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Effects of Random Shadings, Phasing Errors, and Element Failures on the Beam Patterns of Linear and Planar Arrays

Albert H. Nuttall
Special Projects Department

14 March 1980

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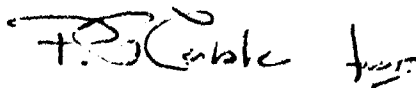Newport, Rhode Island • New London, Connecticut

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PREFACE

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R. W. Hasse
Head, Special Projects Department

The author of this report is located at the New London Laboratory,
Naval Underwater Systems Center, New London, Connecticut 06320

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The moments of a sum of independent complex random variables are derived, up through fourth-order, in complete generality, with no Gaussian assumptions. Applications of these results are made to linear and planar arrays, where the beamforming is subject to random shadings, phasing errors, and element failures. Expressions for the mean and variance of the random power transfer function of the array are presented. Programs and examples of their use are given for all of these results.

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EFFECTS OF RANDOM SHADINGS, PHASING ERRORS, AND ELEMENT FAILURES ON THE BEAM PATTERNS OF LINEAR AND PLANAR ARRAYS

INTRODUCTION

Although an array beamformer with known element positions may be designed for good sidelobes or skirt selectivity behavior by choice of the element shading (weights), the actual array response will undergo degradations due to, e.g., element position movement, delay approximations, element gain quantization, random element gains, element failure, etc. Here we will investigate the effect of all these random perturbations on the response of the array beamformer for a single-frequency plane-wave arrival and for individual array elements with omnidirectional response. Since the power response to a single-frequency plane wave is itself a random variable at each angle of look, we will evaluate its mean and variance as a function of the look angle; the element locations; the statistics of the shading, phasing, and failure perturbations; and the plane-wave arrival frequency, propagation speed, and arrival angle. From these results can be deduced quantitative tolerance limits on the perturbations in order to realize specified sidelobe levels.

Some previous results on the array power response for random perturbations of the element gains alone are given in reference 1. (This reference is also useful for additional background, motivation, and interpretations.) Then, in reference 2, the moments, through order four, of a sum of independent complex random variables were derived, as were the cumulants through order six. However, both of these results were given in terms of the moments of the zero-mean random variables of each component in the sum; this form is rather inconvenient and error prone when calculating array performance. Also, the beamformer application in reference 2 was limited to phasing errors only.

Here we will derive the moments of a sum of independent nonidentically distributed complex random variables, up through the fourth order, in complete generality, with no Gaussian assumptions. Then we will apply these results to both linear and planar arrays and give examples of the performance degradation caused by perturbations in gain, phasing, and element failures. Some results, without the derivations and programs contained herein, have already been presented in reference 3. Additional related results, which, however, do not cover the higher order moments considered here, are presented in references 4 and 5.

DEFINITION OF TERMS AND NOTATION

Let C be a complex random variable. The average value of C is denoted here by two equivalent notations,

$$Av\{c\} = \bar{c},\tag{1}$$

and is a complex quantity. In a similar manner, we have

$$Av\{c^2\} = \overline{c^2}, \qquad Av\{|c|^2\} = \overline{|c|^2}.\tag{2}$$

The variance of complex random variable C is defined as

$$\text{Var}\{c\} = \overline{|c - \overline{c}|^2} = \overline{|c|^2} - |\overline{c}|^2, \tag{3}$$

and is real. Two other averages of interest are

$$\text{Av}\{(c - \overline{c})^2\} = \overline{(c - \overline{c})^2} = \overline{c^2} - \overline{c}^2,$$

$$\text{Av}\{|c|^4\} = \overline{|c|^4}. \tag{4}$$

The final variance of interest is

$$\text{Var}\{|c|^2\} = \overline{\left(|c|^2 - \overline{|c|^2}\right)^2} = \overline{|c|^4} - \overline{|c|^2}^2. \tag{5}$$

The expression in (5) requires the fourth-order moment in (4) and constitutes the major analytical problem addressed herein. We will derive expressions for all the quantities in (1)-(5).

MOMENTS OF SUM OF RANDOM VARIABLES

The particular problem of interest in this section is as follows: $\{z_k\}_1^N$ are statistically independent complex random variables, which are not necessarily identically distributed, nor are they assumed Gaussian. A sum variable C is defined as

$$C = \sum_{k=1}^{N} z_k = \sum_k z_k, \tag{6}$$

and is complex. We wish to evaluate the various averages defined previously in (1)-(5), in terms of the appropriate moments of random variables $\{z_k\}$. These moments of $\{z_k\}$ are presumed known, but they need take no special form; random variable z_k need not have zero mean, for example.

We have immediately, from (6),

$$\text{Av}\{c\} = \overline{C} = \sum_k \overline{z_k} = \sum_k \text{Av}\{z_k\} \tag{7}$$

in terms of the means of $\{z_k\}$. Also there follows

$$\text{Av}\{C^2\} = \overline{C^2} = \sum_{k,n} \overline{z_k z_n} = \sum_k \overline{z_k^2} + \sum_{k \neq n} \overline{z_k}\,\overline{z_n}$$

$$= \sum_k \left(\overline{z_k^2} - \overline{z_k}^2\right) + \left(\sum_k \overline{z_k}\right)^2, \tag{8}$$

where we used the statistical independence of z_k and z_n for $k \neq n$. Continuing on, in an obvious fashion,

$$\text{Av}\{|c|^2\} = \overline{|c|^2} = \overline{C C^*} = \sum_{k,n} \overline{z_k z_n^*} = \sum_k \overline{|z_k|^2} + \sum_{k \neq n} \overline{z_k}\,\overline{z_n^*}$$

$$= \sum_k \left(\overline{|z_k|^2} - |\overline{z_k}|^2\right) + \left|\sum_k \overline{z_k}\right|^2 = \sum_k \left(\overline{|z_k|^2} - |\overline{z_k}|^2\right) + |\overline{C}|^2, \tag{9}$$

2

where we used (7). From (3) and (9), there follows

$$Var\{c\} = \sum_k \left(\overline{|z_k|^2} - |\bar{z}_k|^2 \right) = \sum_k Var\{z_k\} ; \tag{10}$$

and from (4), (8), and (7),

$$Av\left\{ (c - \bar{c})^2 \right\} = \sum_k \left(\overline{z_k^2} - \bar{z}_k^2 \right). \tag{11}$$

An alternative expression for (9) is available by employing (10):

$$Av\left\{ |c|^2 \right\} = \left| Av\{c\} \right|^2 + Var\{c\}. \tag{12}$$

The final quantity of interest is given by (5) in terms of $|C|^4$. For the sum in (6), the latter quantity is given by

$$\overline{|C|^4} = \sum_{k\ell mn} \overline{z_k z_\ell^* z_m z_n^*}. \tag{13}$$

This statistic is evaluated in appendix A. In fact, the more general quantity

$$\sum_{k\ell mn} \overline{a_k b_\ell c_m d_n} \tag{14}$$

is evaluated, where complex random variables a_k, b_k, c_k, d_k are statistically dependent amongst themselves for any k; but these random variables are statistically independent of the random variables a_n, b_n, c_n, d_n for all $n \neq k$. We evaluate (14), rather than (13), for two reasons: First, the notation is simpler and an error in analysis is much easier to detect; secondly, this more general result may have a possible future application, and the analytical effort is no greater. The average in (13) is given by (A-26) in terms of the 16 fundamental sums defined in (A-24), and the variance of $|C|^2$ is given by (A-31). We do not repeat these results here because of their length. A program for evaluating all the above quantities is given in appendix A, table A-1.

As a check on these results, the case of Gaussian complex random variables $\{z_k\}$ is considered in appendix B. A program for this special case is given in appendix B, table B-1.

APPLICATION TO BEAMFORMING

For an ideal array with no perturbations in element positions, gains, phases, delays, or failures, the voltage transfer function to a plane-wave arrival can be expressed as

$$\sum_k v_k, \tag{15}$$

where v_k is a complex quantity that incorporates the plane-wave arrival angle and frequency, the steering angle, and the element parameters such as element position, gain, and phase; see reference 1, equations (1) - (8). When imperfections in the array realization are encountered, they can be included in the array voltage transfer function by replacing v_k in (15) by

$$z_k = v_k \, g_k \, (1 + r_k) \exp(i \phi_k), \tag{16}$$

where g_k, r_k, ϕ_k are real random variables. The random variable g_k represents random element failures, by setting $g_k = 0$ or 1 with specified probabilities; the random variable r_k represents relative

gain perturbations from the desired value of $z_k = v_k$; and random variable ϕ_k represents phase perturbations from the desired value of 0.

The phase perturbations $\{\phi_k\}$ can arise from positional and/or delay perturbations in the array realization. This problem is considered in appendix C for a linear array; the variance of ϕ_k is derived and its dependence on arrival and look angles is made explicit for two situations of knowledge on the part of the array designer.

The random variables g_k, r_k, ϕ_k are dimensionless. We shall assume that they are independent of each other and of all other random variables for different values of k. (This could be generalized at the expense of requiring more detailed knowledge of the joint statistics of these random variables.) We let the moments of these random variables be denoted by

$$
\left.
\begin{array}{l}
\overline{g_k^m} \equiv \mu_m \quad (\text{real}) \\[2mm]
\overline{(1+r_k)^m} \equiv \nu_m \quad (\text{real}) \\[2mm]
\overline{\left(e^{i\phi_k}\right)^m} \equiv \gamma_m \quad (\text{complex})
\end{array}
\right\} \text{ independent of } k.
\tag{17}
$$

The independence of these moments on k could be generalized easily, but is not done herein. Thus, physically, we are presuming an array where all elements are equally random in terms of amplitude perturbations, phase perturbations, and failures.

The necessary statistics that must be evaluated are listed in (A-25). They are, using (16) and (17), given by

$$
\overline{z_k} = v_k \, \mu_1 \, \nu_1 \, \gamma_1
$$

$$
\overline{|z_k|^2} = |v_k|^2 \, \mu_2 \, \nu_2
$$

$$
\overline{z_k^2} = v_k^2 \, \mu_2 \, \nu_2 \, \gamma_2
$$

$$
\overline{|z_k|^2 z_k} = |v_k|^2 v_k \, \mu_3 \, \nu_3 \, \gamma_1
$$

$$
\overline{|z_k|^4} = |v_k|^4 \, \mu_4 \, \nu_4 .
\tag{18}
$$

It is worthwhile noting that only the first two moments γ_1 and γ_2 are required for the phase perturbation. Then the fundamental sums that $\{T_{ij}\}$ in (A-24) depend on are just

$$
\sum_k v_k , \quad \sum_k |v_k|^2 , \quad \sum_k v_k^2 , \quad \sum_k |v_k|^2 v_k , \quad \sum_k |v_k|^4 ,
\tag{19}
$$

which are independent of the perturbation statistics in (17). The reason for this independence is that (17) was presumed independent of k, the element number.

Now we can express the various array responses of interest in terms of the above quantities. We have from (15) (or from (16) for $g_k = 1$, $r_k = 0$, $\phi_k = 0$, all k)

Ideal (Complex) Voltage Response = $\sum_k v_k$.
\tag{20}

Then

Ideal Power Response $= \left| \sum_{k} v_k \right|^2$. (21)

Next, from (16),

Actual Voltage Response $= C = \sum_{k} z_k$ (22)

and finally

Actual Power Response $= |C|^2$. (23)

Then (22), (7), and (18) yield

Average Voltage Response $= Av\{C\} = \mu_1 \nu_1 \gamma_1 \sum_{k} v_k$, (24)

which is a scaled version of the ideal voltage response (20). This simplification results because the moments in (17) were assumed independent of element number k. From (23), (9), (18), and (24), there follows

Average Power Response $= Av\{|C|^2\}$

$= \left(\mu_2 \mu_2 - \mu_1^2 \nu_1^2 |\gamma_1|^2 \right) \sum_{k} |v_k|^2 + \mu_1^2 \nu_1^2 |\gamma_1|^2 \left| \sum_{k} v_k \right|^2$, (25)

the second term of which is a scaled version of the ideal power response (21). The first term of (25) is the variance of the voltage response, as may be seen by combining (22), (10), and (18):

Variance of Voltage Response $= Var\{C\}$

$= \left(\mu_2 \mu_2 - \mu_1^2 \nu_1^2 |\gamma_1|^2 \right) \sum_{k} |v_k|^2$. (26)

Finally, the quantity

Variance of Power Response $= Var\{|C|^2\}$ (27)

is given by (A-31), (A-24), and (18).

EXAMPLE OF ELEMENT PERTURBATIONS

The necessary moments were listed in (17). We now need to specify the probability density functions of $\{g_k\}$, $\{r_k\}$, and $\{\phi_k\}$ in order to evaluate μ_m, ν_m, γ_m. Since g_k is a 0, 1 random variable representing element failures, we let

$$p(g) = Q \delta(g) + (1-Q) \delta(g-1) ;$$ (28)

that is, Q is the probability of element failure. We have taken advantage of the independence of k in (17). Then

$$\mu_m = \overline{g^m} = \int dg \, g^m \, p(g) = 1-Q \quad \text{for all } m > 0.$$ (29)

The second quantity in (17) is

$$\nu_m = \overline{(1+r)^M} .$$ (30)

If we let $\overline{r^m} \equiv \beta_m$, then we have

$$\nu_1 = 1+\beta_1, \quad \nu_2 = 1+2\beta_1+\beta_2, \quad \nu_3 = 1+3\beta_1+3\beta_2+\beta_3, \quad \nu_4 = 1+4\beta_1+6\beta_2+4\beta_3+\beta_4 .$$ (31)

As a special case, if relative perturbation r is zero-mean Gaussian, then

$$p(r) = \frac{1}{\sqrt{2\pi}\,\sigma_r} \exp\left(-\frac{r^2}{2\sigma_r^2}\right) ,$$ (32)

and there follows

$$\beta_1 = 0, \quad \beta_2 = \sigma_r^2, \quad \beta_3 = 0, \quad \beta_4 = 3\sigma_r^4 ,$$ (33)

for which

$$\nu_1 = 1, \quad \nu_2 = 1+\sigma_r^2, \quad \nu_3 = 1+3\sigma_r^2, \quad \nu_4 = 1+6\sigma_r^2+3\sigma_r^4 .$$ (34)

Finally, to evaluate the third quantity in (17), we need

$$\gamma_m = \overline{e^{i\phi m}} = \int d\phi \, e^{i\phi m} \, p(\phi).$$ (35)

The probability density function of ϕ will be taken to be zero-mean Gaussian, in which case,

$$\gamma_m = \int d\phi \, e^{i\phi m} \frac{1}{\sqrt{2\pi}\,\sigma_p} \exp\left(-\frac{\phi^2}{2\sigma_p^2}\right) = \exp\left(-\frac{1}{2}\sigma_p^2 m^2\right).$$ (36)

It should be noticed that γ_m is real for this example; this real property simplifies the programming effort and is used throughout the rest of this report.

EQUISPACED LINEAR ARRAY

GENERAL RESULTS

It was noted earlier that the voltage transfer function of an ideal array to a plane-wave arrival can be written in the form of (15). We now investigate this form for an equispaced linear array; we find, for a symmetric real weight structure $\{w_k\}$ about the center of the array, that (reference 1, page 3)

$$v_k = w_k \exp\left[-i\left(k-\tfrac{1}{2}\right)u\right] \quad \text{for an even number of elements,}$$ (37)

where

$$u = 2\pi f_a \frac{d}{c}\left(\sin\phi_a - \sin\phi_s\right) = 2\pi \frac{d}{\lambda_a}\left(\sin\phi_a - \sin\phi_s\right)$$ (38)

Here f_a, c, λ_a, and ϕ_a are the frequency, speed of propagation, wavelength, and arrival angle (measured from broadside) of the plane-wave arrival; d is the linear array element spacing; and ϕ_s is the look (steering) angle of the array. Extension of (37) to an odd number of elements is readily achieved, but not pursued here. We let N be the total number of elements in the array and express

$$N = 2H, \tag{39}$$

where H is the number of elements in one-half of the array.

The five fundamental sums in (19) now take the form

$$\sum_k v_k = \sum_k w_k \cos\left[(k-\tfrac{1}{2})u\right] = L_1(u)$$

$$\sum_k |v_k|^2 = \sum_k w_k^2 = W_2$$

$$\sum_k v_k^2 = \sum_k w_k^2 \cos\left[(2k-1)u\right] = L_2(u)$$

$$\sum_k |v_k|^2 v_k = \sum_k w_k^3 \cos\left[(k-\tfrac{1}{2})u\right] = L_3(u)$$

$$\sum_k |v_k|^4 = \sum_k w_k^4 = W_4 , \tag{40}$$

where we have taken advantage of the symmetric real weight structure in order to write all the sums as explicitly real quantities. The notation $\sum_k$ here denotes a sum over all nonzero weights $\{w_k\}$ from $k = -H$ to H. In the program to be presented later, advantage is taken of the symmetry in order to decrease the number of terms computed by a factor of 2. If we denote the first function in (40) by $L_1(u)$, then it follows that

$$L_1(-u) = L_1(u), \qquad L_1(u+2\pi) = -L_1(u). \tag{41}$$

Similar useful properties hold for the other L functions in (40); they enable the region, where (27) must be computed, to be reduced to the range $(0,\pi)$.

In terms of the quantities defined in (40), we can now express (20)-(26) as

$$\text{Ideal Voltage Response} = L_1(u)$$

$$\text{Ideal Power Response} = L_1^2(u)$$

$$\text{Average Voltage Response} = C_1 L_1(u)$$

$$\text{Average Power Response} = C_1^2 L_1^2(u) + \left(C_{2m} - C_1^2\right)W_2$$

$$\text{Variance of Voltage Response} = \left(C_{2m} - C_1^2\right)W_2 , \tag{42}$$

where

$$C_1 = \mu_1 \nu_1 \gamma_1 , \qquad C_{2m} = \mu_2 \mu_2 . \tag{43}$$

The variance of the power response is given in appendix D. Examples of (42) and the variance of the power response are deferred until later in this report. A program for calculating the average behavior for a linear array is given in appendix D, table D-1.

DEEP SIDELOBE BEHAVIOR

In the deep sidelobe region, the ideal voltage response is substantially zero; that is, $L_1(u)$ in (42) is approximately zero. Then (42) simplifies to

$$\text{Ideal Voltage Response} \approx 0$$

$$\text{Ideal Power Response} \approx 0$$

$$\text{Average Voltage Response} \approx 0$$

$$\text{Average Power Response} \approx \left(\mu_2 \nu_2 - \mu_1^2 \nu_1^2 \gamma_1^2\right) \sum_k w_k^2$$

$$\text{Variance of Voltage Response} \approx \left(\mu_2 \nu_2 - \mu_1^2 \nu_1^2 \gamma_1^2\right) \sum_k w_k^2 \quad , \tag{44}$$

where we employed (43) and (40). The variance of the power response is given in (D-6) in terms of the other quantities defined there; no simple expression for this variance was obtained. Alternative interpretations of the constants are given in appendix D, (D-9).

Since the peak ideal power response (for positive weights) is given by (see (42) and (40))

$$L_1^2(0) = \left(\sum_k w_k\right)^2 \quad , \tag{45}$$

then it follows from (44) and (45) that

$$\frac{\text{Average Power Response (deep sidelobe region)}}{\text{Peak Ideal Power Response}} = \frac{\mu_2 \nu_2 - \mu_1^2 \nu_1^2 \gamma_1^2}{N_{eff}} = \frac{V}{N_{eff}} \quad , \tag{46}$$

where

$$N_{eff} = \frac{\left(\sum_k w_k\right)^2}{\sum_k w_k^2} \quad . \tag{47}$$

The quantity in (46) requires only second-order statistics of the perturbations and the single summary parameter (47) of the weights. Equation (47) is maximized by equal weights over the entire array.

For the example considered earlier, we use (29), (34), and (36) to evaluate the numerator of (46) as

$$V = (1-Q)(1+\sigma_r^2) - (1-Q)^2 \exp\left(-\sigma_p^2\right). \tag{48}$$

For small phase perturbations, $\sigma_p^2 \ll 1$, and (48) simplifies to

$$V \approx (1-Q)\left(Q + \sigma_r^2 + \sigma_p^2(1-Q)\right). \tag{49}$$

If, additionally, the probability of element failure is small, $Q \ll 1$, then (49) can be further manipulated into a variety of forms:

$$V \approx (1-Q)(Q + \sigma_r^2 + \sigma_p^2)$$

$$= (1-Q)Q + (1-Q)(\sigma_r^2 + \sigma_p^2)$$

$$= \sigma_g^2 + (\sigma_r^2 + \sigma_p^2)\,\text{Prob(element okay)}$$

$$\approx \sigma_g^2 + \sigma_r^2 + \sigma_p^2 \quad . \tag{50}$$

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Then (46) becomes

$$\frac{\text{Average Power Response (deep sidelobe region)}}{\text{Peak Ideal Power Response}} \approx \frac{\sigma_g^2 + \sigma_r^2 + \sigma_p^2}{N_{eff}} . \tag{51}$$

This simple rule-of-thumb is similar to (43) in reference 1 and to (83) in reference 2, but now generalized to include simultaneous element failures and gain and phase perturbations. It indicates that the variances of the three perturbations simply add, at least under the assumptions made.

GAUSSIAN APPROXIMATION IN DEEP SIDELOBE REGION

When the number of elements in the array is large,[*] the sum variable C may be well approximated as a complex Gaussian random variable. However, the statistical description of C is still difficult, since the variances of the real and imaginary parts of C are not equal, in general; see reference 1, appendixes A and B. However, in the deep sidelobe region, we have from (22), (24), (25), and (40),

$$\overline{C} \approx 0, \quad \overline{C^2} \approx 0$$

$$\overline{|C|^2} \approx \left(\mu_2 \mu_2 - \mu_1^2 \nu_1^2 \gamma_1^2 \right) \sum_k w_k^2 \equiv \sigma_c^2 . \tag{52}$$

Then, letting complex random variable

$$C = x + iy, \tag{53}$$

we have

$$\overline{x} = \overline{y} = 0, \quad \overline{xy} = 0, \quad \overline{x^2} = \overline{y^2} = \tfrac{1}{2}\sigma_c^2 . \tag{54}$$

Now, if none of the elements have failed, the deep sidelobe behavior is attainable, and we have, under this conditional situation, the joint Gaussian probability density function

$$p(x,y) \approx \frac{1}{\pi \sigma_c^2} \exp\left(-\frac{x^2 + y^2}{\sigma_c^2} \right) , \tag{55}$$

where, from (52),

$$\sigma_c^2 = \left(\nu_2 - \nu_1^2 \gamma_1^2 \right) \sum_k w_k^2 , \tag{56}$$

since we must set

$$\mu_m = 1 \quad \text{for all } m \tag{57}$$

under this condition of no elements failing. Then

$$Av\{|C|^2\} = \sigma_c^2$$

$$Var\{|C|^2\} = Var\{x^2 + y^2\} = \sigma_c^4$$

$$Std\ Dev\{|C|^2\} = \sigma_c^2 . \tag{58}$$

---

[*]The material in this section applies to any array, not just a linear array; in fact, the larger the number of elements, the better the approximation.

Thus, the standard deviation of power gain, $|C|^2$, equals its mean; this property obtains only when none of the elements have failed. Also

$$\text{Prob}\left(|C|^2 < T\right) = \text{Prob}\left(x^2 + y^2 < T\right) = 1 - \exp\left(-T/\sigma_c^2\right). \tag{59}$$

That is, the probability density function of $|C|^2$ is exponential in the deep sidelobe region. In particular,

$$\text{Prob}\left(|C|^2 < Av\{|C|^2\} + k \cdot \text{Std Dev}\{|C|^2\}\right) = 1 - \exp\left(-1-k\right). \tag{60}$$

Thus, for example, the probability that the actual power gain, $|C|^2$, remains less than its average value is 0.63, while the probability that it is less than the mean plus two standard deviations is 0.95.

PLANAR ARRAY

GENERAL RESULTS

In order to apply the results above to a planar array, it is convenient to use a one-to-one number association between the integer k and the location p, q of a particular element in a planar array presumed to have a grid structure. Thus, instead of (16), we have

$$z_k = v_{pq} \, g_{pq} \left(1 + r_{pq}\right) \exp\left(i \phi_{pq}\right) , \tag{61}$$

and instead of (17), we have

$$\left.\begin{array}{c} \overline{g_{pq}^{m}} = \mu_m \\[2mm] \overline{\left(1 + r_{pq}\right)^{m}} = \nu_m \\[2mm] \overline{\left(e^{i\phi_{pq}}\right)^{m}} = \gamma_m \end{array}\right\} \quad \text{independent of } p, q. \tag{62}$$

The necessary statistics that must be evaluated are identical to (18), except that v_k is replaced everywhere by v_{pq}. Thus, analogous to (19), we need to evaluate the quantities

$$\sum_{pq} v_{pq} \; , \quad \sum_{pq} |v_{pq}|^2 \; , \quad \sum_{pq} v_{pq}^2 \; , \quad \sum_{pq} |v_{pq}|^2 v_{pq} \; , \quad \sum_{pq} |v_{pq}|^4 . \tag{63}$$

Analogous to (20) et seq., we have

$$\text{Ideal (Complex) Voltage Response} = \sum_{pq} v_{pq} \tag{64}$$

for $g_{pq} = 1$, $r_{pq} = 0$, $\phi_{pq} = 0$, all p,q. Then,

$$\text{Ideal Power Response} = \left| \sum_{pq} v_{pq} \right|^2 . \tag{65}$$

Next, from (61),

$$\text{Actual Voltage Response} = C = \sum_{k} z_k = \sum_{pq} v_{pq} \, g_{pq} \left(1 + r_{pq}\right) \exp\left(i \phi_{pq}\right). \tag{66}$$

and

$$\text{Actual Power Response} = |C|^2. \tag{67}$$

Then, (66), (7), (A-24), and (62) yield

$$\text{Average Voltage Response} = Av\{C\} = T_1 = \mu_1 \nu_1 \gamma_1 \sum_{pq} v_{pq}, \tag{68}$$

which is a scaled version of the ideal voltage response (64). This simplification results because the moments in (62) were assumed independent of element location p, q. Also, from (67), (9), (A-30), (A-24), (61), and (62), there follows

$$\text{Average Power Response} = Av\{|C|^2\} = T_{21} - T_{23} + |T_1|^2$$

$$= \left(\mu_2 \nu_2 - \mu_1^2 \nu_1^2 |\gamma_1|^2\right) \sum_{pq} |v_{pq}|^2 + \mu_1^2 \nu_1^2 |\gamma_1|^2 \left| \sum_{pq} v_{pq} \right|^2, \tag{69}$$

the second term of which is a scaled version of the ideal power response (65). The first term of (69) is the variance of the voltage response, as may be seen by combining (66), (10), (A-24), (61), and (62):

$$\text{Variance of Voltage Response} = Var\{C\} = T_{21} - T_{23}$$

$$= \left(\mu_2 \nu_2 - \mu_1^2 \nu_1^2 |\gamma_1|^2\right) \sum_{pq} |v_{pq}|^2. \tag{70}$$

Finally the quantity

$$\text{Variance of Power Response} = Var\{|C|^2\} \tag{71}$$

is given by (A-31), (A-24), and (18) with v_k replaced by v_{pq} everywhere; the quantities that must be evaluated are those listed in (63).

EQUISPACED PLANAR ARRAY WITH MULTIPLICATIVE WEIGHT STRUCTURE

For a planar array with elements equispaced on the x, y plane by distances d_x, d_y, and for a multiplicative weight structure,

$$w_{pq} = w_p^{(x)} w_q^{(y)}, \tag{72}$$

we have, from (21) and (22) of reference 1,

$$v_{pq} = w_p^{(x)} \exp\left[-i\left(p - \tfrac{1}{2}\right)u\right] w_q^{(y)} \exp\left[-i\left(q - \tfrac{1}{2}\right)v\right] \tag{73}$$

for a planar array with an even number of elements in both the x and y coordinates, where the weight structures $\{w_p^{(x)}\}$ and $\{w_q^{(y)}\}$ are assumed symmetric about both $x = 0$ and $y = 0$ ($p = q = 0$), the center of the array. The parameters u and v incorporate look (steering) angle (θ_l, ϕ_l), spacings d_x, d_y, and plane-wave arrival wavelength λ_a and angle (θ_a, ϕ_a):

$$u = 2\pi \frac{d_x}{\lambda_a}\left(\sin\phi_l \cos\theta_l - \sin\phi_a \cos\theta_a\right),$$

$$v = 2\pi \frac{d_y}{\lambda_a}\left(\sin\phi_l \sin\theta_l - \sin\phi_a \sin\theta_a\right). \tag{74}$$

The polar angle (measured from the z-axis) is ϕ, and the azimuthal angle (measured from the x-axis) is θ. The dimensionless parameters required in (74) are relative spacings d_x/λ_a, d_y/λ_a, arrival angle (θ_a, ϕ_a), and look angle (θ_l, ϕ_l).

The five fundamental summations required in (63) take the form

$$\sum_{pq} V_{pq} = \sum_p w_p^{(u)}\cos\left[(p-\tfrac{1}{2})u\right]\cdot\sum_q w_q^{(v)}\cos\left[(q-\tfrac{1}{2})v\right] = L_1^{(u)}(u)\,L_1^{(v)}(v)$$

$$\sum_{pq}|V_{pq}|^2 = \sum_p w_p^{(u)^2}\cdot\sum_q w_q^{(v)^2} = W_2^{(u)}\,W_2^{(v)}$$

$$\sum_{pq} V_{pq}^2 = \sum_p w_p^{(u)^2}\cos\left[(2p-1)u\right]\cdot\sum_q w_q^{(v)^2}\cos\left[(2q-1)v\right] = L_2^{(u)}(u)\,L_2^{(v)}(v)$$

$$\sum_{pq}|V_{pq}|^2 V_{pq} = \sum_p w_p^{(u)^3}\cos\left[(p-\tfrac{1}{2})u\right]\cdot\sum_q w_q^{(v)^3}\cos\left[(q-\tfrac{1}{2})v\right] = L_3^{(u)}(u)\,L_3^{(v)}(v)$$

$$\sum_{pq}|V_{pq}|^4 = \sum_p w_p^{(u)^4}\cdot\sum_q w_q^{(v)^4} = W_4^{(u)}\,W_4^{(v)}. \tag{75}$$

These quantities are all real, due to the symmetry assumptions; so, for the perturbation example in (28)-(36), all the $\{T_{ij}\}$ in (A-24) are real. Furthermore, the summations on negative p and q in (75) can be avoided by multiplying the positive p and q terms by a factor of 2, as done earlier for the linear array.

We can express the desired quantitites in (64)-(71) in terms of (75) and (43):

$$\text{Ideal Voltage Response} = L_1^{(u)}(u)\,L_1^{(v)}(v)$$

$$\text{Ideal Power Response} = L_1^{(u)^2}(u)\,L_1^{(v)^2}(v)$$

$$\text{Average Voltage Response} = C_1\,L_1^{(u)}(u)\,L_1^{(v)}(v)$$

$$\text{Average Power Response} = (C_{2m}-C_1^2)W_2^{(u)}W_2^{(v)} + C_1^2\,L_1^{(u)^2}(u)\,L_1^{(v)^2}(v)$$

$$\text{Variance of Voltage Response} = (C_{2m}-C_1^2)W_2^{(u)}W_2^{(v)}. \tag{76}$$

The variance of the power response is given by (D-11) in appendix D. A program for calculating the average behaviors for the planar array is given in appendix D, table D-2.

EXAMPLES

Four curves are drawn in each of the figures discussed next. The bottommost curve in the deep sidelobe region (the curve with the deep notches) is the ideal power response (21), which would be realized for no element failures, gain perturbations, or phase perturbations; this curve is normalized to 0 dB at its peak where the look angle equals the arrival angle. The second curve from the bottom (in the deep sidelobe region) is the average power response (25) for the particular set of perturbation statistics listed with each figure. The third curve from the bottom is a plot of the average plus one

standard deviation of the power response, that is, (25) plus the square root of (27). The topmost curve is the average plus two standard deviations of the power response, that is, (25) plus two times the square root of (27). In the mainlobe region, the curves can cross each other; however, the ideal power response curve always reaches 0 dB, and the other three average curves always lie in the order relation indicated above. These facts enable the reader to discern the behavior of the four curves in any one figure.

LINEAR ARRAY

We consider an equispaced linear array of N = 20 elements. The single variable, that all the array responses depend on, is the variable u defined in (38),

$$u = 2\pi \frac{d}{\lambda} \left(\sin \phi - \sin \phi_a \right) , \tag{77}$$

for which the range $(0,\pi)$ is sufficient to cover all cases of element spacing, look angle, and arrival angle and wavelength.

In figure 1, a Dolph-Chebyshev −30 dB array design is indicated, for which the effective number of elements, (47), is equal to 17.35. This result is especially useful for evaluating the average power response in the deep sidelobe region; see (46) and (51).

The four parts (A, B, C, D) of figure 1 correspond, respectively, to (A) element failures only; (B) weight perturbations only; (C) phase perturbations only; and (D) combined element failures, weight perturbations, and phase perturbations. Quantitatively, we have

(A) probability of element failure Q = 0.001
(B) variance of relative weight perturbation σ_r^2 = 0.001
(C) variance of phase perturbation σ_p^2 = 0.001
(D) all the above combined.

The parameter values have been chosen so that the three variances are equal $(\sigma_g^2 = Q(1 - Q) \cong 0.001)$; thus the average power response in the deep sidelobe region, (51), should be equal for parts (A), (B), and (C). In fact, (51) gives $0.001/17.35 = -37.6$ dB for the first three parts of figure 1 and $3 \times 0.001/17.35 = -42.4$ dB for part (D). These calculations agree very well with the results plotted in figure 1.

The curves in figures 1B and 1C, for weight and phase perturbations, are virtually identical; the curves in figure 1A, for element failure, indicate slightly poorer performance, about 1 dB larger at the peaks of the sidelobes. Figure 1D, for combined perturbations, is, of course, the poorest of all. The mainlobe response is substantially unchanged in the four parts of figure 1.

Figure 2 is drawn under conditions identical to figure 1 except that the weighting is changed to Hamming. For this N = 20 element array, N_{eff} = 14.68. The results for the average power response in figures 2A, 2B, and 2C are virtually identical; however the standard deviation of the power response for element failures is somewhat greater, thereby leading to poorer performance in figure 2A.

Figure 3 is also drawn under identical conditions except that the weighting is now Hanning, for which N_{eff} = 13.33. The observations for this figure are identical to those for figure 2.
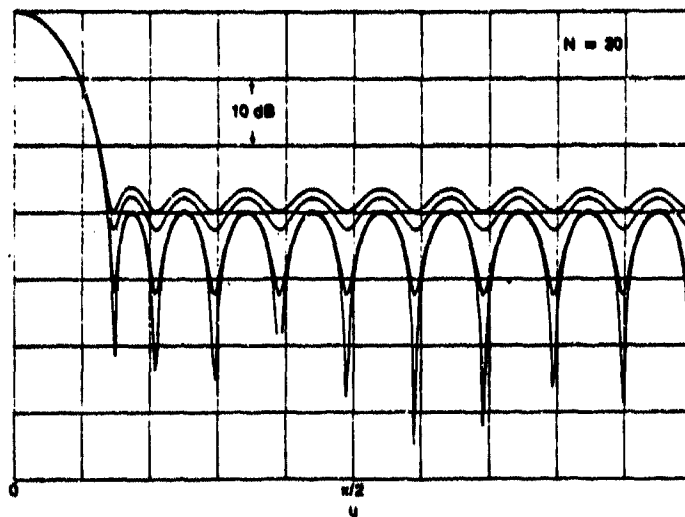
13

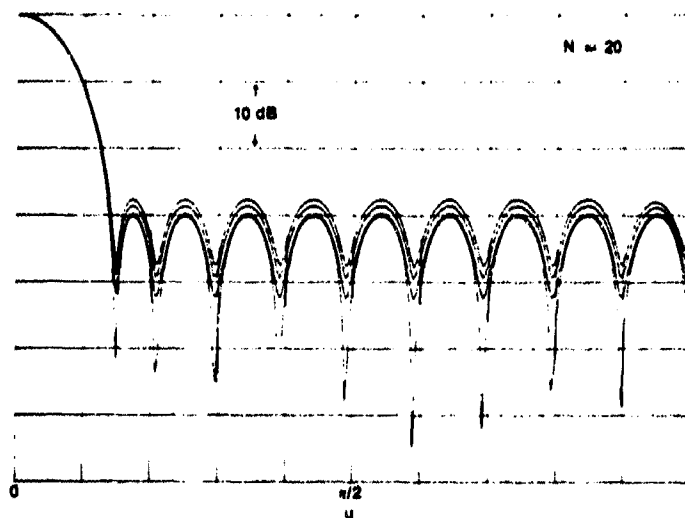Figure 1A. $Q = .001$, $\sigma_r^2 = 0$, $\sigma_p^2 = 0$

Figure 1B. $Q = 0$, $\sigma_r^2 = .001$, $\sigma_p^2 = 0$

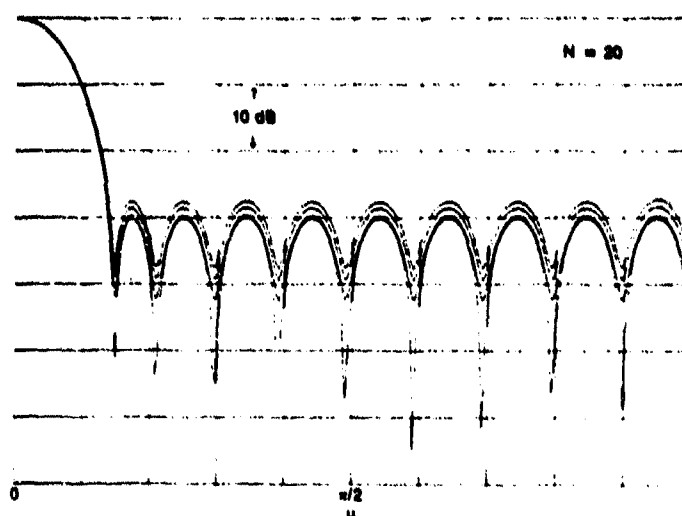Figure 1. Equispaced Linear Array; –30 dB Dolph-Chebyshev Weights
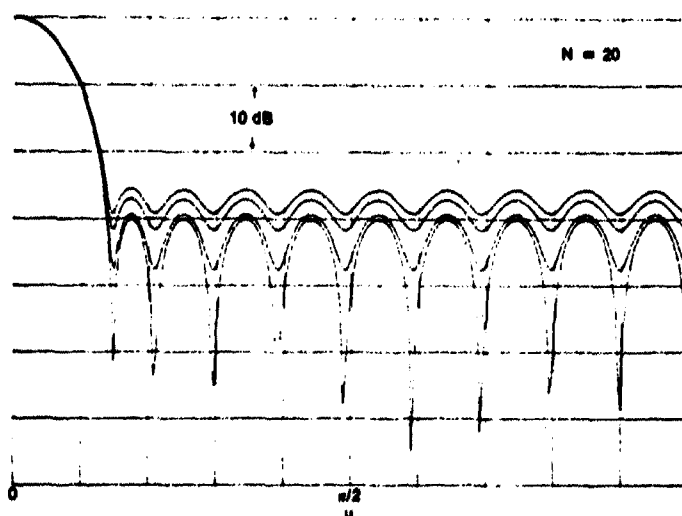
Figure 1C. $Q = 0, \sigma_f^2 = 0, \sigma_p^2 = .001$

Figure 1D. $Q = .001, \sigma_f^2 = .001, \sigma_p^2 = .001$

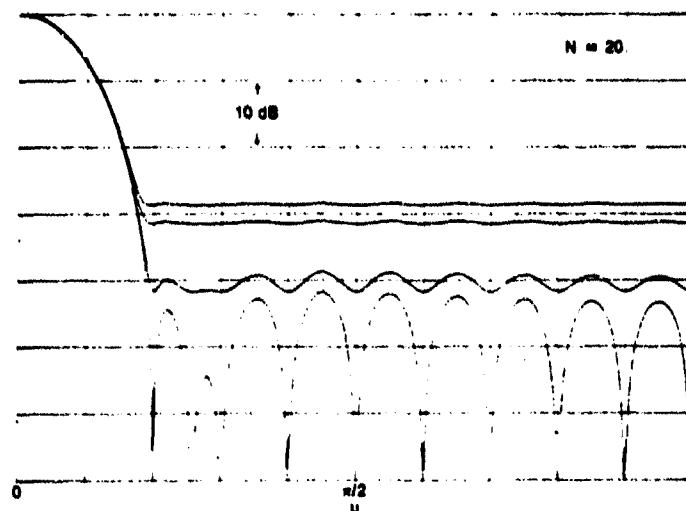Figure 1. (Cont'd) Equispaced Linear Array; –30 dB Dolph-Chebyshev Weights
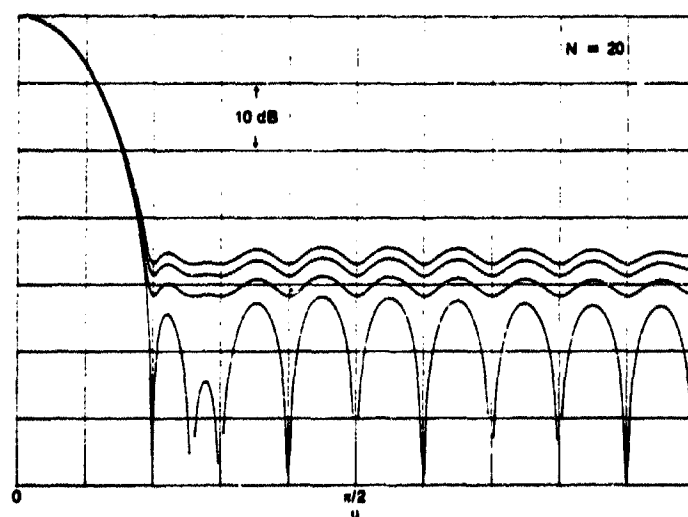
Figure 2A. $Q = .001, \sigma_r^2 = 0, \sigma_p^2 = 0$

Figure 2B. $Q = 0, \sigma_r^2 = .001, \sigma_p^2 = 0$
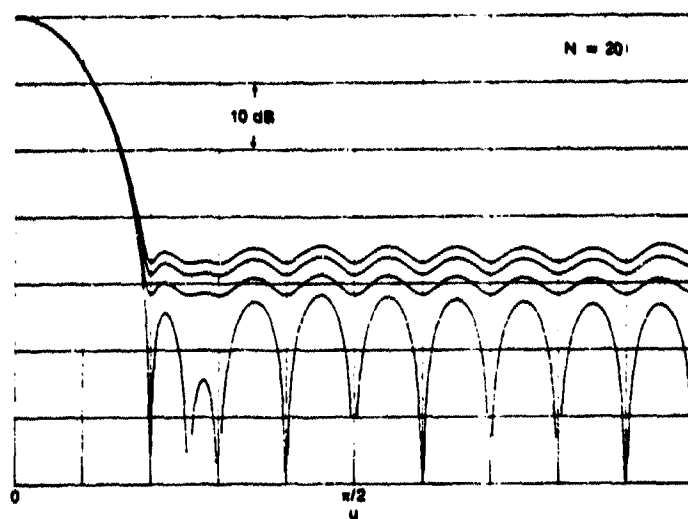
Figure 2. Equispaced Linear Array; Hamming Weights
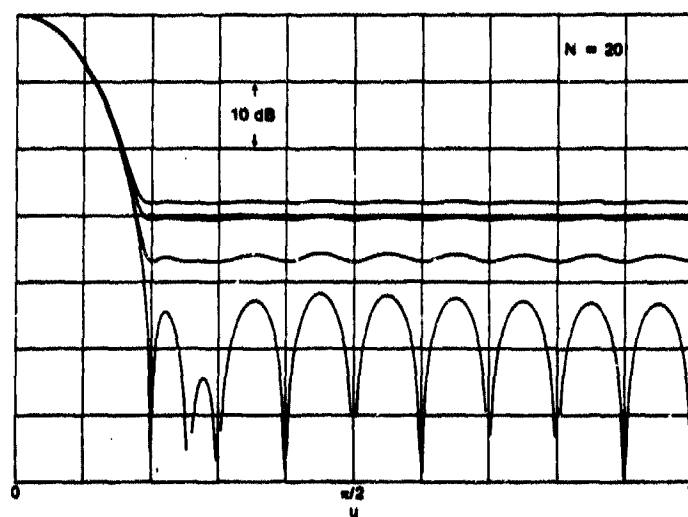
Figure 2C. $Q = 0$, $\sigma_r^2 = 0$, $\sigma_p^2 = .001$

Figure 2D. $Q = .001$, $\sigma_r^2 = .001$, $\sigma_p^2 = .001$

Figure 2. (Cont'd) Equispaced Linear Array; Hamming Weights
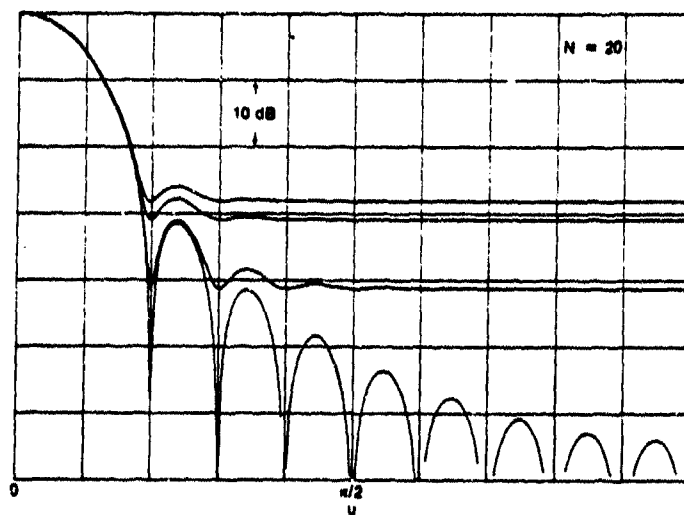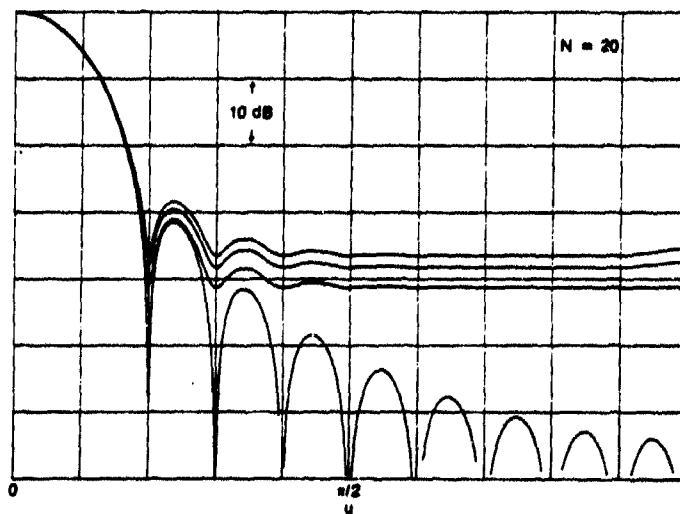
Figure 3A. $Q = .001, \sigma_r^2 = 0, \sigma_p^2 = 0$

Figure 3B. $Q = 0, \sigma_r^2 = .001, \sigma_p^2 = 0$

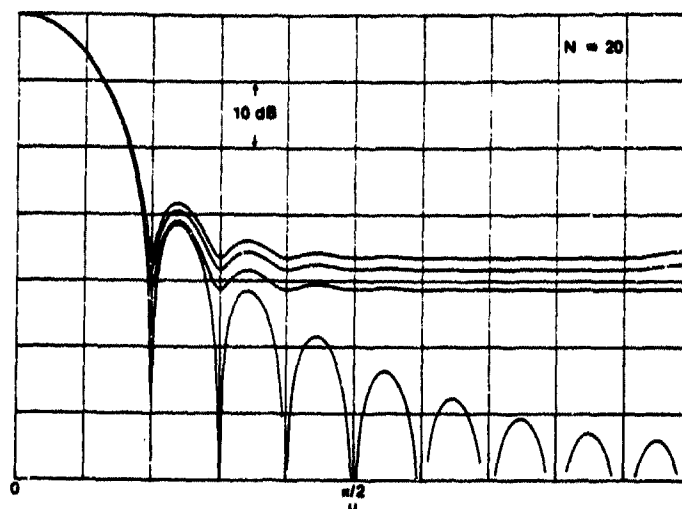Figure 3. Equispaced Linear Array; Hanning Weights

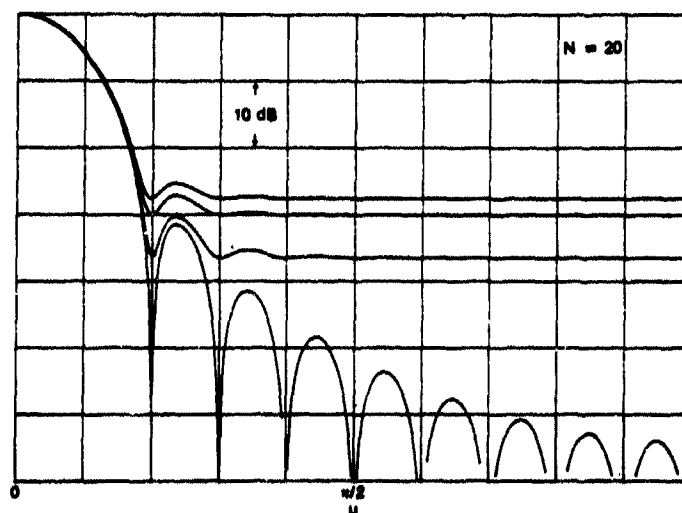Figure 3C. $Q = 0, \sigma_r^2 = 0, \sigma_p^2 = .001$

Figure 3D. $Q = .001, \sigma_r^2 = .001, \sigma_p^2 = .001$

Figure 3. (Cont'd) Equispaced Linear Array; Hanning Weights

PLANAR ARRAY

We consider an equispaced planar array of $N_x \times N_y = 20 \times 18$ elements, for a total of 360 elements. Since there is no simple single parameter like u, (77), in the linear array, we plot the results versus the polar arrival angle or azimuthal arrival angle, for specified polar and azimuthal look angles. The results in figure 4 et seq. employ element spacings in the x and y coordinates of a half wavelength; however the program in table D-2 easily accommodates other spacings of interest. Figure 4 is drawn for a multiplicative weighting of –30 dB Dolph-Chebyshev design in each of the x and y coordinates; the effective numbers of elements in the x and y coordinates are 17.35, 15.57, respectively. The average responses in figure 4 (notice the much larger element perturbations) are plotted versus the polar arrival angle ϕ_a in radians, while the azimuthal arrival angle and the look angles are all zero. The results in figures 4A, 4B, and 4C are substantially the same; the case for combined perturbations in figure 4D is, of course, poorer in terms of performance attainable.

The average power response at the peak of the mainlobe has dropped by about 1 dB in figure 4A because of the effect of element failures, and by about 1/2 dB in figure 4C because of the effect of phase perturbations. The combined effect in figure 4D is such as to lower the average power response at the peak by about 2 dB.

The similarity of results, for weight perturbations or phase perturbations or element failures alone, prompts us to confine further attention only to the case of combined nonzero perturbations. In figure 5, the same array is used as in figure 4, where now we have selected $Q = \sigma_r^2 = \sigma_p^2 = 0.1$. Part A corresponds to a plot of average results versus azimuthal arrival angle θ_a, while part B is for a varying polar arrival angle ϕ_a. The reason that the ideal power response goes below –60 dB is a result of the multiplicative effect of the individual x and y –30 dB patterns. This leads to average responses that are virtually independent of the polar and azimuthal arrival angles.

Figure 6 is drawn under conditions similar to figure 5 except that the weighting is changed to Hanning. The effective numbers of the 20 x 18 elements are 13.33 x 12 in the x and y coordinates. Finally, in figure 7, when we make the polar look and arrival angles ϕ_l, ϕ_a equal to $\pi/2$, we get two large equal responses at $\theta_a = \theta_l$ and $\theta_l + \pi$. This is a result of the element spacings being equal to a half wavelength and, therefore, the array is unable to distinguish or reject arrivals coming endfire to the array from the undesired opposite direction.

Additional results for other equispaced linear or planar arrays are easily available by using the programs in appendix D for whatever set of parameters fits the user's application. Generalizations should be obvious and easily realized from the general results presented in appendix A.

SUMMARY

General equations for the first four moments of a sum of independent complex random variables have been derived with no restrictions on the statistics of the individual variables and no requirement of identical statistics. A program for this case is available in appendix A, table A-1, for those applications requiring the most general case. This program has been thoroughly checked and also compared with the results for Gaussian random variables.

These general equations have been specialized here to a beamforming application, including equispaced linear and planar arrays. Additionally, for ease of programming and investigation, identical statistics were assumed for all the array elements; however this restriction could easily be eliminated by reference to the general results given earlier. To determine allowable tolerances on element weight perturbations, phase perturbations, and element failures, it is recommended that

initial guesses be made at the statistical parameters and, then, the average responses given herein plotted and observed. Then another educated guess can be made and the new responses observed. In this manner one can quickly converge to tolerable limits on the different types of perturbations in order to realize specified performance and sidelobe levels.
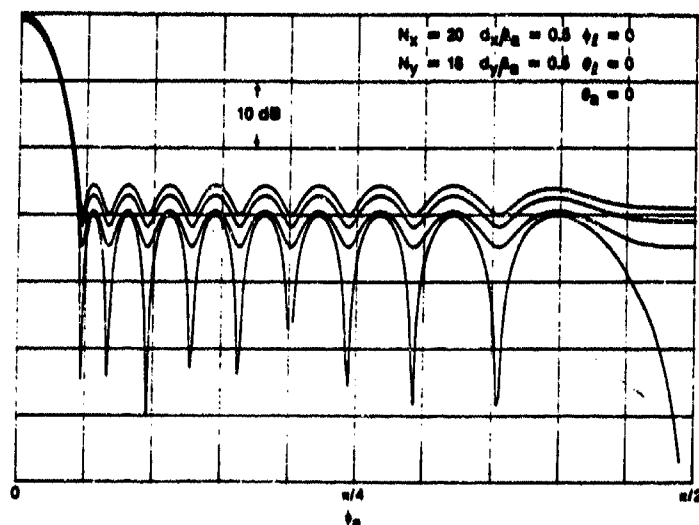
Figure 4A. $Q = .1$, $\sigma_f^2 = 0$, $\sigma_p^2 = 0$

Figure 4B. $Q = 0$, $\sigma_f^2 = .1$, $\sigma_p^2 = 0$

Figure 4. Equispaced Planar Array; –30 dB Dolph-Chebyshev Weights, $\phi_\ell = 0$, $\theta_\ell = 0$
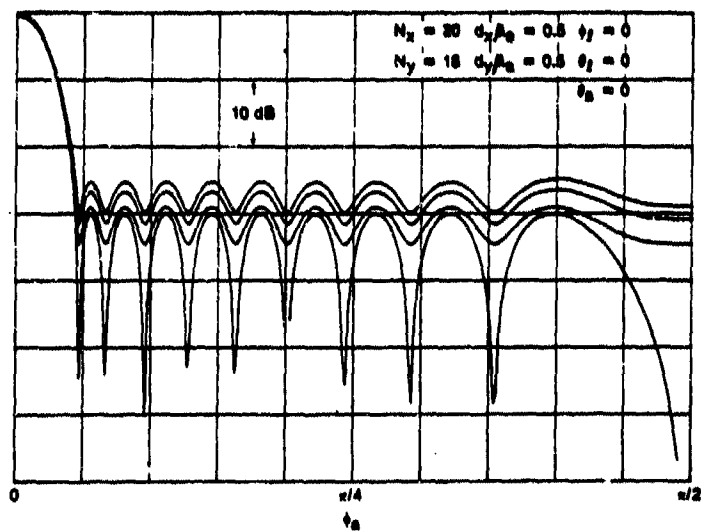
Figure 4C. $Q = 0$, $\sigma_f^2 = 0$, $\sigma_p^2 = .1$

Figure 4D. $Q = .1$, $\sigma_f^2 = .1$, $\sigma_p^2 = .1$

Figure 4. (Cont'd) Equispaced Planar Array; –30 dB Dolph-Chebyshev Weights, $\phi_l = 0$, $\theta_l = 0$

Figure 5A. $Q = .1, \sigma_r^2 = .1, \sigma_p^2 = .1, \phi_a = \pi/4$

Figure 5B. $Q = .1, \sigma_r^2 = .1, \sigma_p^2 = .1, \theta_a = 3\pi/4$

Figure 5. Equispaced Planar Array; –30 dB Dolph-Chebyshev Weights, $\phi_l = \pi/4, \theta_l = 3\pi/4$
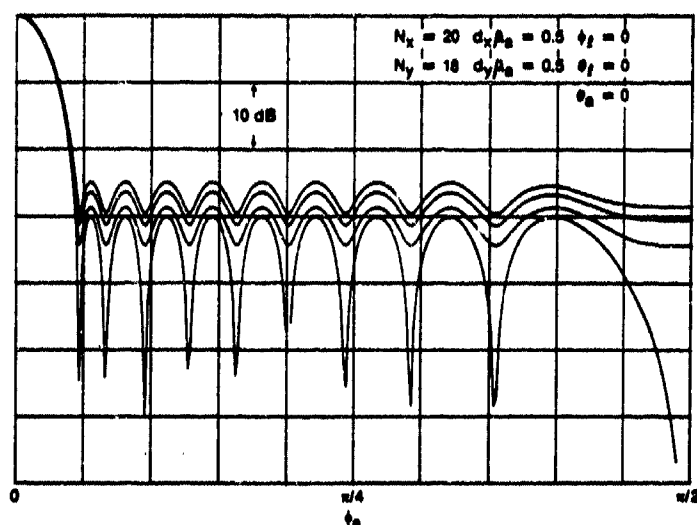
Figure 6A. $Q = .1, \sigma_r^2 = .1, \sigma_p^2 = .1, \phi_a = \pi/6$

Figure 6B. $Q = .1, \sigma_r^2 = .1, \sigma_p^2 = .1, \theta_a = 3\pi/2$

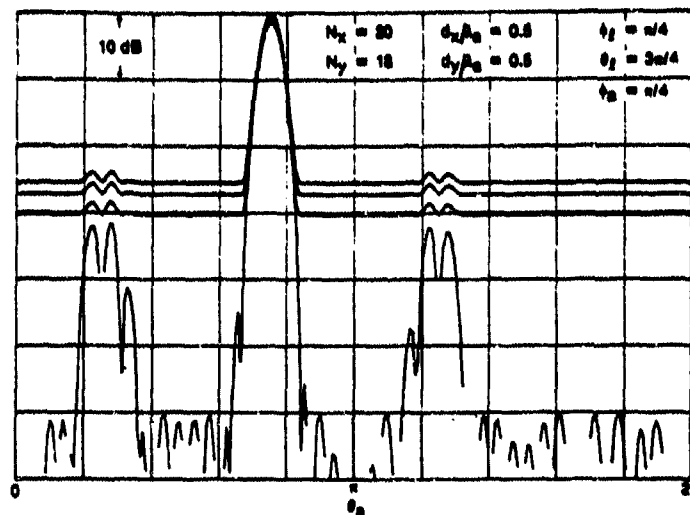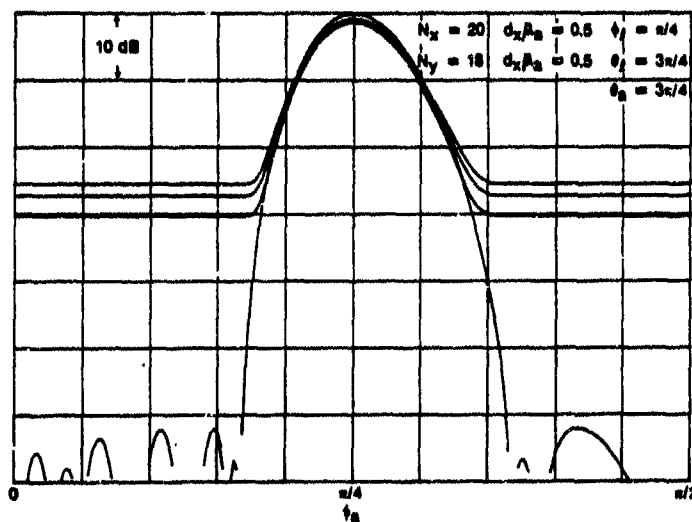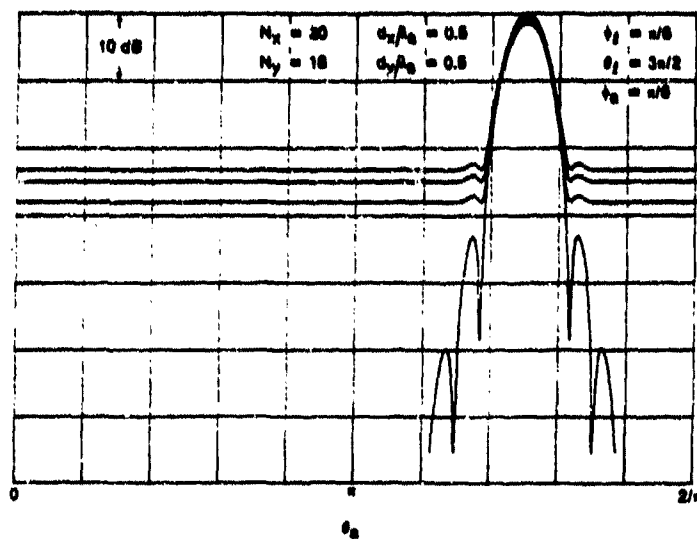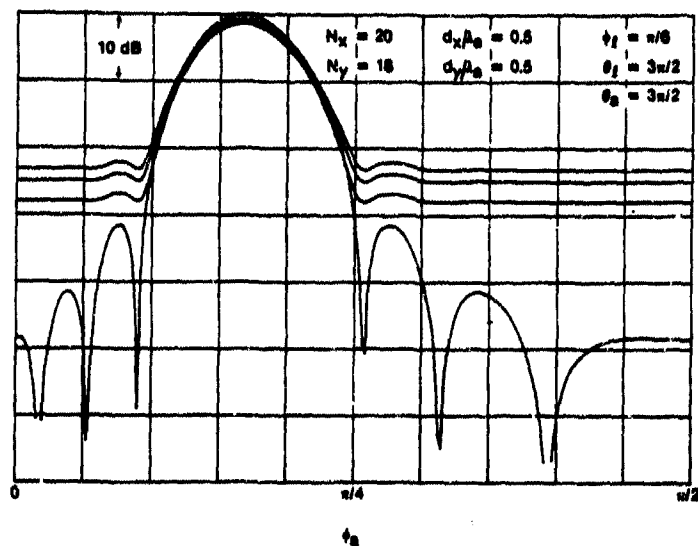Figure 6. Equispaced Planar Array; Hanning Weights, $\phi_l = \pi/6, \theta_l = 3\pi/2$

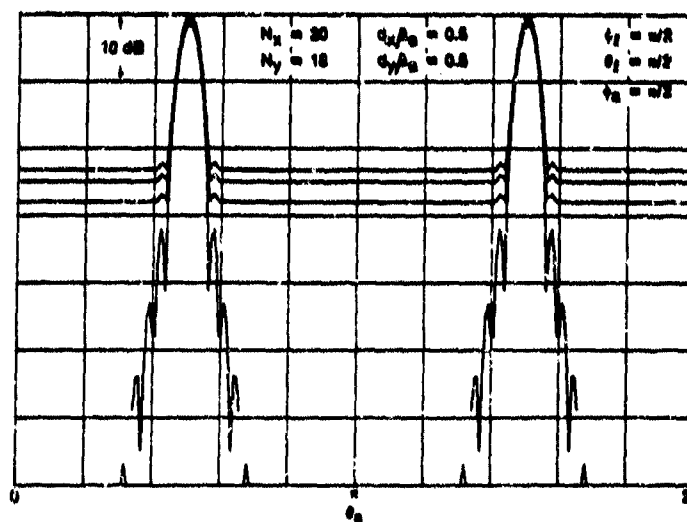Figure 7. Equispaced Planar Array; Hanning Weights, $\phi_\ell = \pi/2$, $\theta_\ell = \pi/2$, $\phi_s = \pi/2$, $Q = .1$, $\sigma_\tau^2 = .1$, $\sigma_\phi^2 = .1$

APPENDIX A
DERIVATION OF FOURTH-ORDER MOMENT

GENERAL RESULTS

Here we will evaluate the complex quantity

$$S = \sum_{k,l,m,n=1}^{N} \overline{a_k \, b_l \, c_m \, d_n} = \sum_{kl=n} \overline{a_k \, b_l \, c_m \, d_n} \,, \tag{A-1}$$

where complex random variables a_k, b_k, c_k, d_k are statistically dependent amongst themselves for any k, but these random variables are statistically independent of the random variables a_n, b_n, c_n, d_n for all $n \neq k$. The fourth-order sum in (A-1) can be broken down into many subcases: (1) all four subscripts equal; (2) three equal and one different; (3) two pairs of two equal, but not to each other; (4) two equal and two others different from each other and the two equal ones; and (5) all different. The values of the components of S in each subcase are

$$S = S_1 + S_2 + S_3 + S_4 + S_5 \,, \tag{A-2}$$

where

$$S_1 = \sum_k \overline{a_k \, b_k \, c_k \, d_k} \,, \tag{A-3}$$

$$S_2 = \sum_U \overline{a_k \, b_k \, c_k \, d_n} + \sum_U \overline{a_k \, b_k \, d_k \, c_m} + \sum_U \overline{a_k \, c_k \, d_k \, b_l} + \sum_U \overline{b_k \, c_k \, d_k \, a_n} \,, \tag{A-4}$$

$$S_3 = \sum_U \overline{a_k \, b_k} \; \overline{c_m \, d_m} + \sum_U \overline{a_k \, c_k} \; \overline{b_m \, d_m} + \sum_U \overline{a_k \, d_k} \; \overline{b_m \, c_m} \,, \tag{A-5}$$

$$S_4 = \sum_U \overline{a_k \, b_k} \; \overline{c_m \, d_n} + \sum_U \overline{a_k \, c_k} \; \overline{b_l \, d_n} + \sum_U \overline{a_k \, d_k} \; \overline{b_l \, c_m}$$
$$+ \sum_U \overline{b_k \, c_k} \; \overline{a_k \, d_n} + \sum_U \overline{b_k \, d_k} \; \overline{a_k \, c_m} + \sum_U \overline{c_m \, d_k} \; \overline{a_k \, b_l} \,, \tag{A-6}$$

$$S_5 = \sum_U \overline{a_k} \; \overline{b_l} \; \overline{c_m} \; \overline{d_n} \,, \tag{A-7}$$

and where $\sum_U$ denotes summation over all subscripts that are unequal in the summands. (As a check, the number of terms in S_1-S_5 are N, 4N(N-1), 3N(N-1), 6N(N-1)(N-2), and N(N-1)(N-2)(N-3), respectively; these add to N^4, as they must.)

The summations in (A-3)-(A-7) must be simplified. To do so, we adopt an abbreviated notation; for example, (A-3) is denoted as

$$S_1 = \sum_k \overline{a_k \, b_k \, c_k \, d_k} = \overline{ABCD} \tag{A-8}$$

In order to simplify (A-4) and (A-5), we develop

$$\sum_U u_k v_n = \sum_k u_k \sum_{m \neq k} v_m = \sum_k u_k \left[\sum_n v_n - v_k \right]$$

$$= \left(\sum_k u_k \right) \left(\sum_n v_n \right) - \sum_k u_k v_k = UV - \overline{UV}. \tag{A-9}$$

Notice that $U \cdot V$ denotes the product of two summations, whereas $\overline{UV}$ denotes one summation. Then (A-4) becomes

$$S_2 = \overline{ABC} \cdot \overline{D} - \overline{ABCD} + \overline{ABD} \cdot \overline{C} - \overline{ABD}\,\overline{C}$$

$$+ \overline{ACD} \cdot \overline{B} - \overline{ACDB} + \overline{BCD} \cdot \overline{A} - \overline{BCDA}, \tag{A-10}$$

while (A-5) yields

$$S_3 = \overline{AB} \cdot \overline{CD} - \overline{ABCD} + \overline{AC} \cdot \overline{BD} - \overline{ACBD} + \overline{AD} \cdot \overline{BC} - \overline{ADBC}. \tag{A-11}$$

Thus, for example, the first two terms in (A-10) are

$$\overline{ABC} \cdot \overline{D} - \overline{ABCD} = \left(\sum_k \overline{a_k b_k c_k} \right) \left(\sum_n \overline{d_n} \right) - \sum_k \overline{a_k b_k c_k}\,\overline{d_k}. \tag{A-12}$$

To simplify (A-6), we use the development

$$\sum_U u_k v_m w_n = \sum_k u_k \sum_{m \neq k} v_m \sum_{\substack{n \neq m \\ n \neq k}} w_n = \sum_k u_k \sum_{m \neq k} v_m \left[\sum_n w_n - w_m - w_k \right]$$

$$= W \cdot \sum_U u_k v_m - \sum_U u_k v_m w_m - \sum_U u_k w_k v_m$$

$$= W \cdot (U \cdot V - \overline{UV}) - (U \cdot \overline{VW} - \overline{UVW}) - (\overline{UW} \cdot V - \overline{UVW})$$

$$= U \cdot V \cdot W - \overline{UV} \cdot W - U \cdot \overline{VW} - V \cdot \overline{UW} + 2\,\overline{UVW}, \tag{A-13}$$

where we used (A-9). Then A-6 becomes

$$S_4 = \overline{AB} \cdot \overline{C} \cdot \overline{D} - \overline{ABC} \cdot \overline{D} - \overline{AB} \cdot \overline{CD} - \overline{C} \cdot \overline{ABD} + 2\overline{ABCD}$$

$$+ \overline{AC} \cdot \overline{B} \cdot \overline{D} - \overline{ACB} \cdot \overline{D} - \overline{AC} \cdot \overline{BD} - \overline{B} \cdot \overline{ACD} + 2\overline{ACBD}$$

$$+ \overline{AD} \cdot \overline{B} \cdot \overline{C} - \overline{ADB} \cdot \overline{C} - \overline{AD} \cdot \overline{BC} - \overline{B} \cdot \overline{ADC} + 2\overline{ADBC}$$

$$+ \overline{BC} \cdot \overline{A} \cdot \overline{D} - \overline{ABC} \cdot \overline{D} - \overline{BC} \cdot \overline{AD} - \overline{A} \cdot \overline{BCD} + 2\overline{BCAD}$$

$$+ \overline{BD} \cdot \overline{A} \cdot \overline{C} - \overline{ABD} \cdot \overline{C} - \overline{BD} \cdot \overline{AC} - \overline{A} \cdot \overline{BDC} + 2\overline{BDAC}$$

$$+ \overline{CD} \cdot \overline{A} \cdot \overline{B} - \overline{ACD} \cdot \overline{B} - \overline{CD} \cdot \overline{AB} - \overline{A} \cdot \overline{BCD} + 2\overline{CDAB}. \tag{A-14}$$

Finally, we need the development

$$\sum_{U} u_k y_\ell w_m x_n - \sum_k u_k \sum_{\ell \neq k} y_\ell \sum_{\substack{m \neq k \\ m \neq \ell}} \left[\sum_n x_n - x_k - x_\ell - x_m \right]$$

$$= X \cdot \sum_{U} u_k y_\ell w_m - \sum_{U} u_k x_k y_\ell w_m - \sum_{U} u_k y_\ell x_\ell w_m - \sum_{U} u_k y_\ell w_m x_m$$

$$= X \cdot \left[U \cdot V \cdot W - UV \cdot W - U \cdot VW - V \cdot UW + 2\, UVW \right]$$

$$- UX \cdot V \cdot W + UVX \cdot W + UX \cdot VW + V \cdot UWX - 2\, UVWX$$

$$- VX \cdot U \cdot W + UVX \cdot W + U \cdot VWX + XV \cdot UW - 2\, UVWX$$

$$- WX \cdot U \cdot V + UV \cdot WX + U \cdot VWX + V \cdot UWX - 2\, UVWX$$

$$= U \cdot V \cdot W \cdot X - \left(UV \cdot X \cdot W + VW \cdot U \cdot X + UW \cdot X \cdot V + UX \cdot V \cdot W + VX \cdot U \cdot W + WX \cdot U \cdot V \right)$$

$$+ 2 \left(UVW \cdot X + UWX \cdot V + UVX \cdot W + VWX \cdot U \right)$$

$$+ UX \cdot VW + XV \cdot UW + UV \cdot WX - 6\, UVWX, \tag{A-15}$$

where we used (A-13). Then (A-7) becomes

$$S_5 = \bar{A} \cdot \bar{B} \cdot \bar{C} \cdot \bar{D} - \left(\overline{AB} \cdot \bar{C} \cdot \bar{D} + \overline{BC} \cdot \bar{A} \cdot \bar{D} + \overline{AC} \cdot \bar{B} \cdot \bar{D} + \overline{AD} \cdot \bar{B} \cdot \bar{C} + \overline{BD} \cdot \bar{A} \cdot \bar{C} + \overline{CD} \cdot \bar{A} \cdot \bar{B} \right)$$

$$+ 2 \left(\overline{ABC} \cdot \bar{D} + \overline{ACD} \cdot \bar{B} + \overline{ABD} \cdot \bar{C} + \overline{BCD} \cdot \bar{A} \right)$$

$$+ \overline{AB} \cdot \overline{CD} + \overline{AC} \cdot \overline{BD} + \overline{AD} \cdot \overline{BC} - 6\, \overline{ABCD}. \tag{A-16}$$

It is not possible to simplify or combine any of the 60 terms in (A-8), (A-10), (A-11), (A-14), and (A-16) because they each employ different statistics. Thus, the general answer to (A-1) is given by (A-2), where the five components are given in the equations just listed above. Repeating the notation, the first and last terms in (A-16) are given by $(\sum_k \bar{a}_k)(\sum_\ell \bar{b}_\ell)(\sum_m \bar{c}_m)(\sum_n \bar{d}_n)$ and $-6 \sum_k \bar{a}_k \bar{b}_k \bar{c}_k \bar{d}_k$, respectively.

SPECIAL CASE

We now set

$$a = z,\ b = z^*,\ c = z,\ d = z^*, \qquad A = Z,\ B = Z^*,\ C = Z,\ D = Z^*, \tag{A-17}$$

in which case (A-1) specializes to

$$S = \sum_{k\ell mn} \overline{z_k z_\ell^* z_m z_n^*} = \overline{\left| \sum_k z_k \right|^4}\ ; \tag{A-18}$$

that is, S is now the mean value of the magnitude-fourth power of the sum. Then (A-8) yields

$$S_1 = \overline{|Z|^4} = \sum_k \overline{|z_k|^4}, \tag{A-19}$$

(A-10) yields

$$S_2 = 2\overline{|z|^2 \vec{z}} \cdot \vec{z}^* - 2\overline{|z|^2 \vec{z}}\,\overline{\vec{z}}^* + \text{complex conjugate}$$

$$= 4\,\mathrm{Re}\left\{\overline{|z|^2 \vec{z}} \cdot \vec{z}^* - \overline{|z|^2}\,\overline{\vec{z}}\,\overline{\vec{z}}^*\right\}$$

$$= 4\,\mathrm{Re}\left\{\sum_k \overline{|z_k|^2 z_k} \cdot \sum_l \vec{z}_l^* - \sum_k \overline{|z_k|^2 z_k}\,\vec{z}_k^*\right\}. \tag{A-20}$$

and (A-11) becomes

$$S_3 = 2\overline{|z|^2} \cdot \overline{|z|^2} - 2\overline{|z|^4} + \vec{z}^2 \cdot \vec{z}^{*2} - |\vec{z}^2|^2$$

$$= 2\left(\sum_k \overline{|z_k|^2}\right)^2 - 2\sum_k \overline{|z_k|^4} + \left|\sum_k \vec{z}_k^2\right|^2 - \sum_k |z_k|^4. \tag{A-21}$$

The quantity S_4 in (A-14) can be developed as follows:

$$S_4 = 4\left\{\overline{|z|^2} \cdot \vec{z} \cdot \vec{z}^* - \overline{|z|^2}\,\vec{z} \cdot \vec{z}^* - \overline{|z|^2} \cdot |z|^2 - \overline{|z|^2}\,\vec{z} \cdot \vec{z} + 2\overline{|z|^2}\,|z|^2\right\}$$

$$+ 2\,\mathrm{Re}\left\{\overline{\vec{z}^2} \cdot \vec{z}^* \cdot \vec{z}^* - 2\overline{\vec{z}^2}\,\vec{z}^* \cdot \vec{z}^* - \overline{\vec{z}^2} \cdot (\vec{z}^*)^2 + 2\overline{\vec{z}^2}(\vec{z}^*)^2\right\}$$

$$= 4\left\{\sum_k \overline{|z_k|^2} \cdot \left|\sum_k \vec{z}_k\right|^2 - 2\,\mathrm{Re}\left\{\sum_k \overline{|z_k|^2 z_k} \cdot \sum_k \vec{z}_k^*\right\} - \sum_k \overline{|z_k|^2} \cdot \sum_k |z_k|^2 + 2\sum_k \overline{|z_k|^2}\,|z_k|^2\right\}$$

$$+ 2\,\mathrm{Re}\left\{\sum_k \overline{\vec{z}_k^2} \cdot \left(\sum_k \vec{z}_k^*\right)^2 - 2\sum_k \overline{\vec{z}_k^2}\,\vec{z}_k^* \cdot \sum_k \vec{z}_k^* - \sum_k \overline{\vec{z}_k^2} \cdot \sum_k \vec{z}_k^* + 2\sum_k \overline{\vec{z}_k^2}\,\vec{z}_k^*\right\}. \tag{A-22}$$

Finally (A-16) yields

$$S_5 = \vec{z} \cdot \vec{z}^* \cdot \vec{z} \cdot \vec{z}^* - 4|z|^2\,\vec{z} \cdot \vec{z}^* - 2\,\mathrm{Re}\left\{\vec{z}^2 \cdot \vec{z}^* \cdot \vec{z}^*\right\}$$

$$+ 8\,\mathrm{Re}\left\{|z|^2\,\vec{z} \cdot \vec{z}^*\right\} + 2|z|^2 \cdot |z|^2 + \vec{z} \cdot \vec{z}^* - 6|z|^4$$

$$= \left|\sum_k \vec{z}_k\right|^4 - 4\sum_k |z_k|^2 \cdot \left|\sum_k \vec{z}_k\right|^2 - 2\,\mathrm{Re}\left\{\sum_k \overline{\vec{z}_k^2} \cdot \left(\sum_k \vec{z}_k^*\right)^2\right\}$$

$$+ 8\,\mathrm{Re}\left\{\sum_k |z_k|^2 \vec{z}_k \cdot \sum_k \vec{z}_k^*\right\} + 2\left(\sum_k |z_k|^2\right)^2 + \left|\sum_k \vec{z}_k^2\right|^2 - 6\sum_k |z_k|^4. \tag{A-23}$$

At this point, it is convenient to define the following 16 fundamental sums that appear above:

$$T_1 = \sum_k \vec{z}_k\;;$$

$$\{T_{2j}\}_1^4 = \sum_k \overline{|z_k|^2}\,,\;\sum_k \vec{z}_k^2\,,\;\sum_k |z_k|^2\,,\;\sum_k \vec{z}_k^2\;;$$

$$\{T_{3j}\}_1^4 = \sum_k \overline{|z_k|^2 z_k}\,,\;\sum_k |z_k|^2 \vec{z}_k\,,\;\sum_k \vec{z}_k^2\,\vec{z}_k^*\,,\;\sum_k |z_k|^2 \vec{z}_k\;;$$

$$\{T_{4j}\}_1^7 = \sum_k \overline{|z_k|^4}\,,\;\sum_k \overline{|z_k|^2 z_k}\,\vec{z}_k^*\,,\;\sum_k \overline{|z_k|^4}\,,\;\sum_k |z_k|^4\,,\;\sum_k \overline{|z_k|^2}\,|z_k|^2,\;\sum_k \vec{z}_k^2\,\vec{z}_k^*,\;\sum_k |z_k|^4. \tag{A-24}$$

It should be observed that the fundamental statistical quantities required of random variables $\{z_k\}$ number only 5, namely

$$\overline{z_k}, \overline{|z_k|^2}, \overline{z_k^2}, \overline{|z_k|^2 z_k}, \overline{|z_k|^4}. \tag{A-25}$$

However, they combine in various ways to yield the 16 sums defined in (A-24), which in turn are encountered in (A-19)-(A-23).

In terms of the sums defined in (A-24), the sum of (A-19)-(A-23) can be written as

$$S = \overline{\left| \sum_k z_k \right|^4} = \sum_{k\ell mn} \overline{z_k \, z_\ell^* \, z_m \, z_n^*}$$

$$= T_{41} - 4 Re\{T_{42}\} - 2 T_{43} - T_{44} + 8 T_{45} + 4 Re\{T_{46}\} - 6 T_{47}$$

$$+ 4 Re\left\{ \left(T_{31} - 2 T_{32} - T_{33} + 2 T_{34} \right) T_1^* \right\}$$

$$+ 2\left(T_{21} - T_{23} \right)^2 + \left| T_{22} - T_{24} \right|^2$$

$$+ 4\left(T_{21} - T_{23} \right) |T_1|^2 + 2 Re\left\{ \left(T_{22} - T_{24} \right) T_1^{*2} \right\} + |T_1|^4. \tag{A-26}$$

This is the general relation for the average of the magnitude-fourth power of the sum (A-18) of independent complex random variables.

Special cases of (A-26) are afforded by

$$S = T_{41} = \overline{|z_1|^4} \quad \text{for} \quad N = 1, \tag{A-27}$$

which is obviously correct, and by

$$S = T_{41} - 2 T_{43} - T_{44} + 2 T_{21}^2 + |T_{22}|^2 \quad \text{for} \quad \overline{z_k} = 0 \text{ for all } k, \tag{A-28}$$

which is identical to (19) in reference 2 for $\bar{c} = 0$ there. That is, (A-28) applies to zero-mean random variables $\{z_k\}$. If we let complex random variable

$$C = \sum_k z_k, \tag{A-29}$$

then (A-26) is obviously an expression for $\overline{|C|^4}$. At the same time, (9) and (A-24) yield

$$\overline{|C|^2} = T_{21} - T_{23} + |T_1|^2. \tag{A-30}$$

Therefore (5), (A-26), and (A-30) yield

$$Var\{|C|^2\} = T_{41} - 4 Re\{T_{42}\} - 2 T_{43} - T_{44} + 8 T_{45} + 4 Re\{T_{46}\} - 6 T_{47}$$

$$+ 4 Re\left\{ \left(T_{31} - 2 T_{32} - T_{33} + 2 T_{34} \right) T_1^* \right\}$$

$$+ \left(T_{31} - T_{23} \right)^2 + \left| T_{22} - T_{24} \right|^2$$

$$+ 2\left(T_{21} - T_{23} \right) |T_1|^2 + 2 Re\left\{ \left(T_{22} - T_{24} \right) T_1^{*2} \right\}. \tag{A-31}$$

This is the general relation for the variance of the magnitude-squared value of the sum (A-29) of independent complex random variables.

Special cases of (A-31) are afforded by

$$Var\{|C|^2\} = T_{41} - T_{21}^2 = \overline{|z_1|^4} - \overline{|z_1|^2}^2 \quad \text{for} \quad N = 1,$$ (A-32)

which is obviously correct, and by

$$Var\{|C|^2\} = T_{41} - 2T_{43} - T_{44} + T_{21}^2 + |T_{32}|^2 \quad \text{for} \quad \overline{z_k} = 0 \quad \text{for all} \quad k,$$ (A-33)

which is identical to (22) in reference 2 for $\bar{c} = 0$ there. That is, (A-33) applies to zero-mean random variables $\{z_k\}$.

A program that evaluates the general expression for the variance in (A-31) is given in table A-1. The fundamental input statistics, (A-25), must be input in lines 90-190. This program has been checked throughly for

1. $N = 1$, arbitrary moments
2. $N = 2$, arbitrary moments
3. $N = 3$, arbitrary moments
4. arbitrary N, complex Gaussian random variables with correlated real and imaginary parts, and nonzero means. (See appendix B for this derivation.)

In terms of the fundamental sums defined in (A-24), the quantities encountered in (7)-(11) are expressible as

$$Av\{C\} = T_1$$

$$Av\{C^2\} = T_{22} - T_{24} + T_1^2$$

$$Av\{|C|^2\} = T_{31} - T_{23} + |T_1|^2$$

$$Var\{C\} = T_{21} - T_{23}$$

$$Av\{(C - \bar{C})^2\} = T_{22} - T_{24} .$$ (A-34)

Table A-1. Program for Variance (A-31)

```
10  !   MOMENTS OF C = SUM OF N INDEPENDENT COMPLEX RANDOM VARIABLES.
20  !   LINES 50-190 GENERATE INPUT STATISTICS.
30  OPTION BASE 1
40  DIM Z1r(9),Z1i(9),Z2m(9),Z2r(9),Z2i(9),Z3r(9),Z3i(9),Z4(9)
50  N=8
60  REDIM Z1r(N),Z1i(N),Z2m(N),Z2r(N),Z2i(N),Z3r(N),Z3i(N),Z4(N)
70  !   TO INPUT THE FUNDAMENTAL STATISTICAL
80  !   QUANTITIES, REPLACE LINES 90-190.
90  RANDOMIZE SQR(.6)
100 FOR K=1 TO N
110 Z1r(K)=RND
120 Z1i(K)=RND
130 Z2m(K)=RND
140 Z2r(K)=RND
150 Z2i(K)=RND
160 Z3r(K)=RND
170 Z3i(K)=RND
180 Z4(K)=RND
190 NEXT K
200 FOR K=1 TO N
210 Q1=Z1r(K)^2
220 Q2=Z1i(K)^2
230 Qp=Q1+Q2
240 Qm=Q1-Q2
250 Qc=2*Z1r(K)*Z1i(K)
260 T1r=T1r+Z1r(K)
270 T1i=T1i+Z1i(K)
280 T21=T21+Z2m(K)
290 T22r=T22r+Z2r(K)
300 T22i=T22i+Z2i(K)
310 T23=T23+Qp
320 T24r=T24r+Qm
330 T24i=T24i+Qc
340 T31r=T31r+Z3r(K)
350 T31i=T31i+Z3i(K)
360 T32r=T32r+Z2m(K)*Z1r(K)
370 T32i=T32i+Z2m(K)*Z1i(K)
380 T33r=T33r+Z2r(K)*Z1r(K)+Z2i(K)*Z1i(K)
390 T33i=T33i+Z2i(K)*Z1r(K)-Z2r(K)*Z1i(K)
400 T34r=T34r+Qp*Z1r(K)
410 T34i=T34i+Qp*Z1i(K)
420 T41=T41+Z4(K)
430 T42r=T42r+Z3r(K)*Z1r(K)+Z3i(K)*Z1i(K)
440 T43=T43+Z2m(K)^2
450 T44=T44+Z2r(K)^2+Z2i(K)^2
460 T45=T45+Z2m(K)*Qp
470 T46r=T46r+Z2r(K)*Qm+Z2i(K)*Qc
480 T47=T47+Qp^2
490 NEXT K
500 R1=T1r^2
510 R2=T1i^2
520 R3=T21-T23
530 R4=T22r-T24r
540 R5=T22i-T24i
550 Rp=R1+R2
560 Rm=R1-R2
570 Rc=2*T1r*T1i
580 V=T41-4*T42r-2*T43-T44+8*T45+4*T46r-6*T47
590 V=V+4*(T31r-2*T32r-T33r+2*T34r)*T1r
600 V=V+4*(T31i-2*T32i-T33i+2*T34i)*T1i
610 V=V+R3^2+R4^2+R5^2
620 V=V+2*(R3*Rp+R4*Rm+R5*Rc)
630 PRINT "N =";N
640 PRINT "Av(C) = ";T1r;"+i(";T1i;")"
650 PRINT "Av(C^2) = ";R4+Rm;"+i(";R5+Rc;")"
660 PRINT "Av(Cmag^2) =";R3+Rp
670 PRINT "Std Dev(Cmag^2) =";SQR(V)
680 PRINT "Var(Cmag^2) =";V
690 PRINT "Av(Cmag^4) =";(R3+Rp)^2+V
700 END

N = 8
Av(C) = 5.40742613063 +i( 4.64537585274 )
Av(C^2) = -4.57145748691 +i( 32.4866283278 )
Av(Cmag^2) = 33.4546032912
Std Dev(Cmag^2) = 6.7210029004
Var(Cmag^2) = 45.1718799872
Av(Cmag^4) = 1164.38236136
```

APPENDIX B
COMPLEX GAUSSIAN RANDOM VARIABLES

For complex Gaussian random variables $\{z_k\}$, the means, variances, and covariances add; thus all we need to do is evaluate these quantities for a typical random variable z. We let the means be noted by

$$\overline{z} = \overline{x} + i\overline{y} = a + ib. \tag{B-1}$$

If we denote the zero-mean components of the real and imaginary parts as

$$\alpha = x - a, \quad \beta = y - b, \tag{B-2}$$

then the variances and covariance are

$$\overline{\alpha^2} = \sigma_x^2, \quad \overline{\beta^2} = \sigma_y^2, \quad \overline{\alpha\beta} = \rho\,\sigma_x\,\sigma_y. \tag{B-3}$$

To check the general results in appendix A, we need the five statistical quantities listed in (A-25). They are

$$Av\{z\} = \overline{z} = a + ib. \tag{B-4}$$

$$Av\{|z|^2\} = \overline{|z|^2} = \overline{x^2} + \overline{y^2} = a^2 + \sigma_x^2 + b^2 + \sigma_y^2. \tag{B-5}$$

$$Av\{z^2\} = \overline{z^2} = \overline{(x+iy)^2} = \overline{x^2} - \overline{y^2} + i\,2\,\overline{xy} = \overline{(a+\alpha)^2} - \overline{(b+\beta)^2}$$
$$+ i\,2\,\overline{(a+\alpha)(b+\beta)} = a^2 + \sigma_x^2 - (b^2 + \sigma_y^2) + i\,2(ab + \rho\,\sigma_x\,\sigma_y). \tag{B-6}$$

$$\overline{|z|^2 z} = \overline{(x^2+y^2)(x+iy)} = \overline{x^3} + \overline{xy^2} + i\,\overline{x^2 y} + i\,\overline{y^3}. \tag{B-7}$$

The components required here are given by

$$\overline{x^3} = \overline{(a+\alpha)^3} = a^3 + 3a\sigma_x^2$$
$$\overline{xy^2} = \overline{(a+\alpha)(b+\beta)^2} = ab^2 + a\sigma_y^2 + 2b\rho\,\sigma_x\,\sigma_y$$
$$\overline{y^3} = b^3 + 3b\sigma_y^2$$
$$\overline{x^2 y} = a^2 b + b\sigma_x^2 + 2a\rho\,\sigma_x\,\sigma_y. \tag{B-8}$$

Hence

$$Av\{|z|^2 z\} = \overline{|z|^2 z} = a^3 + ab^2 + a(3\sigma_x^2 + \sigma_y^2) + 2b\rho\,\sigma_x\,\sigma_y$$
$$+ i\left[b^3 + a^2 b + b(\sigma_x^2 + 3\sigma_y^2) + 2a\rho\,\sigma_x\,\sigma_y\right]. \tag{B-9}$$

Lastly we need

$$\overline{|z|^4} = \overline{(x^2+y^2)^2} = \overline{x^4} + \overline{y^4} + 2\overline{x^2 y^2}, \tag{B-10}$$

for which the components are

$$\overline{x^4} = \overline{(a+\alpha)^4} = a^4 + 6a^2\sigma_x^2 + 3\sigma_x^4$$
$$\overline{y^4} = \overline{(b+\beta)^4} = b^4 + 6b^2\sigma_y^2 + 3\sigma_y^4$$
$$\overline{x^2 y^2} = \overline{(a+\alpha)^2(b+\beta)^2} = \overline{(a^2+2a\alpha+\alpha^2)(b^2+2b\beta+\beta^2)}$$
$$= a^2 b^2 + a^2\sigma_y^2 + 4ab\rho\sigma_x\sigma_y + b^2\sigma_x^2 + \sigma_x^2\sigma_y^2 + 2\rho^2\sigma_x^2\sigma_y^2. \tag{B-11}$$

Then there follows

$$Var\{|z|^2\} = \overline{|z|^4} - \overline{|z|^2}^2 = 2\left(\sigma_x^4+\sigma_y^4+2\rho^2\sigma_x^2\sigma_y^2\right) + 4\left(a^2\sigma_x^2+b^2\sigma_y^2+2\rho a\sigma_x b\sigma_y\right) \tag{B-12}$$

and

$$\overline{|z|^4} = Av^2\{|z|^2\} + Var\{|z|^2\}. \tag{B-13}$$

A program that incorporates the above relations is presented below in table B-1. The Gaussian rules occur in lines 280-320.

Table B-1. Check Program for Complex Gaussian Random Variables

```
10 !    MOMENTS OF C = SUM OF N INDEPENDENT COMPLEX RANDOM VARIABLES.
20 !    GAUSSIAN TEST EXAMPLE.
30   OPTION BASE 1
40   DIM Z1r(9),Z1i(9),Z2m(9),Z2r(9),Z2i(9),Z3r(9),Z3i(9),Z4(9)
50   N=8
60   REDIM Z1r(N),Z1i(N),Z2m(N),Z2r(N),Z2i(N),Z3r(N),Z3i(N),Z4(N)
70   RANDOMIZE SQR(.6)
80   FOR K=1 TO N
90   A=2*RND-1
100  B=2*RND-1
110  Sigx=RND
120  Sigy=RND
130  Rho=2*RND-1
140  A2=A^2
150  B2=B^2
160  Vx=Sigx^2
170  Vy=Sigy^2
180  P=Rho*Sigx*Sigy
190  Z1r(K)=A
200  Z1i(K)=B
210  Z2m(K)=A2+Vx+B2+Vy
220  Z2r(K)=A2+Vx-(B2+Vy)
230  Z2i(K)=2*(A*B+P)
240  Z3r(K)=A*(A2+B2)+A*(3*Vx+Vy)+2*B*P
250  Z3i(K)=B*(A2+B2)+B*(Vx+3*Vy)+2*A*P
260  Var=2*(Vx^2+Vy^2+2*P^2)+4*(A2*Vx+B2*Vy+2*A+B*P)
270  Z4(K)=Z2m(K)^2+Var
280  G=G+A                    ! GAUSSIAN RULES
290  H=H+B                    ! GAUSSIAN RULES
300  Vd=Vd+Vx                 ! GAUSSIAN RULES
310  Ve=Ve+Vy                 ! GAUSSIAN RULES
320  Vc=Vc+P                  ! GAUSSIAN RULES
330  NEXT K
340  FOR K=1 TO N
350  Q1=Z1r(K)^2
360  Q2=Z1i(K)^2
370  Qp=Q1+Q2
380  Qm=Q1-Q2
390  Qc=2*Z1r(K)*Z1i(K)
400  T1r=T1r+Z1r(K)
410  T1i=T1i+Z1i(K)
420  T21=T21+Z2m(K)
430  T22r=T22r+Z2r(K)
440  T22i=T22i+Z2i(K)
450  T23=T23+Qp
460  T24r=T24r+Qm
470  T24i=T24i+Qc
480  T31r=T31r+Z3r(K)
490  T31i=T31i+Z3i(K)
500  T32r=T32r+Z2m(K)*Z1r(K)
510  T32i=T32i+Z2m(K)*Z1i(K)
520  T33r=T33r+Z2r(K)*Z1r(K)+Z2i(K)*Z1i(K)
530  T33i=T33i+Z2i(K)*Z1r(K)-Z2r(K)*Z1i(K)
540  T34r=T34r+Qp*Z1r(K)
550  T34i=T34i+Qp*Z1i(K)
560  T41=T41+Z4(K)
570  T42r=T42r+Z3r(K)*Z1r(K)+Z3i(K)*Z1i(K)
580  T43=T43+Z2m(K)^2
590  T44=T44+Z2r(K)^2+Z2i(K)^2
600  T45=T45+Z2m(K)*Qp
610  T46r=T46r+Z2r(K)*Qm+Z2i(K)*Qc
620  T47=T47+Qp^2
630  NEXT K
640  R1=T1r^2
650  R2=T1i^2
660  R3=T21-T23
```

Table B-1. (Cont'd) Check Program for Complex Gaussian Random Variables

```
670   R4=T22r-T24r
680   R5=T22i-T24i
690   Rp=R1+R2
700   Rm=R1-R2
710.  Rc=2*T1r*T1i
720   V=T41-4*T42r-2*T43-T44+8*T45+4*T46r-6*T47
730   V=V+4*(T31r-2*T32r-T33r+2*T34r)*T1r
740   V=V+4*(T31i-2*T32i-T33i+2*T34i)*T1i
750   V=V+R3^2+R4^2+R5^2
760   V=V+2*(R3*Rp+R4*Rm+R5*Rc)
770   PRINT "N =";N
780   PRINT "Av(C) = ";T1r;"+i(";T1i;")"
790   PRINT "Av(C^2) = ";R4+Rm;"+i(";R5+Rc;")"
800   PRINT "Av(Cmag^2) =";R3+Rp
810   PRINT "Std.Dev(Cmag^2) =";SQR(V)
820   PRINT "Var(Cmag^2) =";V
830   PRINT "Av(Cmag^4) =";(R3+Rp)^2+V
840   PRINT
850   PRINT "USING GAUSSIAN RULES:"
860   PRINT G;"+i(";H;")"
870   PRINT G^2+Vd-(H^2+Ve);"+i(";2*(G*H+Vc);")"
880   PRINT G^2+Vd+H^2+Ve
890   Varc=2*(Vd^2+Ve^2+2*Vc^2)+4*(G^2*Vd+H^2*Ve+2*G*H*Vc)
900   PRINT SQR(Varc)
910   PRINT Varc
920   PRINT (G^2+Vd+H^2+Ve)^2+Varc
930   END
```

```
N = 8
Av(C) = -.81709993443 +i( .81895514559 )
Av(C^2) = -.987717628742 +i(-.76138494979 )
Av(Cmag^2) = 6.17575336444
Std.Dev(Cmag^2) = 6.00940227934
Var(Cmag^2) = 36.1129157551
Av(Cmag^4) = 74.2528453735

USING GAUSSIAN RULES:
-.81709993443 +i( .81895514559 )
-.98771762875 +i(-.761384949778 )
 6.17575336445
 6.00940227937
 36.1129157554
 74.2528453739
```

APPENDIX C
EFFECTS OF POSITION AND DELAY PERTURBATIONS
ON LINEAR ARRAY RESPONSE

Consider a plane-wave arrival, at frequency f_a, from angle θ_a, propagating with speed c, as shown in figure C-1. If the propagation time delay to reach the origin O is defined as zero, then the propagation delay to reach position x on the linear array is

$$\tau(x, \theta_a) = \frac{x}{c} \sin \theta_a . \tag{C-1}$$
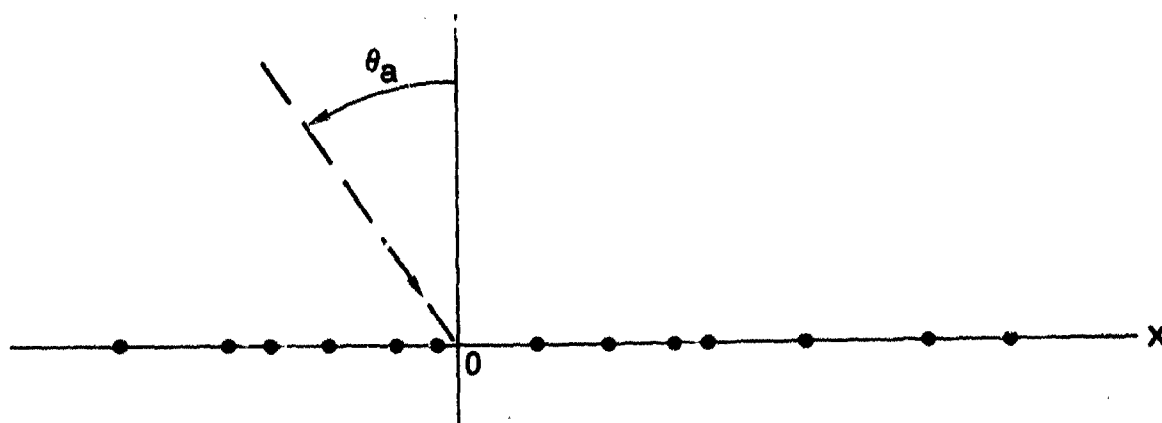
Figure C-1. Linear Array, not Necessarily Equispaced

The processing of interest is depicted in figure C-2. The position of the k-th element in the linear array is x_k; the delay employed in the k-th branch of the beamformer is $D_k(\theta_l)$, which depends on the desired look direction θ_l; and the corresponding element weight is w_k. The voltage transfer function of the propagation delay and beamformer, as applied to the arriving plane-wave, is then

$$V = \sum_k w_k \exp\left[-i 2\pi f_a \tau(x_k, \theta_a) - i 2\pi f_a D_k(\theta_l)\right]. \tag{C-2}$$
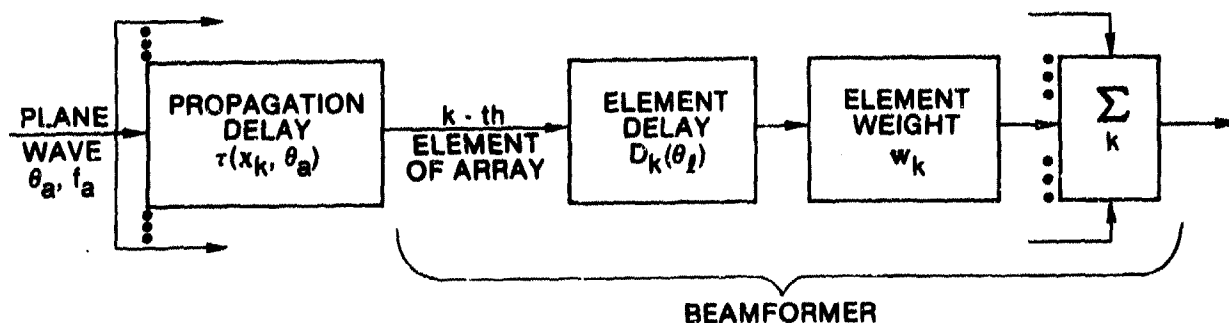
Figure C-2. Processing of Plane-Wave Arrival

Now suppose that the desired position of the k-th element in the linear array is d_k, but that the actual position is

$$x_k = d_k + \Delta_k , \tag{C-3}$$

where Δ_k is a random (distance) variable. Notice that we are restricting consideration to longitudinal element movement only along the linear array; this restriction is removed later.

Also suppose that the desired k-th element time delay to steer the array in direction θ_l is (ignoring a common bulk delay for all elements)

$$- \frac{d_k}{c} \sin \theta_l , \tag{C-4}$$

but that the actually employed delay is

$$D_k(\theta_l) = - \frac{d_k}{c} \sin \theta_l + \delta_k , \tag{C-5}$$

where δ_k is a random (time) variable. (Notice that this choice of element delay takes no account of distance perturbation Δ_k; a case where this distance perturbation is taken into account is considered later.)

The transfer function of the linear array is then*

$$V_l = \sum_k w_k \exp\left[- i \, 2\pi f_a \frac{d_k + \Delta_k}{c} \sin \theta_a + i \, 2\pi f_a \frac{d_k}{c} \sin \theta_l - i \, 2\pi f_a \delta_k \right]$$

$$= \sum_k w_k \exp\left[i \, 2\pi f_a \frac{d_k}{c} \left(\sin \theta_l - \sin \theta_a \right) - i \, 2\pi f_a \left(\frac{\Delta_k}{c} \sin \theta_a + \delta_k \right) \right] . \tag{C-6}$$

The effect of the position and delay perturbations leads to the k-th random phase shift term $\exp[i\phi_k]$, where

$$\phi_k = - 2\pi f_a \left(\frac{\Delta_k}{c} \sin \theta_a + \delta_k \right) = - 2\pi \left(\frac{\Delta_k}{\lambda_a} \sin \theta_a + \frac{\delta_k}{T_a} \right) . \tag{C-7}$$

Here λ_a is the wavelength of the plane-wave arrival, and T_a is the period of the arrival. The fraction of a wavelength movement, Δ_k/λ_a, and the fraction of a period delay, δ_k/T_a, are obviously important parameters.

Now if the position and delay perturbations have the properties†

$$\left. \begin{array}{l} Av\{\Delta_k\} = 0, \ Std \ Dev\{\Delta_k\} = \sigma_d \\[4pt] Av\{\delta_k\} = 0, \ Std \ Dev\{\delta_k\} = \sigma_t \end{array} \right\} \ \text{for all } k, \tag{C-8}$$

and Δ_k is uncorrelated with δ_k, then we find

---

* Weight perturbations and element failures are easily incorporated by replacing w_k by $w_k(1 + r_k)g_k$, where r_k is a zero-mean random variable and g_k is a (0, 1) random variable; these features are not included in this appendix.

† These could be generalized to allow dependence on k.

$$Av\{\phi_k\} = 0, \quad \text{Std Dev}\{\phi_k\} = 2\pi\left[\left(\frac{\sigma_d}{\lambda_a}\right)^2 \sin^2\theta_a + \left(\frac{\sigma_t}{T_a}\right)^2\right]^{\frac{1}{2}} \equiv \sigma_\phi ,\tag{C-9}$$

which is a function of arrival angle θ_a. Thus, if we plot average properties of the beamformer, by varying the look angle θ_ℓ while holding arrival angle θ_a fixed, we can treat σ_ϕ as a constant. However, if we do the reverse, we must take into account a varying σ_ϕ with arrival angle. Of course, if $\sigma_d = 0$ (no position perturbations), then $\sigma_\phi = 2\pi \sigma_t/T_a$ is a constant independent of any angles. We might also note that for broadside arrivals, $\theta_a = 0$, σ_ϕ is a minimum and, in fact, positional perturbations ($\sigma_d \neq 0$) do not affect the transfer function, (C-6).

Instead of choosing element delay $D_k(\theta_\ell)$ without taking account of position perturbation Δ_k, assume that we have this information (perhaps via measurements after construction of each array) and decide to use it by employing, instead of (C-5), the time delay

$$D_k(\theta_\ell) = -\frac{d_k + \Delta_k}{c}\sin\theta_\ell + \delta_k ,\tag{C-10}$$

where δ_k is again a random (time) variable. (Of course $\{\Delta_k\}$ could still be random variables from one array construction to another.) Then the transfer function of the linear array is

$$V = \sum_k w_k \exp\left[-i2\pi f_a \frac{d_k + \Delta_k}{c}\sin\theta_a + i2\pi f_a \frac{d_k + \Delta_k}{c}\sin\theta_\ell - i2\pi f_a \delta_k\right]$$

$$= \sum_k w_k \exp\left[i2\pi f_a \frac{d_k}{c}\left(\sin\theta_\ell - \sin\theta_a\right) - i2\pi f_a\left(\frac{\Delta_k}{c}\left[\sin\theta_a - \sin\theta_\ell\right] + \delta_k\right)\right].\tag{C-11}$$

The effect of the position and delay perturbations then leads to the k-th random phase shift term $\exp[i\phi_k]$, where now

$$\phi_k = -2\pi f_a\left(\frac{\Delta_k}{c}\left[\sin\theta_a - \sin\theta_\ell\right] + \delta_k\right) = -2\pi\left(\frac{\Delta_k}{\lambda_a}\left[\sin\theta_a - \sin\theta_\ell\right] + \frac{\delta_k}{T_a}\right).\tag{C-12}$$

Keeping the same statistical properties as assumed above, we now find

$$Av\{\phi_k\} = 0, \quad \text{Std Dev}\{\phi_k\} = 2\pi\left[\left(\frac{\sigma_d}{\lambda_a}\right)^2\left[\sin\theta_a - \sin\theta_\ell\right]^2 + \left(\frac{\sigma_t}{T_a}\right)^2\right]^{\frac{1}{2}} \equiv \sigma_\phi ,\tag{C-13}$$

which is now a function of both arrival angle θ_a and look direction θ_ℓ. If $\sigma_d = 0$ (no position perturbations), then $\sigma_\phi = 2\pi \sigma_t/T_a$ is independent of any angles. For $\sigma_d \neq 0$, the minimum of σ_ϕ is realized when $\theta_\ell = \theta_a$.

For positional perturbations into the x,y plane (rather than just along the x-axis), we have instead of (C-1), propagation delay

$$\tau(x, y, \theta_a) = \frac{x\sin\theta_a - y\cos\theta_a}{c} .\tag{C-14}$$

Then the voltage transfer function of the linear array becomes

$$V = \sum_k w_k \exp\left[-i2\pi f_a \tau(x_k, y_k, \theta_a) - i2\pi f_a D_k(\theta_\ell)\right].\tag{C-15}$$

Now, instead of (C-3), let element position

$$x_k = d_k + \Delta_k , \quad y_k = \Gamma_k ,\tag{C-16}$$

where Δ_k and γ_k are random (distance) variables. Then, if we still use the same delay as in (C-5),

$$D_k(\theta_k) = -\frac{d_k}{c}\sin\theta_k + \delta_k ,$$

(C-17)

we find the perturbed transfer function to be

$$\tilde{V}_1 = \sum_k w_k \exp\left[i\,2\pi f_a\,\frac{d_k}{c}\left(\sin\theta_k - \sin\theta_k\right)\right]\exp\left(i\phi_k\right) ,$$

(C-18)

where now

$$\phi_k = -2\pi f_a\left(\frac{\Delta_k}{c}\sin\theta_k - \frac{\gamma_k}{c}\cos\theta_k + \delta_k\right)$$

$$= -2\pi\left(\frac{\Delta_k}{\lambda_a}\sin\theta_k - \frac{\gamma_k}{\lambda_a}\cos\theta_k + \frac{\delta_k}{T_a}\right) .$$

(C-19)

Analogous to (C-8), if we now assume

$$\overline{\Delta_k} = 0,\ \overline{\gamma_k} = 0,\ \overline{\delta_k} = 0,\ \ \overline{\Delta_k^2} = \sigma_x^2,\ \overline{\gamma_k^2} = \sigma_y^2,\ \overline{\delta_k^2} = \sigma_t^2,\ \overline{\Delta_k\gamma_k} = \rho\,\sigma_x\sigma_y ,$$

(C-20)

where nonzero ρ corresponds to correlated x,y movement of an element, we find

$$Av\{\phi_k\} = 0,$$

$$Std\ Dev\{\phi_k\} = 2\pi\left[\left(\frac{\sigma_x}{\lambda_a}\right)^2\sin^2\theta_k + \left(\frac{\sigma_y}{\lambda_a}\right)^2\cos^2\theta_k - 2\rho\left(\frac{\sigma_x}{\lambda_a}\right)\left(\frac{\sigma_y}{\lambda_a}\right)\sin\theta_k\cos\theta_k + \left(\frac{\sigma_t}{T_a}\right)^2\right]^{\frac{1}{2}} .$$

(C-21)

This is independent of θ_ℓ, as was (C-9); thus the comments directly under (C-9) are relevant in this case too.

APPENDIX D
VARIANCE OF POWER RESPONSE

LINEAR ARRAY

The variance of the power response was given in (A-31). We now specialize it to the equispaced linear array. Define

$$C_1 = \mu_1 \mu_1 \gamma_1 , \quad C_{2m} = \mu_2 \mu_2 , \quad C_{2r} = \mu_2 \mu_2 \gamma_2 ,$$
$$C_3 = \mu_3 \mu_3 \gamma_1 , \quad C_4 = \mu_4 \mu_4 . \tag{D-1}$$

Then the various quantities $\{T_{ij}\}$ in (A-24) may, with the aid of (18) and (40) and the realness of γ_1, γ_2, be expressed as

$$T_1 = C_1 L_1$$
$$T_{21} = C_{2m} W_2 , \quad T_{22} = C_{2r} L_2 , \quad T_{23} = C_1^2 W_2 , \quad T_{24} = C_1^2 L_2 ,$$
$$T_{31} = C_3 L_3 , \quad T_{32} = C_1 C_{2m} L_3 , \quad T_{33} = C_1 C_{2r} L_3 , \quad T_{34} = C_1^3 L_3 ,$$
$$T_{41} = C_4 W_4 , \quad T_{42} = C_1 C_3 W_4 , \quad T_{43} = C_{2m}^2 W_4 , \quad T_{44} = C_{2r}^2 W_4$$
$$T_{45} = C_1^2 C_{2m} W_4 , \quad T_{46} = C_1^2 C_{2r} W_4 , \quad T_{47} = C_1^4 W_4 \tag{D-2}$$

The only terms in (D-2) that depend on u, are T_1, T_{22}, T_{24}, and $\{T_{3j}\}_1^4$, by virtue of depending on L_1, L_2, L_3. Now (A-31) can be expressed as

$$\text{Var}\{|C|^2\} = T_{41} - 4T_{42} - 2T_{43} - T_{44} + 8T_{45} + 4T_{46} - 6T_{47}$$
$$+ 4(T_{31} - 2T_{32} - T_{33} + 2T_{34}) T_1$$
$$+ (T_{21} - T_{23})^2 + (T_{22} - T_{24})^2 + 2(T_{21} - T_{23} + T_{22} - T_{24}) T_1^2$$

$$= (C_4 - 4C_1 C_3 - 2C_{2m}^2 - C_{2r}^2 + 8C_1^2 C_{2m} + 4C_1^2 C_{2r} - 6C_1^4) W_4$$
$$+ 4C_1 (C_3 - 2C_1 C_{2m} - C_1 C_{2r} + 2C_1^3) L_1 L_3$$
$$+ (C_{2m} - C_1^2)^2 W_2^2 + (C_{2r} - C_1^2)^2 L_2^2$$
$$+ 2C_1^2 \left[(C_{2m} - C_1^2) W_2 + (C_{2r} - C_1^2) L_2 \right] L_1^2 \tag{D-3}$$

in terms of the quantities in (D-1) and (40). Finally we rearrange (D-3) to read

$$\text{Var}\{|C|^2\} = F_4 W_4 + F_3 L_1 L_3 + F_{2a} W_2^2 + F_{2b} L_2^2 + F_{1a} W_2 L_1^2 + F_{1b} L_2 L_1^2 , \tag{D-4}$$

where

$$F_4 = C_4 - 4 C_1 C_3 - 2 C_{2m}^2 - C_{2r}^2 + 8 C_1^2 C_{2m} + 4 C_1^2 C_{2r} - 6 C_1^4$$

$$F_3 = 4 C_1 (C_3 - 2 C_1 C_{2m} - C_1 C_{2r} + 2 C_1^3)$$

$$F_{2a} = (C_{2m} - C_1^2)^2 \quad , \quad F_{2b} = (C_{2r} - C_1^2)^2$$

$$F_{1a} = 2 C_1^2 (C_{2m} - C_1^2) \quad , \quad F_{1b} = 2 C_1^2 (C_{2r} - C_1^2) \tag{D-5}$$

Since L_1 and L_3 only occur in the combinations $L_1 L_3$ and L_1^2, we note, using the properties of L_1, L_2, L_3, that (D-4) is even in u and has period 2π in u. Therefore we need only to evaluate (D-4) in the range $(0, \pi)$ for u for this equispaced linear array. The program for evaluating (42) and (D-4) is presented below.

DEEP SIDELOBE REGION

In the deep sidelobe region, L_1, L_2, and L_3 of (40) are approximately zero; then the only $\{T_{ij}\}$ in (D-2) that are nonzero are T_{21}, T_{23}, and $\{T_{4j}\}$. It then follows from (D-4) that

$$\operatorname{Var}\{|C|^2\} \approx F_4 W_4 + F_{2a} W_2^2 \quad \text{in deep sidelobes,} \tag{D-6}$$

where these quantities are given in (D-5) and (40). The constants needed in (D-5) are given by (D-1). No simpler expression for (D-6) appears possible.

Alternative expressions and interpretations for F_{2a} and F_4 in (D-6) are possible; let

$$z_k = v_k p_k \quad , \quad \text{where} \quad p_k = g_k (1 + r_k) \exp(i \phi_k) , \tag{D-7}$$

where we used (16). Also let

$$\alpha_k = p_k - \overline{p_k} . \tag{D-8}$$

Then we find

$$F_{2a} = \overline{|\alpha_k|^2}^2 \quad , \quad F_4 = \overline{|\alpha_k|^4} - 2 \overline{|\alpha_k|^2}^2 - \left| \overline{\alpha_k^2} \right|^2 . \tag{D-9}$$

Comparing these results with (40) and (44) in reference 2, we see that $F_{2a}^{1/2}$ and F_4 are equal to the cumulants λ_{11} and λ_{22} of random variable p_k.

PLANAR ARRAY

The variance of the power response is given in (A-31); we now specialize it to the planar array. With the aid of (A-24), (18), (D-1), and (75) and the realness of $\gamma_{11} \gamma_2$, there follows

$$T_1 = C_1 \, L_1^{(\mu)}(u) \, L_1^{(\nu)}(v)$$

$$T_{21} = C_{2m} W_2^{(\mu)} W_2^{(\nu)}, \quad T_{22} = C_{2r} \, L_2^{(\mu)}(u) \, L_2^{(\nu)}(v)$$

$$T_{23} = C_1^2 W_2^{(\mu)^2} W_2^{(\nu)^2}, \quad T_{24} = C_1^2 \, L_2^{(\mu)}(u) \, L_2^{(\nu)}(v)$$

$$\{T_{3j}\}_1^4 = \{C_3, C_1 C_{2m}, C_1 C_{2r}, C_1^3\} \, L_3^{(\mu)}(u) \, L_3^{(\nu)}(v)$$

$$\{T_{4j}\}_1^7 = \{C_4, C_1 C_3, C_{2m}^2, C_{2r}^2, C_1^2 C_{2m}, C_1^2 C_{2r}, C_1^4\} W_4^{(\mu)} W_4^{(\nu)} . \qquad \text{(D-10)}$$

The terms that depend on u and v are T_1, T_{22}, T_{24}, and $\{T_{3j}\}_1^4$, by virtue of depending on L_1, L_2, L_3, defined in (75).

Now (A-31) can be expressed as the upper half of (D-3), which, in turn, can be developed, as above, into

$$\text{Var}\{|C|^2\} = F_4 W_4^{(\mu)} W_4^{(\nu)} + F_3 \, L_1^{(\mu)}(u) \, L_1^{(\nu)}(v) \, L_3^{(\mu)}(u) \, L_3^{(\nu)}(v)$$
$$+ F_{2a} W_2^{(\mu)^2} W_2^{(\nu)^2} + F_{2b} \, L_2^{(\mu)^2}(u) \, L_2^{(\nu)^2}(v)$$
$$+ \left[F_{1a} W_2^{(\mu)} W_2^{(\nu)} + F_{1b} \, L_2^{(\mu)}(u) \, L_2^{(\nu)}(v) \right] L_1^{(\mu)^2}(u) \, L_1^{(\nu)^2}(v) , \qquad \text{(D-11)}$$

where the F-constants are given by (D-5). This is the final result, where the necessary functions are defined in (75).

Programs for linear array results, (42) and (D-4), and for planar array results, (76) and (D-11), are presented below in tables D-1 and D-2, respectively. For the linear array case in table D-1, the inputs required of the user are

H in line 20, the number of elements on one-half of the linear array
Q in line 40, the probability of element failure
Sigmar (σ_r) in line 50, the standard deviation of the relative error of the weights
Sigmap (σ_p) in line 60, the standard deviation of the phase perturbation in radians
$\{w_k\}_1^H$ in lines 80-90, the H weights on one half of the array.

Alternative weight structures are available in lines 220-240, if desired.

For the planar array in table D-2, the inputs required of the user are

Phi__ℓ (ϕ_ℓ) in line 20, the polar look angle in radians
Theta__ℓ (θ_ℓ) in line 30, the azimuthal look angle in radians
Slice = 1 or 2 in line 40, depending on polar or azimuthal slice
Theta__a (θ_a) in line 60, the azimuthal arrival angle or Phi__a (ϕ_a) in line 80, the polar arrival angle
H_x, H_y in lines 90 and 110, the number of elements in the halves of the array in the x and y coordinates
Dx__lam, Dy__lam in lines 130 and 140, the ratio of x and y array spacing to the arrival wavelength
$\{w_p^{(x)}\}_1^{H_x}$, $\{w_q^{(y)}\}_1^{H_y}$ in lines 150-200, the weight structures in the x and y coordinates
Q in line 210, the probability of element failure
Sigmar (σ_r) in line 220, the standard deviation of the relative error of the weights
Sigmap (σ_p) in line 230, the standard deviation of the phase perturbations in radians.

Table D-1. Program for Equispaced Linear Array

```
10            ! RANDOM BEAMFORMER; EQUI-SPACED LINE ARRAY
20      H=10  ! NUMBER OF ELEMENTS ON ONE HALF OF ARRAY
30            ! SYMMETRIC WEIGHTS; TOTAL NUMBER OF ELEMENTS = 2H
40      Q=.001            !PROBABILITY OF ELEMENT FAILURE
50      Sigmar=.01        !STANDARD DEVIATION OF RELATIVE ERROR OF WEIGHT
60      Sigmap=.01        !STANDARD DEVIATION OF PHASE (RADIANS)
70            ! DOLPH-CHEBYSHEV WEIGHTS FOR -30 DB SIDELOBES:
80      DATA 2.405798938,2.333866168,2.195114975,1.999277697,1.759768704
90      DATA 1.492415162,1.213997044,.9407573274,.6870419302,.7833503571
100     PRINTER IS 0
110     PRINT "EQUI-SPACED LINE ARRAY             TOTAL NUMBER OF ELEMENTS =";2*H
120     PRINT "Probability of element failure =";Q
130     PRINT "Standard deviation of relative error of weight =";Sigmar
140     PPINT "Standard deviation of phase (in radians) =";Sigmap
150     DIM Mu(1:4),Nu(1:4),Gamma(1:2),W(1:50)
160     DIM Idealpower(0:256),Averagepower(0:256),Sigmapower(0:256)
170     Num=256   ! NUMBER OF PLOTTED POINTS <= 256
180     REDIM W(1:H)
190     READ W(*)
200     T=0
210     FOR K=1 TO H
220 !   W(K)=.54+.46*COS(PI*(K-.5)/H)      ! HANNING WEIGHTS
230 !   W(K)=1+COS(PI*(K-.5)/H)            ! HANNING WEIGHTS
240 !   W(K)=1                             ! UNIFORM WEIGHTS
250     T=T+W(K)
260     NEXT K
270     W2=W4=0
280     FOR K=1 TO H
290     W(K)=W(K)/T
300     P=W(K)^2
310     W2=W2+P
320     W4=W4+P^2
330     NEXT K
340     W2=W2/2
350     PRINT "Effective number of elements =";1/W2
360     PRINT LIN(1)
370     W4=W4/8
380     Vr=Sigmar^2
390     Vp=Sigmap^2
400     Mu(1)=Mu(2)=Mu(3)=Mu(4)=1-Q
410     Nu(1)=1
420     Nu(2)=1+Vr
430     Nu(3)=1+3*Vr
440     Nu(4)=1+6*Vr+3*Vr^2
450     Gamma(1)=EXP(-.5*Vp)
460     Gamma(2)=EXP(-2*Vp)
470     C1=Mu(1)*Nu(1)*Gamma(1)
480     C2m=Mu(2)*Nu(2)
490     C2r=Mu(2)*Nu(2)*Gamma(2)
500     C3=Mu(3)*Nu(3)*Gamma(1)
510     C4=Mu(4)*Nu(4)
520     C12=C1^2
530     Co=2*C2m+C2r
540     F4=C4-4*C1*C3-2*C2m^2-C2r^2+4*C12*Co-6*C12^2
550     F3=4*C1*(C3-C1*Co+2*C12*C1)
560     Aa=C2m-C12
570     Ab=C2r-C12
580     F2a=Aa^2
590     F2b=Ab^2
600     F1a=2*C12*Aa
610     F1b=2*C12*Ab
620     Com=F4*W4+F2a*W2^2
630     F1aw2=F1a*W2
640     Var_volt=Aa*W2
650     P=PI/Num
660     FOR I=0 TO Num
```

Table D-1. (Cont'd) Program for Equispaced Linear Array

```
670   U=I*P
680   L1=L2=L3=0
690   FOR K=1 TO H
700   S=W(K)
710   T=S*COS((K-.5)*U)
720   S=S^2
730   L1=L1+T
740   L2=L2+2*T^2-S
750   L3=L3+S*T
760   NEXT K
770   L2=L2/2
780   L3=L3/4
790   Idealpower(I)=L1^2
800   Averagepower(I)=C12*Idealpower(I)+Var_volt
810   Var_power=Com+F2b*L2^2+F3*L1*L3+(F1aw2+F1b*L2)*L1^2
820   Sigmapower(I)=SQR(Var_power)
830   NEXT I
840   PLOTTER IS "GRAPHICS"
850   GRAPHICS
860   SCALE 0,Num,-7,0
870   GRID Num/10,1
880   PENUP
890   FOR I=0 TO Num
900   A=Idealpower(I)
910   IF A>1E-7 THEN 940
920   PENUP
930   GOTO 950
940   PLOT I,LGT(A)
950   NEXT I
960   PENUP
970   FOR I=0 TO Num
980   PLOT I,LGT(Averagepower(I))
990   NEXT I
1000  PENUP
1010  FOR I=1 TO Num
1020  PLOT I,LGT(Averagepower(I)+Sigmapower(I))
1030  NEXT I
1040  PENUP
1050  FOR I=0 TO Num
1060  PLOT I,LGT(Averagepower(I)+2*Sigmapower(I))
1070  NEXT I
1080  PENUP
1090  PAUSE
1100  DUMP GRAPHICS
1110  PRINT LIN(8)
1120  PRINTER IS 16
1130  END
```

Table D-2. Program for Equispaced Planar Array

```
10          ! RANDOM BEAMFORMER; EQUI-SPACED PLANAR ARRAY
20     INPUT "Polar Look Angle (in radians) = ?",Phi_1
30     INPUT "Azimuthal Look Angle (in radians) = ?",Theta_1
40     INPUT "    For Polar Slice in Arrival Angle, enter 1
            For Azimuthal Slice in Arrival Angle, enter 2",Slice
50     ON Slice GOTO 60,90
60     INPUT "Fixed Azimuthal Arrival Angle (in radians) = ?",Theta_a
70     GOTO 90
90     INPUT "Fixed Polar Arrival Angle (in radians) = ?",Phi_a
90     Hx=10   ! NUMBER OF ELEMENTS ON ONE HALF OF ARRAY, IN X-DIRECTION
100          ! SYMMETRIC WEIGHTS; TOTAL NUMBER OF ELEMENTS IN X = 2 Hx
110    Hy=9    ! NUMBER OF ELEMENTS ON ONE HALF OF ARRAY, IN Y-DIRECTION
120          ! SYMMETRIC WEIGHTS; TOTAL NUMBER OF ELEMENTS IN Y = 2 Hy
130    Dx_lam=.5     ! X-SPACING/WAVELENGTH
140    Dy_lam=.5     ! Y-SPACING/WAVELENGTH
150          ! 10 DOLPH-CHEBYSHEV WEIGHTS FOR -30 DB SIDELOBES:
160    DATA 2.405798938,2.333866168,2.195114975,1.999277697,1.759768704
170    DATA 1.492415162,1.213997044,.9407573274,.6870419302,.7833503571
180          ! 9 DOLPH-CHEBYSHEV WEIGHTS FOR -30 DB SIDELOBES:
190    DATA 2.681154180,2.581677308,2.391494649,2.127093009,1.810713745
200    DATA 1.467761548,1.123942052,.802515104,.825036637
210    Q=.01                IPROBABILITY OF ELEMENT FAILURE
220    Sigmar=.02           ISTANDARD DEVIATION OF RELATIVE ERROR OF WEIGHT
230    Sigmap=.1            ISTANDARD DEVIATION OF PHASE (RADIANS)
240    ! PRINTER IS 0
250    PRINT "Equi-spaced planar array       Total number of elements =";2*Hx;2*Hy
260    PRINT "X-spacing/wavelength =";Dx_lam,"Y-spacing/wavelength =";Dy_lam
270    PRINT "Probability of element failure =";Q
280    PRINT "Standard deviation of relative error of weight =";Sigmar
290    PRINT "Standard deviation of phase (in radians) =";Sigmap
300    DIM Mu(1:4),Nu(1:4),Gamma(1:2),Wx(1:50),Wy(1:50),U(0:256),V(0:256)
310    DIM Idealpower(0:256),Averagepower(0:256),Sigmapower(0:256)
320    Num=256    ! NUMBER OF PLOTTED POINTS <= 256
330    REDIM Wx(1:Hx),Wy(1:Hy)
340    READ Wx(*),Wy(*)
350    T=1/SUM(Wx)
360    W2x=W4x=0
370    FOR K=1 TO Hx
380    Wx(K)=Wx(K)*T
390    P=Wx(K)^2
400    W2x=W2x+P
410    W4x=W4x+P^2
420    NEXT K
430    W2x=W2x/2
440    W4x=W4x/8
450    T=1/SUM(Wy)
460    W2y=W4y=0
.70    FOR K=1 TO Hy
480    Wy(K)=Wy(K)*T
490    P=Wy(K)^2
500    W2y=W2y+P
510    W4y=W4y+P^2
520    NEXT K
530    W2y=W2y/2
540    W4y=W4y/8
550    PRINT "Effective numbers of elements in x,y =";1/W2x;1/W2y
560    PRINT LIN(1)
570    PRINT "Polar Look Angle =";Phi_1
580    PRINT "Azimuthal Look Angle =";Theta_1
590    ON Slice GOTO 600,630
600    PRINT "Fixed Azimuthal Arrival Angle =";Theta_a
610    PRINT "Polar Slice in Arrival Angle from 0 to PI/2"
620    GOTO 650
630    PRINT "Fixed Polar Arrival Angle =";Phi_a
640    PRINT "Azimuthal Slice in Arrival Angle from 0 to 2*PI"
650    Vr=Sigmar^2
```

Table D-2. (Cont'd) Program for Equispaced Planar Array

```
660   Vp=Sigmap^2
670   Mu(1)=Mu(2)=Mu(3)=Mu(4)=1-Q
680   Nu(1)=1
690   Nu(2)=1+Vr
700   Nu(3)=1+3*Vr
710   Nu(4)=1+6*Vr+3*Vr^2
720   Gamma(1)=EXP(-.5*Vp)
730   Gamma(2)=EXP(-2*Vp)
740   C1=Mu(1)*Nu(1)*Gamma(1)
750   C2m=Mu(2)*Nu(2)
760   C2r=Mu(2)*Nu(2)*Gamma(2)
770   C3=Mu(3)*Nu(3)*Gamma(1)
780   C4=Mu(4)*Nu(4)
790   C12=C1^2
800   Co=2*C2m+C2r
810   F4=C4-4*C1*C3-2*C2m^2-C2r^2+4*C12*Co-6*C12^2
820   F3=4*C1*(C3-C1*Co+2*C12*C1)
830   Aa=C2m-C12
840   Ab=C2r-C12
850   F2a=Aa^2
860   F2b=Ab^2
870   F1a=2*C12*Aa
880   F1b=2*C12*Ab
890   Com=F4*W4x*W4y+F2a*(W2x*W2y)^2
900   F1aw2=F1a*W2x*W2y
910   Var_volt=Aa*W2x*W2y
920   S1=SIN(Phi_1)
930   Ax=2*PI*Dx_lam
940   Ay=2*PI*Dy_lam
950   U1=Ax*S1*COS(Theta_1)
960   V1=Ay*S1*SIN(Theta_1)
970   ON Slice GOTO Polar,Azimuthal
980 Polar:  Ua=Ax*COS(Theta_a)
990   Va=Ay*SIN(Theta_a)
1000  P=.5*PI/Num
1010  FOR I=0 TO Num
1020  Phi_a=P*I
1030  Sa=SIN(Phi_a)
1040  U(I)=U1-Ua*Sa
1050  V(I)=V1-Va*Sa
1060  NEXT I
1070  GOTO Common
1080 Azimuthal: Sa=SIN(Phi_a)
1090  Ua=Ax*Sa
1100  Va=Ay*Sa
1110  P=2*PI/Num
1120  FOR I=0 TO Num
1130  Theta_a=P*I
1140  U(I)=U1-Ua*COS(Theta_a)
1150  V(I)=V1-Va*SIN(Theta_a)
1160  NEXT I
1170 Common:FOR I=0 TO Num
1180  L1u=L2u=L3u=0
1190  Ui=U(I)
1200  FOR K=1 TO Hx
1210  S=Wx(K)
1220  T=S*COS((K-.5)*Ui)
1230  S=S^2
1240  L1u=L1u+T
1250  L2u=L2u+2*T^2-S
1260  L3u=L3u+S*T
1270  NEXT K
1280  L2u=L2u^2
1290  L3u=L3u/4
1300  L1v=L2v=L3v=0
1310  Vi=V(I)
```

Table D-2. (Cont'd) Program for Equispaced Planar Array

```
1320 FOR K=1 TO Hy
1330 S=Wy(K)
1340 T=S*COS((K-.5)*Vi)
1350 S=S^2
1360 L1u=L1u+T
1370 L2u=L2u+2*T^2-S
1380 L3u=L3u+S*T
1390 NEXT K
1400 L2u=L2u/2
1410 L3u=L3u/4
1420 L1uv=L1u*L1v
1430 L2uv=L2u*L2v
1440 Idealpower(I)=L1uv^2
1450 Averagepower(I)=C12*Idealpower(I)+Var_colt
1460 Var_power=Com+F3*L1uv*L3u*L3v+F2b*L2uv^2+(F1au+F1b*L2uv)*L1uv^2
1470 Sigmapower(I)=SQR(Var_power)
1480 NEXT I
1490 PLOTTER IS "GRAPHICS"
1500 GRAPHICS
1510 SCALE 0,Num,-7,0
1520 GRID Num/10,1
1530 PENUP
1540 FOR I=0 TO Num
1550 A=Idealpower(I)
1560 IF A>1E-7 THEN 1590
1570 PENUP
1580 GOTO 1600
1590 PLOT I,LGT(A)
1600 NEXT I
1610 PENUP
1620 FOR I=0 TO Num
1630 PLOT I,LGT(Averagepower(I))
1640 NEXT I
1650 PENUP
1660 FOR I=1 TO Num
1670 PLOT I,LGT(Averagepower(I)+Sigmapower(I))
1680 NEXT I
1690 PENUP
1700 FOR I=0 TO Num
1710 PLOT I,LGT(Averagepower(I)+2*Sigmapower(I))
1720 NEXT I
1730 PENUP
1740 PAUSE
1750 DUMP GRAPHICS
1760 PRINT LIN(8)
1770 PRINTER IS 16
1780 END
```

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