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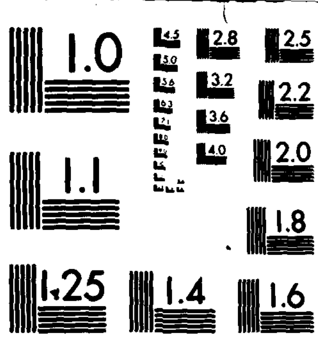

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WEAPONS SYSTEMS RESEARCH LABORATORY

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TECHNICAL REPORT

WSRL-0046-TR

A MODEL FOR THE EFFECT OF TIME DELAY ERRORS ON THE POLAR RESPONSE OF A LINEAR ARRAY OF SENSORS

D.A. GRAY

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

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S U M M A R Y

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This approximation of the required time delays modifies the polar response of the array. In this paper a model describing the degradation in the polar response of a linear array of equispaced sensors at selected angles due to the time delay errors is proposed. At these angles the perturbed polar response is characterized by a number of isolated high sidelobes. These are termed quantization lobes and their position and approximate height are described by the model. Unfortunately the model is limited by its sensitivity to small perturbations in the steered direction.

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In conventional time domain beamformin the receiver outputs are delayed such that the outputs due to a signal arriving from a selected direction are added in phase. If the receiver outputs are sampled time series then all time delays must be integer multiples of the sampling interval. In general the required delays are not exact multiples of the sampling interval and an appropriate choice for the approximating delays is to choose the nearest integer multiple of the sampling interval.

This approximation of the required time delays modifies the polar response of the array. In this paper a model describing the degradation in the polar response of a linear array of equispaced sensors at selected angles due to the time delay errors is proposed. At these angles the perturbed polar response is characterized by a number of isolated high sidelobes. These are termed quantization lobes and their position and approximate height are described by the model. Unfortunately the model is limited by its sensitivity to small perturbations in the steered direction.

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1. INTRODUCTION

Time domain beamforming as implemented by the DIMUS (Digital Multibeam Steering) technique(ref.1) involves the delaying of the sampled outputs of individual receivers. Whether these delays are affected by shift register stages or by the use of random access memories the delays can only be approximated to within the nearest half sampling interval. To ensure such errors are kept to a minimum the ratio of the sampling rate to the highest frequency present should be large. However, the maximum phase error can still be relatively large, e.g. for a frequency 1/8th of the sampling rate the maximum phase error is $\pi/8$.

Anderson(ref.1) has previously proposed a model which predicts the degradation in the height of main beam. His model assumes a random distribution of the errors of the time delays with an equal probability of lying within plus or minus half the sampling interval of the true time delay. However, in many cases the side-lobe response of any array may be an equally if not more important consideration than the height or width of the main beam and is not described by the above model.

A model is proposed in this paper whereby for certain steered angles the perturbed polar response of a linear array with equispaced elements can be calculated exactly. The model, described in Section 3 is valid for those angles at which every qth receiver has the exact time delay. Moreover, for a linear array of N equispaced elements it is common practice to steer N independent beams which at halfwavelength overlap at their 4 dB points. If, for the first beam off broadside the progressive time delay, $\Delta\tau$, is such that the model is applicable then it can easily be shown that the model describes the polar response for all the other N independent beams.

A similar model has been proposed by Allen(ref.2) to describe phase quantization, and an argument similar to that of Allen is used in Section 2 to relate the mean square error of a beam output to that of the time delays. The model proposed by Allen considers only those angles for which the approximating phase delays have steps of equal length and equal height. This condition is relaxed in the proposed model and as a result a closer approximation to the array response is obtained for a larger number of angles.

In Section 4 it is shown how the effect of the time quantization can be incorporated as a perturbing factor on the response of an N/q element linear array.

The perturbed array response, at these selected angles, is characterised by high individual side-lobes. These are termed quantization lobes and their properties are discussed in detail in Section 5. Finally in Section 6 some examples of the model are given.

2. THE RELATIONSHIP BETWEEN TIME DELAY AND PHASE ERRORS

Let $x_j(t)$ represent the time series output of the jth receiver of a linear array of N equispaced receivers. To 'steer' a beam in a direction θ the output of the jth receiver should be delayed by a τ_j, where

$$\tau_j = \frac{j\, d \sin\theta}{c} = j\,\Delta\tau,$$

d is the separation of adjacent receivers and c is the velocity of propagation.

However if the receiver outputs are sampled in time then it is only possible to delay the receiver outputs by integer multiples of the sampling interval. In order to approximate the time delay required for the jth receiver, an integer m_j is chosen such that

$$|m_j \tau_0 - \tau_j|$$

is minimised. It will be shown in the remainder of this section that for N large such a choice minimises the mean square error in the steered beam.

Consider the continuous* beams $y_e(t)$ and $y_a(t)$ corresponding to exact and approximate time delays. It follows that $y_e(t)$ and $y_a(t)$ are given by:

$$y_e(t) = \sum_{j=0}^{N-1} a_j \, x_j(t - \tau_j)$$

and

$$y_a(t) = \sum_{j=0}^{N-1} a_j \, x_j(t - m_j \, \tau_0)$$

where the a's are the shading (or weighting) coefficients. Denoting the Fourier transform of $x_j(t)$ by $X_j(f)$ it follows that the squared error of the beam output at time t, i.e., $|y_a(t) - y_e(t)|^2$ is given by:

$$\iint_{-\infty}^{\infty} e^{2\pi i(f-f')t} \sum_{j,k} X_j(f) X_k^*(f') a_j^* a_k \left(e^{2\pi i f m_j \tau_0} - e^{2\pi i f \tau_j}\right)\left(e^{-2\pi i f' m_k \tau_0} - e^{-2\pi i f' \tau_k}\right) df \, df' . \tag{1}$$

If the process is stationary it follows that

$$\langle X_j^*(f) \; X_k(f') \rangle = S_{jk} \, \delta(f - f')$$

where $\langle \; \rangle$ denotes the ensemble average. Taking the expectation of equation (1), substituting the above equation and making the approximation of only considering the diagonal terms in the double summation it follows that the mean square error is approximated by:

$$\int_{-\infty}^{\infty} \sum_{j} S_{jj}(f) \; a_j^* \; a_j \left| e^{2\pi i f m_j \tau_0} - e^{2\pi i f \tau_j} \right|^2 df.$$

Assuming identical receivers and a homogeneous field the above expression reduces to:

$$\int_{-\infty}^{\infty} S(f) \sum_{j} a_j^* \; a_j \left| e^{2\pi i f m_j \tau_0} - e^{2\pi i f \tau_j} \right|^2 df, \tag{2}$$

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*The following argument can also be applied to sampled beams $y_e(p\tau_0)$ and $y_a(p\tau_0)$ provided a Fourier series expansion is used.

where S(f) is the power spectrum of the omnidirectional field. The expression (2) may further be reduced to:

$$2\int_{-\infty}^{\infty} S(f) \sum_j a_j^* a_j \{1 - \cos 2\pi\, f(m_j\tau_0 - \tau_j)\}\, df$$

which for $2\pi f|m_j\,\tau_0 - \tau_j|$ small may be reduced to:

$$4\pi^2 \sum_j a_j^*\, a_j\, |m_j\tau_0 - \tau_j|^2 \int_{-\infty}^{\infty} f^2\, S(f)\, df$$

Thus under the above assumptions the mean square error of the time delayed beam is related to the average error in the approximating time delays. In particular, since all quantities in the above expression are positive, it follows that choosing the m_j such that $|m_j\tau_0 - \tau_j|$ is minimized is equivalent to minimizing the mean square error of the steered beam.

3. FORMULATION OF THE MODEL

As shown in figure 1 for a linear array the exact time delays corresponding to a particular steered direction are a linear function of receiver number. For selected directions the relative time delays between certain receivers will be integer multiples of the sampling rate and thus the outputs of these receivers will have the correct phase relationship. Since the array elements are equispaced it also follows that if the time delay between the 1st and say the qth, receiver is an exact multiple of the sampling interval then the relative phase relationship between any receivers separated by qd will also be exact. This is illustrated by the first example in figure 1 where the time delay required between every second receiver is equal to exactly one sampling interval. If the time delays for the odd receivers are chosen such that

$$|m_j\,\tau_0 - \tau_j|$$

is minimized the stepwise approximation to the time delays as shown in figure 1 is obtained. While the approximating time delay between adjacent receivers is half a sampling interval (i.e. $\frac{1}{2}\tau_0$) in error the relative delays of all the even numbered receivers are exact and similarly for the odd numbered receivers.

In the second example of figure 1 the steered direction is chosen such that the relative time delay between every fifth receiver is exactly $8\,\tau_0$. Choosing each time delay such that:

$$|m_j\,\tau_0 - \tau_j|$$

is minimized it follows, similarly to the previous example, that the relative delay between any pair of receivers separated by 5d is also $8\,\tau_0$. This stepwise approximation to the time delays is plotted in figure 1. As a consequence of the above considerations the step function, apart from a vertical shift, repeats itself with a periodicity of $8\,\tau_0$.

In general there will always exist angles, denoted by $\theta_{p,q}$, such that the relative time delay between any two receivers separated by a distance of qd will be an integer multiple, p, of the sampling interval. Also, provided the delays are chosen such that $|m_j \tau_0 - \tau_j|$ is minimized for all j then the stepwise approximation to the time delays, apart from vertical shifts, will be periodic with a periodicity of $p\,\tau_0$. This model is a generalization of that proposed by Allen(ref.2) for phase quantization. In Allen's model the width of the steps are restricted to being equal; this restriction is removed in the proposed model and consequently the array response can be evaluated at a larger number of angles.

At any angle the progressive time delay, $\Delta\tau$, can be approximated arbitrarily closely by $p\,\tau_0/q$ where p and q are integers. It then follows that the delay between every qth receiver will be exact and due to the periodicity of the time delays, as discussed above, the array response can be decomposed into a sum of the subarrays with exact phase relationships between their elements. Before discussing the effect of this on the array response some general properties of the factorization and time quantization will be discussed.

(a) Finite Number of Receivers

In general the number, N, of receivers is finite and so the factorization will only hold as an approximation since q does not, in general, divide N. Furthermore for finite N the model will be most useful when q is small compared with N. Also the number of receivers with the correct phase relationship (or time delay) to any given receiver will be N/q and for the model to be most successful the condition $N/q > q$ should hold.

(b) Coarseness of Quantization

The degradation in the array response due to time quantization is a function of the ratio of the frequency of the incident sine wave to the sampling rate. The coarseness of quantization, η, is defined as the ratio of the $f_{\frac{1}{2}}$; the frequency corresponding to the halfwavelength of the array, to $1/\tau_0$; the sampling rate. Thus

$$\eta = f_{\frac{1}{2}}\,\tau_0$$

$$= c\,\tau_0/2d$$

where c is the velocity of propagation.

The above intuitive definition has an alternative interpretation. At a frequency f, the beamwidth, θ_B, is approximated by

$$\theta_B \approx \arcsin\left(\frac{c}{Nd}\,f\right)$$

The delay between adjacent receivers, $\Delta\tau$, to shift the main beam by one beamwidth is then

$$\Delta\tau = \frac{d}{c}\sin\theta_B$$

$$= \frac{1}{Nf}\,.$$

Thus, apart from the 1/N factor, the coarseness of quantization is the ratio of the sampling interval to the delay between adjacent receivers required to shift the main beam by one beamwidth. (Note that this can be defined at any frequency but for the remainder of this paper only $f_{\frac{1}{2}}$ will be considered). This is a convenient measure of degradation as for small η, i.e. fine quantization, the approximating delays will be more accurate.

4. DERIVATION OF THE PERTURBED ARRAY RESPONSE

4.1 Factorization

Consider the array steered in a direction such that the relative delays between receivers separated by a distance of qd is an integer multiple, p, of the sampling interval. As discussed in Section 3 all receivers separated by multiples of qd will have the correct phase relationship. Consequently the array response can be factored into the product of the response of an array of N/q elements with the response of an array of q elements.

Denote by $\theta_{p,q}$ the angle such that the time delay between adjacent receivers ($\Delta\tau$) satisfies

$$q\,\Delta\tau = p\,\tau_0 = \frac{qd \sin\theta_{p,q}}{c} \qquad (3)$$

The response, $P_{p,q}(\phi)$ when the array is steered in a direction $\theta_{p,q}$ to a sine wave incident from a direction ϕ is given by

$$P_{p,q}(\phi) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i f(m_j \tau_0 - (j-1)\Delta t)} \right|^2 \qquad (4)$$

where the m_j are chosen such that $|m_j \tau_0 - \tau_j|$ is minimized and Δt equals $\frac{d \sin\phi}{c}$. From the 'periodicity' of the delays demonstrated in Section 3 it follows that if the delays for receivers 1 to q are $m_1\tau_0$, $m_2\tau_0$, ... $m_q\tau_0$ (where $m_q = p$) then the delays for receivers q + 1 to 2q are $(m_1 + p)\tau_0$, ..., $(m_p + p)\tau_0$ and so on. Substituting these m_j's in equation (4) and summing over every qth receiver $P_{p,q}(\phi)$ can be written as:

$$P_{p,q}(\phi) = \left| \frac{1}{N} \sum_{j=1}^{N/q} e^{2\pi i f(j-1)(p\,\tau_0 - q\,\Delta t)}\, e^{2\pi i\, f\, m_1\, \tau_0} \right.$$

$$+ \sum_{j=1}^{N/q} e^{2\pi i f(j-1)\,(p\,\tau_0 - q\,\Delta t)}\, e^{2\pi i f(m_2\,\tau_0\, -\, \Delta t)}$$

$$\vdots$$

$$\left. + \sum_{j=1}^{N/q} e^{2\pi i f(j-1)\,(p\,\tau_0\, -\, q\,\Delta t)}\, e^{2\pi i f(m_q\,\tau_0 - (q-1)\Delta t)} \right|^2$$

which reduces to

$$P_{p,q}(\phi) = \frac{1}{N^2} \left| \sum_{j=1}^{N/q} e^{2\pi i f(j-1)(p\,\tau_0 - q\,\Delta t)} \right|^2 \left| \sum_{k=1}^{q} e^{2\pi i f(m_k\,\tau_0 - (k-1)\Delta t)} \right|^2$$

From equation (3) the above expression may be further expressed as:

$$P_{p,q}(\phi) = \frac{1}{N^2} \left| \sum_{j=1}^{N/q} e^{2\pi i f(j-1)\,\frac{qd}{c}\,(\sin\theta_{p,q} - \sin\phi)} \right|^2 \left| \sum_{k=1}^{q} e^{2\pi i f(m_k\,\tau_0 - (k-1)\,\Delta t)} \right|^2$$

$$= \frac{\sin^2 \pi\, \frac{fNd}{c}\,(\sin\theta_{p,q}\, -\, \sin\phi)}{(N/q)^2\, \sin^2 \pi f\, \frac{qd}{c}\,(\sin\theta_{p,q}\, -\, \sin\phi)}\; \frac{1}{q^2} \left| \sum_{k=1}^{q} e^{2\pi i f(m_k\,\tau_0\, -\, (k-1)\Delta t)} \right|^2$$

$$= R^e_{N/q}\; R^o_q.$$

From inspection the first factor $R^e_{N/q}$ represents the exact polar diagram of a linear array of N/q elements with a distance qd between adjacent receivers. Furthermore it is the response of an array steered in the direction $\theta_{p,q}$. The second term R^o_q is a sum of q phase factors and contains the effect of the time delay errors in the choice of the m_j. $R^e_{N/q}$ will, in the main, describe the response of the array and the R^o_q will be interpreted as a perturbation upon this response. Often R^o_q (see examples) will have a simple form but in general the phase factors i.e. the $2\pi f m_k$ are not linear in k and as a consequence, a simple expression for R^o_q is not possible.

In particular the peak of R^o_q will not always lie in the direction $\theta_{p,q}$ as it would if the phases were exact. However, it follows from the arguments of Section 2 that the distortion will be minimized since the peak of R^o_q will most closely approximate to $\theta_{p,q}$.

It should also be noted that as a trivial example when p = 1 the above response reduces to the exact polar response. This is due to the fact that at certain selected angles each receiver delay is an exact multiple of the sampling rate and no distortion of the polar response occurs. This illustrates the very strong dependence of the degradation on the angle at which beams are steered.

A more illuminating example is given by considering the polar response of an array for a steering angle $\theta_{1,2}$ where alternate receivers have the correct phases. The factorization of the array response into the product of the response of an N/2 element array with a two element array can be seen intuitively with reference to figure 1. This example is discussed in more detail in Section 6.

4.2 Discussion and interpretation

The array response at $\lambda/2$ will be considered since the response at any other frequency can be derived from it. Since the elements contributing to $R^e_{N/q}$ are separated by qd an incident plane wave whose frequency corresponds to $\lambda/2$ (= d) would be spatially undersampled. A consequence of this is the spatial aliasing of the field and the appearance of q grating lobes in the array response factor $R^e_{N/q}$. This is illustrated in figure 2. To obtain $P_{p,q}(\phi)$ the modulating factor R^o_q must be calculated. As discussed the peak response of R^o_q will most closely approximate that of $\theta_{p,q}$ and a typical R^o_q is illustrated in figure 2. The modulating effect of R^o_q whose magnitude is always less than one, has two major consequences:

(a) The height of the main peak will be reduced. However some simple examples show that unless the quantization is very coarse the main peak will be reduced by less than a few dB.

(b) The height of the grating lobes will be substantially reduced. In general, however, these grating lobes will not be reduced to the height of the other sidelobes and will appear as isolated peaks of high leakage into the main beam. The number, position and height of these grating lobes can be readily calculated and in the following section some expressions for these are obtained. To avoid confusion with the grating lobes due to spatial under-sampling when the exact time delays are used the lobes due to the time quantization will be termed quantization lobes.

R^o_q is termed the 'perturbing factor' since it contains the effect of the quantization errors in its non linear phase terms.

5. DEGRADATION IN ARRAY PERFORMANCE

The degradation in array performance may be quantified by use of the model. Although not every steered angle will correspond to, or be approximated by a $\theta_{p,q}$, there will, in general be a sufficiently large number of steered angles such that broad features, of the degradation may be ascertained. In particular the dependence of features such as the number, position and height of the quantization lobes and the degradation of the main beam on the parameters such as frequency, coarseness of quantization and angle of steer can be determined by reference to this model. In the following sections a quantification of these effects will be attempted.

5.1 Quantization lobes

At an angle $\theta_{p,q}$ and at a frequency $f_{\frac{1}{2}}$ corresponding to the half wavelength of the array there will be (q-1) quantization lobes. At an arbitrary frequency f the number of quantization lobes is $[f/f_{\frac{1}{2}}(q-1)]$ where [x] denotes the integer part of x.

The quantization lobes occur at angles θ_m satisfying:

$$\frac{qd}{\lambda}(\sin\theta_{p,q} - \sin\theta_m) = m$$

for

$$m = 1, 2, \ldots, q-1.$$

Thus

$$\sin\theta_m = \frac{-\lambda m}{qd} + \sin\theta_{p,q} \tag{5}$$

which at halfwavelength reduces to:

$$\sin\theta_m = \frac{-2m}{q} + \sin\theta_{p,q}$$

However

$$\sin\theta_{p,q} = \frac{2p}{q}\eta$$

where η, as defined in Section 3 is a measure of the coarseness of quantization. Consequently equation (5) reduces to

$$\sin\theta_m = \frac{2p\eta - 2m}{q} \text{ for } m = 1, 2, \ldots, q-1. \tag{6}$$

The quantization lobes contained in $R^e_{N/q}$ have unit amplitude and their height in the perturbed polar response is determined by the amplitude of R^o_q at the angles θ_m. Since R^o_q does not in general have an analytic form an approximation to R^o_q is given by the response of a q element array steered at broadside. (See for example figure 1). This is equivalent to assuming the same delay for receivers aq, aq+1, ...(a+1)q. Such an approximation is valid for small values of p, i.e. at angles steered around broadside. Thus, in decibels relative to unity the height of the mth grating lobe is given by:

$$20\log_{10}\left|\frac{\sin\left(\frac{\pi qd}{\lambda}\sin\theta_m\right)}{q\sin\left(\frac{\pi d}{\lambda}\sin\theta_m\right)}\right| \text{ for } m = 1, 2, \ldots, q-1.$$

At half-wavelength $\sin\theta_m$ is given by equation (6) and consequently the above expression reduces to:

$$20\log_{10}\left|\frac{\sin\pi p\eta}{q\sin\left(\frac{\pi p\eta}{q}-\frac{m\pi}{q}\right)}\right| \tag{7}$$

An alternative approximation for R_p^o is to divide the p sampling intervals equally, but to the nearest integer among the q receivers. This results in a linear delay versus receiver number for the q receivers. (See also figure 1). This will be a better approximation than the previous one, although for some angles the two are identical. It follows that $R_q^o(\phi)$ is approximated by:

$$\left|\frac{\sin\frac{\pi qd}{\lambda}(\sin\hat{\theta}-\sin\phi)}{q\sin\frac{\pi d}{\lambda}(\sin\hat{\theta}-\sin\phi)}\right|^2 \tag{8}$$

where $\sin\hat{\theta}$ is given by

$$\sin\hat{\theta} = 2\eta\left\{\frac{p}{q}\right\}^*$$

The height of the mth quantization lobe, h_m, is given by $R_q^o(\theta_m)$, evaluated when $d = \lambda/2$ and where the assumption that this height does not vary greatly with frequency has been made.

Substituting equation (6) for θ_m, the relative height (in dB) of the mth quantization lobe h_m, is given by:

$$h_m = 20\log_{10}\left|\frac{\sin\pi\eta\left(q\left\{\frac{p}{q}\right\}-p\right)}{q\sin\left(\pi\eta\left(\left\{\frac{p}{q}\right\}-\frac{p}{q}\right)+\frac{m\pi}{q}\right)}\right| \tag{9}$$

Note that as the sampling rate increases then η, the quantization measure, approaches zero. As $\eta \to o$ both approximations for the heights of the quantization lobes approach zero which would be expected.

5.2 Main beam

Figure 2 shows that except for the special case of when the delays are exact and thus R_q^o and $R_{N/q}^e$ have their main lobes pointing in the same direction the height of the main beam in the perturbed array response will be reduced. The reduction in the height of the main beam will be a function of the wavenumber separation of the main beams of R_q^o and $R_{N/q}^e$. This in turn will be determined by the coarseness of the quantization. For example for $\theta_{1,2}$ the main beam of R_q^o is always directed towards broadside and that of $R_{N/q}^e$ in the desired steered direction, $\theta_{1,2}$. Now if the quantization is very coarse then $\theta_{1,2}$ may correspond to endire directions and consequently, see figure 2, the reduction in the height of the main peak

* $\{x\}$ has been used to denote the nearest integer to x.

would be large. For fine to medium quantization $\theta_{1,2}$ will correspond to broadside directions and consequently the reduction in the height of the main peak will be small. The reduction in the height of the main peak, Δh, as a function of the coarseness of quantization is given by:

$$\Delta h = 20 \log_{10} \left| \frac{\sin \pi\eta}{2 \sin \frac{\pi\eta}{2}} \right| .$$

To illustrate the dependence of Δh on the coarseness of quantization Δh has been plotted in figure 3 for η varying from 0 to 1. As η, the coarseness of quantization, increases the reduction in the height of the main beam increases.

More generally equation (8) may be used to approximate the perturbing factor and hence the reduction in the height of the main beam, Δh, is approximated by

$$\Delta h \simeq 1 - \left\{ \frac{\sin \pi q \frac{d}{\lambda} (\sin \hat{\theta} - \sin \theta_{p,q})}{q \sin \frac{\pi d}{\lambda} (\sin \hat{\theta} - \sin \theta_{p,q})} \right\}^2$$

Substituting for $\sin \hat{\theta}$ and $\sin \theta_{p,q}$ and evaluating at halfwavelength this reduces to:

$$\Delta h \simeq 1 - \left\{ \frac{\sin \pi q \eta \left(\left\{ \frac{p}{q} \right\} - \frac{p}{q} \right)}{q \sin \pi\eta \left(\left\{ \frac{p}{q} \right\} - \frac{p}{q} \right)} \right\}^2$$

In general the delay approximations are such that $\{\frac{p}{q}\} \approx \frac{p}{q}$ and thus $\Delta h \ll 1$. For medium to fine quantization, η; the ratio of the frequency to the sampling rate will be small and consequently for reasonable values of q the reduction will be less than a few dB. Once again, for a high sampling rate, $\eta \to 0$ and thus $\Delta h \to 0$.

5.3 Shading

From the previous sections it can be seen that the most serious source of degradation in the polar response of the array is the introduction of quantization lobes. These quantization lobes are a consequence of the spatial aliasing incorporated in $R^{e}_{N/q}$ and are unaffected by the array weights. Thus the number and angular position of these grating lobes are the same for a shaded array as for an unshaded one. Furthermore in most schemes of amplitude weighting the shading coefficients vary smoothly from receiver to receiver and thus may be considered to be approximately constant over the subarray described by R^{o}_{q}. It thus follows that R^{o}_{q} for a shaded array is approximately equal to the R^{o}_{q} for an unshaded array. Thus the height of the quantization lobes are to a very good approximation independent of the array shading scheme used.

As would be expected other factors such as the beamwidth of the quantization lobes and the height of the subsidiary sidelobes are determined by the array shading.

6. EXAMPLES

Consider a linear array of N equispaced elements where $f_{\frac{1}{2}}$ is the frequency corresponding to halfwavelength spacing and τ_0 is the sampling interval. If d is the separation of adjacent receivers then the angles $\theta_{p,q}$ as defined in Section 4 satisfy:

$$\sin \theta_{p,q} = \frac{pc\,\tau_0}{qd}$$

where c is the velocity of propagation. As an illustration of the model the following angles are considered.

6.1 $\theta_{1,2}$

At this angle the time quantization is such that the delay between adjacent receivers is exactly one half the sampling interval. Consequently all even and all odd receivers will have the correct phase relationship between them. The time delays are shown in figure 1 as a function of receiver number. The polar response, $P_{1,2}(\phi)$, is given by:

$$P_{1,2}(\phi) = \left\{ \frac{\sin \frac{\pi N d}{\lambda} (\sin \theta_{1,2} - \sin \phi)}{\frac{N}{2} \sin \frac{2\pi d}{\lambda} (\sin \theta_{1,2} - \sin \phi)} \right\}^2 \cos^2\left(\frac{\pi d}{\lambda} \sin \phi\right).$$

The first term is the exact polar response of an array of N/2 equispaced receivers* with adjacent receivers separated by 2d. The N/2 element array is steered in the direction $\theta_{1,2}$. The second factor, R_2^0,

$$\cos^2\left(\frac{\pi d}{\lambda} \sin \theta\right)$$

describes the polar response of a dipole. The effect of the time quantization is to steer this dipole in the broadside direction for all values of $\theta_{1,2}$. If the sampling rate is sufficiently high, i.e. the quantization is fine, then $\theta_{1,2}$ will necessarily correspond to a direction near broadside and little degradation will occur. However for a low sampling rate $\theta_{1,2}$ will approach endfire and so considerable distortion could be expected.

At $f_{\frac{1}{2}}$ the spacing of the N/2 element array is λ and so an aliased lobe will appear. Using equation (6) the position of this quantization lobe, in degrees relative to broadside, is given by

$$\sin \theta_1 = \eta - 1.$$

The height of this lobe is given by:

$$20 \log_{10} \left| \sin \frac{\pi \eta}{2} \right|$$

which is a special case of equation (7).

---

* For N odd the above approximation will still be valid provided N is sufficiently large.

The perturbed response corresponding to N = 30, $f_{½}$ = 320 Hz and a sampling rate of 2048 Hz is plotted in figure 4. The polar diagrams in this, and all subsequent figures were calculated by choosing the delay for the jth receiver so as to minimize

$$|m_j \tau_0 - \tau_j|$$

as discussed in Section 2. The coarseness of quantization, η, has the value 5/32.

The angle $\theta_{1,2}$ is 8.9°, and from figure 4 the position of the quantization lobe is -56° and its height is -12.4 dB relative to the peak. Using equations (6) and (9) the predicted position and height of the lobe are -58° and -12.3 dB respectively, and show good agreement with the results of figure 4.

6.2 $\theta_{8,5}$

This angle corresponds to every fifth receiver having the correct time delays relative to each other. The delays for each subset of five receivers are chosen so as to most closely approximate the linear delay relationship and are also shown in figure 1. Assuming the total number of receivers to be a multiple of five the perturbed array response is given by:

$$P_{8,5}(\phi) = \left| \frac{\sin \frac{N\pi d}{\lambda} (\sin \theta_{8,5} - \sin \phi)}{\frac{N}{5} \sin \pi \frac{5d}{\lambda} (\sin \theta_{8,5} - \sin \phi)} \right|^2 R_5^0(\phi)$$

where from figure 1, $R_5^0(\phi)$ is given by:

$$R_5^0(\phi) = \frac{1}{25} \left| 1 + z + z^2 e^{i\psi} + z^3 e^{i\psi} + z^4 e^{2i\psi} \right|^2$$

where

$$z = e^{2\pi if(2\tau_0 - \frac{d \sin \phi}{c})}$$

and $\psi = 2\pi if\tau_0$. Once again the effects of the time quantization are incorporated in the R_q^0 factor. At a frequency of $f_{½}$ the spacing of the N/5 element array is $\frac{5\lambda}{2}$ and consequently four quantization lobes will appear. These, in contrast to the exact array response do not appear at angles corresponding to the nulls of $R_5^0(\phi)$ and consequently have a finite height.

In figure 5 the polar diagram for N = 30, $f_{½}$ = 320 Hz and τ_0 = 1/2048 s is plotted where $\theta_{8,5}$ is 30°. The position and heights of the quantization lobes appearing in this figure are tabulated in the table below. In order to calculate the predicted values of these lobes, the $R_5^0(\phi)$ term, as suggested in Section 5, has been approximated by:

$$\hat{R}^{o}_{s} = \frac{1}{25}\,|\,1 + z + z^2 + z^3 + z^4\,|^2$$

$$= \frac{\sin^2\left(\frac{\pi 5d}{\lambda}(\sin\phi - \sin\hat{\theta})\right)}{25\sin^2\left(\frac{\pi d}{\lambda}(\sin\phi - \sin\hat{\theta})\right)},$$

where $\sin\hat{\theta} = 4\eta$. Using equation (6) the position of the mth quantization lobe is given by:

$$\sin\theta_m = \frac{16\eta}{5} - \frac{2m}{5}$$

$$= \frac{1}{2} - \frac{2m}{5}, \text{ for } m = -1,\ 1,\ 2 \text{ and } 3.$$

Equation (9) for the relative height of the mth quantization lobe reduces to

$$h_m = 20\log_{10}\left|\frac{\sin\pi 2\eta}{5\sin\pi\frac{2\eta+m}{5}}\right|.$$

In the table these values have been calculated from the above formulae and are compared with those measured from figure 5. The angular positions of the quantization lobes are in good agreement whilst their heights only show agreement at angles well removed from the main beam.

The position and heights of the quantization lobes for $\theta_{16,5}$ are also tabulated in the table. The polar response for $\theta_{16,5}$ is shown in figure 6 where the same parameters as for figures 4 and 5 have been used. Once again the predicted and observed quantization lobe parameters show good agreement.

TABLE 1. POSITION AND HEIGHT OF QUANTIZATION LOBES

m	Measured		Predicted	
	Position (degrees)	Height*	Position (degrees)	Height*
$\theta_{8,5}$				
-1	66	-19	64	-8
1	7	-17.5	6	-13
2	-17	-17	-17	-15.5
3	-43	-14.5	-44	-14.5
$\theta_{16,5}$				
-1	36	-14	37	-15
1	12	-19	11.5	-20
2	-10	-20	-11.5	-20
3	-37	-16	-37	-17

*Height in decibels relative to unity

6.3 Shading and angular perturbations

In figure 7 the perturbed array response corresponding to figure 5 but with a Hanning window incorporated is plotted. The fact, discussed in Section 5, that shading only significantly affects the width of the quantization lobes but not their number, angular position or heights is well illustrated by this figure.

Unfortunately the values of p and q may radically change for a small perturbation of the steered angle. In the above example if the steered angle is varied by 2^o to 28^o then the values of p and q are altered to 3 and 2 respectively. As illustrated by figure 8, the array response corresponding to this new angle will have drastically different characteristics. Also the examples given have been chosen to show well defined quantization lobes. However not all angles will correspond to reasonable values of p and q. The polar response of the line array of the previous examples is plotted in figure 9 for an arbitrary angle of 34^o. The grating lobes are no longer so well defined and the response is characterized by a general increase in the sidelobe level. This is perhaps the most serious shortcoming of the model.

7. CONCLUSIONS

A model has been proposed to predict the degradation in the response of an N element linear array at selected angles due to the finite sampling interval. It is capable of predicting the number, the angular position and approximate height of a number of high isolated side-lobes introduced by the time delay errors. Furthermore the dependence of these quantization lobes on the parameters of frequency, sampling rate and array shading can be deduced from the model.

Unfortunately the model is sensitive to relatively small perturbations in the angle of steer.

8. ACKNOWLEDGEMENTS

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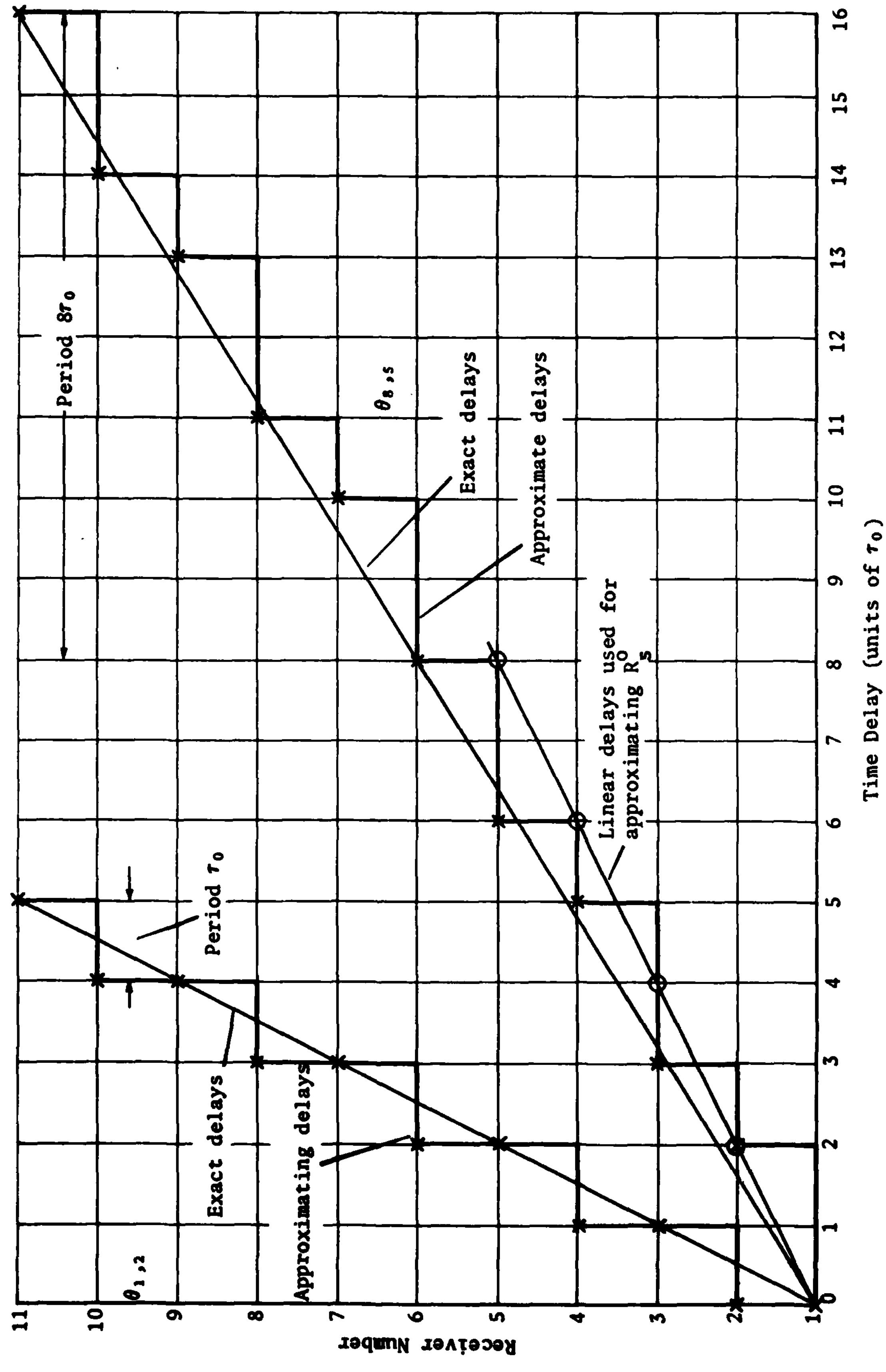

Figure 1. Exact and approximating time delays

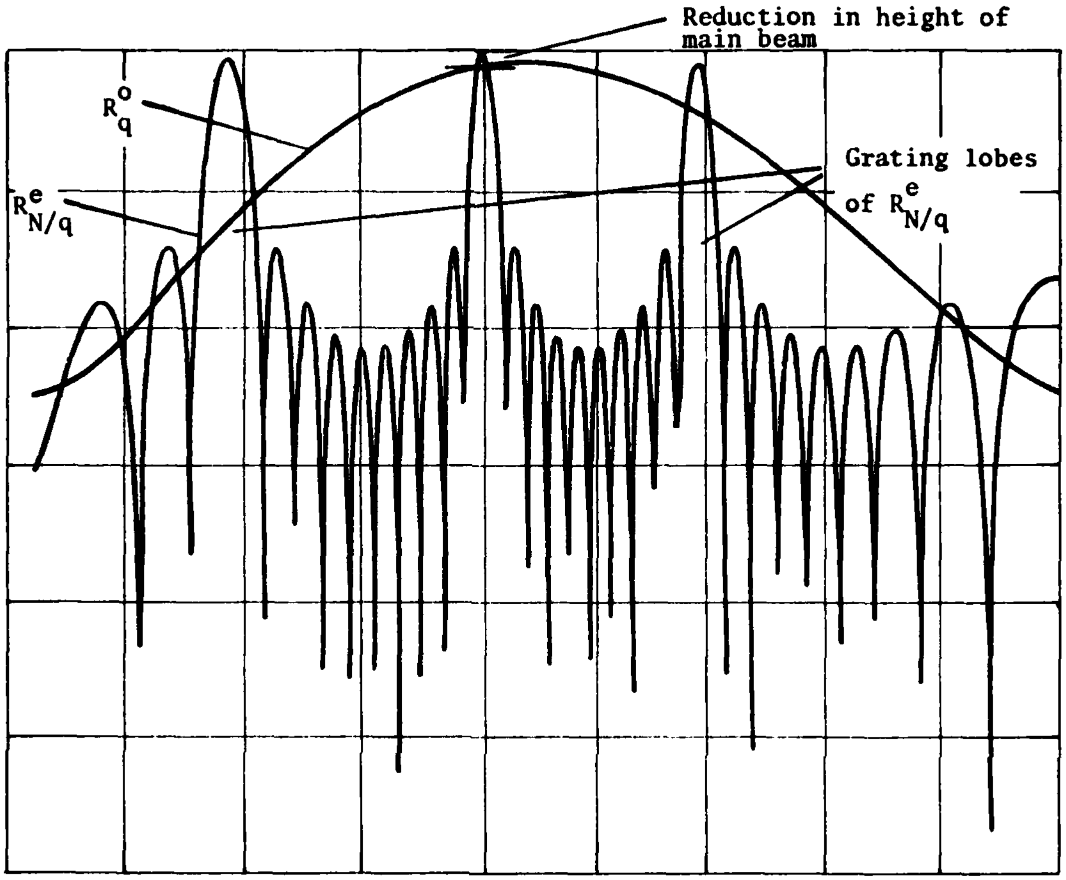

Figure 2. Perturbed array factors

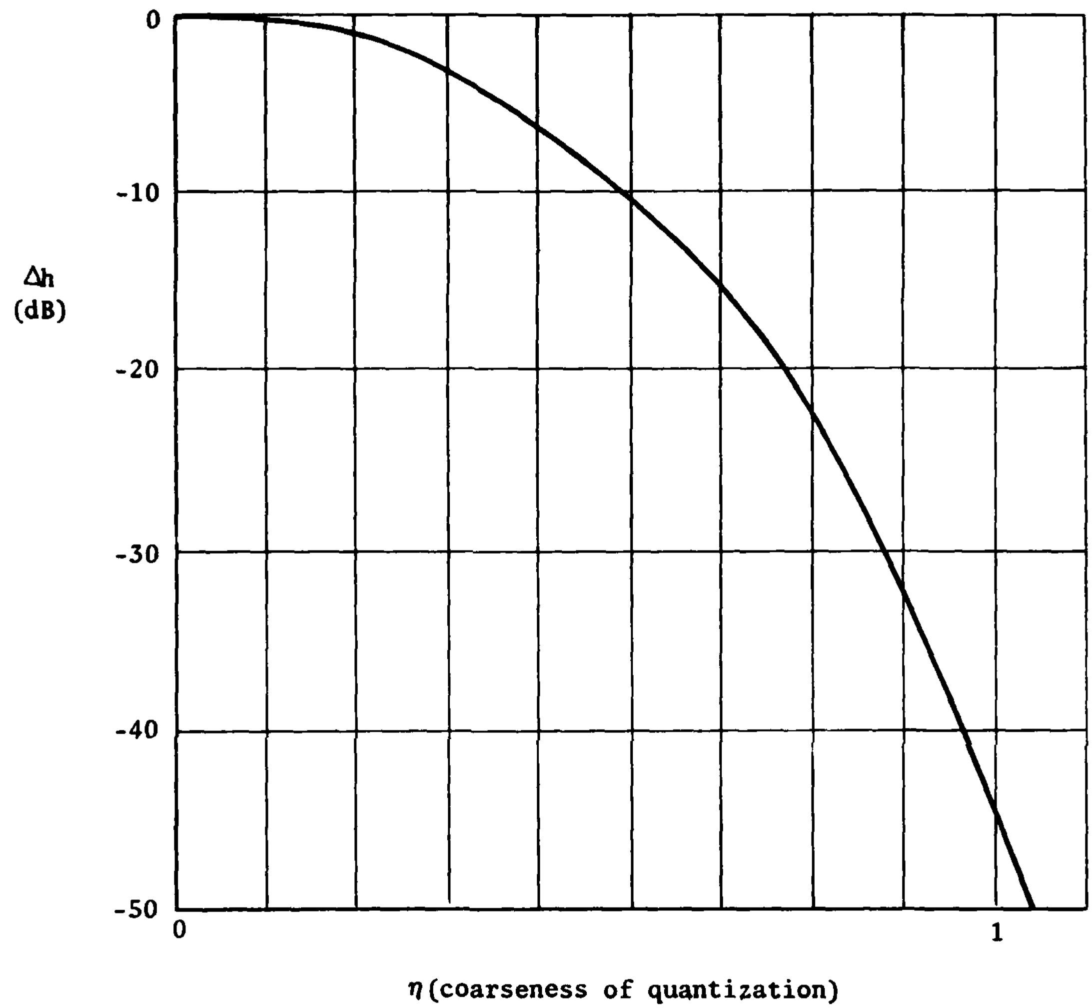

Figure 3. Reduction in height of main beam

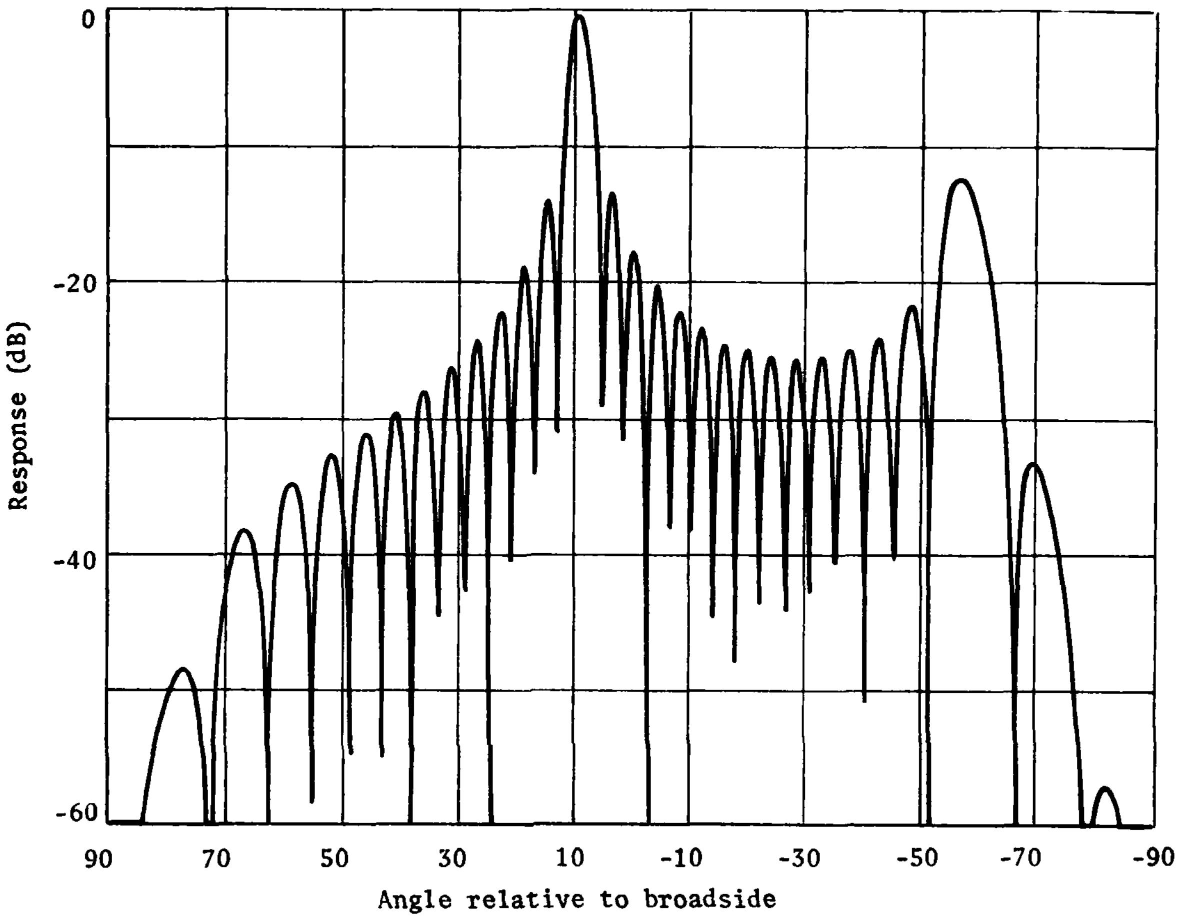

Figure 4. Perturbed response at 8.9° ($\theta_{1,2}$)

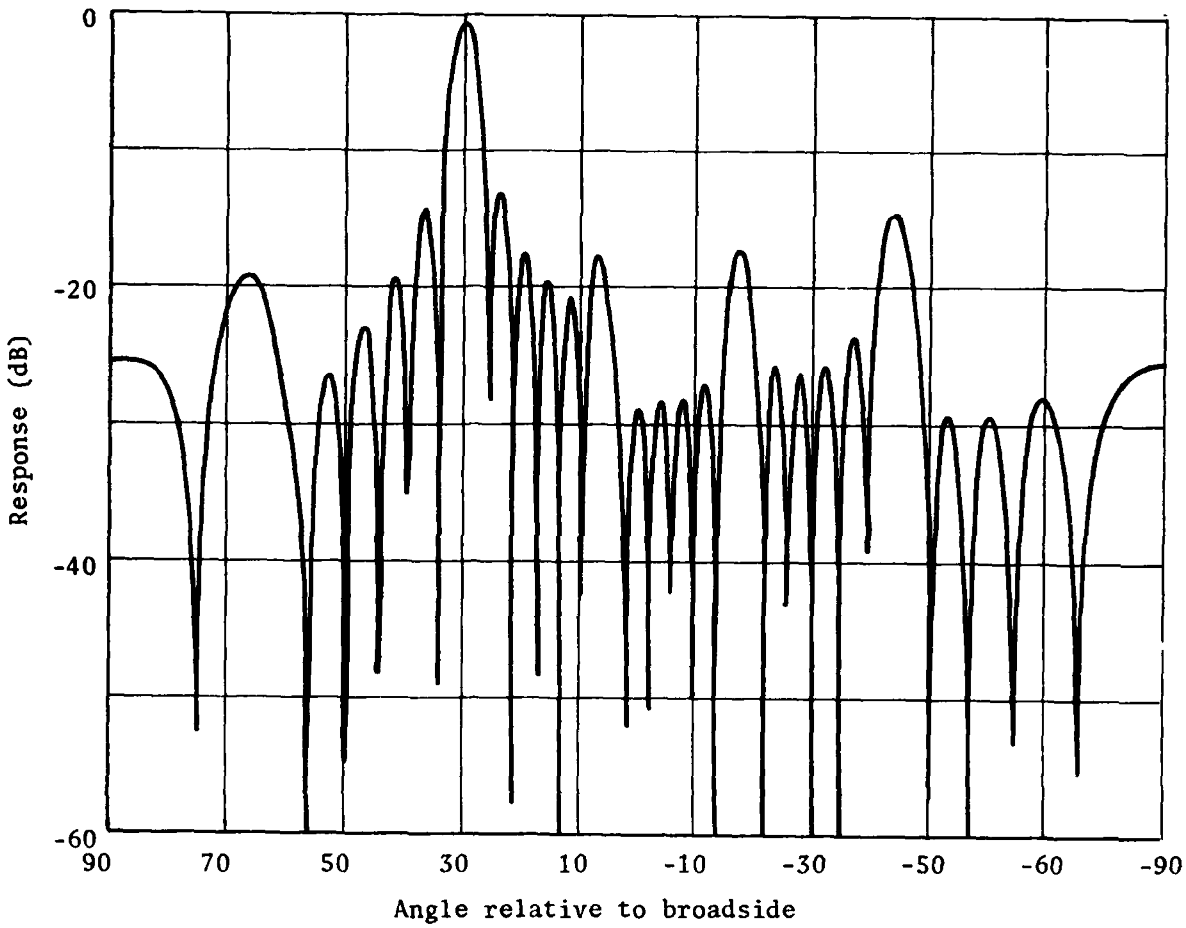

Figure 5. Perturbed response at 30° ($\theta_{8,5}$)

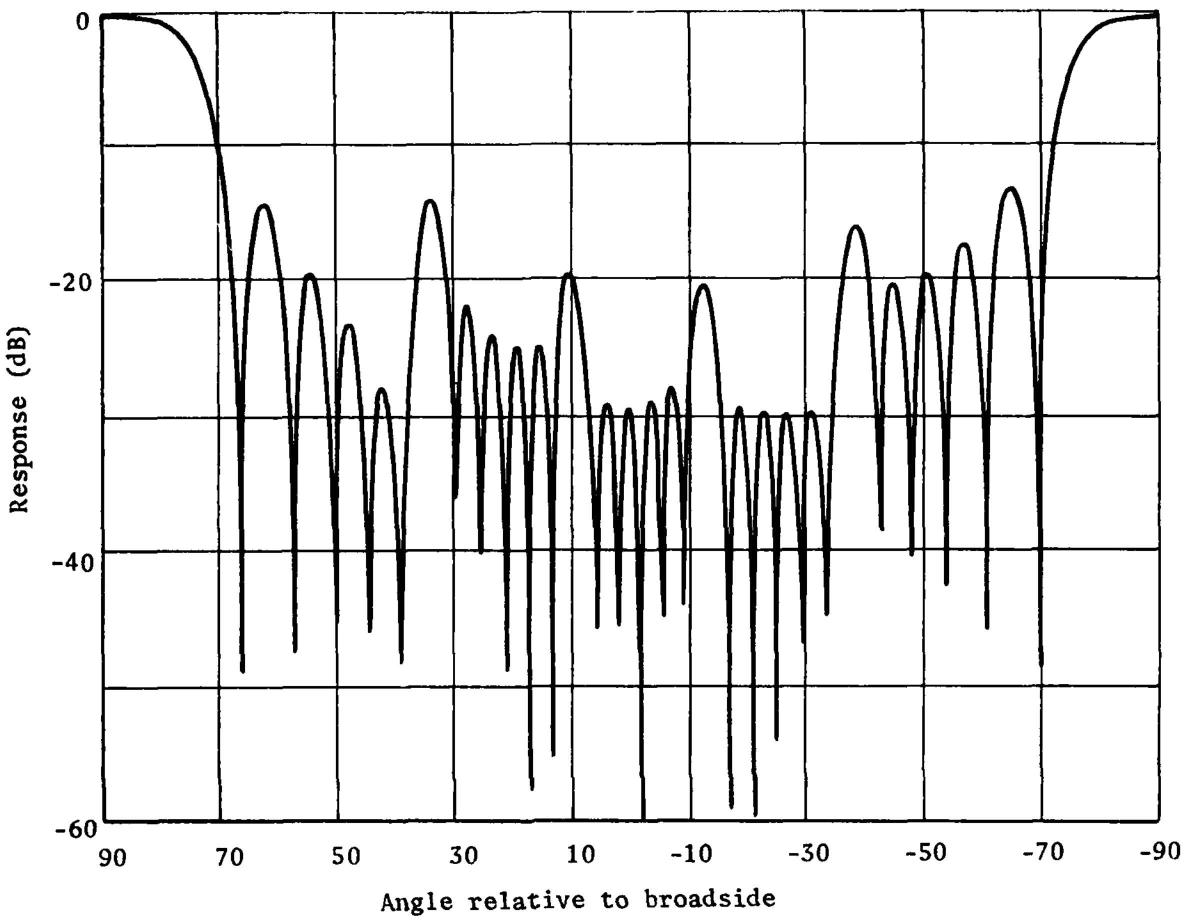

Figure 6. Perturbed response at 90° ($\theta_{16,5}$)

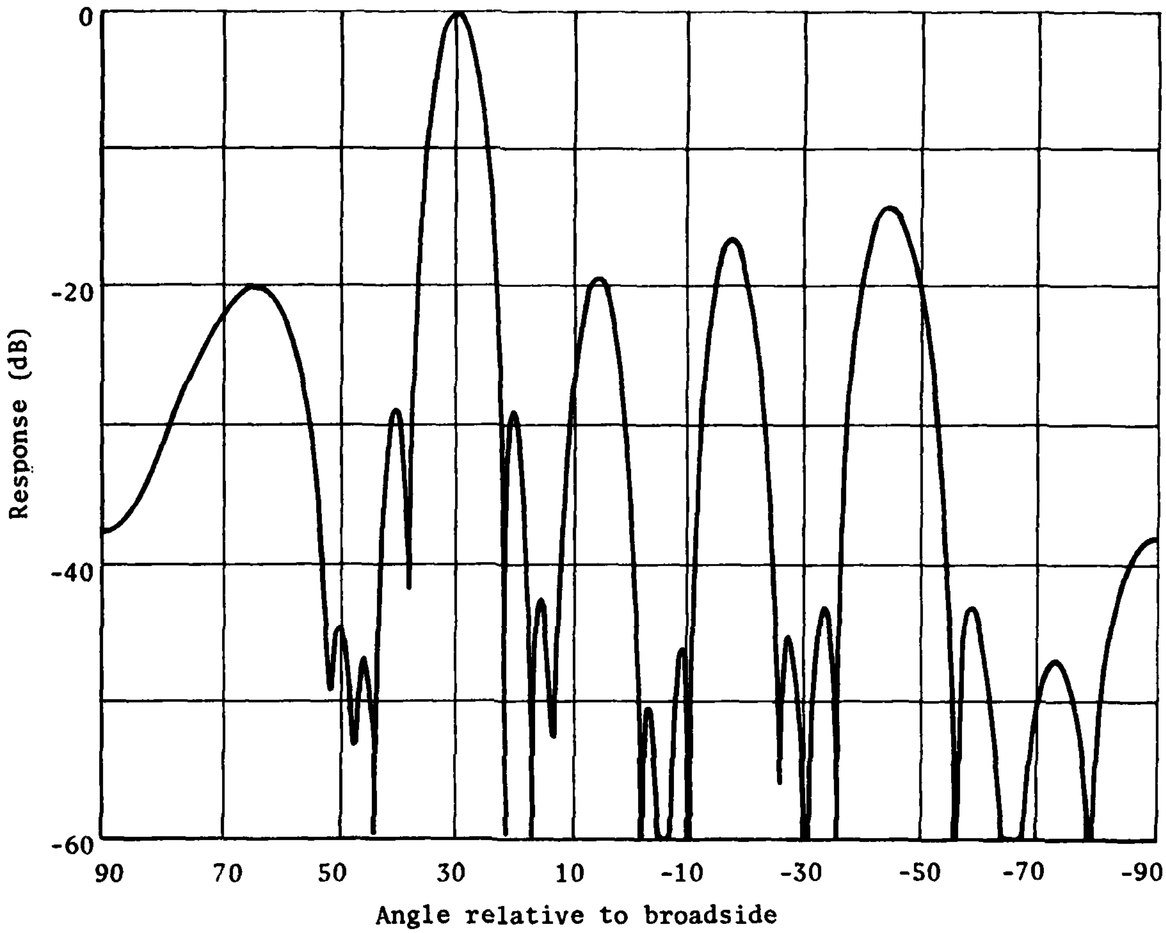

Figure 7. Perturbed Hanning response at 30° ($\theta_{8,5}$)

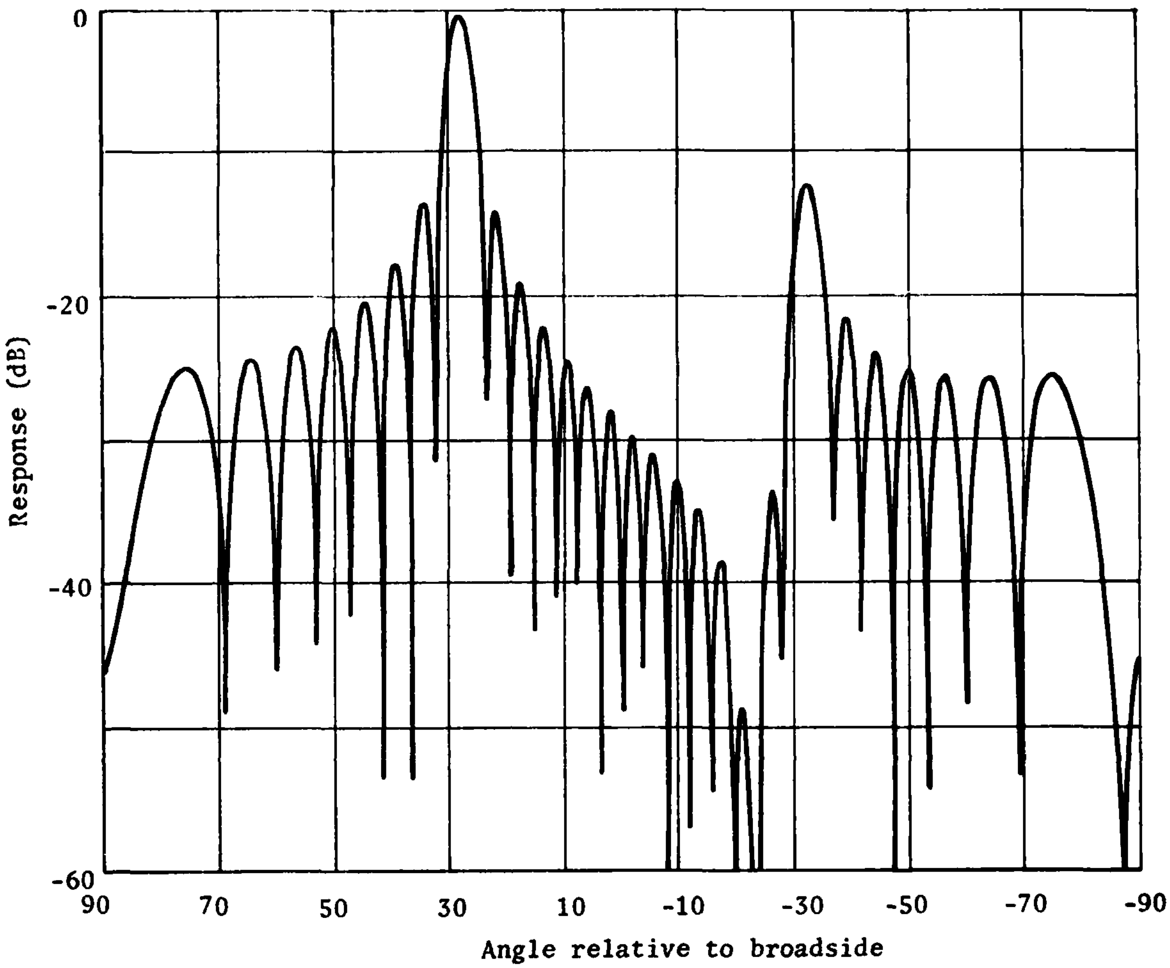

Figure 8. Perturbed response at 28° ($\theta_{3,2}$)

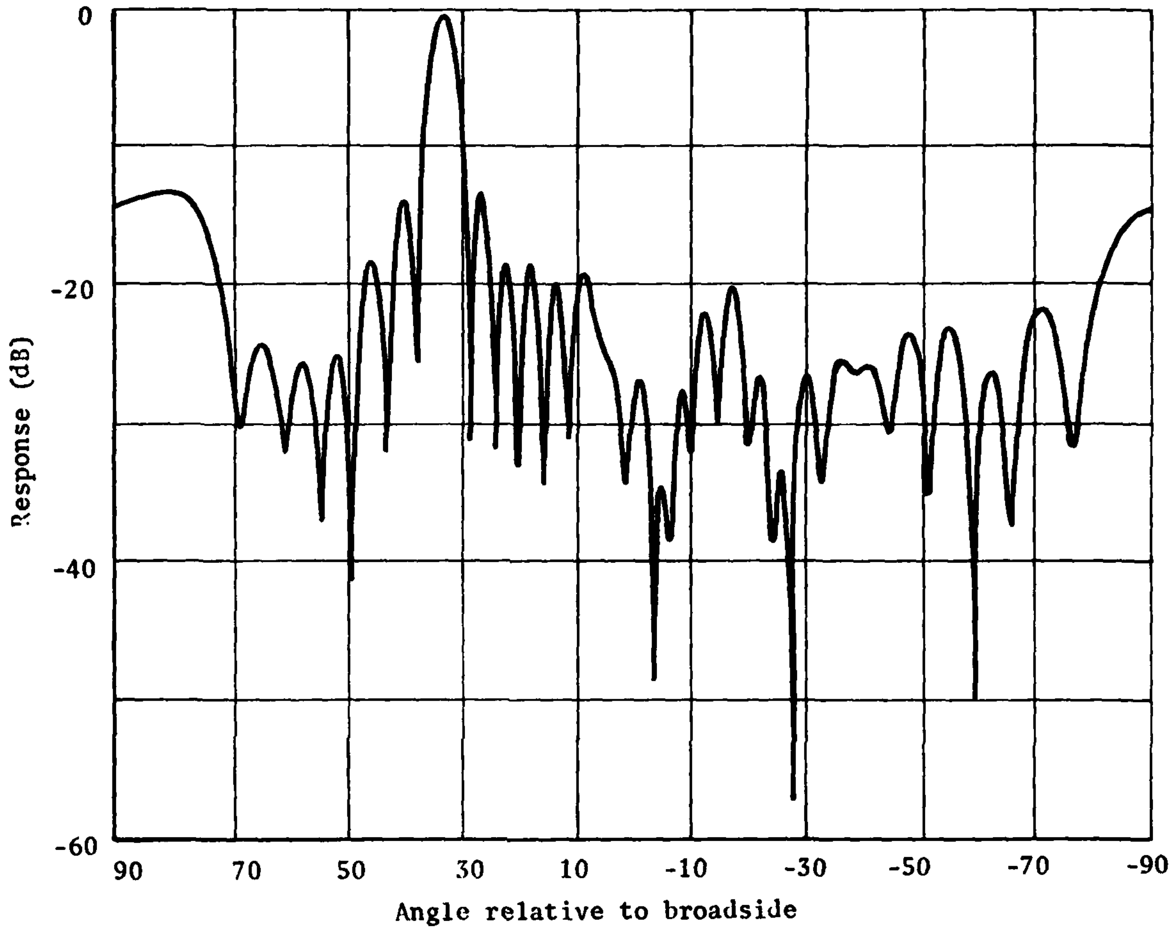

Figure 9. Perturbed response at 34°

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