Uniform approximation theory with constraints, strong uniqueness.

It is shown that the strong uniqueness theorem need not hold in its standard form when constraints are imposed on a uniform approximation problem. In particular, it is shown that when approximation with polynomials subject to a monotone constraint ($p^+(x) > 0$) and Hermite-Birkhoff interpolating constraints, a best possible result is the inequality

$$ ||f - p|| > ||f - p_f|| + \delta \left(||p - p_f||^{1/2m}\right) $$

where $p$ is any approximating polynomial satisfying the constraints and the additional condition that (OVER)

\[ \| P \| \leq M \quad (M \text{ fixed}) \quad \text{and} \quad P_f \text{ is the unique best approximation to } f \text{ from the given class of approximants.} \]
ON THE EXISTENCE OF STRONG UNICITY
OF ARBITRARILY SMALL ORDER

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The strong unicity theorem, first given by Newman and Shapiro (4), may
be described as follows: Given $C[a, b]$ and $W$ an $n$-dimensional Haar sub-
space of $C[a, b]$. Let $f \in C[a, b]$ and $p_f \in W$ be the best approximation to $f$
from $W$. Then there exists a positive constant $\gamma$, depending only on $f$, such
that
\[ \| f - p \| \geq \| f - p_f \| + \gamma \| p - p_f \| \]  
for all $p \in W$ where $\| h \| = \max\{|h(t)| : t \in [a, b]\}$, $h \in C[a, b]$. The extension
of this theorem to the setting of monotone approximation has recently been
studied by Fletcher and Roulier (3) and Schmidt (5). Specifically, fix an
interval $[a, b]$, integers $1 \leq r_0, \ldots, r_k$, signs $\epsilon_i = \pm 1$, $i = 0, \ldots, k$
and define
\[ K = K(r_0, \ldots, r_k; \epsilon_0, \ldots, \epsilon_k) \]  
by
\[ K = \{ p \in \Pi_n : \epsilon_j p(r_j)(x) \geq 0, a < x < b, j = 0, 1, \ldots, k \text{ with } k < n \} \]  
where $\Pi_n$ denotes the class of all real algebraic polynomials of degree $\leq n$.
The study of approximation of $C[a, b]$ by $K$ is called the monotone

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approximation problem. Professor G.G. Lorentz has played a major role in the development of the theory for this problem. See (2) for a brief expository treatment of this problem and an extensive bibliography.

In (3), Fletcher and Roulie constructed an example in $K=\{p \in \Pi_3 : p'(0) \geq 0\}$ on $[-1,1]$ which shows that the best result of form (1.1) that could hold in this setting would be where $\|p-p_f\|$ is replaced by $\|p-p_f\|^2$. Also, some positive results were given that were extended by Schmidt (5). In (5) it is proved that given $f \in C[a,b]$, $K$ as defined in (1.2), $p_f \in K$ the best monotone approximation to $f$ and a positive constant $M$, there exists $\gamma > 0$ depending only on $f$ and $M$ such that

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\|^2$$

(1.3)

for all $p \in K$ satisfying $\|p\| \leq M$.

In (5) one has the following definition: If $p_f$ is the best uniform approximation to $f \in C[a,b]$ from $W$ a subset of $C[a,b]$, we say that $p_f$ is strongly unique of order $\alpha$ $(0 < \alpha \leq 1)$ if for each $M > 0$ there is a constant $\gamma > 0$ such that

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\|^{1/\alpha}$$

for all $p \in W$ satisfying $\|p\| \leq M$. Thus, these two papers taken together show that in monotone approximation strong unicity of order $1/2$ holds and this is a best possible result.

In this paper we shall show that by taking an appropriate combination of interpolatory constraints with a monotone constraint one obtains an approximation problem in which strong unicity of order $\frac{1}{2m}$, $m$ a positive integer, holds and that this is also a best possible result.

Thus, fix $m$ a positive integer and define $K \subseteq \Pi_n$ by

$$K = \{p \in \Pi_n : p^{(1)}(x) \geq 0, a \leq x \leq b \text{ and } p^{(2)}(x_0) = \ldots = p^{(2m-1)}(x_0) = 0 \text{ for } x_0 \in (a,b) \text{ fixed, } n \geq 2m+1\}.$$  

(1.4)

Now, by referring to the general theory of (1), one can prove that corresponding to each $f \in C[a,b]$, there exists a unique best approximation,
p" from K to f. The basic tools of this theory are extreme linear functionals (extremals) of the dual of \( \Pi_n \) corresponding to f and a given p \( \in K \).

In this particular setting the extremals are as follows. Given f \( \in C([a,b]) \) and p \( \in K \), define for x \( \in [a,b] \), \( e_x^0 \) on \( C([a,b]) \) by \( e_x^0(g) = g(x) \) for all \( g \in C([a,b]) \) (point evaluation) and for x \( \in [a,b] \), and \( 1 \leq j \leq 2m \), \( e_x^j \) on \( \Pi_n \) by \( e_x^j(q) = q^{(j)}(x) \) for all \( q \in \Pi_n \). The linear functional \( e_x^0 \), x \( \in [a,b] \), is said to be an extremal for f and p provided \( |e_x^0(f - p)| = \|f - p\| \). The linear functional \( e_x^1 \), x \( \in [a,b] \) is said to be extremal for f and p provided \( e_x^1(p) = 0 \). Whenever \( e_x^1 \) is an extremal for f and p and x \( \in (a,x_0,b) \) then an additional extremal called an augmented extremal is also present; namely, the extremal \( e_x^2 \) for which \( e_x^2(p) = 0 \) must also hold (since \( p^{(1)}(x) \geq 0 \)). If \( e_x^1 \) is an extremal for f and p, then the linear functional \( e_x^{2m} \) is an augmented extremal for f and p with \( e_x^{2m}(p) = 0 \) holding (since \( p^{(1)}(x) \geq 0 \)). If one starts with an extremal set for f and p (which contains \( e_x^0, ..., e_x^{2m-1} \)) and adds all possible augmented extremals (as described above) to this set, then one has the augmented set of extremals for f and p corresponding to the original extremal set. Observing that these augmented extremal sets always correspond to Hermite-Birkhoff interpolation problems in which every supported block is even, it is relatively straightforward to prove that the maximal augmented extremal set for f and its best approximation, \( p_f \), from K must have \( n+2 \) elements which span the dual of \( \Pi_n \). Thus, K is generalized Haar and uniqueness of best approximations holds (1). In addition, suppose \( p_f \) is the best approximation to f from K. Then there exists \( k \leq n+2 \) extremals (e.g. (2)), \( E = \{e_i\}_{i=1}^k \), none of which are augmented extremals, for which \( E \) belongs to the convex hull of \( \{\sigma(e)_e : e \in E\} \) where \( \sigma(e) = \text{sgn}(f(x) - p_f(x)) \) if \( e = e_x^0 \) for some \( x \in [a,b] \), \( \sigma(e) = 1 \) if \( e = e_y^1 \) for some \( y \in [a,b] \) and \( \sigma(e_y^j) = 1 \), \( j = 2, ..., 2m - 1 \). Then, by adjoining to E the set \( E^a = \{ \text{all augmented extremals corresponding to elements of } E \} \) we must have that the set \( E_{\text{aug}} = E \cup E^a \) contains at least \( n+2 \) elements of \( \Pi_n^* \) which will necessarily span \( \Pi_n^* \) by the fact that every
supported block in the corresponding Hermite-Birkhoff problem is even.
Likewise, we must have that there exists $e \in E$ for which $e = e_x^0$ some $x \in (a, b]$ as otherwise $E$ is also an extremal set for $f$ and $p_f + c$, $c$ any constant, for
which $Q$ is in the convex hull of $\{\sigma(e): e \in E\}$ violating uniqueness of best approximation. Using these observations we can now prove

**Theorem.** Let $f \in C[a, b]$ and $p_f \in K$ be the best approximation to $f$ from $K$.
Given $M > 0$ there exists $\gamma = \gamma(f, M) > 0$ such that for $p \in K$ satisfying $\|p\| \leq M$,

$$\|f - p\| \geq \|f - p_f\| + \gamma \|p - p_f\|^{2m}$$

(i.e. strong unicity of order $1/2m$) and this inequality is best possible.

**Proof:** The proof is an extension of the techniques of Fletcher and Roulier and Schmidt. If $f \in K$ then $\gamma = (2M)^{1-2m}$ suffices. Thus, assume $f \notin K$. Let $E = \{e_j\}_{j=1}^k$ be a set of $k$ extremal, which contains $\{e_j\}_{j=0}^{2m-1}$ but contains no augmented extremals, for which $Q$ is in the convex hull of $\{e(e): e \in E\}$. Set $E^{\text{aug}} = E \cup E^a$. Further, define $E^0, E^1 \subseteq E$, where $e \in E$ is in $E^0$ if $e = e_x^0$ for some $x \in (a, b]$ and $e \in E^1$ if $e = e_y$ for some $y \in [a, b]$. Define the semi-norm $\|\cdot\|$ on $E$ by $\|q\| = \max\{|e(q)|: e \in E\}$. Set $Q = \{q = \frac{p_f - p}{\|p_f - p\|}: \|p_f - p\| \neq 0$ and $p \in K\}$. We claim that $\inf_0 \max_{e \in E^0} \sigma(e)e(q) = \tau > 0$. Indeed, if there exists $q \in Q$ with $q \in Q$ e \in E^0 \max_{e \in E^0} \sigma(e)e(q) < 0$. Then from $q = \frac{p_f - p}{\|p_f - p\|}$ with $\|p_f - p\| \neq 0$ and $p \in K$ we see that $e(q) \neq 0$ for some $e \in E$ and $e(q) < 0$ for all $e \in E^1$. Thus, $\sigma(e)e(q) < 0$ for all $e \in E$ with strict inequality holding at least once. This violates the fact that $Q$ belongs to the convex hull of $\{\sigma(e): e \in E\}$. Using this lower bound, we have for $p \in K$ with $\|p_f - p\| \neq 0$ that there exists $e \in E^0$ for which

$$\sigma(e)(p_f - p) \geq \tau \|p_f - p\|'.$$

Now observe that (as usual)

$$\|f - p\| \leq \|f - p_f\| + \gamma \|p_f - p\|'.$$

As this inequality holds for $\|p_f - p\|' = 0$, we have a strong uniqueness-type result for the seminorm $\|\cdot\|'$. Next, the norm, $\|p\| *= \max_{e \in E^{\text{aug}}} |e(p)|$, is introduced. Thus, there exists a constant $\lambda > 0$ such that $\|p\| * \geq \lambda \|p\|$ for all $p \in K$.

Finally, we claim that there exists $A > 0$ for which $\|p_f - p\|' \geq A (\|p_f - p\|')^{2m}$, $\forall p \in K$ satisfying $\|p\| \leq M$. First observe that $\|p_f - p\|' = 0$ with $p \in K$ implies
e(p_f-p)=0 \forall \text{ } e \in E^{\text{aug}} \text{ so that } \|p_f-p\|^{\ast}=0. \text{ Now, for } e \in E, \text{ there exists a constant } K_1 \text{ for which } |e(p_f-p)| \geq K_1 |e(p_f-p)|^{2m} \text{ as } \|p\| \leq M. \text{ Let } e \in E^{\text{aug}} \cap E \text{ and assume that } e = e_0^{2m} \text{ (the augmented extremal corresponding to } e_0^{1}). \text{ We claim that there exists } K_2 > 0 \text{ for which } |e_0^{1}(p_f-p)| \geq K_2 |e_0^{2m}(p_f-p)|^{2m} \text{ for } p \in K \text{ satisfying } \|p\| \leq M. \text{ If this is not the case, then corresponding to each integer } v > 0 \text{ there exists } q_v \in K \text{ with } \|q_v\| \leq M \text{ for which } |q_v^{1}(x_0)| + |q_v^{2m}(x_0)|^{2m}. \text{ Now we may assume that } q_v \text{ converges uniformly to } q \in K. \text{ Clearly, } q'(x_0) = 0. \text{ We can write } q_v^{(2m)}(x_0) = q_v^{(2m)}(x_0) + s_v(x)(x-x_0)^{2m-1} + s_v(x)(x-x_0)^{2m} = \beta_v + \alpha_v(x-x_0)^{2m-1} \text{ for } x \in [a,b], \text{ some } M_1 \text{ independent of } v \text{ and } q_v^{(2m)}(x_0) \geq 0 \forall x \in [a,b]. \text{ For } v \text{ sufficiently large (so that } x \in (a,b)), \text{ set } x-x_0 = -\frac{1}{2m^{2m-1}} \text{. This gives } \frac{(2m-1)}{M_1} \beta_v - \alpha_v^{2m} \text{ or that there exists a constant } K_3 \text{ independent of } v \text{ (sufficiently large) such that } |q_v^{(2m)}(x_0)| \geq K_3 |q_v^{(2m)}(x_0)|^{2m} \text{ which is our desired contradiction. Finally, if } e \in E^{\text{aug}} \cap E \text{ is of the form } e = e_y \text{ some } y \in (a,b)^{(x_0)}, \text{ the above argument (modified) shows that there exists } K_3 \text{ for which } |e_y^{1}(p_f-p)| \geq K_3 |e_y^{2m}(p_f-p)|^{2m} \text{ for } p \in K \text{ satisfying } \|p\| \leq M \text{ where } K_3 \text{ is independent of } p. \text{ By taking } A \text{ to be the smallest of the constants produced above, we have that } \|p_f-p\|^{\ast} \geq A(\|p_f-p\|^{\ast})^{2m} \text{ implying } \|f-p\| \geq \|f-p_f\| + \gamma \|p_f-p\|^{2m} \text{ for } p \in K \text{ satisfying } \|p\| \leq M \text{ with } \gamma = \gamma(M,f) > 0 \text{ independent of } p. \text{ To show this result is best possible we construct an example. Fix } m \text{ a positive integer and let } r_1, r_2, r_3 \text{ denote the three roots of } p_0(x) = 2m+1 \text{ 2x}^{2m-1} \text{ (note } -2 < r_1 < -1, \text{ } r_2 = -1, \text{ } 0 < r_3 < 1). \text{ Define } K = \{ p \in \mathbb{P}_{2m+1} : p'(x) > 0, x \in [r_1, r_3], \text{ } p(0) = \ldots = p(2m-1)(0) = 0 \} = \{ p \in \mathbb{P}_{2m+1} : p(0) = a_0x^{2m+1} + a_1x^{2m} + a_2x + a_3, \text{ } p'(x) > 0 \text{ on } [r_1, r_3] \}. \text{ Define } g \in C[r_1, r_3] \text{ by } g(r_1) = \frac{1}{2}, \text{ } g(-1) = \frac{1}{2}, \text{ } g(r_3) = \frac{1}{2} \text{ and extend } g \text{ linearly to all } [r_1, r_3]. \text{ Set } f = g + 2x^{2m+1} \text{ and } p_f(x) = 2x^{2m+1}. \text{ Note that } (-e_0^{1}, e_0^{1}, -e_0^{1}, e_0^{1}) \text{ is an extremal set for } f \text{ and } p_f \text{ whose convex hull contains the zero of } V^*, V = \{ a_0x^{2m+1} + a_1x^{2m} + a_2x + a_3 \}. \text{ (Coefficients are: } a_1 = 1, \text{ } a_2 = 1 + a_3 \).
Thus, $p_f$ is the desired best approximation to $f$ from $K_2$. Next, define $p_{\alpha}(x) = p_f(x) + \alpha p_0(x) + 4\alpha^2 m x$, for $0 < \alpha \leq \alpha_0$ where $\alpha_0$ is chosen so small that $|f-p_{\alpha}| = |g-\alpha[p_0+4\alpha^2 m x]|$ decreases as $x$ moves away from $r_i$ in a neighborhood of $(r_1, r_2, r_3)$ for all $\alpha$ ($0 < \alpha \leq \alpha_0$).

This can be done since $|g|$ decreases linearly as $x$ moves away from $r_i$.

Hence $\alpha_0$ can be chosen so small that $\|f-p_{\alpha}\| = \max_{i=1,2,3} |(f-p_{\alpha})(r_i)|$ for $0 < \alpha \leq \alpha_0$.

Also, $\|f-p_f\| = \frac{1}{2}$, $\|p_f-p_{\alpha}\| \geq |p_f(0)-p_{\alpha}(0)| = \alpha$, and

$p_{\alpha}'(x) = 2(2m+1)x m + 2m + 4mx^{2m-1} + 4\alpha^2 m x$. Now, for $x > 0$, $p_{\alpha}'(x) > 0$; for $x \in [r_1, -\alpha]$, the term $2(2m+1)x^2 m$ dominates showing that $p_{\alpha}'(x) > 0$ here; and for $x \in [-\alpha, 0]$ the term $4\alpha^2 m x$ again implying that $p_{\alpha}'(x) > 0$. Thus $p_{\alpha} \in K$ and $(\|f-p_{\alpha}\| - \|f-p_f\|)/\|p_f-p_{\alpha}\| \leq \frac{4\alpha^2 m}{\alpha^2 m}$. This implies that we must have $\beta > 2m$ in order for the strong unicity theorem to hold for this $f$ and $p_f$.

By suitably selecting $g$, it can be shown that this weaker strong unicity result holds for an $f$ which also satisfies all the constraints of $K$. Additional results on this topic will appear elsewhere.

REFERENCES

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