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A QUICK METHOD FOR CALCULATING TEMPERATURES ARISING FROM EXTEND--ETC(U)
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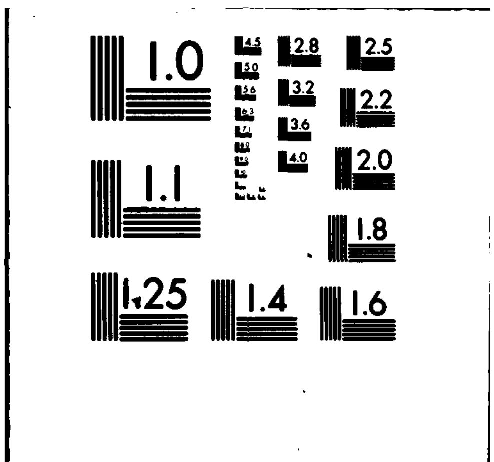

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A Quick Method for Calculating Temperatures arising from Extended Microcircuit Heat Sources

D J Dean

Recommended for issue by

A J Martin, Superintendent

Approved by

G C Ellis, Senior Superintendent

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CONTENTS

SUMMARY

In microelectronics the rapid degradation in reliability which occurs with modest rises in temperature makes it important that temperatures resulting from different geometries can be simply evaluated. One particular geometry which is becoming increasingly common is that of a long narrow strip heat source mounted upon the upper surface of a thin substrate which has a heat sink on its rear face.

This report presents a series summation computation method suitable for pocket calculators, by which the state of convergence (SOC), in terms of maximum possible error, is calculated at each stage in the summation, thus enabling the calculation to be terminated as soon as the required accuracy has been obtained.

1. INTRODUCTION

One of the major advantages of solid state microelectronic circuits is their inherent high reliability. This reliability deteriorates rapidly with elevated temperatures in the active devices. Thermal design, therefore, becomes more important as the technology advances and the power densities of the heat dissipating elements increase (1). Of the methods (2) that are available for the calculation of temperature distributions, numerical computer methods are generally the most useful as they can accommodate the wide range of parameters which exist in modern electronic assemblies.

There is also, however, a need for the circuit designer to be able to make, at his own desk, a quick estimate of the temperature distributions in specific regions (3). One of the techniques presented in reference (3) is the evaluation of spreading resistance from a small isolated heat source attached to a substrate cooled by conduction to a rear-mounted heat sink. This geometry was analysed by equating a fixed angle solution to that obtained by integration and superposition (4) for the heat flow into a semi-infinite body. the result being the well known "45^{o}-Model".

There is now an increasing need in both microwave and power device circuits to be able to carry out a similar analysis for long strip heat sources also mounted on rear-cooled heat sinks. This report presents a solution to this problem, suitable for pocket calculators, by which the greatest error in successive summations is calculated, thus enabling the computation to be terminated when the required accuracy has been reached.

2. PROBLEM DESCRIPTION

Many temperature distribution problems in microelectronics are of the general form of a long narrow strip heat generator located on one surface of a thin substrate which has a heat sink mounted upon its rear face. In these examples it is usually possible to assume that there is no heat loss except to the heat sink which maintains a uniform temperature over its entire surface.

3. PREVIOUS EVALUATION

Using an analogy between fringe capacitance and heat flow spreading H F Cooke has published a useful approximate solution of undefined accuracy and based upon the elliptical integrals of hyperbolic functions (5). All of Cooke's published values are compared in section 7.2 with results obtained by the technique developed in this present report.

4. PRINCIPLE OF REFLECTED HEAT SOURCES

The temperature distributions arising from most heat source configurations are readily calculated when their cooling is entirely by conduction into an infinite body. It is the introduction of boundaries which generally make the solutions complicated. Conveniently, microelectronic geometries generally have mutually normal boundaries. It has already been proposed (6), and is shown in appendix A, that such boundaries can be represented by planes of symmetry in the case of insulated boundaries, and by planes of inverted symmetry in the case of heat sunk surfaces. This enables temperature distributions to be obtained by the combination of infinite body problem calculations and the principle of superposition (4).

5. MATHEMATICAL ANALYSIS

An analysis is now presented for the evaluation of the temperatures reached on a long narrow heat dissipating element mounted on an insulating face of an infinite substrate of finite thickness and having a heat sink mounted upon the rear face. This analysis is a summary; the more detailed analysis is given in appendix B.

As the heat source is long compared to its width, the central and hottest area will experience heat flow parallel to planes which are normal to the longest dimension of the heat source, and the analysis of the maximum temperature may be treated as a two-dimensional problem.

The analysis technique consists of obtaining the temperature at the middle of a strip, mounted on a semi-infinite solid, from the expression for heat flow into an infinite solid and by the use of superposition. A further expression is then obtained for the contribution from a reflected strip at a distance s, and this expression is used to sum all such contributions to infinity.

5.1 Heat flow from a strip heat source in an infinite solid

Consider a long strip heat source of width b in an infinite solid. The maximum temperature θ will be at the centre of that strip and is shown in section B2 of appendix B to be equal to

$$\theta_1 = \frac{\dot{Q}}{2\pi k}\left[C + 1 - L_n(\tfrac{b}{2})\right],$$

where $\dot{Q}$ is the total power dissipation per unit length of the strip,

k is the thermal conductivity of the solid,

C is a constant.

5.2 Maximum temperature of strip mounted upon a semi-infinite solid

It has been shown in appendix A that mirror image sources are equivalent to an insulated boundary. Hence the placing of an equal power dissipating strip upon the original will result in the same temperature value θ_2 as would be obtained for the original strip mounted upon a semi-infinite solid having its plane surface insulated. By superposition this doubles the temperature of the original strip acting on its own, therefore

$$\theta_2 = \frac{\dot{Q}}{\pi k}\left[C + 1 - L_n\left(\frac{b}{2}\right)\right].$$

5.3 Temperature contributions from a reflected strip source at distance s

The temperature contribution θs from the reflection of a strip reflected in a line of symmetry parallel to the original strip and the line of symmetry being at a distance s/2 from the original strip is shown in section B4 of appendix B to be

$$\theta_s = \frac{\dot{Q}}{\pi k}\left[C + 1 - L_n\left(\sqrt{\frac{b^2}{4} + s^2}\right) - \frac{2s}{b}\tan^{-1}\frac{b}{2s}\right].$$

5.4 Temperature of strip mounted upon a substrate of finite thickness

Reference (6) proposed the use of negative mirror image sources to represent a heat sink at the line of inverted symmetry. We can, therefore, obtain an expression for the final temperature θ_F by superimposing all of the contributions from the original source and all of the positive and negative reflections to infinity. Section B5 of appendix B shows this to result in the expression

$$\theta_F = \frac{\dot{Q}}{\pi k}\left\{L_n\left(\sqrt{1 + w^2}\right) + w\tan^{-1}\frac{1}{w}\right.$$

$$+ \sum_{n=1}^{n=\infty}\left[L_n\left(\sqrt{\frac{(1 + (2n - 1)^2w^2)(1 + (2n + 1)^2w^2)}{(1 + (2n)^2w^2)^2}}\right)\right.$$

$$+ (2n - 1)\,w\tan^{-1}\left(\frac{1}{(2n - 1)w}\right) - 2(2n)w\tan^{-1}\left(\frac{1}{2nw}\right)$$

$$\left.\left. + (2n + 1)w\tan^{-1}\left(\frac{1}{(2n + 1)w}\right)\right]\right\},$$

where w = 4 times the ratio of substrate thickness to strip width, ie,

$$w = \frac{4t}{b},$$

6. SUMMATION TECHNIQUE

The relation given in section 5.4 can be summed over many terms but with no actual knowledge of the state of convergence (SOC), except by experience or excessive over summation. This report will now present a technique to evaluate the SOC at each step in the summation and by so doing enable the number of terms summed to be kept to a minimum and the evaluation taken to any predetermined degree of accuracy.

6.1 Underlying principle

The technique depends upon the absolute convergence of two series to their own values and the relative convergence of the individual terms of both series.

A simple graphical analysis (3) shows that in any problem having finite power input, finite dimensions and at least one heat sink boundary, the resultant temperature will also be finite. Hence any series obtained in a similar manner to the one presented in this report must be convergent, provided that at least one plane of negative symmetry exists.

By examination of the principle of multiple reflections it can be seen that the terms of the general series, which results from taking the reflected sources geometry into account, also converge upon the terms of the particular series which would be obtained if all the heat was generated at a point. As this results in a large w (= 4t/b), the particular series can be obtained from the general series by making w large. This particular series is then evaluated, and by comparison of each equivalent term, it is possible to determine the maximum possible error which would result from taking the remainder of the, already evaluated, particular series, instead of the balance of the more complex general and accurate series.

6.2 Application of the state of convergence (SOC) technique to strip heat source problems

The detailed derivations of the terms are given in appendix C.

The particular summation series which is obtained by making w large in the summation part of the expression obtained in section 5.4 is equal on being summed to Ln (2/π).

The nth term of this series is

$$P_n = L_n\left(1 - \frac{1}{(2n)^2}\right).$$

This is compared with the nth term in the general series

$$G_n = L_n\left(\sqrt{\frac{(1 + (2n - 1)^2 w^2)(1 + (2n + 1)^2 w^2)}{(1 + (2n)^2 w^2)^2}}\right)$$

$$+ (2n - 1)w \tan^{-1}\left(\frac{1}{(2n - 1)w}\right) - 2(2n)w \tan^{-1}\left(\frac{1}{2nw}\right)$$

$$+ (2n + 1)w \tan^{-1}\left(\frac{1}{(2n + 1)w}\right).$$

As the corresponding terms of the general series converge upon the terms of the particular series, the ratio of the sum of the subsequent terms will be less than the ratio of the two equivalent terms at all stages of the summation, ie,

$$\left(\left| \frac{\sum_{n=m}^{n=\infty} (G_n - P_n)}{\sum_{n=n}^{n=\infty} G_n} \right| < \left| \frac{G_n - P_n}{G_n} \right| \right).$$

Hence by comparison with the state of the summation at that time we can evaluate a maximum possible error.

6.3 Technique of summation

At the commencement of the summation the particular series is added to the non-summed part of the expression in section 5.4 to give

$$\frac{\pi k \theta_o}{Q} = L_n \left(\frac{2}{\pi}\sqrt{1 + w^2}\right) + w \tan^{-1} \left(\frac{1}{w}\right).$$

At each stage in the summation the corresponding term in the particular series is replaced by that of the general series. At the same time the ratio of these two terms multiplied by the ratio of the balance of the Ln(2/n) series to the value of the summation at that time is evaluated to give the maximum error, ie, after m summation terms,

$$\frac{\pi k}{Q} \theta_m = L_n \left(\frac{2}{\pi}\sqrt{1 + w^2}\right) + w \tan^{-1} \left(\frac{1}{w}\right) + \sum_{n=1}^{n=m} (G_n - P_n)$$

and the maximum possible error given by

$$\frac{G_m - P_m}{P_m} \cdot \frac{L_n\left(\frac{2}{\pi}\right) - \sum_{n=1}^{n=m} P_n}{\frac{\pi k}{Q} \theta_m}.$$

This is evaluated at each stage of the summation and the value of θ_m obtained when an acceptable SOC, as indicated by the maximum possible error, has been reached.

7. RESULTS

7.1 General

A wide range of calculations has been carried out for strip widths from 8×10^{-4} to 40 times the substrate thickness. Apart from the narrow width evaluations, which were used to compare with Cooke's analysis, all the summations were carried out in excess of 500 terms in order to relate the maximum possible error with the actual error. In all cases the actual error was found to be less than one-third of the calculated maximum error. Figure 1 presents the actual errors for different strip widths from the basic evaluation and after one and two summation exchanges.

7.2 Particular examples

The maximum temperatures reached by a heat source which is very wide compared to the substrate thickness is that obtained by parallel flow calculation. Figure 2 shows the ratio of the results from the heat spreading calculation to those of parallel flow calculations plotted against summation exchanges. The calculated maximum error is also shown in the same figure. After 20 exchanges the calculated maximum error is less than 1% and the actual error 0.297%. In practice such a problem would be tackled by parallel flow, but 20 exchanges take very little time on a pocket calculator and provide a stated confidence in the result.

Cooke (5) presented values of thermal resistance over the range W = 125 → 5000. Table 1 presents the ratio of Cooke's results to those of the present work, together with the indicated maximum error of the present results after one summation exchange. As w is so large the summation approximates to

$$\theta_F = \frac{\dot{Q}}{\pi k}\left(1 + L_n\left(\frac{2w}{\pi}\right)\right).$$

The error obtained in using this equation, compared with the detailed analysis, is also presented in table 1.

TABLE 1

W	Ratio $\frac{\text{Cooke}}{\text{Dean}}$	Maximum Possible Error, % (Dean)	Error in Simple Equation, cf, Full Summation, %
125	0.926	6.9×10^{-5}	3.2×10^{-4}
500	0.957	3.4×10^{-6}	1.6×10^{-5}
1000	0.958	7.7×10^{-7}	3.6×10^{-6}
5000	0.965	2.6×10^{-8}	1.3×10^{-7}

8. TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

Alumina is a commonly used substrate material in microelectronics and, together with beryllia, it undergoes a rapid decrease in thermal conductivity with increases in temperature in the range encountered in microcircuits. The principle of superposition cannot be directly applied to bodies having a temperature dependent thermal conductivity. However, results obtained from superposition analysis assuming constant thermal conductivity can readily be corrected by use of a transformation function derived from the thermal conductivity-temperature relationship (7).

9. FUTURE WORK

In principle this technique should be applicable to any problem which has a finite solution and can be represented by multiple reflections in mutually normal boundaries.

It is intended in the first instance to use the same technique to produce an expression for the heat flow spreading from small circular and rectangular heat sources and to evaluate the accuracy of the 45^{o} model over its whole range. It is then intended to evaluate the effect of multiple heat sources.

Consideration will also be given to developing pocket calculator programs.

10. DISCUSSION

A new technique for the evaluation of temperature distributions has been presented and shown to give very rapid accurate results from strip heat sources. In addition, a novel approach to the evaluation of the SOC has been developed which gives confidence at all stages of the summation.

The ability to evaluate other models and approximations is an important feature of the method. In the particular case of the strip heat source it shows that a parallel flow calculation is correct to within 1% for all strip widths greater than 5.6 times the substrate thickness and 10% for strip widths greater than 2.7 times the substrate thickness.

11. CONCLUSIONS

A new analytical expression has been obtained for the analysis of temperature distributions from a strip heat source. In addition to this analysis, a new evaluation technique has also been developed. This allows the status of the calculation to be determined by a state of convergence expression being obtained after each summation exchange. The resulting confidence in the result at each stage of the summation is, it is believed, unique in this type of solution.

APPENDIX A

PRINCIPLE OF REFLECTED HEAT SOURCES

A1. INTRODUCTION

The temperature distributions arising from most heat source configurations are readily calculated when their cooling is entirely by conduction into an infinite body. It is the introduction of boundaries which generally make the solutions complicated. Conveniently, microelectronic geometries generally have mutually normal boundaries. It has already been proposed (6) that such boundaries can be represented by planes of symmetry, within the conductive solid, in the case of insulated boundaries, and by planes of inverted symmetry in the case of heat sunk surfaces.

This approach allows solutions to be obtained by the combination of infinite body problem calculations and using the principle of superposition (4).

A2. INSULATED BOUNDARIES

There can be no heat flow normal to an insulating boundary. Hence the temperature gradient normal to such a boundary is zero. Similarly, if a surface within a body can be maintained in such a state that the temperature gradient normal to that surface is zero, that surface can represent an insulating boundary.

Consider the temperature distribution from a small spherical source in an infinite solid. The temperature at a distance s is given by

$$\theta_s = \frac{\dot{Q}}{4\pi ks} + \text{const}$$

and the gradient by

$$\left(\frac{d\theta}{dr}\right)_s = -\frac{\dot{Q}}{4k\pi s^2},$$

therefore for a source at point (a, b, c) we have

$$\theta_{x,y,z} = \frac{\dot{Q}}{4\pi k\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} + \text{const}$$

and the gradient in the x direction

$$\left(\frac{\partial\theta}{\partial x}\right)_{x,y,z} = -\frac{\dot{Q}}{4\pi k}\left(\frac{(x-a)}{\sqrt{((x-a)^2 + (y-b)^2 + (z-c)^2)^3}}\right).$$

Consider an insulated boundary in the yz plane and at a distance d in the x direction. At this boundary the gradient in the x direction must be zero. For it to be possible to represent this boundary by a plane of symmetry the superimposed gradients in the x direction from both the original source and its reflected source must sum to zero over the whole plane.

The gradients in the x direction due to the original source acting on its own are at the plane of symmetry:-

$$\left(\frac{\partial\theta}{\partial x}\right)_{((d+a),y,z)} = \frac{-\dot{Q}}{4\pi k}\left(\frac{((d + a) - a)}{\sqrt{(((d + a) - a)^2 + (y - b)^2 + (z - c)^2)^3}}\right),$$

$$= \frac{-\dot{Q}}{4\pi k}\left(\frac{d}{\sqrt{(d^2 + (y - b)^2 + (z - c)^2)^3}}\right)$$

and from the reflected source at ((a + 2d), b, c)

$$\left(\frac{\partial\theta}{\partial x}\right)_{((d+a),y,z)} = \frac{-\dot{Q}}{4\pi k}\left(\frac{(a + d) - (a + 2d)}{\sqrt{(((a + d) - (a + 2d))^2 + (y - b)^2 + (z - c)^2)^3}}\right),$$

$$= \frac{-\dot{Q}}{4\pi k}\left(\frac{-d}{\sqrt{(d^2 + (y - b)^2 + (z - c)^2)^3}}\right).$$

Hence, by superposition the gradients normal to the plane of symmetry are zero for all values of y and z. Similarly, by superposition this is true for multiple heat source configurations.

Hence it can be stated that an insulated boundary can be replaced by a plane of symmetry by the use of reflected mirror image sources.

A3. HEAT SUNK BOUNDARIES

By a similar analysis to that just used to show that an insulated boundary can be represented by a line of positive symmetry, it can also be shown, by superimposing the temperature expressions, that a line of negative symmetry will produce a plane of constant temperature. It can also be shown that, if the constants in the temperature expression are equal, the temperature of this plane will be zero and the resulting calculated temperatures are excess temperatures over the sink.

APPENDIX B

ANALYSIS OF HEAT FLOW SPREADING FROM A STRIP HEAT SOURCE

B1. INTRODUCTION

Many thermal resistance problems in microelectronics are of the general form of a long narrow strip heat generator formed on one surface of a thin substrate which has a heat sink mounted upon its rear face. It is usually possible to assume that there is no heat loss, except to the heat sink which maintains a uniform temperature over its entire surface. An analysis of such a problem is now presented.

As the heat source is long the central and hottest area will experience heat flow normal to the longest dimension of the heat source and the analysis of the maximum temperature may be treated as a two-dimensional problem.

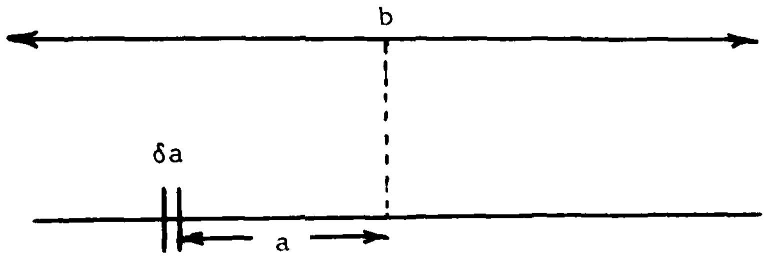
B2. HEAT FLOW FROM A LINE SOURCE IN AN INFINITE SOLID

First consider the two-dimensional flow from a small linear element (δa) in an infinite solid. The element dissipates at a rate of $\dot{q}$ per unit width and the system is taken to be in dynamic equalibrium

$$\dot{q}\delta a = - k2\pi r\frac{d\theta}{dr},$$

$$\int\frac{dr}{r} = - \int\frac{2\pi k}{\dot{q}\delta a}\, d\theta,$$

$$\text{therefore } L_n(r) = - \frac{2\pi k}{\dot{q}\delta a}\theta + C. \qquad \text{.......(B1)}$$



Consider a strip of width b in an infinite solid. The maximum temperature θ_1 will be at the centre of that strip and by superposition (4), will be the sum of the temperatures at that point due to all of the elements δa at distance r = a.

The sum of these contributions is equal to

$$\sum\theta = 2\int_0^{\frac{b}{2}} \frac{\dot{q}}{2\pi k}(C - L_n(a))da,$$

$$= \frac{\dot{q}}{\pi k}\left[aC - a(L_n(a) - 1)\right]_0^{\frac{b}{2}},$$

NOTE: $0.L_n(0) = 0$ Consider rate of convergence to zero,

$$= \frac{\dot{q}}{\pi k}\left[\frac{b}{2}C - \frac{b}{2}(L_n(\frac{b}{2}) - 1)\right],$$

$$= \frac{\dot{q}b}{2\pi k}\left[C - (L_n(\frac{b}{2}) - 1)\right].$$

$\dot{q}b = \dot{Q}$ = power dissipation per unit length of strip, therefore the total temperature

$$\theta_1 = \frac{\dot{Q}}{2\pi k}\left[C + 1 - L_n(\frac{b}{2})\right].$$

B3. MAXIMUM TEMPERATURE OF STRIP MOUNTED UPON A SEMI-INFINITE SOLID

It has been shown in appendix A that mirror image sources are equivalent to an insulated boundary. Hence the placing of an equal power dissipating strip upon the original will result in the same temperature value as the original strip mounted upon a semi-infinite solid having its plane surface insulated. By superposition this doubles the temperature of the original strip acting on its own, therefore

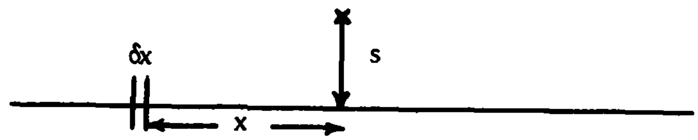
$$\theta_2 = \frac{\dot{Q}}{\pi k}\left[C + 1 - L_n(\frac{b}{2})\right]. \quad ....(B2)$$

B4. TEMPERATURE CONTRIBUTION FROM A REFLECTED STRIP SOURCE AT DISTANCE s

Consider a strip source parallel to the original source and having a separation of s from it.

From equation (B1) the temperature contribution at the centre of the original source from any element δx a distance x along the new source would be

$$\delta\theta = \frac{\dot{q}\delta x}{2\pi k}\,(C - L_n\,(\sqrt{x^2 + s^2})).$$



The total temperature contribution θ_s at the centre of the original strip by a new strip at distance s is therefore

$$\theta_s = 2\int_0^{\frac{b}{2}} \frac{\dot{q}}{2\pi k}\,(C - L_n(\sqrt{x^2 + s^2}))dx,$$

$$= \frac{\dot{q}}{\pi k}\left[Cx + x(1 - L_n(\sqrt{x^2 + s^2})) - s\,\tan^{-1}\frac{x}{s}\right]_0^{\frac{b}{2}},$$

$$= \frac{\dot{q}}{\pi k}\left[\frac{Cb}{2} + \frac{b}{2}(1 - L_n(\sqrt{\frac{b^2}{4} + s^2})) - s\,\tan^{-1}\frac{b}{2s}\right].$$

NOTE: $\int L_n(\sqrt{x^2 + s^2})dx$,

let $w = \sqrt{x^2 + s^2}$, therefore $x = \sqrt{w^2 - s^2}$,

$$dx = \frac{w\,dw}{\sqrt{w^2 - s^2}}.$$

$$\int L_n\,(\sqrt{x^2 + s^2})dx = \int (L_n(w))\,\frac{w}{\sqrt{w^2 - s^2}}\,dw,$$

$$= (L_n(w))(\sqrt{w^2 - s^2}) - \int\frac{\sqrt{w^2 - s^2}}{w}\,dw,$$

$$= \sqrt{w^2 - s^2}\;L_n(w) - \sqrt{w^2 - s^2} + s.\cos^{-1}\,(\frac{s}{w}),$$

$$= x\,(L_n(\sqrt{x^2 + s^2}) - 1) + s\,\cos^{-1}\,(\frac{s}{\sqrt{x^2 + s^2}}),$$

$$= x\,(L_n(\sqrt{x^2 + s^2}) - 1) + s\,\tan^{-1}\,(\frac{x}{s}),$$

In the particular example of sources mounted upon an insulating surface the power input is from two superimposed strips each contributing $\dot{Q}$,

therefore $\dot{q}b = 2\dot{Q}$,

therefore $\theta_s = \frac{\dot{Q}}{\pi k}\left[C + 1 - L_n\left(\sqrt{\frac{b^2}{4} + s^2}\right) - \frac{2s}{b}\tan^{-1}\frac{b}{2s}\right].$

B5. TEMPERATURE OF STRIP MOUNTED UPON A SUBSTRATE OF FINITE THICKNESS

Appendix A proposed the use of negative mirror image sources to represent a heat sink at the line of inverted symmetry.

In this case a negative heat source would have to be equal in magnitude to the sum of the original and its reflected source and would have to be at a distance 2t where t is the substrate thickness. Hence from the equation in section B4 the contribution of the negative heat source at a distance 2t establishing a heat sink is

$$\theta_{2t} = -\frac{\dot{Q}}{\pi k}\left[C + 1 - L_n\left(\sqrt{\frac{b^2}{4} + 4t^2}\right) - \frac{4t}{b}\tan^{-1}\frac{b}{4t}\right]. \qquad \text{...(B3)}$$

As the original heat source is on a line of positive symmetry this new heat source also has to be reflected contributing a further θ_{2t}. This is further reflected in the heat sink and hence gives rise to an infinite series for the total temperature by the addition of the temperature contributions of the positive and negative reflected heat sources (equation (B2) + (B3), etc). Hence,

$$\theta_F = \frac{\dot{Q}}{\pi k}\{C + 1 - L_n(\tfrac{b}{2}) + 2\sum_{n=1}^{n=\infty}\Big[-1^n\Big[(C + 1) - L_n\left(\sqrt{\frac{b^2}{4} + (2nt)^2}\right)$$

$$- \frac{4nt}{b}\tan^{-1}\left(\frac{b}{4nt}\right)\Big]\Big]\}. \qquad \text{.....(B4)}$$

In order that a zero temperature is achieved at the heat sink there must be an equal number of heat sources on each side of the plane of inverted symmetry. We therefore take the last term at infinity to be a single negative term.

Hence, equation (B4) is re-written as

$$\theta_F = \frac{Q}{\pi k}\{L_n(\sqrt{\frac{b^2}{4} + 4t^2}) + \frac{4t}{b}\tan^{-1}(\frac{b}{4t}) - L_n(\frac{b}{2})$$

$$+ \sum_{n=1}^{n=\infty}\Big[L_n(\sqrt{\frac{b^2}{4} + 4(2n-1)^2t^2}) + \frac{4(2n-1)t}{b}\tan^{-1}(\frac{b}{4(2n-1)t})$$

$$- 2(L_n(\sqrt{\frac{b^2}{4} + 4(2n)^2t^2}) - \frac{4(2nt)}{b}\tan^{-1}(\frac{b}{4(2nt)}))$$

$$+ L_n(\sqrt{\frac{b^2}{4} + 4(2n+1)t^2}) + \frac{4(2n+1)t}{b}\tan^{-1}(\frac{b}{4(2n+1)t})\Big]\}.$$

$$= \frac{Q}{\pi k}\{L_n(\sqrt{1 + (\frac{4t}{b})^2}) + \frac{4t}{b}\tan^{-1}\frac{b}{4t}$$

$$+ \sum_{n=1}^{n=\infty}\Big[L_n(\sqrt{\frac{(1 + ((2n-1)(4\frac{t}{b}))^2)(1 + ((2n+1)(4\frac{t}{b}))^2)}{(1 + ((2n)(\frac{4t}{b}))^2)^2}})$$

$$+ (2n-1)\frac{4t}{b}\tan^{-1}(\frac{b}{4(2n-1)t}) - 2.2n.\frac{4t}{b}\tan^{-1}(\frac{b}{.2n,4t})$$

$$+ (2n+1)\frac{4t}{b}\tan^{-1}(\frac{b}{4(2n+1)t})\Big]\}.$$

Let w = 4 times the ratio of substrate thickness to strip width, ie, 4t/b, hence

$$\theta_F = \frac{Q}{\pi k}\{L_n(\sqrt{1 + w^2}) + w\tan^{-1}\frac{1}{w}$$

$$+ \sum_{n=1}^{n=\infty}\Big[L_n(\sqrt{\frac{(1 + (2n-1)^2w^2)(1 + (2n+1)^2w^2)}{(1 + (2n)^2w^2)^2}})$$

$$+ (2n-1)w\tan^{-1}(\frac{1}{(2n-1)w}) - 2(2n)w\tan^{-1}(\frac{1}{2nw}) + (2n+1)w\tan^{-1}(\frac{1}{(2n+1)w})\Big]\} \qquad .....(B5)$$

APPENDIX C

SUMMATION TECHNIQUE

C1. INTRODUCTION

The expression obtained in appendix B for the temperature at the centre of a strip heat source can be summed over many terms but with no actual knowledge of the SOC, except by experience or excessive over summation. The technique presented in this appendix allows the SOC to be evaluated at each step in the summation and by so doing enables the number of terms summed to be kept to a minimum and the evaluation taken to any predetermined degree of accuracy.

C2. UNDERLYING PRINCIPLE

The technique depends upon the absolute convergence of two series to their own values and the relative convergence of the individual terms of both series.

By examination of the principle of multiple reflections it can be seen that the terms of the general series converge upon the terms of the particular series which would be obtained if all the heat were generated at a point. This particular series can be obtained from the general series by making w large.

C3. APPLICATION TO STRIP HEAT SOURCE PROBLEM

Reference equation (B5), in appendix B,

$$\theta_F = \frac{\dot{Q}}{\pi k}\{L_n(\sqrt{1 + w^2}) + w \tan^{-1} \frac{1}{w}$$

$$+ \sum_{n=1}^{n=\infty}\Big[L_n(\sqrt{\frac{(1 + (2n - 1)^2w^2)(1 + (2n + 1)^2w^2)}{(1 + (2n)^2w^2)^2}})$$

$$+ (2n - 1)w \tan^{-1} (\frac{1}{(2n - 1)w}) - 2(2n)w \tan^{-1} (\frac{1}{2nw})$$

$$+ (2n + 1)w \tan^{-1} (\frac{1}{(2n + 1)w})\Big]\},$$

if w is large, the general series becomes the particular series

$$\sum_{n=1}^{n=\infty}\Big[L_n(\frac{(2n - 1)(2n + 1)}{(2n)^2}) + (2n - 1)w \frac{1}{(2n - 1)w} - 2(2n)w \frac{1}{2nw}$$

$$+ (2n + 1)w \frac{1}{(2n + 1)w},$$

$$\text{if } y \text{ is large } \tan^{-1} (\frac{1}{y}) \to \frac{1}{y},$$

therefore the series then becomes a logarithm of a product series

$$L_n\left(\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^2}\right)\right) \text{ which equals } L_n\left(\frac{2}{\pi}\right).$$

NOTE: $\dfrac{\sin\psi}{\psi} = \prod_{n=1}^{\infty}\left(1-\left(\dfrac{\psi}{n\pi}\right)^2\right).$
Let $\dfrac{\psi}{n\pi} = \dfrac{1}{2n}$,
therefore $\psi = \dfrac{\pi}{2}$,
therefore $\prod_{n=1}^{\infty}\left(1-\dfrac{1}{(2n)^2}\right) = \dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \dfrac{2}{\pi}.$

Subtract the Ln(2/π) term from each of the series terms and add Ln (2/π) to the initial term of the summation, ie,

$$\theta_F = \frac{\dot{Q}}{\pi k}\left\{L_n\left(\frac{2}{\pi}\left(\sqrt{1+w^2}\right)\right) + w\tan^{-1}\frac{1}{w}\right.$$

$$+ \sum_{n=1}^{n=\infty}\left[L_n\left(\sqrt{\frac{(1+(2n-1)^2w^2)(1+(2n+1)^2w^2)(2n)^4}{(1+(2n)^2w^2)^2(2n-1)^2(2n+1)^2}}\right)\right.$$

$$+ (2n-1)w\tan^{-1}\left(\frac{1}{(2n-1)w}\right) - 2(2n)w\tan^{-1}\left(\frac{1}{2nw}\right)$$

$$\left.\left. + (2n+1)\,w\tan^{-1}\left(\frac{1}{(2n+1)w}\right)\right]\right\}.$$

$$\text{Let } \frac{\pi k}{Q} \theta_m = \{L_n\left(\frac{2}{\pi}\sqrt{1 + w^2}\right) + w \tan^{-1} \frac{1}{w}$$

$$+ \sum_{n=1}^{n=m} \left[L_n\left(\sqrt{\frac{(1 + (2n - 1)^2 w^2)(1 + (2n + 1)^2 w^2)(2n)^4}{(1 + (2n)^2 w^2)^2 (2n - 1)^2 (2n + 1)^2}}\right)\right.$$

$$+ (2n - 1)\, w \tan^{-1} \left(\frac{1}{(2n - 1)w}\right) - 2(2n)w \tan^{-1} \left(\frac{1}{2nw}\right)$$

$$\left.+ (2n + 1)w \tan^{-1} \left(\frac{1}{(2n + 1)w}\right)\right]\},$$

that is $\theta_m = \theta_F$ summed over m terms.

$$\text{Let } G_n = L_n\left(\sqrt{\frac{(1 + (2n - 1)^2 w^2)(1 + (2n + 1)^2 w^2)}{(1 + (2n)^2 w^2)^2}}\right) + (2n - 1)w \tan^{-1} \left(\frac{1}{(2n - 1)w}\right)$$

$$- 2(2n)w \tan^{-1} \left(\frac{1}{2nw}\right) + (2n + 1)w \tan^{-1} \left(\frac{1}{(2n + 1)w}\right).$$

That is the nth term of the general series and

$$P_n = L_n\left(1 - \frac{1}{(2n)^2}\right).$$

That is the nth term of the particular series.

C4. TECHNIQUE OF SUMMATION

At the commencement of the summation the particular series is added to the non-summed part of the expression for θ_F obtained in appendix B to give

$$\frac{\pi k}{Q} \theta_o = L_n\left(\frac{2}{\pi}\sqrt{1 + w^2}\right) + w \tan^{-1} \left(\frac{1}{w}\right).$$

At each stage in the summation the corresponding term in the particular series is replaced by that in the general series, and at the same time the ratio of these two terms multiplied by the ratio of the balance of the particular series to the value of the summation at that time is evaluated to give the maximum error or SOC.

After m summation terms

$$\frac{\pi k}{Q}\theta_m = L_n\left(\frac{2}{\pi}\sqrt{1 + w^2}\right) + w \tan^{-1}\left(\frac{1}{w}\right) + \sum_{n=1}^{n=m}(G_n - P_n)$$

and the maximum error is given by

$$\frac{G_m - P_m}{P_m} \cdot \frac{L_n\left(\frac{2}{\pi}\right) - \sum_{n=1}^{n=m} P_n}{\frac{\pi k}{Q}\theta_m} .$$

This is evaluated at each stage of the summation and when an acceptable SOC is indicated by this maximum possible error the value of θ_m is then computed.

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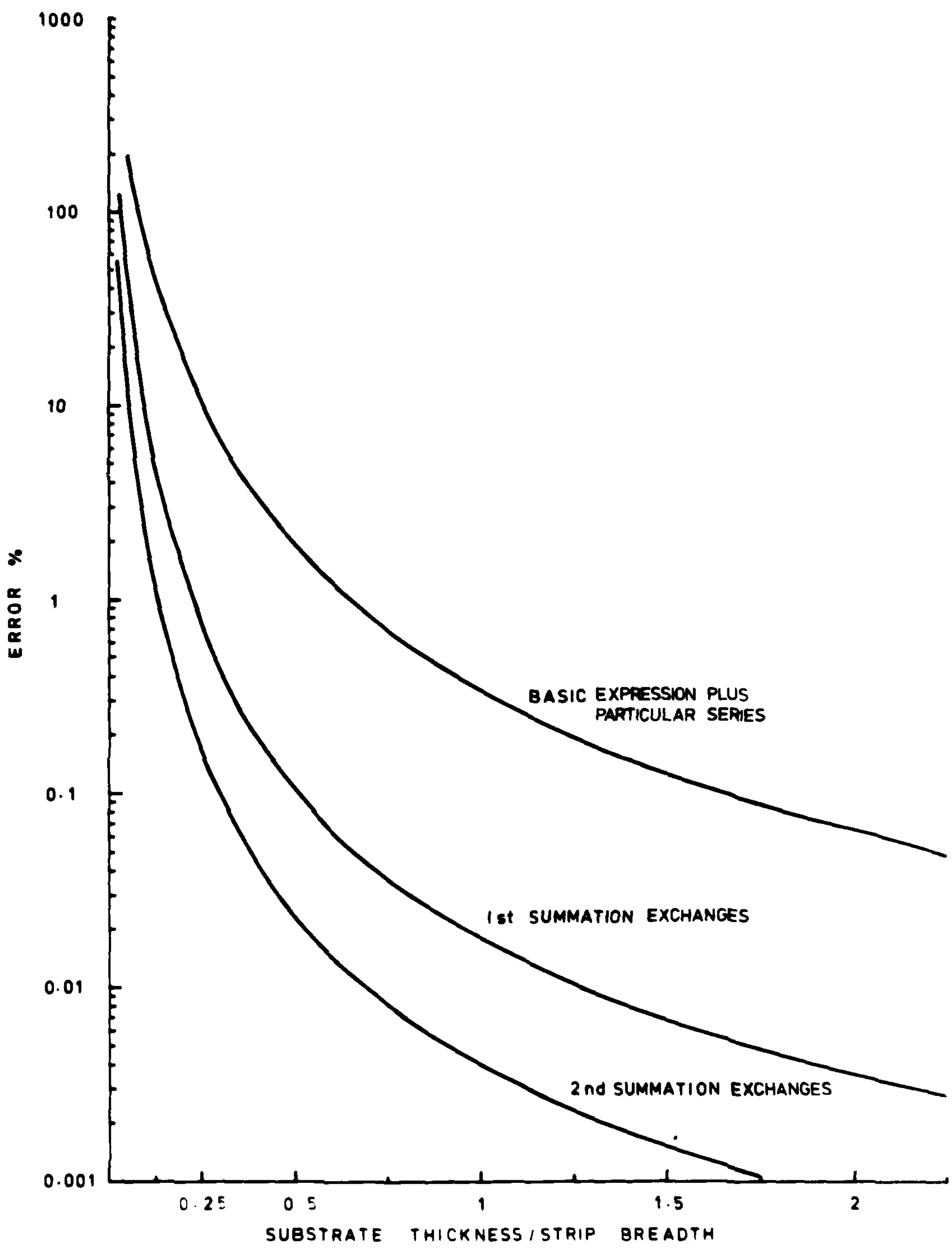

FIGURE 1 Errors in Heat Spreading Calculation

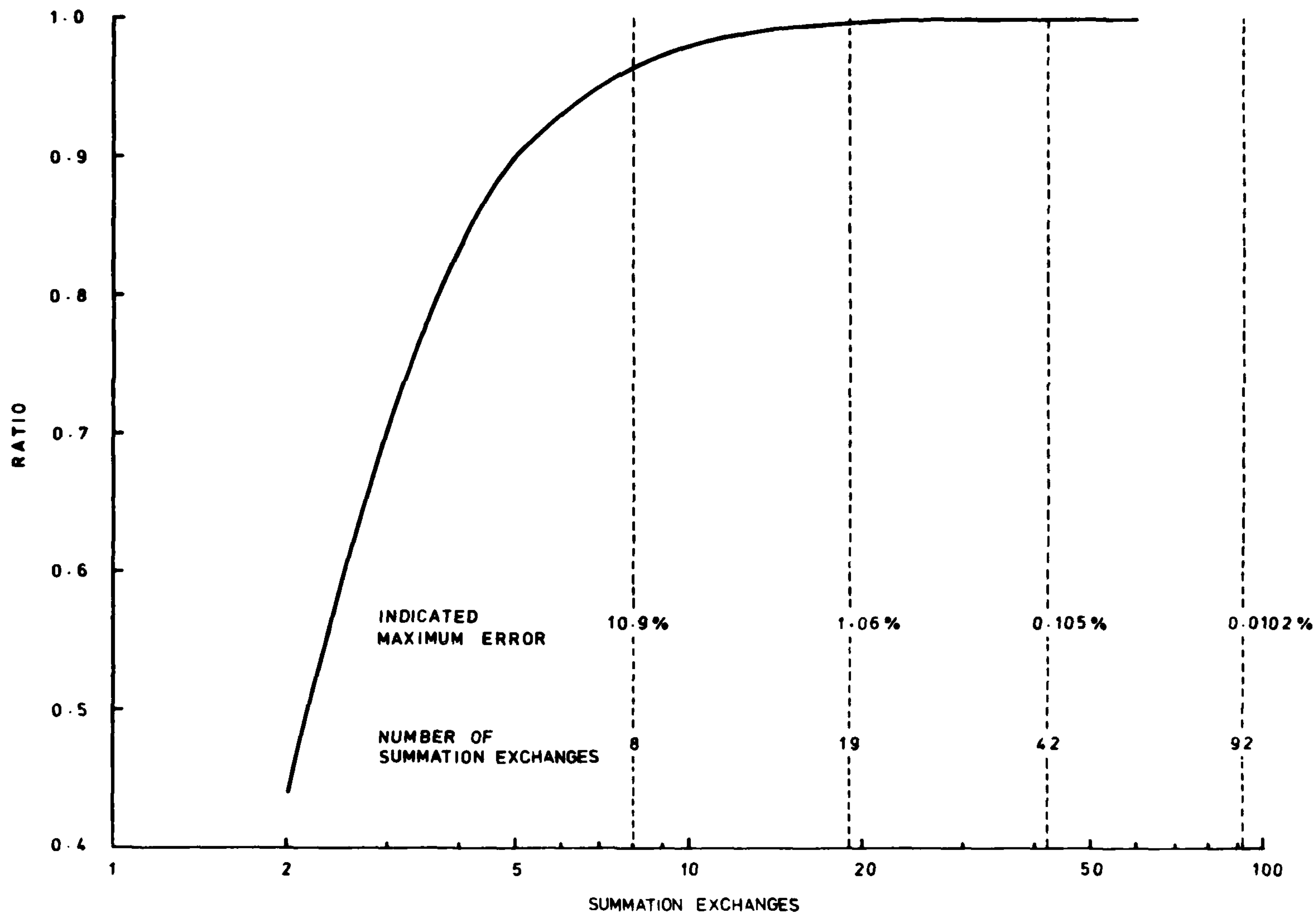

FIGURE 2 Ratio of Heat Spreading Calculation to Parallel Flow Calculation

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Abstract

In microelectronics the rapid degradation in reliability which occurs with modest rises in temperature makes it important that temperatures resulting from different geometries can be simply evaluated. One particular geometry which is becoming increasingly common is that of a long narrow strip heat source mounted upon the upper surface of a thin substrate which has a heat sink on its rear face. This report presents a series summation computation method suitable for pocket calculators, by which the state of convergence (SOC), in terms of maximum possible error, is calculated at each stage in the summation thus enabling the calculation to be terminated as soon as the required accuracy has been obtained.

Some Metric and SI Unit Conversion Factors

(Based on DEF STAN 00-11/2 "Metric Units for Use by the Ministry of Defence", DS Met 5501 "AWRE Metric Guide" and other British Standards)

Quantity	Unit	Symbol	Conversion
Basic Units			
Length	metre	m	1 m = 3.2808 ft 1 ft = 0.3048 m
Mass	kilogram	kg	1 kg = 2.2046 lb 1 lb = 0.45359237 kg 1 ton = 1016.05 kg
Derived Units			
Force	newton	$N = kg\ m/s^2$	1 N = 0.2248 lbf 1 lbf = 4.44822 N
Work, Energy, Quantity of Heat	joule	J = N m	1 J = 0.737562 ft lbf 1 J = 9.47817×10^{-4} Btu 1 J = 2.38846×10^{-4} kcal 1 ft lbf = 1.35582 J 1 Btu = 1055.06 J 1 kcal = 4186.8 J
Power	watt	W = J/s	1 W = 0.238846 cal/s 1 cal/s = 4.1868 W
Electric Charge	coulomb	C = A s	-
Electric Potential	volt	V = W/A = J/C	-
Electrical Capacitance	farad	F = A s/V = C/V	-
Electric Resistance	ohm	$\Omega = V/A$	-
Conductance	siemen	$S = 1\ \Omega^{-1}$	-
Magnetic Flux	weber	Wb = V s	-
Magnetic Flux Density	tesla	$T = Wb/m^2$	-
Inductance	henry	H = V s/A = Wb/A	-
Complex Derived Units			
Angular Velocity	radian per second	rad/s	1 rad/s = 0.159155 rev/s 1 rev/s = 6.28319 rad/s
Acceleration	metre per square second	m/s^2	1 m/s^2 = 3.28084 ft/s^2 1 ft/s^2 = 0.3048 m/s^2
Angular Acceleration	radian per square second	rad/s^2	-
Pressure	newton per square metre	N/m^2 = Pa	1 N/m^2 = 145.038×10^{-6} lbf/in^2 1 lbf/in^2 = 6.89476×10^3 N/m^2
	bar	bar = 10^5 N/m^2	- 1 in. Hg = 3386.39 N/m^2
Torque	newton metre	N m	1 N m = 0.737562 lbf ft 1 lbf ft = 1.35582 N m
Surface Tension	newton per metre	N/m	1 N/m = 0.0685 lbf/ft 1 lbf/ft = 14.5939 N/m
Dynamic Viscosity	newton second per square metre	N s/m^2	1 N s/m^2 = 0.0208854 lbf s/ft^2 1 lbf s/ft^2 = 47.8803 N s/m^2
Kinematic Viscosity	square metre per second	m^2/s	1 m^2/s = 10.7639 ft^2/s 1 ft^2/s = 0.0929 m^2/s
Thermal Conductivity	watt per metre kelvin	W/m K	-
Odd Units*			
Radioactivity	becquerel	Bq	1 Bq = 2.7027×10^{-11} Ci 1 Ci = 3.700×10^{10} Bq
Absorbed Dose	gray	Gy	1 Gy = 100 rad 1 rad = 0.01 Gy
Dose Equivalent	sievert	Sv	1 Sv = 100 rem 1 rem = 0.01 Sv
Exposure	coulomb per kilogram	C/kg	1 C/kg = 3876 R 1 R = 2.58×10^{-4} C/kg
Rate of Leak (Vacuum Systems)	millibar litre per second	mb l/s	1 mb = 0.750062 torr 1 torr = 1.33322 mb

*These terms are recognised terms within the metric system.