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ON THE SINGULARITY EXPANSION METHOD APPLIED TO APERTURE PENETRA--ETC(U)
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UNCLASSIFIED NOTE-134 NL

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Interaction Notes

Note-134

May 1973

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On the Singularity Expansion
Method Applied to Aperture Penetration.
Part I, Theory

by

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ABSTRACT

A formulation for determining the electromagnetic field penetration through a circular aperture is developed using Babinet's principle and the singularity expansion method. Computational procedures for determining the penetration field of the aperture are discussed and a low frequency check of the procedures is proffered.

INTRODUCTION

In the study of various problems of electromagnetic compatibility it is necessary to determine the penetration field of an aperture. At low frequencies a quasi static solution is available,[1] and at high frequencies the Kirchhoff's solution is available.[2] Also a rigorous solution developed by Flammer is available.[3] He applies Babinet's principle and determines the scattering from the disk equivalent of an aperture in a conducting plate by considering the disk as a limiting form of an oblate spheroid and constructs vector wave function solutions to the Helmholtz wave equation in oblate spheroidal coordinates. However no numerical results are presented; but it is expected that the greatest amount of energy penetrates the aperture near the first resonances of the aperture, the intermediate frequency region.

Recently the singularity expansion technique has been found to be useful in the study of electromagnetic pulse interaction.[4] In principle one should be able to take Flammer's solution and develop singularity expansions for the induced current and charge as Baum[4] did in treating the sphere. However, identifying and calculating the natural frequencies and modes may be difficult. An alternative treatment used here is to develop an integral equation for the induced current density on a Babinet equivalent disk, solve the integral equation numerically using the method of moments, and apply the singularity expansion method numerically similar to Tesche.[5]

With the singularity expansion solution for the induced current density on the Babinet equivalent disk the scattered fields are readily determined. An application of Babinet's principle then yields the field penetrating the aperture in a conducting plate. Computational procedures for determining the penetration field are discussed and a low frequency check of the procedures is proffered.

ANALYSIS

The solution for the electromagnetic field penetrating an aperture in a conducting plane may be determined by solving the complimentary disk problem and applying Babinet's principle.[3] The electromagnetic form of Babinet's principle states that if $(\vec{E}_1^s, \vec{H}_1^s)$ is the scattered field when $(\vec{E}_1^i, \vec{H}_1^i)$ is the field incident in the positive z direction on a perfectly conducting disk lying in the plane z=0, then

$$\left.\begin{array}{llll} \vec{E}_2^s = -\zeta\,\vec{H}_1^s & , \quad \vec{H}_2^s = \frac{1}{\zeta}\,\vec{E}_1^s & , & z \geq 0 \\ \vec{E}_2^s = \zeta\,\vec{H}_1^s & , \quad \vec{H}_2^s = -\frac{1}{\zeta}\,\vec{E}_1^s & & z \leq 0 \end{array}\right\} \qquad (1a)$$

are the diffracted fields when the wave

$$\vec{E}_2^i = \zeta\,\vec{H}_1^i \quad , \quad \vec{H}_2^i = -\frac{1}{\zeta}\,\vec{E}_1^i \qquad (1b)$$

is incident in the positive z direction on the complimentary perfectly conducting screen with an aperture. Here $\zeta = \sqrt{\mu/\varepsilon}$ is the intrinsic wave impedance of the medium surrounding the aperture or disk. In the case of the aperture the total field in the half space $z \leq 0$ is formed from the superposition of the incident wave, the reflected wave in the absence of an aperture, and the diffracted field $(\vec{E}_2^s, \vec{H}_2^s)$. In the half space $z \geq 0$, $(\vec{E}_2^s, \vec{H}_2^s)$ is the total field penetrating the aperture.

To determine the scattered or diffracted field from the aperture it is necessary to obtain the induced surface current density on the disk. The integral equation for the surface current density, $\vec{J}_s$, on a disk centered at the origin and in the z=0 plane is[6]

$$\hat{t}_n' \cdot \vec{E}_1^i(\vec{r}') = \frac{j\omega\mu}{4\pi k^2}\, \hat{t}_n' \cdot \left[k^2 \int_S \vec{J}_s(\vec{r}) G(\vec{r}|\vec{r}')\, dS \right.$$

$$\left. + \text{grad}' \int_S \text{div}\, \vec{J}_s(\vec{r}) G(\vec{r}|\vec{r}')\, dS \right] \qquad (2)$$

where $\hat{t}_1' = \hat{r}'$ and $\hat{t}_2' = \hat{\phi}'$ with

$$G(\vec{r}|\vec{r}') = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

The incident field may be expanded in transverse magnetic (TM) and transverse electric (TE) cylindrical modes. For the TM case there is no magnetic field component along the z-axis and for the TE case there is no electric field component along the z-axis. An arbitrary electromagnetic field may be expressed as the linear combination of a TM part and a TE part (see Appendix I). These two cases are treated separately.

TM Mode Excitation

The form of the TM mode expansion for the incident plane wave field is[6]

$$\vec{E}_1^i(\vec{r}) = \sum_{m=0}^{\infty} \left[E_{1,m}^e(r) \cos m\phi\, \hat{r} \right.$$

$$\left. + E_{2,m}^o(r) \sin m\phi\, \hat{\phi} \right] e^{-jkz \cos \theta_i} \qquad (3)$$

where θ_i is the angle of incidence, the angle between the direction of propagation and the direction of the positive z-axis. The corresponding form of the induced current distribution on the disk is[6]

$$\vec{J}_s(\vec{r}) = \sum_{m=0}^{\infty} \left[K_{1,m}^e(r) \cos m\phi\, \hat{r} + K_{2,m}^o(r) \sin m\phi\, \hat{\phi} \right] \qquad (4)$$

Using the representations (3) and (4) in (2) yields coupled integral equations for $K^e_{1,m}(r)$ and $K^o_{2,m}(r)$ to be solved for each value of m. They are

$$
\begin{aligned}
E^e_{1,m}(r') = j\frac{\zeta k}{4\pi}\Bigg\{ & \int_o^a \left[G_{m-1}(r|r') + G_{m+1}(r|r')\right] K^e_{1,m}(r)rdr \\
& + \frac{2}{k^2}\frac{d}{dr'}\int_o^a G_m(r|r')(r\frac{d}{dr} + 1)K^e_{1,m}(r)dr \\
& - \int_o^a \left[G_{m-1}(r|r') - G_{m+1}(r|r')\right] K^o_{2,m}(r)rdr \\
& + \frac{2m}{k^2}\frac{d}{dr'}\int_o^a G_m(r|r')K^o_{2,m}(r)dr \Bigg\} \qquad (5)
\end{aligned}
$$

and

$$
\begin{aligned}
E^o_{2,m}(r') = j\frac{\zeta k}{4\pi}\Bigg\{ & -\int_o^a \left[G_{m-1}(r|r') - G_{m+1}(r|r')\right] K^e_{1,m}(r)rdr \\
& - \frac{2m}{k^2r'}\int_o^a G_m(r|r')(r\frac{d}{dr} + 1)K^e_{1,m}(r)dr \\
& + \int_o^a \left[G_{m-1}(r|r')+G_{m+1}(r|r')\right]K^o_{2,m}(r)rdr \\
& - \frac{2m^2}{k^2r'}\int_o^a G_m(r|r')K^o_{2,m}(r)dr \Bigg\} \qquad (6)
\end{aligned}
$$

where

$$G_m(r|r') = \int_0^{\pi} \frac{e^{-jk\sqrt{(r-r')^2+2rr'(1-\cos\phi)}}}{\sqrt{(r-r')^2+2rr'(1-\cos\phi)}} \cos m\phi d\phi \tag{7}$$

Apparently G_m as shown above possesses a logarithmic singularity at r-r'. Since (7) must be evaluated numerically a more convenient form is needed. After some mathematical manipulation (7) becomes

$$G_m(r|r') = 2\int_0^{\pi/2} \left\{ \left[e^{-jkR} \cos(2m\theta) - 1 \right] / R \right\} d\theta$$

$$+ \frac{2}{r+r'} K \left[1 - \left(\frac{r-r'}{r+r'} \right)^2 \right] \tag{8}$$

where

$$R = \sqrt{(r-r')^2 + 4rr' \sin^2 \theta}$$

and K is the complete elliptic integral of the first kind. Note that the integral in (8) does not possess a singularity at r=r' and may be evaluated numerically.

A different singularity appears to occur when r'=0 and r=0 unless a restriction is placed on the current expansion. The evaluation of (7) at r'=0 yields

$$G_m(r|0) = \pi \frac{e^{-jkr}}{r} \delta_{mo} \tag{9}$$

where δ_{mo} is the usual Kronecker delta. In order for the respective integrals in (5) and (6) to remain integrable at r=0 the following is required

$$K^e_{1,0}(r) \xrightarrow[r \to 0]{} (\text{const.})r \tag{10}$$

The physical equivalent to (10) is the requirement of a finite charge distribution at r=0.

Another restriction to be satisfied by the current distribution is the radial component of the current must vanish at the edge of the disk. That is,

$$K^{e}_{1,m}(a) = 0 \tag{11}$$

for all m. Further note that one may set

$$K^{o}_{2,0}(r) = 0 \tag{12}$$

Because of the singularities in G_m the derivatives of the integrals must be evaluated carefully. If the derivatives are taken inside the integrals the integration is performed by using Cauchy principal values. To avoid this additional complication the derivatives of the integrals will be evaluated numerically by using finite differences.

TE Mode Excitation

The form of the TE mode expansion for the incident plane wave field is[6]

$$\vec{E}^{i}_{1}(\vec{r}) = \sum_{m=0}^{\infty} \left[E^{o}_{1,m}(r) \sin m\phi \, \hat{r} - E^{e}_{2,m}(r) \cos m\phi \, \hat{\phi} \right] e^{-jkz \cos \theta_i} \tag{13}$$

And the corresponding form of the induced current distribution on the disk is[6]

$$\vec{J}_s(\vec{r}) = \sum_{m=0}^{\infty} \left[K^{o}_{1,m}(r) \sin m\phi \hat{r} - K^{e}_{2,m}(r) \cos m\phi \, \hat{\phi} \right] \tag{14}$$

Using the representations (13) and (14) in (2) yields coupled integral equations for $K^{o}_{1,m}(r)$ and $K^{e}_{2,m}(r)$ to be solved for each value of m. The form of the integral equations is exactly the same as (5) and (6) when the following transformations are made:

$$\left.\begin{aligned} K^{e}_{1,m}(r) &\longrightarrow K^{o}_{1,m}(r) \\ K^{o}_{2,m}(r) &\longrightarrow K^{e}_{2,m}(r) \\ E^{e}_{1,m}(r) &\longrightarrow E^{o}_{1,m}(r) \\ E^{o}_{2,m}(r) &\longrightarrow E^{e}_{2,m}(r) \end{aligned}\right\} \qquad (15)$$

In order to have a finite charge distribution at r=0

$$K^{e}_{2,o}(r) \xrightarrow[r \to 0]{} (\text{const.})r \qquad (16)$$

Another restriction to be satisfied by the current expansion is

$$K^{o}_{1,m}(a) = 0 \qquad (17)$$

For convenience one may set

$$K^{o}_{1,0}(r) = 0 \qquad (18)$$

for all r.

NUMERICAL SOLUTION

Because of the inherent complexity of the integral equations for the induced surface current density an analytic solution appears to be virtually impossible to obtain. Therefore a numerical solution technique, namely the method of moments,[7] is employed. Moment expansions that satisfy the restrictions on the induced surface current densities are

$$K_{1,m}(r) = \sum_{p=1}^{N} \frac{\alpha_{p+1}^{(m)}(r-r_p)+\alpha_p^{(m)}(r_{p+1}-r)}{r_{p+1}-r_p} U(r;r_{p+1},r_p) \tag{19}$$

and

$$K_{2,m}(r) = \sum_{p=1}^{N} \frac{\beta_{p+1}^{(m)}(r-r_p)+\beta_p^{(m)}(r_{p+1}-r)}{r_{p+1}-r_p} U(r;r_{p+1},r_p) \tag{20}$$

where

$$U(r;r_{p+1},r_p) = \begin{cases} 1 & r_p \le r < r_{p+1} \\ 0 & \text{otherwise} \end{cases}$$

$$r_p = (p-1)\Delta r \quad , \quad \Delta r = a/N$$

$$\alpha_{N+1}^{(m)} = 0$$

$$\alpha_1^{(0)} = 0$$

$$\beta_1^{(0)} = 0$$

If (19) and (20) are substituted into (5) and (6), and the resulting equations are satisfied at $r=r_p$, $p=1,2,\ldots N$, a system of linear equations are obtained for the α_p's and the β_p's. This procedure is sometimes referred to as point matching or colocation. The resulting system of equations is

$$\sum_{J=1}^{2N+1} \Pi(m,I,J)F(m,J) = \Gamma(m,I) \tag{21}$$

where I=1,N and J=1,N

$$\begin{aligned}\Pi(m,I,J) = {} & \frac{k}{\Delta r}\Big[F1(m-1,I,J) + F1(m+1,I,J) \\ & + F2(m-1,I,J) + F2(m+1,I,J)\Big] \\ & + \frac{2}{k(\Delta r)^2}\Big\{2\Big[F5(m,I+1,J) - F5(m,I,J) \\ & + F6(m,I+1,J) - F6(m,I,J)\Big] \\ & + r_{J-1}\Big[F3(m,I+1,J) - F3(m,I,J)\Big] \\ & - r_{J+1}\Big[F4(m,I+1,J) - F4(m,I,J)\Big]\Big\}\end{aligned}$$

for I=1,N and J=1,N+1

$$\begin{aligned}\Pi(m,I,J+N) = {} & -\frac{k}{\Delta r}\Big[F1(m-1,I,J) - F1(m+1,I,J) \\ & + F2(m-1,I,J) - F2(m+1,I,J)\Big] \\ & + \frac{2m}{k(\Delta r)^2}\Big[F3(m,I+1,J) - F3(m,I,J) \\ & + F4(m,I+1,J) - F4(m,I,J)\Big]\end{aligned}$$

for I=1,N+1 and J=1,N

$$\Pi(m,I+N,J) = -\frac{kr_I}{\Delta r}\Big[F1(m-1,I,J) - F1(m+1,I,J) + F2(m-1,I,J) - F2(m+1,I,J)\Big] - \frac{2m}{k\Delta r}\Big\{2\Big[F5(m,I,J) + F6(m,I,J) + r_{J-1}\ F3(m,I,J) - r_{J+1}\ F4(m,I,J)\Big]\Big\}$$

for I=1,N+1, J=1,N+1

$$\Pi(m,I+N,J+N) = \frac{k}{\Delta r}\, r_I\Big[F1(m-1,I,J) + F1(m+1,I,J) + F2(m-1,I,J) + F2(m+1,I,J)\Big] - \frac{2m^2}{k\Delta r}\Big[F3(m,I,J) + F4(m,I,J)\Big]$$

and

$$F(m,J) = \begin{cases} \alpha_J^{(m)} & J=1,N \\ \beta_{J-N}^{(m)} & J=N+1,2N+1 \end{cases} \tag{22}$$

$$\Gamma(m,I) = -j\frac{4\pi}{\zeta}\begin{cases} E_{1,m}(r_I) & I=1,N \\ E_{2,m}(r_{I-N}) & I=N+1,2N+1 \end{cases} \tag{23}$$

The integral functions F1, F2, F3, F4, F5 and F6 are defined in Appendix II. It should be observed that the foregoing applies for both TM and TE excitation. The appropriate modal expansion for the incident fields are given in Appendix I.

At this point the induced surface current density may be obtained by solving the foregoing system of linear equations using the digital computer. Obviously a system of equations must be solved for each modal current. Andreasen[6] who treats the body of revolution suggests that the maximum number of modes needed is

$$m\Big|_{max} \sim ka \sin \theta_i + 6 \tag{24}$$

for ka sin $\theta_i \gtrsim 3$ and $m|_{max} \to 1$ as $\theta_i \to 0$.

SINGULARITY EXPANSION METHOD

According to the singularity expansion method the natural frequencies may be obtained by searching for the zeros of the determinant of the foregoing system matrix $\Pi(m,k)$. From this point forward in the analysis the Laplace transform frequency variable $s = jck$ is used for the frequency. Thus the natural frequencies are obtained from

$$\det \left[\Pi(m,s) \right]_{s=s_\alpha} = 0 \tag{25}$$

for frequencies s_α independent of the index m. Note that the natural frequencies for the TE modes are the same as for the TM modes.

The solution for the induced current distribution is

$$F(m,s) = \Pi^{-1}(m,s)\Gamma(m,s) \tag{26}$$

for each mode. Applying the singularity expansion method (26) becomes[9*]

$$F(m,s) = \sum_\alpha \frac{M_\alpha(m)}{s-s_\alpha} C_\alpha^T(m)\Gamma(m,s) \tag{27}$$

where $M_\alpha(m)$ is defined as the natural mode vector and is the solution to the equation

$$\Pi(m,s_\alpha)M_\alpha(m) = 0 \tag{28}$$

and $C_\alpha(m)$ is referred to as the coupling vector, and satisfies the equation

$$\Pi^T(m,s_\alpha)C_\alpha(m) = 0 \tag{29}$$

The $M_\alpha(m)$ and $C_\alpha(m)$ are normalized according to[4]

$$C_\alpha^T(m) \left[\frac{d}{ds} \Pi(m,s) \right]_{s=s_\alpha} M_\alpha(m) = 1 \tag{30}$$

*Here the class II form for the coupling coefficient is used.

Actually F(m,s) represents the mth cylindrical mode contribution to the current distribution. Summing the cylindrical mode contributions, see (4) and (14), yields the final form for the current distribution. The foregoing (25)-(30) apply for both TE and TM excitation. For TE excitation use the TE mode field components in Γ as defined in (13) and (23) and for TM excitation use the TM mode field components in Γ as defined in (3) and (23).

To obtain the time domain response of the induced current distribution on the disk the appropriate Laplace transform of (27) is evaluated. It is[5]

$$F(m,t) = \sum_{\alpha} U(t) M_{\alpha}(m) C_{\alpha}^{T}(m) \Gamma(m,s_{\alpha}) \frac{e^{s_{\alpha} t}}{s_{\alpha}} \tag{31}$$

where U(t) is a diagonal square matrix of unit Heaviside functions which serves to enforce the requirements of causality.

SCATTERED FIELD

Once the induced current distribution of the disk is known then the scattered field may be readily determined. The electric field is[8]

$$\vec{E}_1^s(\vec{r}\,') = -j\frac{\zeta}{4\pi k}\int_S \left[(\vec{J}_s\cdot\nabla)\nabla + k^2\vec{J}_s\right]\frac{e^{-jkR}}{R}\,ds \tag{32}$$

and the magnetic field

$$\vec{H}_1^s(\vec{r}\,') = \frac{1}{4\pi}\int_S \vec{J}_s \times \nabla\left(\frac{e^{-jkR}}{R}\right)ds \tag{33}$$

where

$$\nabla\left(\frac{e^{-jkR}}{R}\right) = \left(jk + \frac{1}{R}\right)\frac{e^{-jkR}}{R}\,\hat{R} \tag{34}$$

and

$$(\vec{J}_s\cdot\nabla)\nabla\,\frac{e^{-jkR}}{R} = \left\{ -k^2(\vec{J}_s\cdot\hat{R})\hat{R} + \frac{1}{R}\left(jk+\frac{1}{R}\right) \cdot\left[3(\vec{J}_s\cdot\hat{R})\hat{R}-\vec{J}_s\right]\right\}\frac{e^{-jkR}}{R} \tag{35}$$

with $\vec{R}=\vec{r}\,'-\vec{r}$ and $\hat{R}=\vec{R}/R$. Note that $\vec{J}_s$ is given by (4) or (14) depending upon the mode of excitation. The far field approximations to (32) and (33) may be readily determined by neglecting the R^{-2} and R^{-3} terms appearing in (34) and (35).

In order to verify the numerical procedures that are employed, low frequency excitation may be considered and the equivalent dipole moments of the scattered field obtained. These are

$$\vec{P}_o = \frac{1}{j\omega}\int_S \vec{J}_s\,ds \tag{36}$$

$$\vec{M}_o = \frac{1}{2} \int_S \vec{r} \times \vec{J}_s ds \tag{37}$$

From using a quasi-static approximation it is found that[1]

$$\vec{P}_o = \frac{8\varepsilon_o}{3} a^3 \left[(\hat{x} \cdot \vec{E}^{inc})\hat{x} + (\hat{y} \cdot \vec{E}^{inc})\hat{y} \right] \tag{38}$$

$$\vec{M}_o = -\frac{4}{3} a^3 (\hat{z} \cdot \vec{H}^{inc})\hat{z} \tag{39}$$

substituting the current representations (4) and (14) into (36) and (37) yields

$$P_{ox} = \frac{\pi}{j\omega} \int_o^a \left[K^e_{1,1}(r) - K^o_{2,1}(r) \right] rdr \tag{40}$$

$$P_{oy} = \frac{\pi}{j\omega} \int_o^a \left[K^o_{1,1}(r) - K^e_{2,1}(r) \right] rdr \tag{41}$$

$$M_{oz} = -\pi \int_o^a K^e_{2,0}(r) rdr \tag{42}$$

At low frequency (40)-(42) may be compared with (38) and (39) to check the numerical solutions for the components of $\vec{J}_s$.

The field penetrating an aperture in an infinite plate may be determined by applying the electromagnetic form of Babinet's principle. As discussed previously the field transformations (1a) and (1b) are used with the scattered field from the Babinet equivalent disk to determine the field diffracted by an aperture. The scattered field from the disk is given by (32) and (33).

CONCLUSION

The problem of the electromagnetic field penetration through an aperture in a perfectly conducting plate is formulated by using Babinet's principle. A numerical solution for the induced current distribution on the Babinet equivalent disk is developed. This numerical solution is formulated both as a direct moment method solution and a singularity expansion solution. Finally computational procedures for determining the penetration field of the aperture are discussed and a low frequency check on the computational procedures is proffered.

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APPENDIX I

For convenience consider a plane wave propagating in the direction $\theta = \Pi - \theta_i$ and to be polarized in the direction forming an angle θ_p with the unit vector $-\hat{\theta}$. Here θ is the usual polar angle of the spherical coordinate system. Therefore the incident electric field may be resolved into two components

$$\vec{E}_1^i = - E_1^{\parallel} \hat{\theta} + E_1^{\perp} \hat{y} \tag{A1}$$

where

$$E_1^{\parallel} = E_o \cos \theta_p \, e^{-jk(z \cos \theta_i - x \sin \theta_i)} \tag{A2}$$

$$E_1^{\perp} = E_o \sin \theta_p \, e^{-jk(z \cos \theta_i - x \sin \theta_i)} \tag{A3}$$

Then by the principle of superposition each component of $\vec{E}^i$ is considered to be a plane wave. The magnetic field associated $E_1^{\parallel}$ will have no z component - a TM wave. And the z component of $E_1^{\perp}$ will be zero - a TE wave.

<u>TM case</u>

If the electric field $E_1^{\parallel}$ is expanded in a Fourier series and $E_1^{\perp}$ set equal zero, then (A1) yields

$$\vec{E}_1^i(\vec{r}) = \sum_{m=0}^{\infty} \Big[E_{1,m}^e(r) \cos m\phi \, \hat{r} + E_{2,m}^o(r) \sin m\phi \, \hat{\phi} \Big] e^{-jkz \cos \theta_i}$$

where

$$E^{e}_{1,m}(r) = -\cos\theta_p \cos\theta_i\, \varepsilon_m j^{m+1} J'_m(kr\sin\theta_i)\, E_o$$

$$E^{o}_{2,m}(r) = \cos\theta_p \cos\theta_i\, \varepsilon_m j^{m+1} m \frac{J_m(kr\sin\theta_i)}{kr\sin\theta_i} E_o$$

with (r,ϕ,z) as the usual cylindrical coordinates and ε_m as Neumann's number.

TE case

If the electric field $E_1^{\perp}$ is expanded in a Fourier series and $E_1^{\parallel}$ set equal zero, then (A1) yields

$$\vec{E}_1^{i}(\vec{r}) = \sum_{m=0}^{\infty} \Big[E^{o}_{1,m}(r) \sin m\phi\, \hat{r}$$

$$+ E^{e}_{2,m}(r) \cos m\phi\hat{\phi} \Big] e^{-jkz\cos\theta_i}$$

where

$$E^{o}_{1,m}(r) = -\sin\theta_p\, \varepsilon_m j^{m+1} m \frac{J_m(kr\sin\theta_i)}{kr\sin\theta_i} E_o$$

$$E^{e}_{2,m}(r) = \sin\theta_p\, \varepsilon_m j^{m+1} J'_m(kr\sin\theta_i)\, E_o$$

APPENDIX II

The various integral functions used in $\Pi(m,I,J)$ are defined here.

<u>F1(m,I,J)</u>

For $J \neq I+1$, $I>1$

$$F1(m,I,J) = \int_0^{\Delta r} (u+r_{J-1}) G_m(r_I | u+r_{J-1}) u du$$

$$+2 \int_0^{\Delta r} \frac{(u+r_{J-1})}{r_I+r_{J-1}+u} K \left[\frac{r_I - r_{J-1} - u}{r_I + r_{J-1} + u} \right] du$$

where

$$G_m(u|v) = 2 \int_0^{\pi/2} \left\{ \left[e^{-jkR} \cos(2m\theta) - 1 \right] / R \right\} d\theta$$

$$R = \sqrt{(u-v)^2 + 4uv \sin^2\theta}$$

$$K(\zeta) = \left[a_o + a_1\zeta^2 + a_2\zeta^4 + a_3\zeta^6 + a_4\zeta^8 \right]$$

$$- \left[b_o + b_1\zeta^2 + b_2\zeta^4 + b_3\zeta^6 + b_4\zeta^8 \right] \ln\zeta^2$$

$$r_I = (I-1)\Delta r$$

$$r_J = (J-1)\Delta r$$

$$\Delta r = a/N$$

a_o = 1.386 294 4	b_o = 0.5
a_1 = 0.096 663 443	b_1 = 0.124 985 94
a_2 = 0.035 900 924	b_2 = 0.068 802 486
a_3 = 0.037 425 637	b_3 = 0.033 283 553
a_4 = 0.014 511 962	b_4 = 0.004 417 870

For J=I+1, I>1

$$F1(m,I,J) = \int_0^{\Delta r} (u+r_{J-1}) G_m(r_I | u+r_{J-1}) u du$$

$$+2 \int_0^{(1-\delta)\Delta r} \frac{(u+r_{J-1})u}{r_I+r_{J-1}+u} K\left[\frac{r_I-r_{J-1}-u}{r_I+r_{J-1}+u}\right] du$$

$$+ (\Delta r)^2 q_I$$

where

$$q_I = \left\{ a_o - 2b_o \left[\ell n \left(\frac{\Delta r \delta}{2r_I} \right) - 1 \right] \right\} \delta$$

$$\delta \lesssim 0.1$$

For I=1

$$F1(m,I,J) = \begin{cases} \left[\frac{e^{-jk\Delta r}(1+jk\Delta r)-1}{k^2} \right] e^{-jkr_{J-1}} & m=0 \\ 0 & \text{otherwise} \end{cases}$$

<u>F2(m,I,J)</u>

For J ≠ I+1, I>1

$$F2(m,I,J) = \int_0^{\Delta r} (r_{J+1}-u) G_m(r_I | r_{J+1}-u) u du$$

$$+2 \int_0^{\Delta r} \frac{(r_{J+1}-u)u}{r_I+r_{J+1}-u} K\left[\frac{r_I-r_{J+1}+u}{r_I+r_{J+1}-u}\right] du$$

For J=I+1, I>1

$$F2(m,I,J) = \int_{o}^{\Delta r} (r_{J+1}-u)G_m(r_I|r_{J+1}-u)udu$$

$$+ 2\int_{o}^{(1-\delta)\Delta r} \frac{(r_{J+1}-u)u}{r_I+r_{J+1}-u} K\left[\frac{r_I-r_{J+1}+u}{r_I+r_{J+1}-u}\right] du$$

$$+ (\Delta r)^2 q_I$$

For I=1

$$F2(m,I,J) = \begin{cases} \dfrac{\left[e^{jk\Delta r}(1-jk\Delta r)-1\right]}{k^2} e^{-jkr_{J+1}} & m=0 \\ 0 & \text{otherwise} \end{cases}$$

<u>F3(m,I,J)</u>

For $J \neq I$ and $J \neq I+1$

$$F3(m,I,J) = \int_{o}^{\Delta r} G_m(r_I|u+r_{J-1})du$$

$$+2\int_{o}^{\Delta r} \frac{1}{r_I+r_{J-1}+u} K\left[\frac{r_I-r_{J-1}-u}{r_I+r_{J-1}+u}\right] du$$

For J=I+1

$$F3(m,I,J) = \int_{o}^{\Delta r} G_m(r_I|u+r_{J-1})du$$

$$+2\int_{\delta\Delta r}^{\Delta r} \frac{1}{r_I+r_{J-1}+u} K\left[\frac{r_I-r_{J-1}-u}{r_I+r_{J-1}+u}\right] du$$

$$+ \frac{\Delta r}{r_I} q_I$$

For J=I

$$F3(m,I,J) = \int_o^{\Delta r} G_m(r_I | u + r_{J-1})\,du$$

$$+\ 2\int_o^{(1-\delta)\Delta r} \frac{1}{r_I + r_{J-1} + u}\ K\left[\frac{r_I - r_{J-1} - u}{r_I + r_{J-1} + u}\right] du$$

$$+\ \frac{\Delta r}{r_I}\ q_I$$

<u>F4(m,I,J)</u>

For $J \neq I$ and $J \neq I+1$

$$F4(m,I,J) = \int_o^{\Delta r} G_m(r_I | r_{J+1} - u)\,du$$

$$+\ 2\int_o^{\Delta r} \frac{1}{r_I + r_{J+1} - u}\ K\left[\frac{r_I - r_{J+1} + u}{r_I + r_{J+1} - u}\right] du$$

For J=I

$$F4(m,I,J) = \int_o^{\Delta r} G_m(r_I | r_{J+1} - u)\,du$$

$$+\ 2\int_{\delta\Delta r}^{\Delta r} \frac{1}{r_I + r_{J+1} - u}\ K\left[\frac{r_I - r_{J+1} + u}{r_I + r_{J+1} - u}\right] du$$

$$+\ \frac{\Delta r}{r_I}\ q_I$$

For J=I+1

$$F4(m,I,J) = \int_{o}^{\Delta r} G_m(r_I | r_{J+1} - u)\,du$$

$$+\ 2\int_{o}^{(1-\delta)\Delta r} \frac{1}{r_I + r_{J+1} - u}\, K\left[\frac{r_I - r_{J+1} + u}{r_I + r_{J+1} - u}\right] du$$

$$+\ \frac{\Delta r}{r_I}\, q_I$$

<u>F5(m,I,J)</u>

For J ≠ I, I>1

$$F5(m,I,J) = \int_{o}^{\Delta r} G_m(r_I | r_{J-1} + u)\,u\,du$$

$$+\ 2\int_{o}^{\Delta r} \frac{1}{r_I + r_{J-1} + u}\, K\left[\frac{r_I - r_{J-1} - u}{r_I + r_{J-1} + u}\right] u\,du$$

For J=I, I>1

$$F5(m,I,J) = \int_{o}^{\Delta r} G_m(r_I | r_{J-1} + u)\,u\,du$$

$$+\ 2\int_{o}^{(1-\delta)\Delta r} \frac{1}{r_I + r_{J-1} + u}\, K\left[\frac{r_I - r_{J-1} - u}{r_I + r_{J-1} + u}\right] u\,du$$

$$+\ \frac{(\Delta r)^2}{r_I}\quad q_I$$

For I=1

$$F5(m,I,J) = \begin{cases} \displaystyle\int_o^{\Delta r} \frac{e^{-jk|r_{J-1}+u|}}{|r_{J-1}+u|}\, udu & m=0 \\ 0 & \text{otherwise} \end{cases}$$

<u>F6(m,I,J)</u>

For $J \neq I$, $I>1$

$$F6(m,I,J) = \int_o^{\Delta r} G_m(r_I|r_{J+1}-u)udu$$

$$+ 2\int_o^{\Delta r} \frac{1}{r_I+r_{J+1}-u} K\left[\frac{r_I-r_{J+1}+u}{r_I+r_{J+1}-u}\right]udu$$

For $J=I$, $I>1$

$$F6(m,I,J) = \int_o^{\Delta r} G_m(r_I|r_{J+1}-u)udu$$

$$+ 2\int_o^{(1-\delta)\Delta r} \frac{1}{r_I+r_{J+1}-u} K\left[\frac{r_I-r_{J+1}+u}{r_I+r_{J+1}-u}\right]udu$$

$$+ \frac{(\Delta r)^2}{r_I} q_I$$

For I=1

$$F6(m,I,J) = \begin{cases} \displaystyle\int_o^{\Delta r} \frac{e^{-jk(r_{J+1}-u)}}{r_{J+1}-u}\, udu & m=0 \\ 0 & \text{otherwise} \end{cases}$$