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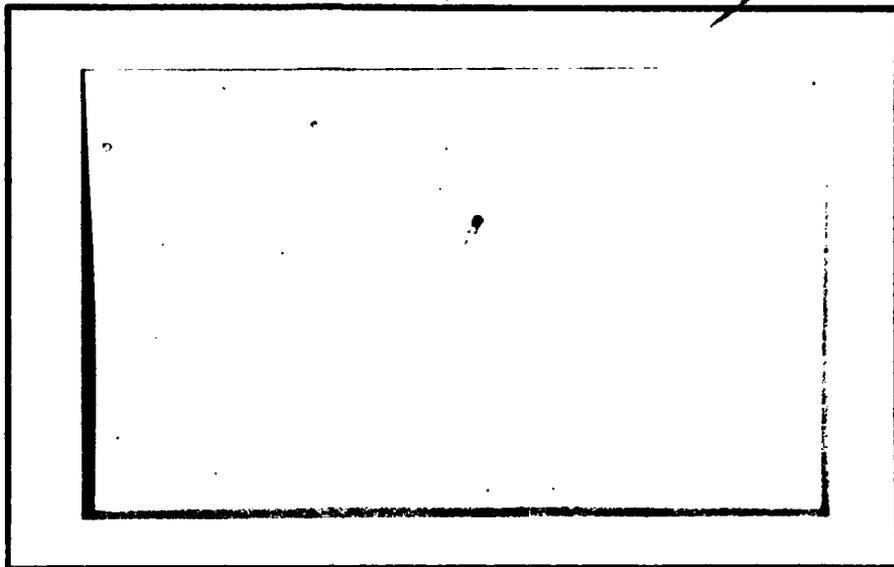


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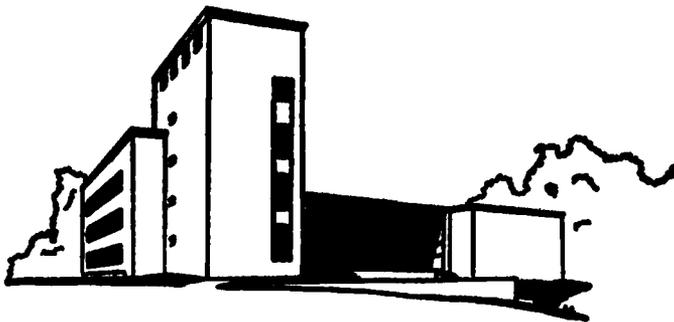
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6 NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL OF QUASILINEAR PARTIAL DIFFERENTIAL SYSTEMS.

by

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NECESSARY AND SUFFICIENT CONDITIONS
FOR OPTIMAL CONTROL OF QUASILINEAR
PARTIAL DIFFERENTIAL SYSTEMS

by

N.A. Derzko, S.P. Sethi, and G.L. Thompson

ABSTRACT



This paper studies the problem of optimal control of quasilinear partial differential systems. A maximum principle necessary condition is derived and shown to be sufficient when the Hamiltonian and boundary conditions satisfy a convexity condition. The derivation is made using Stokes theorem and a generalized version of Green's theorem. Management science applications are suggested to problems of cattle ranching, perishable inventories, manpower planning, and advertising.



KEY WORDS:

- Optimal control
- Distributed parameter systems
- Quasilinear differential systems
- Maximum principle

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1. Introduction

In this paper we deal with the optimal control problem

$$\max J(u) = \int_{\Omega} F(x, u, t) d\omega + \int_{\partial\Omega} G(x, t) \cdot d\sigma \quad (1.1)$$

subject to

$$A(t)x = f(x, u, t) \quad (1.2)$$

where $t \in \Omega \subseteq \mathbb{R}^n$, $u(t) \in K \subset \mathbb{R}^m$, $A(t)$ is a linear partial differential operator and $\partial\Omega$ denotes the boundary of Ω as usual. We assume x is scalar valued for simplicity in exposition. Generalization to vector valued x is possible. Suitable boundary conditions for (1.2) will be introduced later. We call (1.2) quasilinear because it is linear in the derivatives of x (but x itself can appear nonlinearly). We derive the maximum principle for this problem using methods which treat all the components of t on an equal basis, thereby allowing more generality in certain directions than the usual approach to distributed parameter systems. In particular Ω does not need to be a cylinder set. Furthermore this approach uses Stokes Theorem and a generalized version of Green's theorem. It leads to a clean concise methodology which can take advantage of existence theory when available, and yields useful formulas even when existence theory is not available. Finally, by making a convexity

assumption on the Hamiltonian and boundary conditions as in [2, 7, 8] we obtain a sufficiency result as well.

There are numerous classical optimal control problems currently handled as distributed parameter systems - optimal heating of solids, efficient extraction of crude oil from oil fields, optimal drying processes, and optimally controlled large scale chemical reactions, to name a few. Governing differential equations and typical objective functions for such problems are described in [4]. The results of this paper will allow solution for more general geometries and time profiles than hitherto possible.

In addition this theory is applicable to several new areas in management science, for example: cattle ranching [5]; perishable inventories [3]; manpower planning [Gaimon-Thompson, unpublished]; and advertising problems [Seidman-Sethi-Derzko, unpublished]. Many other applications are possible given the generality of the partial differential operator and objective functional used in this work.

Preliminaries

To make this paper more generally accessible we review some concepts and results of general geometric integration theory [9].

In this paper we shall deal with a *standard domain* $\Omega \subset \mathbb{R}^n$, i.e. one to which the general Stokes theorem applies; see [9]. The definition of standard domain includes that Ω must be connected. We impose the additional restriction that the boundary $\partial\Omega$ is made up of a finite number of pieces, each of which is a differentiable image of a connected open set in \mathbb{R}^{n-1} . This requirement provides us with a parametrization of the hypersurface $\partial\Omega$, useful for evaluating integrals on $\partial\Omega$. Of course further generalization of Ω is possible but will not be considered in this paper.

We denote a point in Ω by $t = (t_1, \dots, t_n)$. The volume integral over Ω of a scalar valued function $\phi(t)$ will be written $\int_{\Omega} \phi(t) d\omega$, where $d\omega = dt_1 dt_2 \dots dt_n$. The surface integrals we consider are generalizations of the common two dimensional formula $\iint_S \Psi \cdot \nu d\sigma$ where ν is the usual outward normal. In the $n-1$ dimensional case hypersurface integrals (called simply surface integrals henceforth) are written

$$\int_{\partial\Omega} \Psi \cdot d\sigma \quad (2.1)$$

Here $\Psi \cdot d\sigma$ is an $n-1$ differential form on $\partial\Omega$. More

specifically,

$$\Psi \cdot d\sigma = \sum_{j=1}^n \Psi_j(t) d\sigma_j \quad (2.2)$$

where each $\Psi_j(t)$ is a scalar valued function on $\partial\Omega$,
 $d\sigma = (d\sigma_1, d\sigma_2, \dots, d\sigma_n)$ and $d\sigma_j$ is the symbol

$$d\sigma_j = (-1)^j dt_1 dt_2 \dots dt_{j-1} dt_{j+1} \dots dt_n \quad (2.3)$$

Evaluation of (2.1) requires us to choose a parametrization of $\partial\Omega$ (or a piece of $\partial\Omega$) say $t(s)$ where $s \in S \subset \mathbb{R}^{n-1}$. Then the term $\int_{\partial\Omega} \Psi_j d\sigma_j$ is evaluated by the formula

$$\int_{\partial\Omega} \Psi_j d\sigma_j = (-1)^j \int_S \Psi_j(t(s)) v_j(s) ds_1 \dots ds_{n-1}$$

where

$$v_j(s) = \frac{\partial(t_1 \dots t_{j-1} t_{j+1} \dots t_n)}{\partial(s_1 \dots s_{n-1})} \quad (2.4)$$

is a Jacobian. The order of appearance of the functions $t_j(s)$ in (2.4) is determined by the orientation of $\partial\Omega$ as a manifold. Again we refer the reader to [W] for details.

Stokes Formula is

$$\int_{\Omega} d\phi = \int_{\partial\Omega} \phi \quad (2.5)$$

where $\phi = \Psi \cdot d\sigma$ is a differential form on $\partial\Omega$. Corresponding

to $\phi = \Psi \cdot d\sigma$ the derivative $d\phi$ is defined to be so that

$$d\phi = \left(\sum_{j=1}^n \partial_j \Psi_j(t) \right) d\omega. \quad (2.6)$$

The symbol $\partial_j \equiv \frac{\partial}{\partial t_j}$ is the partial derivative with respect to t_j . Thus we are dealing with the n dimensional divergence theorem which, using (2.2) and (2.5) and (2.6), can be written as

$$\int_{\Omega} \left(\sum_{j=1}^n \partial_j \Psi_j(t) \right) d\omega = \int_{\partial\Omega} \sum_{j=1}^n \Psi_j(t) d\sigma_j. \quad (2.7)$$

In the sequel we shall not use this theorem directly, but rather the following derivate of it resembling the two dimensional Green's theorem.

Generalized Green's Theorem

Let $x(t)$ and $y(t)$ be sufficiently differentiable scalar functions on \mathbb{R}^n and $\Omega, \partial\Omega$ be as in (2.7). Then it follows from (2.7) that

$$\begin{aligned} & \int_{\Omega} \left[(\partial_{k_1} \dots \partial_{k_r} x) y + (-1)^{r-1} x (\partial_{k_1} \dots \partial_{k_r} y) \right] d\omega \\ &= \int_{\partial\Omega} \sum_{i=1}^r (-1)^{i-1} (\partial_{k_1} \dots \partial_{k_{i-1}} y) (\partial_{k_{i+1}} \dots \partial_{k_r} x) d\sigma_{k_i} \end{aligned} \quad (2.8)$$

where

$$\partial_{k_1 k_2 \dots k_r} = \partial_{k_1} \partial_{k_2} \dots \partial_{k_r}. \quad (2.9)$$

Note that the first term corresponding to $i=1$ in the integrand on the right side is

$$y(\partial_{k_2 \dots k_r} x) d\sigma_{k_1}.$$

Proof outline

We have from (2.6)

$$d[xy d\sigma_j] = \partial_j(xy) d\sigma_j = [(\partial_j x)y + x(\partial_j y)] d\omega \quad (2.10)$$

We begin with the identity

$$\begin{aligned} & (\partial_{k_1 \dots k_r} x)y + (-1)^{r-1} x(\partial_{k_1 \dots k_r} y) \\ &= (\partial_{k_1} \partial_{k_2 \dots k_r} x)y + (\partial_{k_2 \dots k_r} x)(\partial_{k_1} y) \\ & \quad - (\partial_{k_2 \dots k_r} x)(\partial_{k_1} y) - (\partial_{k_3 \dots k_r} x)(\partial_{k_1 k_2} y) \\ & \quad + (\partial_{k_3 \dots k_r} x)(\partial_{k_1 k_2} y) + (\partial_{k_4 \dots k_r} x)(\partial_{k_1 k_2 k_3} y) \\ & \quad + (-1)^{r-1} [(\partial_{k_r} x)(\partial_{k_1 \dots k_{r-1}} y) + x(\partial_{k_1 \dots k_r} y)] \end{aligned} \quad (2.11)$$

If we multiply (2.1) by $d\omega$ and apply (2.10) to each pair of

terms on the right we obtain

$$\begin{aligned} & \left[(\partial_{k_1 \dots k_r} x) y + (-1)^{r-1} x (\partial_{k_1 \dots k_r} y) \right] d\omega \\ & = d \left[\sum_{j=1}^r (-1)^{j-1} (\partial_{k_1 \dots k_{j-1}} y) (\partial_{k_{j+1} \dots k_r} x) d\sigma_{k_j} \right] \end{aligned} \quad (2.12)$$

which when substituted in (2.5) or (2.7) yields the theorem.

Next we introduce linear partial differential operators, their adjoints and boundary forms. For this it is convenient to mention the standard notation for a high order partial derivative. If $J = (j_1 \dots j_n)$ is an n -vector of nonnegative integers we let

$$\partial^J = \partial_1^{j_1} \partial_2^{j_2} \dots \partial_n^{j_n}. \quad (2.13)$$

It is understood that ∂_k^0 means no derivative. The order $|J|$ of the derivative ∂^J is defined as $j_1 + j_2 + \dots + j_n$. A linear partial differential operator A is given by

$$Ax(t) = \sum_J a_J(t) \partial^J x(t) \quad (2.14)$$

where J ranges over a finite set S . The order of A is $\max_{J \in S} |J|$.

Hilbert spaces constitute a useful tool for dealing with linear partial differential operators. We let $L^2(\Omega)$ denote the usual Hilbert space of square integrable functions

on Ω . If $f(t)$ and $g(t)$ are real, their L^2 inner product is denoted

$$(f, g) = \int_{\Omega} f(t)g(t)dw. \quad (2.15)$$

For our subsequent work we derive the final specialization of Stokes Theorem. Suppose in (2.8) we let

$$y(t) = a_J(t)\lambda(t) \quad (2.16)$$

and assume that $a_J(t)$ is sufficiently smooth to present no problems in taking the required partial derivatives.

Substituting (2.16) into (2.8) we obtain

$$\begin{aligned} & \int_{\Omega} \left[\lambda a_J (\partial^J x) + (-1)^{|J|-1} \partial^J (a_J \lambda) x \right] dw \\ &= \int_{\partial\Omega} \left[\sum_{j=1}^{|J|} (-1)^{i-1} (\partial_{k_j \dots k_{j-1}} a_J \lambda) (\partial_{k_{j+1} \dots k_{|J|}} x) d\sigma_{k_j} \right] \\ &\triangleq \int_{\partial\Omega} \Gamma_J(a_J \lambda, x) \cdot d\sigma \end{aligned} \quad (2.17)$$

where the last expression defines Γ_J .

Summation over J yields the form of Stokes Theorem to be used in the next section,

$$\begin{aligned} & \int_{\Omega} [\lambda(Ax) - (A'\lambda)x] dw \\ &= \int_{\partial\Omega} \Gamma_A(\lambda, x) \cdot d\sigma \end{aligned} \quad (2.18)$$

where the formal adjoint operator A' is defined by

$$A'\lambda = \sum_J (-1)^{|J|} \partial^J (a_J \lambda) \quad (2.19)$$

and the boundary form Γ_A is given by

$$\Gamma_A(\lambda, x) = \sum_J \Gamma_J(a_J \lambda, x). \quad (2.20)$$

We note that Γ_A is bilinear in (λ, x) and each component of Γ_A is of the form

$$\sum_{I_1, I_2, I_3} \partial^{I_1} a_J \partial^{I_2} \lambda \partial^{I_3} x$$

where $|I_2| + |I_3| < \text{order}(A)$.

Finally, we shall need the following lemma in proving our sufficiency result in section 5. Special cases of the result are well known as envelope theorems.

Lemma. Let $h(x, y)$, with $x \in R$, $y \in S \subset R^k$, and S compact be continuous in (x, y) and continuously differentiable in x for each fixed y . Define

$$h^0(x) = \max\{h(x, y) : y \in S\} = h(x, y^0(x)), \quad (2.21)$$

and assume that $h^0(x)$ is also continuously differentiable.

Then

$$h^{0'}(x) = \frac{\partial h}{\partial x}(x, y^0(x)), \quad x \in S. \quad (2.22)$$

Proof:

From the definition of derivative we have

$$\begin{aligned} h^{0'}(x) &= \frac{\partial h}{\partial x}(x, y^0(x)) \\ &= \lim_{x' \rightarrow x} \frac{h(x', y^0(x')) - h(x', y^0(x))}{x' - x}. \end{aligned} \quad (2.23)$$

If we let L denote the limit and DQ' denote the difference quotient in (2.23), our uniformity assumptions enable us to conclude that given $\epsilon > 0$, there exists $\delta > 0$, such that $|x' - x| < \delta$ implies $|DQ' - L| < \epsilon$. Now (2.21) implies

$$h(x', y^0(x')) - h(x', y^0(x)) \geq 0$$

for all x', x . Consequently if we choose x'' such that $x'' - x = -(x' - x)$ and let DQ'' denote the corresponding difference quotient, then $|DQ'' - L| < \epsilon$ and DQ', DQ'' have opposite sign. It follows that

$$|DQ'' - DQ'| \leq |DQ'' - L| + |L - DQ'| < 2\epsilon$$

implying both

$$DQ'' < 2\epsilon \quad \text{and} \quad DQ' < 2\epsilon.$$

We conclude

$$L < |L - DQ'| + |DQ'| = 3\varepsilon$$

for every $\varepsilon > 0$ and therefore $L = 0$, completing the proof.

3. The Optimal Control Problem

In this section we give a precise statement of the optimal control problem (1.1) and (1.2) using the notation of section 2.

In (1.2) let A be a linear partial differential operator defined in the domain $\Omega \subset \mathbb{R}^n$ as specified in section 2. That is, if $x \in C^\infty(\Omega)$ then

$$Ax(t) = \sum_J a_J(t) \partial^J x(t). \quad (3.1)$$

for all $t \in \Omega$ where we assume $a_J \in C^{|\mathcal{J}|}(\Omega)$, the space of functions on Ω continuously differentiable up to order $|\mathcal{J}|$. The set of admissible controls $u(t)$ is specified by requiring that $u(\cdot)$ be measurable and map $\Omega \rightarrow K \subset \mathbb{R}^m$. We assume that the function $f(x, u, t)$ is continuously differentiable on $\mathbb{R} \times K \times \Omega$. To complete the specification of (1.2) we introduce the boundary conditions

$$B_k x(t) = g_k(t) \quad 1 \leq k \leq r \quad t \in \partial\Omega \quad (3.2)$$

where each B_k is a linear partial differential operator of order less than A and defined for $t \in \partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω as is customary. It is often convenient to use the vector notation $Bx(t) = g(t)$ instead of (3.2). Finally, we assume that the boundary value problem defined by (1.2) and (3.2) has a unique solution for each admissible control $u(\cdot)$. In the objective function (1.1) we assume that the scalar function F and the \mathbb{R}^n valued function G are continuously differentiable on their domains, and x is the solution of (1.1) and (3.2) for each admissible u .

We have now completed the specification of the optimal control problem. To summarise, we restate the problem:

$$\max_u J(u) = \int_{\Omega} F(x, u, t) d\omega + \int_{\partial\Omega} G(x, t) \cdot d\sigma \quad (3.3)$$

subject to

$$A(t)x = f(x, u, t) \quad (3.4)$$

with boundary condition

$$B(t)x = g(t) \quad \text{on } \partial\Omega \quad (3.5)$$

with admissible controls

$$u : \Omega \rightarrow K \subset \mathbb{R}^n. \quad (3.6)$$

Necessary conditions for (3.3) - (3.6) are derived in the next section.

4. Derivation of Necessary Conditions

In this section A' denotes the adjoint operator to A and $\Gamma_A(\cdot, \cdot)$ is the corresponding boundary form as defined in section 2. The Hamiltonian is defined by the function

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t) \quad (4.1)$$

Analogously to the adjoint equations in optimal control theory we shall find that the adjoint function $\lambda(t)$, $t \in \Omega$ satisfies

$$A'\lambda(t) = \partial_x H(x(t), u(t), \lambda(t), t) \quad (4.2)$$

where $u(t)$ and $x(t)$, $t \in \Omega$ are the optimal control and

corresponding state functions. In addition, λ satisfies the adjoint boundary condition given by

$$\partial_x G(\xi, t) \xi = \Gamma_A(\lambda, \xi), \quad t \in \partial\Omega \quad (4.3)$$

for all functions $\xi(t)$ defined in a neighbourhood of $\partial\Omega$ and satisfying

$$B\xi(t) = 0, \quad t \in \partial\Omega \quad (4.3')$$

where B is defined in (3.5). Note that both $\partial_x G$ and Γ_A are n vectors.

The actual derivation proceeds as follows. We let J_a denote the augmented objective function

$$\begin{aligned} J_a(u) &= J(u) + \int_{\Omega} \lambda(t) [f(x(t), u(t), t) - Ax(t)] d\omega \\ &= \int_{\Omega} H d\omega + \int_{\partial\Omega} G \cdot d\sigma - (\lambda, Ax) \end{aligned} \quad (4.4)$$

where we note that the derivation of the second expression uses (3.3) and (4.1). If x satisfies (3.4) and (3.5) then, of course, $J_a = J$. We assume also that λ satisfies (4.2) and (4.3). Then using (2.15) and the generalized Green's theorem (2.18), we rewrite (4.4) as

$$\begin{aligned} J_a &= \int_{\Omega} H d\omega + \int_{\partial\Omega} (G - \Gamma_A) \cdot d\sigma - (A'\lambda, x) \\ &= \int_{\Omega} [H - (A'\lambda)x] d\omega + \int_{\partial\Omega} (G - \Gamma_A) d\sigma. \end{aligned} \quad (4.5)$$

Finally consider the first order variation δJ_a of J_a produced by a change δu in u , uniformly small for each t . For each t , we shall denote the first order change in H due to a change δx in x by $\partial_x H \delta x$ (u held constant), and the change in H due to the change in u by ΔH (x held constant). Likewise $\partial_x G \delta x$ denotes the first order part of the change in G due to a change in x . Then

$$\begin{aligned} \delta J_a &= \int_{\Omega} [(\partial_x H - A' \lambda) \delta x + \Delta H] d\omega \\ &+ \int_{\partial \Omega} [\partial_x G(x(t), t) \delta x(t) - \Gamma_A(\lambda(t), \delta x(t))] d\sigma \\ &= \int_{\Omega} \Delta H d\omega \end{aligned} \quad (4.6)$$

after using (3.4), (3.5), (4.2) and (4.3). Suppose now that u is an optimal controller and \bar{u} differs from u only within a small neighbourhood of the given point t . Then $\delta J_a \leq 0$ if u corresponds to a maximum of J and standard techniques [4] can be used to show

$$\Delta H(x(t), u(t), \lambda(t), t) \leq 0. \quad (4.7)$$

This implies that if $u(t)$ is an optimal control function and $x(t), \lambda(t)$ are corresponding state and adjoint functions then (4.7) implies

$$H(x, u, \lambda, t) = \max_{\bar{u}} H(x, \bar{u}, \lambda, t) \quad (4.8)$$

which is the maximum principle. To summarise, the necessary optimality conditions are (3.4), (3.5), (3.6), (4.1), (4.2), (4.3), (4.3'), and (4.8).

5. Sufficiency Conditions

In this section we derive sufficiency conditions which generalize sufficiency conditions in ordinary control problems [2, 7, 8, etc.]. The assumptions required to prove these results quite often hold in economics problems.

The key assumption is the concavity of the derived Hamiltonian H^0 , where

$$H^0(x, \lambda, t) = \max_u H(x, u, \lambda, t). \quad (5.1)$$

Specifically we assume that H^0 is concave as a function of x , for any given (λ, t) , that is

$$H^0(x, \lambda, t) - H^0(x^*, \lambda, t) \leq \partial_x H^0(x^*, \lambda, t) (x - x^*). \quad (5.2)$$

In fact, as we shall see shortly, we only need this inequality when x^* is the optimum trajectory, but since such a trajectory is not normally known at the outset, a concavity hypothesis is more useful. We assume also that the function G is *concave-like*, that is

$$G(x^*, t) - G(x, t) \leq \partial_x G(x - x^*, t) (x^* - x) \quad (5.3)$$

for each t, x, x^* . (5.3) holds trivially if $G(x,t)$ is linear in x . Combining (5.1) and (5.2) we obtain

$$H(x,u,\lambda,t) \leq H^0(x,\lambda,t) \leq H^0(x^*,\lambda,t) + \partial_x H^0(x^*,\lambda,t)(x-x^*). \quad (5.5)$$

Under the assumption that both H and H^0 are continuously differentiable functions of x we have from the lemma in Section 2

$$\partial_x H^0(x,\lambda,t) = \partial_x H(x,u(x),\lambda,t) \quad (5.6)$$

where $u(x)$ is the maximising point in (5.1). We now substitute the expression for the Hamiltonian (4.1) in (5.5) and note (5.6) and (4.2) to write

$$\begin{aligned} & F(x^*,u^*,t) - F(x,u,t) \\ & \geq \lambda^* [f(x,u,t) - f(x^*,u^*,t)] \\ & \quad - \partial_x H(x^*,u^*,\lambda^*,t)(x-x^*) \\ & = \lambda^* A(x(t) - x^*(t)) - A' \lambda^*(t)(x(t) - x^*(t)) \end{aligned} \quad (5.7)$$

where x^*, u^*, λ^* denote values at optimum. Integrating over Ω and adding the terms $\int_{\partial\Omega} G(x^*,t) \cdot d\sigma - \int_{\partial\Omega} G(x,t) \cdot d\sigma$ to both sides we obtain

$$\begin{aligned}
& J(u^*) - J(u) \\
& \geq (\lambda^*, A(x-x^*)) - (A'\lambda^*, x-x^*) \\
& \quad + \int_{\partial\Omega} G(x^*, t) \cdot d\sigma - \int_{\partial\Omega} G(x, t) \cdot d\sigma \\
& \geq \int_{\partial\Omega} \Gamma_A(\lambda^*, x-x^*) \cdot d\sigma \\
& \quad - \int_{\partial\Omega} \partial_x G(x-x^*, t) (x-x^*) \\
& = 0
\end{aligned}$$

where we have used (2.8), (5.3) and (4.3). Thus $J(u^*) \geq J(u)$ which completes the argument.

6. Extensions

I. This work can be extended to the state x , to be a vector. Then A becomes a matrix whose entries A_{ij} are linear partial differential operators. The formal adjoint A' is then the transposed matrix of formal adjoints $\{A'_{ji}\}$.

II. In many practical problems the adjoint boundary condition (4.3), (4.3') can be reduced to a condition of the form $B^{\wedge}\lambda(t) = 0$ where B^{\wedge} is a vector of partial differential operators. Such a simplification is very useful in simplifying the handling of the adjoint equations.

III. Extension of this paper to include boundary controls is currently under investigation.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper studies the problem of optimal control of quasilinear partial differential systems. A maximum principle necessary condition is derived and shown to be sufficient when the Hamiltonian and boundary conditions satisfy a convexity condition. The derivation is made using Stokes theorem and a generalized version of Green's theorem. Management science applications are suggested to problems of cattle ranching, perishable inventories, manpower planning, and advertising.		

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