A PREDICTION INTERVAL FOR A FIRST ORDER GAUSSIAN MARKOV PROCESS--ETC(U)

APR 80  T JAYACHANDRAN, T S MURTHY

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by

Toke Jayachandran

and

T.S. Murthy

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A PREDICTION INTERVAL FOR A FIRST ORDER GAUSSIAN MARKOV PROCESS

Let \( x(t) = \{ x_1, x_2, \ldots \} \) be a stationary Gaussian Markov process of order one with
\[
E(x_t) = \mu_t \quad \text{and} \quad \text{Cov}(x_t, x_{t+k}) = \sigma^2_k.
\]
We derive a prediction interval for \( x_{2n+1} \) based on the preceding \( 2n \) observations \( x_1, x_2, \ldots, x_{2n} \).
A PREDICTION INTERVAL FOR A FIRST ORDER GAUSSIAN MARKOV PROCESS

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Let $x_t (t = 1, 2, \ldots)$ be a stationary Gaussian Markov process of order one with $E(x_t) = \mu$ and $\text{Cov}(x_t, x_{t+k}) = \sigma^2 \rho^k$. We derive a prediction interval for $x_{2n+1}$ based on the preceding $2n$ observations $x_1, x_2, \ldots, x_{2n}$.

1. INTRODUCTION

Consider a stationary Gaussian Markov process of order one with $E(x_t) = \mu$ and $\text{Cov}(x_t, x_{t+k}) = \sigma^2 \rho^k$. For $\mu = 0$ such a process can be generated from an autoregressive model

$$x_t = \rho x_{t-1} + \varepsilon_t \quad t = 1, 2, \ldots \tag{1.1}$$

with $(\varepsilon_t)$ a sequence of independent and identically distributed random variables with normal distributions $N(0, \sigma^2)$, $|\rho| < 1$ and $x_0 = 0$. The process has many applications such as in modelling certain economic and meteorological time series. From a set of sample observations $x_1, x_2, \ldots, x_{2k}$, in this paper, we construct a conditional prediction interval for $x_{2k+1}$ treating one half of the observations as conditioning variables. The effect of the parameters $\sigma^2, \rho, k$ and the prediction coefficient $\alpha$ on the prediction interval is also investigated.

2. DERIVATION OF PREDICTION INTERVAL

For the stochastic process defined above it can be shown [1] that when $x_{2k-1}, k = 1, 2, \ldots, n+1$ are fixed, $x_{2k}, k = 1, 2, \ldots, n$ are conditionally independent and are normally distributed with mean $\mu_{2k} = \alpha + \beta x'_k$ and variance
\[ \sigma_o^2 \text{ where} \]

\[
a = m(1 - \rho)^2 / (1 + \rho^2)
\]
\[
b = 2\rho / (1 + \rho^2)
\]
\[
x'_k = (x_{2k-1} + x_{2k+1}) / 2
\]
\[
\sigma_o^2 = \sigma^2 (1 - \rho^2) / (1 + \rho^2).
\]

Conditionally, it may therefore be assumed that the \( x_{2k} \) satisfy the simple linear regression model

\[ x_{2k} = a + bx'_k + e_k \quad k = 1, 2, \ldots, n \]

where \( \{e_k\} \) are i.i.d. \( N(0, \sigma_o^2) \).

Given the sample observations \( x_1, x_2, \ldots, x_{2n}, x_{2n+1} \), from standard regression theory the parameters \( a, b \) and \( \sigma_o^2 \) in (2.1) can be estimated using the first 2n-1 observations as

\[
\hat{b} = \frac{s_{xy}}{s_{xx}}
\]
\[
\hat{a} = \overline{x}_2 - \hat{b} \overline{x}'
\]
\[
\hat{\sigma}_o^2 = \frac{s_{yy} - \hat{b}^2 s_{xx}}{(n-3)}
\]
\[ s_y = \frac{n-1}{
\sum_{k=1}^{n} x_k^2 - (n-1)x_2^2 \]
\[ s_{xy} = \frac{n-1}{\sum_{k=1}^{n} x'_k x_k^2 - (n-1)x'_2 x_2} . \]

If \( \hat{x}_{2n} \) is the least squares predictor of \( x_{2n} \), i.e., \( \hat{x}_{2n} = a + bx' \), then

\[ \frac{x_{2n} - \hat{x}_{2n}}{\sigma_0 \left[ 1 + \frac{1}{n-1} + \frac{(x'_n - \bar{x'})^2}{s_{xx}} \right]^{\frac{1}{2}}} \]

has a student's t-distribution with \( n-3 \) degrees of freedom; hence

\[ P \left\{ \left| x_{2n} - \hat{x}_{2n} \right| < t \sigma_0 \left[ 1 + \frac{1}{n-1} + \frac{(x'_n - \bar{x'})^2}{s_{xx}} \right]^{\frac{1}{2}} \right\} = 1 - \alpha \]

where \( t \) is the \( 100(1 - \frac{\alpha}{2}) \)th percentage point of the student's t-distribution with \( n-3 \) degrees of freedom. The above probability statement can be converted into a prediction interval for \( x_{2n+1} \), by noting that \( \hat{x}_{2n} \) is a function of \( x'_n = (x_{2n-1} + x_{2n+1})/2 \), as shown below.

Squaring the inequality and rearranging terms the above probability statement can be expressed as

\[ P \left\{ \left( b^2 - \frac{t^2 \sigma_0^2}{s_{xx}} \right) (x'_n - \bar{x'})^2 - 2b(x_{2n} - \bar{x}_2)(x'_n - \bar{x'}) + (x_{2n} - \bar{x}_2)^2 - \frac{nt^2 \sigma_0^2}{n-1} < 0 \right\} = 1 - \alpha \]

or

\[ P \left\{ A(x'_n - \bar{x'})^2 + B(x'_n - \bar{x'}) + C < 0 \right\} = 1 - \alpha \]

(2.4)
where

\[ A = \hat{b}^2 - \frac{t^2_o \sigma^2}{s_{xx}} \]
\[ B = -2b(x_{2n} - \bar{x}_2) \]
\[ C = (x_{2n} - \bar{x}_2)^2 - \frac{nt^2_o \sigma^2}{n-1} \]  \hspace{1cm} (2.5)

A prediction "interval" for \( x_n' = (x_{2n-1} + x_{2n+1})/2 \) and in turn for \( x_{2n+1} \) is now obtainable in terms of the roots of the quadratic expression in (2.4). If \( B^2 - 4AC < 0 \) i.e., the roots are complex the prediction interval will be taken to be \((-\infty, \infty)\); in the other cases the "interval" can turn out to be a two sided interval, a one sided interval or the union of two one sided intervals. The different possibilities will now be examined in detail.

3. PROPERTIES OF THE PREDICTION INTERVAL

**Case 1:** \( A > 0 \)

Let

\[ F = \frac{b^2 s_{xx}}{t^2 \sigma_o^2} \]

Then, \( A > 0 \Rightarrow F > 1 \).

Also, \( B^2 - 4AC > 0 \) \( \Leftrightarrow \)

\[ F > 1 - \frac{(n-1)(x_{2n} - \bar{x}_2)^2}{nt^2_o \sigma^2/n} \]

Hence, \( A > 0 \Rightarrow B^2 - 4AC > 0 \) and the prediction interval for \( x_{2n+1} \) will be of the form \((D-E, D+E)\) where

\[ D = 2\bar{x'} - x_{2n-1} + 2\hat{b}(x_{2n} - \bar{x}_2)/(b^2 - t^2_o \sigma^2/s_{xx}) \]
\[ E = 2t \sigma_o \left[ \frac{(x_{2n} - \bar{x}_2)^2}{s_{xx}} + \frac{n}{n-1} \left( b^2 - \frac{t^2_o \sigma^2}{s_{xx}} \right) \right]^{1/2} \left( b^2 - \frac{t^2_o \sigma^2}{s_{xx}} \right) \]  \hspace{1cm} (3.1)
Case 2: $A < 0$, $B^2 - 4AC > 0$

$A < 0$ and $B^2 - 4AC > 0 \implies$

$$1 - \frac{n-1}{n} \frac{(x_{2n} - \bar{x}_2)^2}{\frac{2}{\sigma^2}} < F < 1$$

and the prediction interval will be the union of two non-overlapping intervals $(-E, D+E)$ and $(D-E, \infty)$. Note that in this case $E < 0$.

Case 3: $A \neq 0$, $B^2 - 4AC < 0$

As indicated earlier, the roots will be complex and the prediction interval is defined to be $(-a, \infty)$. We will call the prediction intervals resulting from the above three cases a type 1, type 2 and a type 3 interval, respectively. There are two other cases viz., $A = 0$ and $B^2 - 4AC = 0$ and we have ignored these possibilities since their probability of occurrence is zero. It should be clear that the following identity holds for the prediction coefficient $1 - \alpha$.

$$\sum_{i=1}^{3} P[\text{an interval of type } i \text{ is obtained}] \cdot P[\text{the interval will contain } x_{2n+1}] = 1 - \alpha.$$

To study the effect of the parameters $n, \rho, \sigma^2$ and $\alpha$ on the probability of occurrence of the different types of intervals we conducted a simulation. For each choice of the parameter values, $2n+1$ samples are generated from the autoregressive process (1.1) and the prediction interval for $x_{2n+1}$ is calculated using the first $2n$ values. We then calculated the empirical frequencies of the three types of intervals, in 1000 replications, and also for each type of interval the frequency of inclusion of $x_{2n+1}$. In Table 1 we present the
results for $\alpha = .05$, $\sigma = 1.0$, $n = 14, 22, 30, 38$ as $\rho$ takes on the values .1, .3, .5, .7, .9. Table 2 shows the effect of increasing $n$ as the other parameters are held fixed. In Table 3 the standard deviation $\sigma$ is varied from 1 to 5 while the other parameter values are fixed. Some of the results are also presented in graphical form in figures 1-5.

The following general conclusions can be drawn from the results of the simulation. The probability of obtaining a type 1 interval increases with $\rho$, $n$ and $\alpha$. For $n > 15$ (30 or more samples) and $\rho > .5$ the probability of a type 1 interval is of the order of .85. The standard deviation $\sigma$ does not appear to have any effect on this probability.

4. AN EXAMPLE

The following data represents the monthly Dow-Jones industrial averages for the years 1966-67.

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<tr>
<th></th>
<th>1966</th>
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<tr>
<td>Jan</td>
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<tr>
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<td>933.68</td>
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<tr>
<td>May</td>
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<td>Jun</td>
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</tr>
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<td>Jul</td>
<td>847.38</td>
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</tr>
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<td>Aug</td>
<td>788.41</td>
<td>Aug 31</td>
</tr>
<tr>
<td>Sep</td>
<td>774.22</td>
<td>Sep 29</td>
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<tr>
<td>Oct</td>
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<tr>
<td>Nov</td>
<td>791.59</td>
<td>Nov 30</td>
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<tr>
<td>Dec</td>
<td>785.69</td>
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Assuming that the data is generated by a Gaussian Markov process of order one (a calculation of lagged correlations supports the assumption with $\rho = .8$) we computed prediction intervals for March 1967, May 1967, July 1967, September
1967 and November 1967 based on all the preceding data and the results are presented below.

<table>
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<tr>
<th>Month</th>
<th>n</th>
<th>Lower Prediction Limit</th>
<th>Upper Prediction Limit</th>
<th>Length of Interval</th>
<th>True Value</th>
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<td>598.39</td>
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<td>302.16</td>
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<td>Jul 67</td>
<td>9</td>
<td>708.70</td>
<td>1009.10</td>
<td>300.40</td>
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<td>Sep 67</td>
<td>10</td>
<td>573.59</td>
<td>864.87</td>
<td>291.28</td>
<td>926.66</td>
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<tr>
<td>Nov 67</td>
<td>11</td>
<td>633.12</td>
<td>899.73</td>
<td>266.61</td>
<td>875.81</td>
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</table>

All the intervals except for September 1967 contain the true value. As is to be expected the length of the interval decreases with an increase in sample size.

REFERENCES

PROBABILITIES OF OCCURRENCE OF PREDICTION INTERVALS
OF TYPES 1, 2, 3 IN 1000 REPLICATIONS

TABLE I

<table>
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<tr>
<th>n</th>
<th>ρ</th>
<th>P(type 1)</th>
<th>P(type 2)</th>
<th>P(type 3)</th>
<th>Empirical Prediction Coefficient</th>
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The numbers in parentheses are the probabilities that $x_{2n+1}$ is contained in the interval; for a type 3 interval this probability is always 1.
PROBABILITIES OF OCCURRENCE OF PREDICTION INTERVALS
OF TYPES 1, 2, 3 IN 1000 REPLICATIONS

TABLE II

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10
Table II (Continued)

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<td>.369</td>
<td>.119</td>
<td>.512</td>
<td>.949</td>
<td></td>
</tr>
<tr>
<td>(0.902)</td>
<td>(0.874)</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td>.586</td>
<td>.095</td>
<td>.319</td>
<td>.942</td>
<td></td>
</tr>
<tr>
<td>(0.920)</td>
<td>(0.884)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.791</td>
<td>.071</td>
<td>.138</td>
<td>.939</td>
<td></td>
</tr>
<tr>
<td>(0.934)</td>
<td>(0.873)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>.840</td>
<td>.065</td>
<td>.095</td>
<td>.944</td>
<td></td>
</tr>
<tr>
<td>(0.940)</td>
<td>(0.908)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>.914</td>
<td>.028</td>
<td>.058</td>
<td>.953</td>
<td></td>
</tr>
<tr>
<td>(0.951)</td>
<td>(0.929)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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PROBABILITIES OF OCCURRENCE OF PREDICTION INTERVALS OF TYPES 1, 2, 3 IN 1000 REPLICATIONS

**TABLE III**

\( \alpha = .05 \)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( P(\text{type 1}) )</th>
<th>( P(\text{type 2}) )</th>
<th>( P(\text{type 3}) )</th>
<th>Empirical Prediction Coefficient</th>
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<tr>
<td>1</td>
<td>.369 (.902)</td>
<td>.119 (.874)</td>
<td>.512</td>
<td>.949</td>
</tr>
<tr>
<td>n=8</td>
<td>.352 (.901)</td>
<td>.123 (.886)</td>
<td>.525</td>
<td>.951</td>
</tr>
<tr>
<td>( \rho = .7 )</td>
<td>.354 (.898)</td>
<td>.122 (.885)</td>
<td>.524</td>
<td>.950</td>
</tr>
<tr>
<td>3</td>
<td>.354 (.898)</td>
<td>.121 (.884)</td>
<td>.525</td>
<td>.950</td>
</tr>
<tr>
<td>4</td>
<td>.353 (.898)</td>
<td>.122 (.885)</td>
<td>.525</td>
<td>.950</td>
</tr>
<tr>
<td>5</td>
<td>.483 (.909)</td>
<td>.110 (.818)</td>
<td>.407</td>
<td>.936</td>
</tr>
<tr>
<td>n=12</td>
<td>.446 (.901)</td>
<td>.115 (.826)</td>
<td>.439</td>
<td>.936</td>
</tr>
<tr>
<td>( \rho = .5 )</td>
<td>.445 (.899)</td>
<td>.117 (.828)</td>
<td>.438</td>
<td>.935</td>
</tr>
<tr>
<td>3</td>
<td>.445 (.899)</td>
<td>.117 (.828)</td>
<td>.438</td>
<td>.935</td>
</tr>
<tr>
<td>4</td>
<td>.446 (.899)</td>
<td>.115 (.826)</td>
<td>.439</td>
<td>.935</td>
</tr>
<tr>
<td>5</td>
<td>.444 (.901)</td>
<td>.115 (.817)</td>
<td>.441</td>
<td>.935</td>
</tr>
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Fig. 1. $n$ versus $P$(Intervals of types 1, 2, 3)

$p = .5 \quad \sigma = 3.0 \quad \alpha = .05$
Fig. 2. \( n \) versus \( P(\text{Intervals of types } 1, 2, 3) \)

\[ \rho = 0.7 \quad \sigma = 1.0 \quad \alpha = 0.05 \]
Fig. 3. \( p \) versus \( P(\text{Intervals of types 1,2,3}) \)

\[ n = 5 \quad \sigma = 3 \quad \alpha = .05 \]
Fig. 4. $\rho$ versus $P$ (Intervals of types 1, 2, 3)

$n = 15 \quad \sigma = 1.0 \quad \alpha = .05$

```
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
```

```
\rho +
```

```
P(type 1) P(type 2) P(type 3)
```

```
PROB. OF TYPES OF INT. +
```

```
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
```

```
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.
```

```
\rho +
```
Fig. 5. \( n \) versus \( P(\text{type 1 interval}) \)

\[ a = .02, .05, .10 \quad \rho = .9 \quad \sigma = 1.0 \]
APPENDIX

PROGRAM TO CALCULATE PROBABILITY OF TYPE 1, TYPE 2, TYPE 3, TYPE 4 INTERVALS FOR SIMULATED SAMPLES.
PROGRAMMER T S MURTHY SEP 1979.

DIMENSION Z(55), X(55), V(100, 55), XI(50), YI(50), S(55), LI(5), IR(5), IPC(5), STAT(10, 7), VS(10, 10, 7), IVS(10)

CALL OVFLOW
INDEX=1
SIGMA=1.0

READ(5,2) K, T
WRITE (6,2) K, T

FORMAT (1X, I2, 2X F5.3)
IF (K .EQ. 0 ) GO TO 460
ROW=0.10
DO 300 I3=1, 5
DO 10 J=1, 5
B=(1-ROW**2)**.5
C
SIMULATION OF SAMPLES
ISEED=12345

10 IPC(J)=0
DO 250 IA=1, 10
DO 5C M=1, 100
CALL SNORM(ISEED, Z, K)
X(1)=SIGMA*Z(1)
C
J=2, K
30 X(J)=ROW*X(J-1)+3*SIGMA*Z(J)
DO 40 L=1, K
40 V(M, L)=X(L)
50 CONTINUE
DO 60 II=1, 5
60 IR(II)=0
DO 200 I=1, 10C
DO 70 J=1, K
70 S(IJ)=V(I, J)
K=K-5
KK=K/2

CALL OVFLOW
INDEX=2
SIGMA=1.0

READ(5,2) K, T
WRITE (6,2) K, T

FORMAT (1X, I2, 2X F5.3)
IF (K .EQ. 0 ) GO TO 460
ROW=0.10
DO 300 I3=1, 5
DO 10 J=1, 5
B=(1-ROW**2)**.5
C
SIMULATION OF SAMPLES
ISEED=12345

10 IPC(J)=0
DO 250 IA=1, 10
DO 5C M=1, 100
CALL SNORM(ISEED, Z, K)
X(1)=SIGMA*Z(1)
C
J=2, K
30 X(J)=ROW*X(J-1)+3*SIGMA*Z(J)
DO 40 L=1, K
40 V(M, L)=X(L)
50 CONTINUE
DO 60 II=1, 5
60 IR(II)=0
DO 200 I=1, 10C
DO 70 J=1, K
70 S(IJ)=V(I, J)
K=K-5
KK=K/2

CALL OVFLOW
INDEX=3
SIGMA=1.0

READ(5,2) K, T
WRITE (6,2) K, T

FORMAT (1X, I2, 2X F5.3)
IF (K .EQ. 0 ) GO TO 460
ROW=0.10
DO 300 I3=1, 5
DO 10 J=1, 5
B=(1-ROW**2)**.5
C
SIMULATION OF SAMPLES
ISEED=12345

10 IPC(J)=0
DO 250 IA=1, 10
DO 5C M=1, 100
CALL SNORM(ISEED, Z, K)
X(1)=SIGMA*Z(1)
C
J=2, K
30 X(J)=ROW*X(J-1)+3*SIGMA*Z(J)
DO 40 L=1, K
40 V(M, L)=X(L)
50 CONTINUE
DO 60 II=1, 5
60 IR(II)=0
DO 200 I=1, 10C
DO 70 J=1, K
70 S(IJ)=V(I, J)
K=K-5
KK=K/2

CALL OVFLOW
INDEX=4
SIGMA=1.0

READ(5,2) K, T
WRITE (6,2) K, T

FORMAT (1X, I2, 2X F5.3)
IF (K .EQ. 0 ) GO TO 460
ROW=0.10
DO 300 I3=1, 5
DO 10 J=1, 5
B=(1-ROW**2)**.5
C
SIMULATION OF SAMPLES
ISEED=12345

10 IPC(J)=0
DO 250 IA=1, 10
DO 5C M=1, 100
CALL SNORM(ISEED, Z, K)
X(1)=SIGMA*Z(1)
C
J=2, K
30 X(J)=ROW*X(J-1)+3*SIGMA*Z(J)
DO 40 L=1, K
40 V(M, L)=X(L)
50 CONTINUE
DO 60 II=1, 5
60 IR(II)=0
DO 200 I=1, 10C
DO 70 J=1, K
70 S(IJ)=V(I, J)
K=K-5
KK=K/2
DO EG L=1, KK
80 YI(L)=S(2*L)
N=KK-1
DO 50 LL=1, N
90 XI(LL)=(S(2*LL-1)+S(2*LL+1))/2.0
XSUM=0.0
YSUM=0.0
SXX=0.0
SXY=0.0
SYY=0.0
DO 100 KL=1, 5
100 IC(KL)=0
DO 110 M=1, N
YSUM=YSUM+YI(M)
XSUM=XSUM+XI(M)
CONTINUE
110 XB=XSUM/N
YB=YSUM/N
DO 120 M=1, N
SXX=SXX+(XI(M)-XB)**2
SYY=SYY+(YI(M)-YB)**2
SXY=SXY+(XI(M)-XB)*(YI(M)-YB)
CONTINUE
120 VRES=(SYY-((SXY**2)/SXX))/(N-2)
EH=SXY/SXX
AH=YB-BH*XB
SS=(T**2)/VRES
A=(EH**2)-SS/SXX
P=S(2*KK)-YB
B=-(2*BH*P)
C=(P**2)-(SS*(N+1))/N
F=(B**2)*SXX/SS
SSQ=1.0-N*(P**2)/(SS*(N+1))
IF (F_LT_ SSQ) GO TO 150
IF (F_EQ_1.0) GO TO 500
D=(2.*XB)-S(K-1)+2.*BH*P/A
E=\left(\frac{2}{A}\right) \times \left((SS*(P**2)/SXX)+(N+1)*A*SS/N)**.5\right)
FIL=0-E
PIR=0+E
PVAL=S(K+1)
IF(F.GT.1.0) GO TO 130
IC(2)=IC(2)+1
IF(PVAL.LE.PIR.OR. PVAL.GE.PIL) IC(5)=IC(5)+1
GO TO 160
130 IC(1)=IC(1)+1
IF(PIL.LE.PVAL.AND. PIR.GE.PVAL) IC(4)=IC(4)+1
GO TO 160
150 IC(3)=IC(3)+1
160 07 170 J=1,5
170 IR(J)=IR(J)+IC(J)
K=K+5
200 CONTINUE
CJ 220 J=1,5
220 IPC(J)=IPC(J)+IR(J)
250 CONTINUE
C PRINT STATISTICS
STAT(IB,1)=0
STAT(IB,2)=IPC(1)/1000.0
STAT(IB,3)=IPC(2)/1000.0
STAT(IB,4)=IPC(3)/1000.0
IF(STAT(IB,2).EQ.0.0) GO TO 275
STAT(IB,5)=IPC(4)/(STAT(IB,2)*1000.0)
GO TO 280
275 STAT(IB,5)=IPC(4)
280 IF(STAT(IB,3).EQ.0.0) GO TO 285
STAT(IB,6)=IPC(5)/(STAT(IB,3)*1000.0)
GO TO 290
285 STAT(IB,6)=IPC(5)
290 SIG=STAT(IB,2)*STAT(IB,5)+STAT(IB,3)*STAT(IB,6)
1 +STAT(IB,4)
STAT(IB,7)=SIG
RJW=ROW+0.2
300 CONTINUE
DO 305 IQ=1,5
CJ 305 JQ=1,7
VS(INDEX,IQ,JQ)=STAT(IQ,JQ)
305 CONTINUE
IVS(INDEX)=N
INDEX=INDEX+1
ISZ=K-5
WRITE(6,310)ISZ,SIGMA,N
WRITE (6,325)
WRITE(6,350)((STAT(K,L),L=1,7),K=1,5)
310 FORMAT(1X,' SAMPLE SIZE = ',ISZ,' SIGMA = ',F5.0, ' N = ',ISZ,')
325 FORMAT(1X,' ROW TYPE 1 TYPE 2 TYPE 3
1PROB.1 PROB.2 CON. REG ',/,'/70(---))
350 FORMAT( 7(F8.3,2X),/) 
WRITE(6,485)
GJ TO 1
460 CCRR=0.1
INDEX=INDEX-1
DO 470 J=1,5
WRITE (6,472) CCRR,SIGMA
WRITE(6,475)
DO 471 I=1,INDEX
WRITE(6,480)(IVS(I),(VS(I,J,K),K=2,7))
471 CONTINUE
CCRR=CCRR+0.2
470 CONTINUE
472 FORMAT( ' CORR.COEFF. = ',F5.3,' SIGMA = ',F5.0,')
475 FORMAT(' SAMPLE SIZE TYPE 1 TYPE 2 TYPE 3 PRO
18.1 PROB.2 CON. REG ',/,'/70(---))
480 FORMAT(15,5X,6(F8.3,2X),')
500 STOP
END
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