MULTI-BODY DYNAMICS
INCLUDING THE EFFECTS OF
FLEXIBILITY AND COMPLIANCE

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ABSTRACT

New and recently developed concepts and ideas useful in obtaining efficient computer algorithms for solving the equations of motion of multi-body mechanical systems with flexible links are presented and discussed. These ideas include the use of Euler parameters, Lagrange's form of d'Alembert's principle, generalized speeds, quasi-coordinates, relative coordinates, structural analysis techniques and body connection arrays. The mechanical systems considered are linked bodies forming a tree structure, but with no "closed loops" permitted. An explicit formulation of the equations of motion is presented.
INTRODUCTION

This report discusses the development of new methods for including the effects of flexibility and link and joint compliance in the governing dynamical equations of multi-body systems. Specifically, new, computationally-oriented techniques with potential for efficient, automated, and comprehensive analyses of multi-body system dynamics are presented and discussed.

The development of equations of motion of multi-body mechanical systems has received considerable attention of analysts in recent years. There are several reasons for this: Foremost, is the fact that many mechanical systems and devices can be effectively modelled by systems of linked bodies. But, another reason is the fact that it has just recently been possible with the aid of high-speed digital computers, to obtain efficient numerical solutions of the governing dynamical equations. Hence, the emphasis of researchers and analysts working with multi-body systems has been the formulation of equations of motion which can easily be developed into numerical algorithms for a computer code.

Most of the recent efforts in obtaining these dynamic formulations and their corresponding computational algorithms has been with systems of linked rigid bodies. Recently, however, a few researchers have attempted to include the effects of flexibility, compliance, and relative translation of the links by using a variety of approaches such as quasi-static methods, finite-element methods, modal analysis, and the strategic positioning of the flexible bodies, (for example, to the extremities of the system). Many of these efforts and the corresponding methodologies have been stimulated and motivated by specific application areas such as mechanism vibration and flexible satellite oscillations. In this report, these ideas are used and extended in the outline of new procedures for efficiently modelling the dynamics of multi-body systems with flexible links and joints.
If a mechanical system consists of connected bodies such that no closed loops or circuits are formed, the system is called a "general-chain", "open-chain", or "open-tree" system. Figure 1 depicts such a system. References [1-81]* provide summary of approaches taken to obtain efficient, computer-oriented formulation of equations of motion for such systems and related systems. If the mechanical system model of Figure 1 is generalized to include translation and compliance at the joints, it might appear as shown in Figure 2. References 2-143 provide a summary of approaches taken to include the effects of flexibility, translation, and link and joint compliance of these systems.

In one of these approaches it is shown [28, 29, 37, 38, 39, 51] that it is possible to obtain expressions for the governing equations in a form where the coefficients are directly related through computer algorithms. This approach uses Lagrange's form of d'Alembert's principle, as exposited by Kane and others [30, 52, 144, 145, 146] together with body connection arrays [37, 38, 39] and relative orientation coordinates [31, 32, 37, 50] to obtain the governing equations. Lagrange's form of d'Alembert's principle - a virtual work type approach combines the computational advantages of both Newton's laws and Lagrange's equations. That is, it has the advantage of automatic elimination of non-working internal constraint forces but without the introduction of tedious differentiation or other similar calculations.

Recently, it has been suggested by Huston, et. al. [32, 33, 37] that further efficiencies could be obtained through the use of Euler parameters as described by Whittaker [146] and Kane and Likins [147], together with the quasi-coordinates suggested by Kane and Wang [148]. Specifically, it is claimed [32, 33, 37] that using Euler parameters together with relative angular velocity components as generalized coordinate derivatives allows for the avoidance of geometrical singu-

* Numbers in brackets refer to References at the end of the report.
larities encountered with using Euler angles or dextral orientation angles to
define the relative orientation of the bodies.

The use of Lagrange's form of d'Alembert's principle, body connection arrays,
relative orientation coordinates, quasi-coordinates, and Euler parameters also
promises to provide an effective and efficient approach in the modelling, govern-
ing equation formulation, and analysis of multi-body systems with flexible links
and joints. The exposition of these ideas is a primary objective of this report.

The balance of the report is divided into five parts with the first two
parts containing the geometrical and kinematical development. The governing
dynamical equations are developed in the third part. This is followed by an
analysis of the flexibility and compliance effects in the fourth part. The
final part contains a discussion of the developed procedure together with con-
cluding remarks.
PRELIMINARY GEOMETRICAL CONSIDERATIONS

Body Connection Array

Consider a mechanical system such as depicted in Figure 1. To develop an accounting routine for the system's geometry arbitrarily select one of the bodies as a reference body and call it $B_1$. Next, number or label the other bodies of the system in ascending progression away from $B_1$ as shown in Figure 1. Now, although this numbering procedure does not lead to a unique labeling of the bodies, it can nevertheless be used to describe the chain structure or topology through the "body connection array" as follows: Let $L(k)$, $k=1, ..., N$ be an array of the adjoining lower numbered body of body $B_k$. For example, for the system shown in Figure 1., $L(k)$ is:

$$L(k) = (0,1,1,3,1,5,6,7,6)$$  \hspace{1cm} (1)

where

$$ (k) = (1,2,3,4,5,6,7,8,9) $$  \hspace{1cm} (2)

and where 0 refers to an inertial reference frame $R$. It is not difficult to see that, given $L(k)$, one could readily describe the topology of the system. That is, Figure 1. could be drawn by simply knowing $L(k)$. It is shown in the sequel that $L(k)$ is useful in the development of expressions of kinematical quantities needed for analysis of the system's dynamics.
Transformation Matrices

Next, consider a typical pair of adjoining bodies such as $B_j$ and $B_k$ as shown in Figure 3. The general orientation of $B_k$ relative to $B_j$ may be defined in terms of the relative orientation of the dextral orthogonal unit vector sets $n_{ji}$ and $n_{ki}$ ($i=1,2,3$) fixed in $B_j$ and $B_k$ as shown in Figure 2. Specifically $n_{ji}$ and $n_{ki}$ are related to each other as

$$n_{ji} = S_{JK}\cdot n_{km}$$

(3)

where $S_{JK}$ is a $3\times3$ orthogonal transformation matrix defined as [47]:

$$S_{JK}\cdot n_{km}$$

(4)

(Regarding notation, the $J$ and $K$ in $S_{JK}$ and the first subscripts on the unit vectors refer to bodies $B_j$ and $B_k$, and repeated indices, such as the $m$, in Equation (3) signify a sum over the range (eg. 1,...,3) of that index. Thus, with a computer $S_{JK}\cdot n_{km}$ would be the array $S_{JK}(I,M)$.)

From Equation (3), it is easily seen that with three bodies $B_j$, $B_k$, $B_l$, the transformation matrix obeys the following chain and identity rules:

$$S_{JL} = S_{JK} \cdot S_{KL}$$

(5)
Figure 3. Two Typical Adjoining Bodies
and

\[
SJ = I = SJK SKJ = SJK SJK^{-1} \quad (6)
\]

where I is the identity matrix.

These expressions allow for the transformation of components of vectors referred to one body of the system into components referred to any other body of the system and, in particular, to the inertial reference frame, R. For example, if a typical vector, \( \mathbf{v} \), is expressed as

\[
\mathbf{v}_i = v_i^{(k)} n_{ki} = v_i^{(0)} n_{0i} \quad (7)
\]

then

\[
\mathbf{v}_i^{(0)} = SOK_{ij} v_j^{(k)} \quad (8)
\]

where 0 refers to the inertial frame, R.

Since these transformation matrices play a central role throughout the analysis, it is helpful to also have an algorithm for their derivative, especially the derivative of SOK. Using Equation (3), and noting that \( n_{0i} \) are fixed in R, the following is obtained:

\[
d(SOK_{ij})/dt = n_{0i} \cdot ^R d n_{kj}/dt \quad (9)
\]
where the $R$ in $\frac{d n_{kj}}{dt}$ indicates that the derivative is computed in $R$. However, since the $n_{kj}$ are fixed in $B_k$, their derivatives may be written as $\omega_k \times n_{kj}$ where $\omega_k$ is the angular velocity of $B_k$ in $R$.

Equation (9) may then be written as:

$$\frac{d(SOK_{ij})}{dt} = -e_{imn} \omega_n n_m \cdot n_{kj}$$

or as

$$\frac{d(SOK)}{dt} = WOK SOK$$

where $WOK$ is a matrix defined as

$$WOK_{im} = -e_{imn} \omega_n$$

and where $\omega_n$ are the components of $\omega_k$ referred to $n_{on}$ and $e_{imn}$ is the standard permutation symbol [150]. ($WOK$ is simply the matrix whose dual vector [150] is $\omega_k$.). Equation (11) thus shows that the transformation matrix derivative may be computed by a simple matrix multiplication.

**Euler Parameters**

Finally, consider describing the relative orientation of $B_j$ and $B_k$ by using the so-called Euler parameters as discussed by Whittaker [147] and Kane and Likins [148]. It is well known [147] that $B_k$ may be brought into any
general orientation relative to B_j by means of a single rotation about an appropriate axis. If \( \lambda_k \) is a unit vector along this axis and if \( \theta_k \) is the rotation angle, the four Euler parameters describing the orientation of B_k relative to B_j may be defined as:

\[
\begin{align*}
\varepsilon_{k1} &= \lambda_{k1} \sin(\theta_k/2) \\
\varepsilon_{k2} &= \lambda_{k2} \sin(\theta_k/2) \\
\varepsilon_{k3} &= \lambda_{k3} \sin(\theta_k/2) \\
\varepsilon_{k4} &= \cos(\theta_k/2)
\end{align*}
\]

where the \( \lambda_{ki} \) \((i=1,2,3)\) are the components of \( \lambda_k \) referred to \( \mathbf{n}_{ji} \), the unit vectors fixed in B_j. Clearly, the \( \varepsilon_{ki} \) \((i=1,2,3,4)\) are not independent since:

\[
\varepsilon_{k1}^2 + \varepsilon_{k2}^2 + \varepsilon_{k3}^2 + \varepsilon_{k4}^2 = 1
\]

These parameters may be related to angular velocity components by using the transformation matrices as follows: It is shown in [147, 148] that SJK may be expressed in terms of these parameters as:
\[
\begin{bmatrix}
\varepsilon_{k1}^2 & -\varepsilon_{k2}^2 + \varepsilon_{k3}^2 + \varepsilon_{k4}^2 & 2(\varepsilon_{k1}^2 \varepsilon_{k2} - \varepsilon_{k3}^2 \varepsilon_{k4}) & 2(\varepsilon_{k1}^2 \varepsilon_{k3} - \varepsilon_{k2}^2 \varepsilon_{k4}) \\
2(\varepsilon_{k1}^2 \varepsilon_{k2} - \varepsilon_{k3}^2 \varepsilon_{k4}) & 2(\varepsilon_{k1}^2 \varepsilon_{k3} - \varepsilon_{k2}^2 \varepsilon_{k4}) & -\varepsilon_{k1}^2 - \varepsilon_{k2}^2 + \varepsilon_{k3}^2 + \varepsilon_{k4}^2 & 2(\varepsilon_{k2}^2 \varepsilon_{k3} - \varepsilon_{k1}^2 \varepsilon_{k4}) \\
2(\varepsilon_{k1}^2 \varepsilon_{k3} - \varepsilon_{k2}^2 \varepsilon_{k4}) & 2(\varepsilon_{k2}^2 \varepsilon_{k3} - \varepsilon_{k1}^2 \varepsilon_{k4}) & \varepsilon_{k1}^2 - \varepsilon_{k2}^2 + \varepsilon_{k3}^2 + \varepsilon_{k4}^2 & \end{bmatrix}
\]

Now, by solving Equations (11) and (12) for the angular velocity components, one obtains:

\[
\omega_{k1} = S\omega_{k1} \cdot S\omega_{k1} + S\omega_{k2} \cdot S\omega_{k2} + S\omega_{k3} \cdot S\omega_{k3}
\]

\[
\omega_{k2} = S\omega_{k1} \cdot S\omega_{k1} + S\omega_{k2} \cdot S\omega_{k2} + S\omega_{k3} \cdot S\omega_{k3}
\]

\[
\omega_{k3} = S\omega_{k1} \cdot S\omega_{k1} + S\omega_{k2} \cdot S\omega_{k2} + S\omega_{k3} \cdot S\omega_{k3}
\]

where the dot designates time differentiation. By using Equation (15), these expressions may be used to express the \(\eta_{ji}\) components of the angular velocity of \(B_k\) relative to \(B_j\) in terms of the Euler parameters as:

\[
\hat{\omega}_{k1} = 2(\varepsilon_{k4} \hat{\varepsilon}_{k1} - \varepsilon_{k3} \hat{\varepsilon}_{k2} + \varepsilon_{k2} \hat{\varepsilon}_{k3} - \varepsilon_{k1} \hat{\varepsilon}_{k4})
\]

\[
\hat{\omega}_{k2} = 2(\varepsilon_{k3} \hat{\varepsilon}_{k1} + \varepsilon_{k4} \hat{\varepsilon}_{k2} - \varepsilon_{k1} \hat{\varepsilon}_{k3} - \varepsilon_{k2} \hat{\varepsilon}_{k4})
\]

\[
\hat{\omega}_{k3} = 2(-\varepsilon_{k2} \hat{\varepsilon}_{k1} + \varepsilon_{k1} \hat{\varepsilon}_{k2} + \varepsilon_{k4} \hat{\varepsilon}_{k3} - \varepsilon_{k3} \hat{\varepsilon}_{k4})
\]
(Regarding notation, in the sequel "hats" refer to relative angular velocity vectors or their components. That is the \( \hat{\omega}_k \) represent the angular velocity of \( B_k \) in \( R \) and \( \hat{\omega}_k \) represent the angular velocity of \( B_k \) relative to \( B_{k-1} \), its adjoining lower numbered body.) Equation (17) may now be solved for the \( \hat{\omega}_{k1} \) \((i=1,...,4)\) in terms of the \( \hat{\omega}_{k1} \) leading to the expressions:

\[
\begin{align*}
\hat{\omega}_{k1} &= \frac{1}{4}(\epsilon_{k4} \hat{\omega}_{k1} + \epsilon_{k3} \hat{\omega}_{k2} - \epsilon_{k2} \hat{\omega}_{k3}) \\
\hat{\omega}_{k2} &= \frac{1}{4}(-\epsilon_{k3} \hat{\omega}_{k1} + \epsilon_{k4} \hat{\omega}_{k2} + \epsilon_{k1} \hat{\omega}_{k3}) \\
\hat{\omega}_{k3} &= \frac{1}{4}(\epsilon_{k2} \hat{\omega}_{k1} - \epsilon_{k1} \hat{\omega}_{k2} + \epsilon_{k4} \hat{\omega}_{k3}) \\
\hat{\omega}_{k4} &= \frac{1}{4}(-\epsilon_{k1} \hat{\omega}_{k1} - \epsilon_{k2} \hat{\omega}_{k2} - \epsilon_{k3} \hat{\omega}_{k3})
\end{align*}
\]

This solution is quickly obtained by observing that if Equation (14) is differentiated and placed with Equation (17), the resulting set of equations could be written in the matrix form:

\[
\begin{bmatrix}
\epsilon_{k1} \\
\epsilon_{k2} \\
\epsilon_{k3} \\
\epsilon_{k4}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\epsilon_{k4} & -\epsilon_{k3} & \epsilon_{k2} & -\epsilon_{k1} \\
\epsilon_{k3} & \epsilon_{k4} & -\epsilon_{k1} & -\epsilon_{k2} \\
-\epsilon_{k2} & \epsilon_{k1} & \epsilon_{k4} & -\epsilon_{k3} \\
\epsilon_{k1} & -\epsilon_{k2} & \epsilon_{k3} & \epsilon_{k4}
\end{bmatrix} \begin{bmatrix}
\hat{\omega}_{k1} \\
\hat{\omega}_{k2} \\
\hat{\omega}_{k3} \\
\hat{\omega}_{k4}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\epsilon_{k1} \\
\epsilon_{k2} \\
\epsilon_{k3} \\
\epsilon_{k4}
\end{bmatrix}
\]

\[(19)\]
where $\omega_{k4}$ is equal to the derivative of Equation (14) and has the value zero. The square matrix in Equation (19) is seen to be orthogonal (i.e., the inverse is the transpose) and hence, Equations (18) follow immediately from (19) upon letting $\omega_{k4}$ be zero.
Coordinates

A multibody system of \( N \) bodies, with translation permitted between the bodies will, in general, have \( 6N \) degrees of freedom. Let these be described by \( 6N \) generalized coordinates \( x_2 \) \((i=1,\ldots,6N)\) and let the first \( 3N \) of these be divided into \( N \) triplets describing the relative orientation of the successive bodies of the system. Let the remaining \( 3N \) \( x_2 \) also be divided into \( N \) triplets representing the relative displacement of the successive bodies of the system. As before, let \( B_k \) be a typical body of the system and let \( B_j \) be its adjacent lower numbered body, as in Figure 3. The angular velocity of \( B_k \) relative to \( B_j \) (that is, the relative rate of change of orientation) may then be written as:

\[
\omega_k = \omega_k^{1} n_j^{1} + \omega_k^{2} n_j^{2} + \omega_k^{3} n_j^{3}
\]  

(20)

where \( n_j^{i} \) \((j=1,\ldots,N, i=1,2,3)\) are mutually perpendicular dextral unit vectors fixed in \( B_j \). Next, let these bodies be displaced relative to each other with the displacement measured by the vector \( \xi_k \) as shown in Figure 4., where \( O_j \) and \( O_k \) are arbitrarily selected reference points of \( B_j \) and \( B_k \). \( Q_k \), which is fixed in \( B_j \), is the connection point or "origin" of \( B_k \). Then \( \xi_k \) may be written in the form:
Figure 4. Reference Points and Position Vectors of Two Adjacent Bodies
\[ \xi_k = \xi_{k,1} \eta_{j_1} + \xi_{k,2} \eta_{j_2} + \xi_{k,3} \eta_{j_3} \]  \hspace{1cm} (21)

Following Kane and Wang [149] introduce 6N parameters \( y_\ell \) (\( \ell = 1, \ldots, 6N \)) defined as:

\[ y_\ell = x_\ell \quad \forall \ell = 1, \ldots, 6N \]  \hspace{1cm} (22)

where the first 3N of these are called "generalized speeds", are

\[ y_{3k-2} = \hat{\omega}_{k1} \]

\[ y_{3k-1} = \hat{\omega}_{k2} \]  \hspace{1cm} (23)

\[ y_{3k} = \hat{\omega}_{k3} \]

and the remaining 3N are:

\[ y_{3(N+k)-2} = \xi_{k1} \]

\[ y_{3(N+k)-1} = \xi_{k2} \]  \hspace{1cm} (24)

\[ y_{3(N+k)} = \xi_{k3} \]
In general, Equations (23) are non-integrable. That is, they cannot be integrated to obtain generalized orientation coordinates $x_{3k-2}$, $x_{3k-1}$, $x_{3k}$. Thus, explicit parameters $x_{3k-2}$, $x_{3k-1}$, and $x_{3k}$ do not in general exist—hence, the name "quasi-coordinates". However, since parameters are needed to relate the relative orientation of the bodies to the respective relative angular velocities, let the Euler parameters introduced in the foregoing section be used for this purpose. Hence, if the orientation of a typical body $B_k$ relative to $B_j$ is described by the four parameters $\epsilon_{ki}$ ($i=1,\ldots,4$), the geometry and kinematics of the entire system may be expressed in terms of the $4N$ Euler parameters $\epsilon_{ki}$ ($k=1,\ldots,N; i=1,\ldots,4$), the $3N$ relative angular velocity components $\dot{\omega}_{ki}$ ($k=1,\ldots,N; i=1,2,3$), and the $3N$ displacement components $\xi_{ki}$ ($k=1,\ldots,N; i=1,2,3$).

Angular Velocity

The angular velocity of a typical body $B_k$ in the inertial frame $R$ is readily obtained by the addition formula as [145]

$$\dot{\omega}_k = \dot{\omega}_1 + \ldots + \dot{\omega}_k$$  \hspace{1cm} (25)

where the relative angular velocities on the right side of this expression are each with respect to the respective adjacent lower numbered bodies and where the sum is taken over the bodies of the chain from $B_1$ outward through the branch containing $B_k$. The $L(k)$ array introduced in the foregoing section can be useful in computing this sum: Consider for example, the system shown in Figure 1. The angular velocity of $B_9$ is:
\[ \omega_9 = \hat{\omega}_1 + \hat{\omega}_5 + \hat{\omega}_6 + \hat{\omega}_9 \]  

(26)

The subscript indices (i.e., 9, 6, 5, 1) may be obtained from \( L(k) \) as follows: Consider \( L(k) \) as a function mapping the \( (k) \) array (See Equation (2)) into the \( L(k) \) array. Then, using the notation that \( L^0(k) = (k) \), \( L^1(k) = L(L^0(k)) \), \( L^2(k) = L(L^1(k)) \), \( \ldots \), \( L^j(k) = L(L^{j-1}(k)) \), it is seen (see Equation (1)) that:

\[ L^0(9) = 9, \quad L^1(9) = 6, \quad L^2(9) = 5, \quad L^3(9) = 1 \]  

(27)

Therefore, \( \omega_9 \) may be written as:

\[ \omega_9 = \sum_{p=0}^{3} \hat{\omega}_q \quad q = L^p(9) \]  

(28)

Hence, in general, the angular velocity of \( B_k \) may be written as:

\[ \omega_k = \sum_{p=0}^{r} \hat{\omega}_q \quad q = L^p(k) \]  

(29)

where \( r \) is the index such that \( L^r(k) = 1 \) and it is obtained by comparing \( L^p(k) \) to 1. The index \( r \) represents the number of bodies from \( B_1 \) to \( B_k \) in that branch of the chain system \( B_k \). For example, for the system of Figure 1., if \( k=9, \ r=3 \). Equation (29) is thus an algorithm for determining \( \omega_k \) once \( \omega_k \) and \( L(k) \) are known.
By examining Equations (20, (23), and (25) it is seen that \( \omega_k \) may be written in the form

\[
\omega_k = \omega_{klm} y_{km} \tag{30}
\]

where there is a sum over the repeated indices and where \( \omega_{klm} \) (\( k=1,...,N; \ l=1,...,3N; \ m=1,2,3 \)) form a block array of coefficients needed to express \( \omega_k \) in terms of \( \xi_{om} \). In view of Equations (3), (16), (20), and (23), it is seen that the elements of the \( \omega_{klm} \) array may be obtained from the SOK transformation matrices. Moreover, it can be shown that the matching between the elements of the \( \omega_{klm} \) and SOK arrays is solely dependent upon the body connection array \( L(k) \).

To see this, consider for example the angular velocity of \( B_4 \) of the system of Figure 1: From Equation (25), \( \omega_4 \) is

\[
\omega_4 = \hat{\omega}_1 + \hat{\omega}_3 + \hat{\omega}_4 \tag{31}
\]

where from Equations (3), (20), and (23) \( \hat{\omega}_1, \ hat{\omega}_3, \) and \( \hat{\omega}_4 \) may be written as:

\[
\hat{\omega}_1 = y_{1}^0 n_{01} + y_{2}^0 n_{02} + y_{3}^0 n_{03} = y_{j}^0 \delta_{mj} n_{0m} \tag{32}
\]

\[
\hat{\omega}_3 = y_{7}^0 n_{11} + y_{8}^0 n_{12} + y_{9}^0 n_{13} = y_{g+j}^0 S_{01} m_{h} n_{0m} \tag{33}
\]

\[
\hat{\omega}_4 = y_{10}^0 n_{31} + y_{11}^0 n_{32} + y_{12}^0 n_{33} = y_{g+j}^0 S_{03} m_{j} n_{0m} \tag{34}
\]
Hence, the $\omega_{4lm}$ are:

\[
\begin{align*}
\delta_{m}^{\ell} & = 1,2,3 \\
0 & = 4,5,6 \\
\omega_{4lm} = S01_{m\ell-9} & = 7,8,9 \quad \ell = 1,2,3 \\
S03_{m\ell-9} & = 10,11,12 \\
0 & = \ell > 12
\end{align*}
\]

where $\delta_{ij}$ are the identity matrix components [150].

Next, consider that the results such as Equation (35) may be obtained for the entire system of Figure 1. or Figure 2. from a table such as Table 1., where the "m" entries of the $\omega_{k\ell m}$ array are the column of the transformation matrices. Finally, note that the non-zero entries in a typical row, say the $k^{th}$ row of Table 1. are obtained as follows: Let $P = L(k)$. Then SOP is placed in the $k^{th}$ column of triplets of $\dot{\theta}_k$. Next, let $Q = L(P)$. The SOQ is placed in the $P^{th}$ column to triplets of $\dot{\theta}_k$, etc. That is, SOM is placed in column $L^{k-1}(k)$ where $M = L^{j}(k)$, $j=1,...,r+1$ with $r$ determined from $L^r(k) = 1$.

Finally, it is interesting to note that the elements of the $\omega_{k\ell m}$ array (and hence, the transformation matrix columns of Table 1.) are components of the "partial angular velocity vectors" as originally defined by Kane [144].
| $k$  | 1, 2, 3 | 4, 5, 6 | 7, 8, 9 | 10, 11, 12 | 13, 14, 15 | 16, 17, 18 | 19, 20, 21 | 22, 23, 24 | 25, 26, 27 | 28, 29, 30 | 31, 32, 33 | 34, 35, 36 | 37, 38, 39 | 40, 41, 42 | 43, 44, 45 | 46, 47, 48 | 49, 50, 51 | 52, 53, 54 | 55, 56, 57 | 58, 59, 60 | 61, 62, 63 | 64, 65, 66 | 67, 68, 69 | 70, 71, 72 | 73, 74, 75 | 76, 77, 78 | 79, 80, 81 | 82, 83, 84 | 85, 86, 87 | 88, 89, 90 | 91, 92, 93 | 94, 95, 96 | 97, 98, 99 | 100, 101, 102 |
|-----|--------|--------|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\theta_k$ |        |        |        |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |            |
Angular Acceleration

The angular acceleration of $B_k$ in $R$ may be obtained by differentiating Equation (30). Noting that the $\omega_{kom}$ are constant, this leads to:

$$\ddot{\omega}_{km} = (\omega_{km} \gamma_k + \omega_{km} \gamma_k) \omega_{kom}$$  \hspace{1cm} (36)

A table containing the $\omega_{km}$ can be constructed directly from the corresponding table for the $\omega_{kom}$. For example, for the system of Figure 1, such a table is shown in Table 2.

Mass Center Velocities

The velocity and acceleration of the mass center $G_k$ of a typical body $B_k$ ($k=1,\ldots,N$) may be obtained as follows: Let $x_k$ locate $G_k$ relative to $Q_k$ as shown in Figure 4. Since $Q_k$ is located relative to $Q_k$ by $x_k$ and if $Q_k$ is located relative to $O_j$ by the vector $q_k$ (See Figure 4.), then by continuing this procedure, $G_k$ may ultimately be located relative to a fixed point $0$ in $R$, the inertial reference frame. For example, for Body $B_8$ of Figure 2, the position vector $P_8$ of $G_8$ relative to $0$ is:

$$P_8 = \xi_1 + q_5 + \xi_5 + q_6 + \xi_6 + q_7 + \xi_7 + q_8 + \xi_8 + r_8$$  \hspace{1cm} (37)

In general, for Body $B_k$, the position vector $P_k$ of $B_k$ relative to $0$ is:

23
\[ P_k = \left[ S_{0k}^h \right] r_{kh} + \sum_{q=0}^{u} S_{0sh}^h (q_{sh} + \xi_{sh}) n_{oi} \]  

(38)

where \( s = L^q(k) \), \( S = L^{q+1}(k) \), and \( u \) is the index such that \( L^u(k) = 1 \), and where \( q_1 \) is 0. By differentiating, the velocity of \( G_k \) in \( R \) is obtained as:

\[
\begin{align*}
\nu_k &= \left[ S_{0k}^h \right] r_{ks} + \sum_{q=0}^{u} \left[ \sum_{q=0}^{S_{0sh}^h} (q_{sh} + \xi_{sh}) 
\right.

+ S_{0sh}^h \xi_{sh} \right] n_{oi} \\
\end{align*}
\]

(39)

By using Equations (11), (12), and (30), \( \nu_k \) may be written in the form:

\[
\nu_k = v_{km} \nu_{l_n} \quad (40)
\]

where \( v_{km} \) (\( k=1,\ldots,N; \ l=1,\ldots,6N; \ m=1,2,3 \)) form a block array of coefficients needed to express \( \nu_k \) in terms of \( n_{om} \). In view of Equation (39), the non-zero \( v_{km} \) are:

\[
\begin{align*}
v_{km} &= W_{km}^h r_{kj} + \sum_{q=0}^{u} W_{km}^h (q_{sh} + \xi_{sh}) \\
\end{align*}
\]

\[ (k=1,\ldots,N; \ l=1,\ldots,3N; \ m=1,2,3) \]

(41)
where $W_{mk}$ is defined as:

$$W_{mk} = \frac{\partial SOK_p}{\partial y_k} = \omega_{mki} SOK_{ph}$$  \hspace{1cm} (42)$$

and

$$v_k(3N+1)m = \omega_{k,m} \hspace{1cm} (k=1, \ldots, N; l=1, \ldots, 3N; m=1,2,3). \hspace{1cm} (43)$$

The elements of the $v_{km}$ array are components of the "partial velocity vectors" as originally defined by Kane [144].

**Mass Center Accelerations**

Similarly, by differentiation of Equations (40), the acceleration of $G_k$ in $R$ is

$$z_k = (v_{km} y^* + v_{km} y_k^*)_{cm} \hspace{1cm} (44)$$

where the non-zero $v_{km}$ are, by Equations (41) to (43),

$$v_{km} = \tilde{W}_{mk}^{\varepsilon_{kh}^{*}} + \frac{\tilde{W}_{mk}^{\varepsilon_{kh}^{*}}}{q=0} \left[ W_{mk}^{\varepsilon_{kh}^{*}} (\xi_{sh} + q_{sh}) + W_{mk}^{\varepsilon_{kh}^{*}} \xi_{sh} \right] (k=1, \ldots, N; \hspace{1cm} q=1, \ldots, 3N; m=1,2,3) \hspace{1cm} (45)$$

where $\tilde{W}_{mk}$ is:

$$\tilde{W}_{mk} = -\omega_{mki} (\omega_{kli} SOK_{ph} + \omega_{kli} SOK_{ph}) \hspace{1cm} (46)$$
\[ \dot{v}_{k(3N+2)} = \omega_{k,m} \quad (k=1, \ldots, N; \, \ell=1, \ldots, 3N, \, m=1, 2, 3) \quad (47) \]
Consider again a general chain system such as shown in Figure 2., and imagine the system to be subjected to an externally applied force field. Let the force field on a typical body \( B_k \), be replaced by an equivalent force field consisting of a single force \( F_k \), passing through \( G_k \) together with a couple with torque \( M_k \). Then Lagrange's form of d'Alembert's principle leads to governing dynamical equations of motion of the form [38]:

\[
F_L + F_L^* = 0 \quad l = 1, \ldots, 6N
\] (48)

\( F_L \) \((l=1,\ldots,6N)\) is called the generalized active force and is given by:

\[
F_L = v_{k,m} F_{km} + \omega_{k,m} M_{km}
\] (49)

where there is a sum from 1 to \( N \) on \( k \) and from 1 to 3 on \( m \), and where \( F_{mk} \) and \( M_{km} \) are the components of \( F_k \) and \( M_k \) with respect to \( \omega_{om} \). \( F_L^* \) \((l=1,\ldots,6N)\) is called the generalized inertia force and is given by:

\[
F_L^* = v_{k,m} F_{km}^* + \omega_{k,m} M_{km}
\] (50)
where the indices follow the same rules as in Equation (48), and where $F_{km}^*$ and $M_{km}^*$ are components of inertia forces, $F_k^*$, and inertia torques, $M_k^*$, given by [145].

\[ F_{km}^* = -m_k a_{km} \]  
\[ (51) \]

and

\[ M_{km}^* = -I_k \cdot a_{km} - \omega_k \times (I_k \cdot \omega_k) \]  
\[ (52) \]

where $m_k$ is the mass of $B_k$ and $I_k$ is the inertia dyadic of $B_k$ relative to $G_k$ ($k=1,\ldots,N$). ($F_{km}^*$, with line of action passing through $G_k$ together with $M_{km}^*$ are equivalent to the inertia forces on $B_k$ [145]. Through use of the shifter transformation matrices, $I_k$ may be written in the form:

\[ I_k = I_{kmn} n_{omn} \]  
\[ (53) \]

By substituting Equations (36) and (44) into Equations (51) and (52) and ultimately into Equation (47), the equations of motion may be written in the form:
\[ a_{lp} \dot{y}_p = f_\ell \quad (\ell = 1, \ldots, 6N) \]  

(54)

where there is a sum from 1 to 6N on p and where \( a_{lp} \) and \( f_\ell \) are given by:

\[ a_{lp} = m_k v_{kpm} v_{klm} + I_{kmm} \omega_{kpm} \omega_{klm} \]  

(55)

and

\[ f_\ell = -(F_\ell + m_k v_{klm} v_{kqm} y_q + I_{kmm} \omega_{klm} \omega_{kqn} y_q + e_{nmh} I_{kmr} \omega_{kqn} \omega_{krs} \omega_{klm} y_q y_s) \]  

(56)

where there is a sum from 1 to N on k, from 1 to 6N on q and s, and from 1 to 3 on the other repeated indices.

Recall that the first 3N \( y_p \) are relative angular velocity components. These may be related to the Euler parameters by N sets of first order equations of the form of Equations (18).

Equations (54), (20), and the 4N equations of the form of Equations (18) form a set of 13N simultaneous first-order differential equations for the 6N \( y_p \), the 3N \( \xi_{ki} \), and the 4N Euler parameters \( \epsilon_{ki} \) \((k=1, \ldots, N; i=1, \ldots, 4)\). Since the coefficients \( a_{lp} \) and \( f_\ell \) in Equations (54) are algebraic functions of the physical parameters and the four block arrays \( \omega_{klm}, \omega_{klm}, v_{klm}, v_{kln} \), computer algorithms can be written for the numerical development of these governing equations. Moreover, once these
arrays are developed, the system of equations consisting of Equations (54), (20), and 4N equations of the form of Equations (18), may also be solved numerically by using one of the standard numerical integration routines and a linear equation solver.

The development of these computer algorithms and the numerical development of Equations (54) might proceed as follows: First, let the body connection array \( L(k) \) (See Equation (1)) together with the geometrical and physical parameters \( r_k, x_k, l_k, \) and \( m_k \) (See Equations (38), (51), and (52).) and the applied forces and moments \( F_k \) and \( M_k \) (See Equation (48).) be read into the computer. (Let \( r_k, x_k, l_k, \) and, if desired, \( F_k \) and \( M_k \) be expressed in terms of \( \eta_{ki} \).) Next, from assumed initial values of \( \epsilon_{ki} \) form the transformation matrix arrays \( SOK \) using Equations (15) and (5). Use these arrays to express \( r_k, x_k, l_k \) and possibly \( F_k \) and \( M_k \) in terms of \( \eta_{ok} \). Next, using \( L(k) \) and \( SOK \) write an algorithm, with Tables 1. and 2. as a guide, to form \( \omega_{k1m} \) and \( \omega_{k1m}' \). For example, to obtain the non-zero \( \omega_{k1m} \), observe that if \( L(k) = p \), then \( \omega_{k1m} = SOP_{m} \) (\( m = 1,2,3; l = 3p+1, 3p+2, 3p+3 \)). Then, if \( L(p) = q, L^2(k) = q \) and \( \omega_{k1m} = SOQ_{m} \) (\( m = 1,2,3; l = 3q+1, 3q+2, 3q+3 \)). This assignment procedure is continued until unity is reached or \( r \) times where \( r \) is given by \( L^r(k) = 1 \) (See the remark following Equation (29).).

\( v_{kim} \) and \( \dot{v}_{kim} \) may then be obtained using Equations (40) to (47). Finally, numerical values of the coefficients \( a_{lp} \) and \( f_{l} \) of the governing differential equations (54) may then be obtained from Equations (55) and (56). These equations may then be integrated numerically to obtain incremental values to the initial values of the parameters \( y_p, \epsilon_{ki}, \) and \( x_q \) (\( p=1,\ldots,3N+3; k=1,\ldots,N; i=1,2,3, \) and \( q=1,2,3 \)), at the end of a time interval, say \( t_1 \). New values of the
transformation matrix arrays \( SOK \) may then be obtained and the entire process repeated until a history of the configuration and motion of the system is determined.

The application of these expressions and ideas in an analysis of the flexibility and compliance effects is developed in the following part of the report.
EFFECTS OF COMPLIANCE AND FLEXIBILITY

Compliance

Let the term "compliance" refer to the yielding or deformation of the system due to the externally applied forces and due to the inertia forces. If the assumption is made that the compliance of a link or joint is "small" compared with the general dimensions of the system, then the effects of the compliance can be determined directly from the integration of Equations (54).

To see this, consider a typical integration step as described at the end of the foregoing section. If the \( v_{klm}, \dot{v}_{klm}, \omega_{klm}, \dot{\omega}_{klm}, \epsilon_{kl}, \gamma_k, \) and \( \dot{\gamma}_k \) are known, all the kinematics is known. That is, by using Equations (30) and (36) the angular velocity and angular acceleration of each body is determined. Similarly, by using Equations (40) and (44), the velocities and accelerations of the mass centers of each of the bodies is determined. Then, by using Equations (51) and (52) the equivalent inertia force system on each body is determined. Hence, since the externally applied force field on each body is also known, the entire force system on each body is determined. Therefore, by taking successive free body diagrams of the bodies of the system, starting with the Nth body and working backward through the chain, the force system transmitted across each connection joint may be determined. Finally, by knowing the complete force system acting on each body, including the forces transmitted across the connection joints, the physical force-deformation relations may be used to determine the compliance. Then, by addition and superposition the compliance of the entire system is determined.

To illustrate this procedure in more detail, assume, for example, that the bodies of the system are long slender members which can be modeled as beams with uniform cross section. Hence, the system of Figures 1. and 2. might appear as
shown in Figure 5., and a typical member of this system might be depicted as in Figure 6., where a rectangular shape is assumed and where an axes system is introduced. As before \( O_j \) is the connection point of the adjacent lower numbered body \( B_i \) and \( Q_k \) is the connection point with the adjacent higher numbered body \( B_k \).

Let the forces exerted on \( B_j \) by \( B_i \) and \( B_k \) at the connection joints \( O_j \) and \( Q_k \) be represented by single forces \( f_{i/j} \) and \( f_{k/j} \) passing through \( O_j \) and \( Q_k \) together with couples with torques \( \tau_{i/j} \) and \( \tau_{k/j} \). Similarly, let the externally applied force system on \( B_j \) together with the inertia force system of \( B_j \) be represented by equivalent force systems at the ends \( O_j \) and \( Q_k \) of \( B_j \). Hence, let the resultant force system exerted on \( B_j \) at \( Q_k \) be represented by the single force \( f_{k/j} \) passing through \( Q_k \), together with a couple with torque \( m_{k/j} \) and let \( f_{k/j} \) and \( m_{k/j} \) be expressed in the forms:

\[
f_{k/j} = f_{j1} n_{j1} + f_{j2} n_{j2} + f_{j3} n_{j3}
\]

and

\[
m_{k/j} = m_{j1} n_{j1} + m_{j2} n_{j2} + m_{j3} n_{j3}
\]

(Note, that from equilibrium considerations, the force system exerted on \( B_j \) at \( O_j \) is equivalent to a single force \(-f_{k/j}\) passing through \( O_j \) together with a couple with torque \(-m_{k/j}\).)

Let the displacement of \( Q_k \) relative to \( O_j \) due to the beam compliance, or deformation, be represented by \( u_j \). Let \( u_j \) be written in the form:

\[
u_j = u_{j1} n_{j1} + u_{j2} n_{j2} + u_{j3} n_{j3}
\]

Similarly, let the rotation of the beam cross section at \( Q_k \) relative to the cross section at \( O_j \), due to the beam compliance, be represented by \( \phi_j \). Let \( \phi_j \) be
Figure 5. A General Chain System of Slender Members
Figure 6. A Typical Member of a Beam Chain System
written in the form:

$$\phi_j = \phi_{j1} n_{j1} + \phi_{j2} n_{j2} + \phi_{j3} n_{j3}$$  \hspace{1cm} (60)

By following the procedures of matrix structural analysis [15] \(u_{ji}\) and \(\phi_{ji}\) may be expressed in terms of \(f_{ji}\) and \(m_{ji}\) \((i = 1, 2, 3)\) as:

\[
\begin{align*}
    u_{j1} &= \left( \frac{L_j}{A_j E_j} \right) f_{j1} \\
    u_{j2} &= \left( \frac{L_j^2}{3 E_j J_{j2}} \right) f_{j2} + \left( \frac{L_j^2}{2 E_j J_{j2}} \right) m_{j3} \\
    u_{j3} &= \left( \frac{L_j^2}{3 E_j J_{j3}} \right) f_{j3} - \left( \frac{L_j^2}{2 E_j J_{j3}} \right) m_{j2} \\
    \phi_{j1} &= \left( \frac{L_j}{J_{j1} G_j} \right) m_{j1} \\
    \phi_{j2} &= \left( \frac{L_j^2}{2 E_j J_{j2}} \right) f_{j2} + \left( \frac{L_j}{E_j J_{j2}} \right) m_{j3} \\
    \phi_{j3} &= \left( \frac{L_j^2}{2 E_j J_{j3}} \right) f_{j3} - \left( \frac{L_j}{E_j J_{j3}} \right) m_{j2}
\end{align*}
\]

where \(L_j\) is the axial length of \(B_j\), \(A_j\) is the cross-sectional area, \(J_{ji}\) \((i = 1, 2, 3)\) are the centroidal second moments of area of the cross section relative to the \(X_{ji}\) axes, \(E_j\) is the elastic modulus, and \(G_j\) is the shear modulus.

\(u_j\) and \(\phi_j\) thus represent the compliance as yielding of member \(B_j\) due to the holding and system motion. In an automated analysis as outlined in the foregoing section, \(u_j\) and \(\phi_j\) would be calculated and then used to adjust the geometrical parameters at each integration step. Specifically, \(g_k\) (See Figure 4.) and SOK are adjusted as:

\[
\begin{align*}
    g_k &= g_k + u_j \\
    \text{SOK} &= \text{SOK} \cdot CJ
\end{align*}
\]

where \(CJ\) is

$$CJ = \begin{bmatrix} 1 & -\phi_{j3} & \phi_{j2} \\ \phi_{j3} & 1 & -\phi_{j1} \\ \phi_{j2} & \phi_{j1} & 1 \end{bmatrix}$$  \hspace{1cm} (69)
(The development of Equation (69) follows from the successive multiplications of matrices of the form of Equation (15) for $\theta$ being $\phi_{jk}$, $\phi_{j2}'$, and $\phi_{j3}$ (small angles) about the $X_{j1}$, $X_{j2}$, and $X_{j3}$ axes respectively.) The integration would then proceed with the adjusted values of $q_k$ and $SOK$.

**Vibration and Impact Response**

The above compliance analysis is a quasi static analysis and as such it does not directly account for oscillatory or vibration phenomena due to the flexibility of the system and the externally applied (for example, impact) and inertia forces. If, as before, it is assumed that the vibrations have relatively small amplitude, then a modelling and description of the vibration phenomena may be obtained through torsion and translation springs introduced at the connection points, or joints, of the system.

To illustrate this, consider again a system consisting of long slender members which can be modelled as beams as in Figure 5. Consider two typical adjoining members of such a system as shown in Figure 7. The contribution to the systems oscillation due to the flexibility of $B_j$ can be modelled by 1) three torsion springs connecting the surfaces of $Q_k$ and $O_k$ with spring constants $G_j J_{j1}/\lambda_j$, and $E_j J_{j3}/\lambda_j$ and governing the relative rotation of $B_j$ and $B_k$ about axes parallel to $n_{j1}$, $n_{j2}$ and $n_{j3}$ respectively; and by 2) three translation springs connecting $Q_k$ and $O_k$ with spring constants $A_j E_j/\lambda_j$, $3E_j J_{j3}/\lambda_j^3$ and $3E_j J_{j2}/\lambda_j^3$ and governing the relative translation of $B_j$ and $B_k$ along axes parallel to $n_{j1}$, $n_{j2}$, $n_{j3}$. (These constants are determined from elementary structural analysis as in Equations (61) to (66).)

**Discussion**

Both of the above analyses involve the effects of forces and moments transmitted across connection joints. The dynamics analysis of the preceding part is
particularly well suited for accommodating the introduction of these forces and moments and for obtaining their contributions to the generalized forces. That is, although the compliance procedure above suggests the use of successive free body diagrams to obtain the force and moment components, and although this procedure could be automated, these components as well as the spring and moment components of the above oscillation analysis, can be readily obtained and directly incorporated into the governing equations by using the partial velocity and partial angular velocity vectors of the preceding dynamics analysis.

To see this, consider again two typical adjoining bodies such as $B_j$ and $B_k$ of Figures 4. and 7. As in Figure 4., let $\xi_k$ measure the displacement of $Q_k$ relative to $Q_k$. Then by differentiation of $\xi_k$ [145] the velocity of $Q_k$ may be expressed as:

$$\dot{Q}_k = \dot{Q}_k + \omega_j \times \xi_k + \hat{\xi}_k n_j \tag{70}$$

As before, let the orientation and rotation of $B_k$ relative to $B_j$ be defined in terms of Euler parameters and relative angular velocity components. Then, from Equation (25) the angular velocity of $B_k$ may be expressed as:

$$\omega_k = \omega_j + \hat{\omega}_k = \omega_j + \hat{\omega}_k n_j \tag{71}$$

Let the force system which $B_k$ exerts on $B_j$ be equivalent to a single force $\xi_{k/j}$ passing through $Q_k$ together with a couple with torque $m_{k/j}$, as in Equations (57) and (58). Then by the law of action-reaction, the force system which $B_j$ exerts on $B_k$ is equivalent to a single force $-\xi_{k/j}$ passing through $Q_k$ together with a couple with torque $-m_{k/j}$.

Let $y_l (l=1, \ldots, 6N)$ be the generalized coordinate derivatives as defined by Equations (23) and (24). Let the contribution to the generalized active force $F_l$ by these forces transmitted across the connecting joint be $\hat{F}_l$. Then $\hat{F}_l$ is given by the expression [145]:

$$\hat{F}_l$$
Consider the following cases:

Case 1: \( y_k \) is not equal to either \( \xi_{ki} \) or \( \hat{\xi}_{ki} \). In this case the partial velocities and partial angular velocities of \( Q_k \), \( O_k \), \( B_j \), and \( B_k \) may be expressed by using Equations (70) and (71) as:

\[
\frac{\partial y^Q_k}{\partial y_L} = \frac{\partial y^Q_k}{\partial y_L}
\]

and

\[
\frac{\partial \omega_k}{\partial y_L} = \frac{\partial \omega_k}{\partial y_L}
\]

Then, by Equation (72), \( \hat{F}_L \) becomes:

\[
\hat{F}_L = 0
\]

Case 2: \( y_k \) is equal to one of the \( \xi_{ki} \) (i=1, 2, 3). In this case, the partial velocities and partial angular velocities of \( Q_k \), \( O_k \), \( B_j \), and \( B_k \) become:

\[
\frac{\partial y^Q_k}{\partial y_L} = \frac{\partial y^Q_k}{\partial y_L} \xi_{ki} = 0
\]

\[
\frac{\partial y^O_k}{\partial y_L} = \frac{\partial y^O_k}{\partial y_L} \xi_{ki} = B_j
\]

\[
\frac{\partial \omega_j}{\partial y_L} = \frac{\partial \omega_j}{\partial y_L} \xi_{ki} = 0
\]

and

\[
\frac{\partial \omega_k}{\partial y_L} = \frac{\partial \omega_k}{\partial y_L} \xi_{ki} = 0
\]

Hence, by Equation (72) and (57), \( \hat{F}_L \) becomes:

\[
\hat{F}_L = \hat{F}_{ji} \times \frac{-\xi_{kj}}{\partial/\partial y_L} = -\hat{F}_{ji}
\]
Case 3: \( y_\ell \) is equal to one of the \( \hat{\omega}_{ki} \) (i=1, 2, 3). In this case, the partial velocities and partial angular velocities of \( Q_k, \theta_k \), \( B_j \), and \( B_k \) become:

\[
\begin{align*}
\frac{\partial y^k}{\partial y_\ell} &= \frac{\partial \hat{\omega}_k}{\partial \hat{\omega}_{ki}} = 0 \\
\frac{\partial y^k}{\partial y_\ell} &= \frac{\partial \hat{\omega}_k}{\partial \hat{\omega}_{ki}} = 0 \\
\frac{\partial \omega_j}{\partial y_\ell} &= \frac{\partial \omega_j}{\partial \hat{\omega}_{ki}} = 0 \\
\frac{\partial \omega_k}{\partial y_\ell} &= \frac{\partial \omega_k}{\partial \hat{\omega}_{ki}} = \nu_{ji}
\end{align*}
\]

Hence, by Equation (72) and (58), \( \hat{F}_\ell \) becomes:

\[ \hat{F}_\ell = \nu_{ji} \cdot (\frac{-m_k}{j}) = -\nu_{ji} \]  

The above three cases include the contribution to the generalized active forces for each of the \( y_\ell \) (\( \ell = 1, \ldots, 6N \)). Moreover, each of the non-zero contributions (from Equations (80) and (85) occurs individually; that is, each contribution occurs separately in one of the governing equations. Hence, if in a particular configuration or motion of the system, \( y_\ell \) is specified (for example, \( y_\ell \) is zero) then the \( \ell \)th governing equation becomes an uncoupled linear expression for the unknown restraining force or moment, thus determining the compliance. Conversely, if \( y_\ell \) is an unknown variable (representing a degree of freedom) the contribution to \( F_\ell \) due to the flexibility as modelled by the translation and torsion springs is determined directly by Equations (61) to (66), (80), and (85).
CONCLUDING REMARKS

The results of numerically solving the governing differential equations (54) where the coefficients are given by Equations (55) and (56) are reported and discussed in References [1, 32, 34, 35, 36, 77, 152-155] for a number of physical systems and configurations (e.g. human-body models, head-neck models, and flexible cables).

The application of Equations (54) with these systems, however, is based on the use of relative orientation angles between the respective bodies of the system as the generalized coordinates ($\mathbf{x}_j$) as opposed to the use of Euler parameters, quasi-coordinates, and generalized speeds as outlined herein. A problem which arises in the numerical solution of Equations (54) where orientation angles are used is that there always exists values of the angles and hence, configurations of the system, for which the determinant of $\mathbf{a}_{kj}$ is zero. A numerical solution will, of course, fail to converge at these singular configurations of the system, and convergence is very slow for configurations in the vicinity of a singularity. This problem is avoided by using Euler parameters to relate the orientation geometry to the angular velocity.

The advantages of using Lagrange's form of d'Alembert's principle to obtain the governing equations of motion for multi-body mechanical systems has been exposited in detail in References [28-30]. Basically, this principle has the advantages of Lagrange's equations or of virtual work in that non-working internal constraint forces, between the bodies of the system, are automatically eliminated from the analysis, and may therefore be ignored in the formulation of the governing equations. The principle, however, has the additional advantage of avoiding the
differentiation of scalar energy functions. Indeed, the differentiation required to obtain velocities and accelerations are performed by vector cross products and multiplication algorithms — procedures which are ideally suited for numerical computation. As with Lagrange's equations, Lagrange's form of d'Alembert's principle requires the use of generalized coordinates to define the system geometry. The use of Euler parameters to avoid problems with singularities, as discussed above, leads naturally to the use of generalized speeds — that is, relative angular velocity components as the generalized coordinate derivatives. This, in turn, leads to additional computational advantages as observed by Kane and Wang [149] and Likins [122]. Specifically, by using generalized speeds (relative angular velocity components) as the principle parameters of the analysis, the coefficient matrices in the governing equations can be obtained directly from the body connection array L(k) (see Tables 1. and 2.).

The use of "relative" coordinates, that is, angular velocity components of the bodies with respect to their adjoining bodies, as opposed to "absolute" coordinates, (for example, angular velocity components in inertial space) also contributes to the computational advantage. In applications with specific geometrical configurations [1, 31, 32, 34-36, 50, 77, 152-155], it is seen that the geometry is more easily described in terms of relative coordinates.

The generalization to allow translation between the bodies of the system makes the analysis applicable to a much broader class of problems than was possible with these previous analyses which are restricted to linked multi-body systems. For example, with the head-neck systems of References [152, 154, 155] the use of translation variables between the vertebrae is necessary to obtain satisfactory models of such systems. But, and of perhaps greater significance, the generalization to include translation between the bodies of the system is necessary for an efficient
analysis of the flexibility and compliance effects as discussed earlier.

In this regard, the compliance can be modelled with a quasi-static approach whereas the oscillations and impact response require a dynamic analysis with the introduction of additional degrees of freedom. In both of these cases the analysis outlined herein (using Lagrange's form of d'Alembert's principle together with the use of generalized speeds) accommodates the effects of flexibility in an extremely efficient manner. That is, the forces and moments transmitted across the system joints are directly determined and incorporated into the governing equations. Moreover, the modelling may be made as detailed as necessary by introducing non-linearities through the elastic springs and dampers and by increasing the number of joints and bodies of the system.

Finally, the entire analysis outlined herein is developed with the intent of obtaining efficiencies in a computer oriented development of the governing dynamical equations. As such, its most productive application will be with large multi-body systems such as finite-segment models of the human body, chains, cables, robots, manipulators, and teleoperators.
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**Title:** Multi-body Dynamics Including the Effects of Flexibility and Compliance

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**Abstract:**

New and recently developed concepts and ideas useful in obtaining efficient computer algorithms for solving the equations of motion of multi-body mechanical systems with flexible links are presented and discussed. These ideas include the use of Euler parameters, Lagrange's form of d'Alembert's principle, generalized speeds, quasi-coordinates, relative coordinates, structural analysis techniques and body connection arrays. The mechanical systems considered are linked bodies forming a tree structure, but with no "closed loops" permitted. An explicit formulation of the equations of motion is presented.