working form utilizing the radians to degrees conversion factor (RADDEG). An index (ITEMPA) is next computed by dividing the value of azimuth by 15 and adding one to it so the lower value of the azimuth band can be determined. (The azimuth bands run from the nose to the tail (0° to 180°) in 15° bandwidths.)

Next an increment term is set to one (IN = 1) for later use. If the azimuth is equal to 180° then this increment is set to zero.

TEMPB (an interpolation angle based upon the size of the angle formed by the azimuth with the lower azimuth band angle), is now computed. The difference between the azimuth and the lower azimuth band angle is calculated, and this difference is multiplied by 12 to place it into a convenient size. The band is 15° wide and if the azimuth equals the lower band angle, TEMPB will be zero; if the difference is 7.5°, TEMPB will be 90°; and if the difference is 15°, TEMPB will be 180°. TEMPA is then set to the cosine of TEMPB times one half.

Data points from the azimuth array are now selected for an azimuth interpolation. TEMPC is chosen by the computed IRNG and ITEMPA where ITEMPA is the lower azimuth band. TEMPD next selects the next higher azimuth value at the same range as TEMPC. If the azimuth is 180°, IN, which increments ITEMPA, is zero so that an inappropriate value is not selected from the data matrix. (There is no data point for azimuth greater than 180°.)

The interpolation now takes place at the lower range band and a lower azimuth band value (BNDAL) is computed:

BNDAL = (TEMPC + TEMPD)*HALF + (TEMPC - TEMPD)*TEMPA

or equivalently:

$$\text{BNDAL} = \frac{AZ_L \text{ Value} + AZ_U \text{ Value}}{2} + \frac{AZ_L \text{ Value} - AZ_U \text{ Value}}{2} \cos(\text{TEMPB}) \ .$$

If TEMPB is 0° then cosine(TEMPB) is equal to one and BNDAL reduces to the lower azimuth value. If TEMPB is 90° then cosine is equal to zero and BNDAL becomes one-half the lower plus one-half the upper azimuth values. If TEMPB is 180° the cosine is equal to (-1) and BNDAL reduces to the upper azimuth value. For angles between these values, an appropriate ratio of the azimuth values is computed.

Next an upper limit (BNDAU) based on the next range data (IRNG + 1) is computed in a similar manner. The same azimuth indices are used but IRNG is incremented by one into the next outer range bin.

Elevation

The next series of computations are to arrive at a pair of elevation data points which correspond to the azimuth position. In other words, the IR signature can be thought to be similar along a conical path around the source.

The first statement determines if the elevation is negative and if it is, adds 13 to the ITEMPA index to reference the negative elevation data array. Values for TEMPC and TEMPD are selected as previously discussed for azimuth, and a similar interpolation is done for an inner range value BNDEL and an outer range value BNDEU.

The final interpolation based upon angles is accomplished by first computing an index based upon the elevation angle (TEMPA). The value of the elevation is divided by 15 and one is added to it so a lower value of the elevation band can be determined. (The elevation bands run from the plane of the wings to the perpendicular axis of the plane (0° to 90°) in 15° bandwidths.)

TEMPB, an interpolation angle, is now computed, similar to the azimuth interpolation angle. The cosine of the angle (TEMPB) is then computed and the final interpolation between the azimuth and elevation values takes place. The values computed are a lower (BNDL) and an upper (BNDU) signal strength.

Final Interpolation

The last interpolation combines the signal values at distances which bracket the actual distance at the correct azimuth and elevation. The missile seeker will receive a signal somewhere in between the values calculated at BNDL and BNDU. The infrared signal is a function of one over range squared (Ref 6), so what the program does now is compute a reference signal at the close range (BNDL) and then a reference signal at the far range (BNDU). The two reference signal strengths are then combined to form a new reference signal strength at the actual range, and finally this reference is divided by the range squared (TEMPD) to give the interpolated signal. Let

BNDU $= S_U =$ Upper tabulated value (Interpolated)

BNDL $= S_L =$ Lower tabulated value (Interpolated)

Then

$$S_U = \frac{S_{rU}}{R_U^2} \quad \text{and} \quad S_L = \frac{S_{rL}}{R_L^2}$$

where S_{rU} = the source reference signal upper and S_{rL} = the source reference signal lower.

$$S_{rU} = \text{BNDU*RBNDSQ(IRNG + 1)}$$

$$S_{rL} = \text{BNDL*RBNDSQ(IRNG)}$$

Using the calculated reference strengths, and the interpolated signal strengths a delta signal is calculated as a ratio of the change in reference to the change in interpolated signals.

$$\Delta S = \frac{(S_{rL} - S_{rU})}{(S_U - S_L)}$$

or

TEMPB = (BNDL*TEMPC - BNDU*RBNDSQ(IRNG + 1))/BNDU - BNDL)

TEMPA is now computed:

TEMPA = BNDL*(TEMPC + TEMPB)

which is equivalent to:

$$\text{TEMPA} = S_L * (R_L^2 + \Delta S) \ .$$

The signal is then calculated by:

RADANC = TEMPA/(TEMPD + TEMPB)

or,

$$\text{RADANC} = S_L \left(\frac{R_L^2 + \Delta S}{R^2 + \Delta S} \right)$$

which is the value of the signal at the inner range decreased by the slope determined by the ratio of the ranges plus the decreased value of the signal from the inner band to the outer band.

The interpolated signal strength is then converted to Phasey decibles by:

TENLNH = 10.0*ALOG(RADANC*PHASEY/TEMPD) .

where Phasey decibles are ten times the natural log of the signal and PHASEY/TEMPD is a factor to convert the interpolated RADANC from watts/steradian to pico-watts/cm^2. The comparison logic uses the Phasey decible strength to determine the maximum track rate of the missile seeker.

Example Problem

For an example of how the azimuth and elevation interpolation works, suppose the target and launch aircraft have a range of 4000 feet (IRNG=2). Further, let the aspect measured from the target's nose be 150° and the elevation be -45°.

The first value determined is ITEMPA which would be equal to 150 divided by 15 or 10 plus one or ITEMPA would equal 11. Since 150° is less than 180°, IN would equal one.

The interpolation angle (TEMPB) is now computed by:

$$TEMPB = (TEMPA - BNDAZ(ITEMPA))*12*DEGRAD ,$$

or

$$TEMPB = (150 - 150)*12*DEGRAD .$$

(DEGRAD is the transformation constant from degrees to radians just as RADDEG is the transformation constant from radians to degrees.) This is equivalent to:

$$TEMPB = 0 .$$

Next TEMPA is computed as the cosine of TEMPB or

$$TEMPA = \cos(0)*(.5) = 0.5$$

Now the first interpolation begins. TEMPC and TEMPD are the values which bracket the azimuth or AZARAY (IRNG,ITEMPA) and AZARAY (IRNG,ITEMPA + 1). Those correspond to the values 165 and 180. BNDAL is now computed by:

$$BNDAL = \frac{(TEMPC + TEMPD)}{2} + \frac{TEMPC - TEMPD}{2} \cos(0°)$$

or

$$BNDAL = \frac{165 + 180}{2} + \frac{165 - 180}{2} = 165 .$$

BNDAU is computed similarly but at the next outer range values (IRNG + 1).

$$BNDAU = \frac{TEMPC + TEMPD}{2} \div \frac{TEMPC - TEMPD}{2}$$

where TEMPC = AZARAY (3,11) = 100 and TEMPD = AZARAY (3,12) = 110. So

$$BNDAU = \frac{100 + 110}{2} + \frac{100 - 110}{2} = 100 .$$

The elevation values are now computed in a similar manner, but the elevation is less than 0° (-45°) so,

$$ITEMPA = ITEMPA + 13$$

or

$$ITEMPA = 24 .$$

The lower elevation value is:

$$BNDEL = \frac{ELARAY(2,24) + ELARAY(2,25)}{2}$$

$$+ \frac{ELARAY(2,24) - ELARAY(2,25)}{2}$$

or

$$BNDEL = \frac{157 + 168}{2} + \frac{157 - 168}{2} = 157 .$$

Similarly for the outer range:

$$BNDEU = \frac{105 + 109}{2} + \frac{105 - 109}{2} = 105 .$$

The final computation for BNDL and BNDU consists of first computing an interpolation angle as a function of the elevation. The index ITEMPA is computed:

$$ITEMPA = (45/15) + 1 = 4 .$$

The interpolation angle TEMPB is now computed:

$$TEMPB = (45 - 45)*12*DEGRAD$$

and the factor TEMPA:

$$TEMPA = \cos(0)*(.5) = 0.5 .$$

BNDL is now computed:

$$BNDL = \frac{165 + 157}{2} + \frac{165 - 157}{2} = 165$$

and

$$BNDU = \frac{100 + 105}{2} + \frac{100 - 105}{2} = 100 .$$

These values are then finally interpolated for range as previously discussed and the value for RADANC is:

$$RADANC = 117.33 \text{ watts/steradian} .$$

Appendix E

DRIVER

The model is an adaptation of a dynamic simulation utilized by the ACMR/I range which involved real time data transmission to and from the participating aircraft and the simulation computers. To enable the model to work interactively on a time sharing terminal, a DRIVER program had to be developed which would initialize the necessary parameters normally supplied by the aircraft telemetry pods. In addition, DRIVER also sets up the quaternions relating the U (earth surface) coordinate system to the T (target) coordinate system.

Tilt from U to T

The tilt quaternion is constructed in Section 2 through the work variables TEMPA; TEMPB; WORKAR; and WORKAI, as follows:

$$\text{TEMPA} = \text{VT} + \text{VTU1}$$

$$\text{TUTR} = \text{VTUR/TEMPA}$$

$$\text{TUTI} = \text{VTUI/TEMPA}$$

The variables used here are VT, the magnitude of the target velocity; VTU1, the x-coordinate of target velocity; VTUR, the z-coordinate of target velocity; and VTUI, the y-coordinate of the target velocity. These lines of code are equivalent to:

$$\text{TEMPA} = \text{VT} + \text{VTU1} \qquad \text{VTU1} = v_1$$

$$t_3 = \text{TUTR} = \frac{v_2}{\text{VT} + v_1} \qquad \text{VTUR} = v_2$$

$$t_2 = \text{TUTI} = \frac{v_3}{\text{VT} + v_1} \qquad \text{VTUI} = v_3 \ .$$

Since the tilt,

$$t = (1 = kT) \quad ,$$

and

$$T = t_3 + it_2$$

where

$$t_2 = \frac{v_3}{\text{VT} + v_1} \quad \text{and} \quad t_3 = \frac{v_2}{\text{VT} + v_1}$$

so

$$T = \text{TUTR} + \text{TUTI} = \text{ZTUT} \ .$$

<u>Roll from U to T</u>

The roll relation is formed by the following lines of code:

```
TEMPA  =  TUTI**2
DUT   =  TWO/(ONE + TUTR**2 + TEMPA)
WORKAR  =  ONE - DUT*TEMPA
WORKAI  =  -DUT*TUTI*TUTR
TEMPA  =  CABS(ZWORKA)
RUTR   =  -WORKAR/TEMPA
RUTI   =  -WORKAI/TEMPA
```

TEMPA is defined:

$$\text{TEMPA} = \frac{v_3^2}{(\text{VT} + v_1)^2} \ .$$

The magnitude of T is:

$$|T| = (t_2^2 + t_3^2)^{1/2}$$

and DUT is therefore:

$$d = DUT = \frac{2}{1 + |T|^2}$$

$$= \frac{2}{1 + t_2^2 + t_3^2}$$

$$= \frac{2}{1 + (TUTI)^2 + (TUTR)^2} \quad .$$

The additional lines of code follow from the roll transformation

$$e^{i\phi} = \frac{A}{|A|} \quad \text{from Appendix B} \quad .$$

The complex number A:

$$A = 1 - i\ d\ Im(T)\bar{T}$$

where again

$$T = t_3 + it_2$$

$$\bar{T} = t_3 - it_2$$

and $$d = DUT \quad .$$

ZRUT which is equivalent to $e^{i\phi}$ is found by:

$$WORKAR = Re(A) = 1 - dt_2^2$$

$$WORKAI = Im(A) = -dt_2t_3$$

and

$$A = ZWORKA = WORKAR + iWORKAI \quad .$$

Therefore

$$A = 1 - dt_2^2 - idt_2t_3$$

$$= 1 - d(t_2^2 + it_2t_3) \quad .$$

where

$$iIm(T)\overline{T} = it_2(t_3 - it_2) = t_2^2 + it_2t_3 \quad .$$

Since

$$TEMPA = CABS(ZWORKA) = |A|$$

and

$$RUTR = -\frac{Re(A)}{|A|} = -\frac{WORKAR}{TEMPA}$$

$$RUTI = -\frac{Im(A)}{|A|} = -\frac{WORKAI}{TEMPA}$$

then

$$ZRUT = RUTR + RUTI = \frac{A}{|A|} \quad .$$

Tilt Derivative

In Section three of DRIVER, a derivative of tilt from U coordinates to T coordinates is calculated based upon a roll angle ϕ from the target to the maneuvering plane; NT, the magnitude of the normal acceleration; and VT, the magnitude of the target velocity. The code begins by redefining TEMPA:

$$\text{TEMPA} = \text{TUTI**2}$$

which is equivalent to:

$$\text{TEMPA} = t_2^2 \ .$$

DUT is then calculated:

$$\text{DUT} = \text{TWO/(ONE + TUTR**2 + TEMPA)}$$

or

$$d = \text{DUT} = \frac{2}{1 + t_3^2 + t_2^2} = \frac{2}{1 + |T|^2} \ .$$

WORKAR and WORKAI are recalculated:

$$\text{WORKAR} = \text{ONE - DUT*TEMPA}$$
$$\text{WORKAI} = \text{-DUT*TUTI*TUTR}$$

or

$$\text{WORKAR} = 1 - dt_2^2$$

and

$$\text{WORKAI} = -dt_2t_3 \ .$$

This again is equivalent to

$$A = 1 - i\ d\ \text{Im}(T)\overline{T}$$

or

$$ZWORKA = WORKAR + iWORKAI$$

$$= 1 - dt_2^2 - idt_2 t_3 \quad .$$

The roll relation from U to T is therefore:

$$ZRUT = RUTR + RUTI = \frac{A}{|A|}$$

$$= \frac{ZWORKA}{CABS(ZWORKA)} \quad .$$

The derivative of the tilt is now calculated:

TUTDR = (CPHITM*RUTR - SPHITM*RUTI)*TEMPA

TUTDI = (-SPHITM*RUTR - CPHITM*RUTI)*TEMPA

This sequence can be broken up into

CPHITM*RUTR - SPHITM*RUTI

where

$$CPHITM = \cos\phi$$

and

$$SPHITM = \sin\phi$$

ϕ being the angle between the target coordinate plane and the maneuver plane. This portion of the code projects the target roll relation onto the maneuvering coordinate frame where

CPHITM*RUTR - SPHITM*RUTI

is the real axis projection and

$$-SPHITM*RUTR - CPHITM*RUTI$$

is the imaginary axis projection. The sum of the two projections form the complex number O' from equation (19) in Appendix B. The tilt derivative is formed by:

$$\dot{T} = \frac{O'}{d} \quad .$$

This is accomplished by multiplying the terms of O' by TEMPA where:

$$TEMPA = -NT/(VT*DUT)$$

where the factor $(-\frac{NT}{VT})$ forms the derivative of O' with respect to time for numerical integration and

$$DUT = d$$

which completes the denominator of the tilt derivative:

$$\dot{T} = \frac{O'}{d} = \frac{TUTDR + TUTDI}{DUT} (-\frac{NT}{VT})$$

```
C
C
C
C
C
C
C  DRIVER SECTION TWO ABSTRACT
C
C
      TEMPA=VT+VTUI
      TUTR=VTUR/TEMPA
      TUTI=VTUI/TEMPA
      TEMPA=VTUI**2
      OUT=TWO/(ONE+TUTR**2+TEMPA)
      WORKAR=ONE-OUT*TEMPA
      WORKAI=-OUT*TUTI*TUTR
      TEMPA=CABS(ZWORKA)
      RUTR=-WORKAR/TEMPA
      RUTI=-WORKAI/TEMPA
C
C
C
C
C
C  DRIVER SECTION THREE ABSTRACT
C
C
      TEMPA=TUTI**2
      OUT=TWO/(ONE+TUTR**2+TEMPA)
      WORKAR=ONE-OUT*TEMPA
      WORKAI=-OUT*TUTI*TUTR
      TEMPA=CABS(ZWORKA)
      RUTR=WORKAR/TEMPA
      RUTI=WORKAI/TEMPA
      TEMPA=-NT/(VT*OUT)
      TUTOR=(CPHITM*RUTR-SPHITM*RUTI)*TEMPA
      TUTOI=(-SPHITM*RUTR-CPHITM*RUTI)*TEMPA
```

Fig. 13. DRIVER Abstract

VITA

William John Thome was born on 29 August 1949, in Cleveland, Ohio. He graduated from Garfield Heights High School in 1967, and obtained his Bachelor of Science degree in Aeronautical Engineering from Purdue University in 1971. After entering active duty, he completed Undergraduate Navigator Training at Mather AFB, California in 1972, and embarked on a rated career in the F-4 fighter. Upon completing RTU at George AFB, California in 1973; he was assigned to Udorn RTAFB, Thailand as a Weapons System Officer in the 421 TFS. He was next assigned to the 614 TFS at Torrejon AB, Spain in 1974 and became an instructor and later a stan eval flight examiner in the 401 TFW. In 1978, he entered the Air Force Institute of Technology.

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The air-to-air missile model used by the Cruise Missile Independent Survivability Team did not contain data useful to the cruise missile. The objective of this study was to modify the subroutine SIGNAL in the model to incorporate the cruise missile data. The modification required an understanding of quaternion algebra utilized within the model to represent three-dimensional motion. These quaternions allow real time outputs from the model for use by tactical ranges. The study contains a discussion of quaternions and their algebra.

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