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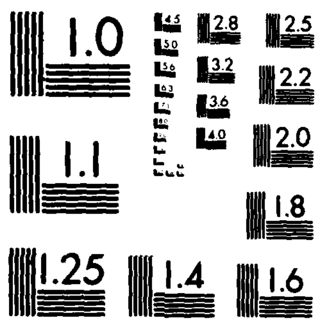

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A SEMIGROUP APPROACH TO PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY+

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The objective of this investigation is to give sufficient conditions on f and g such that local semigroups can be associated with the equation (1). This will imply representation formulas for the solutions of (1), and secondly to discuss various notions of solutions which have arisen in the study of (1). Although only the autonomous equation is considered here, many of the results remain true if f and g depend on t.

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Although the theory of functional differential equations in $\mathbb{R}^n$ is very well developed, comparatively little is known about these equations, when the right hand side contains unbounded operators. Since semigroup methods have proved to be a powerful tool in treating functional differential equations in $\mathbb{R}^n$ [1,9,12], it seems desirable to extend semigroup theory methods also to the more general situation of partial differential equations with delay. The present paper is intended to make a contribution in this sense.

We shall consider the functional differential equation

$$(1) \qquad \frac{d}{dt} x(t) = f(x(t)) + g(x(t),x_t)$$

in a reflexive Banach space Y with norm $|\cdot|$. As usual for a function $x:[-r,T) \to Y$, we let $x_t(s) = x(t+s)$ for $s \in [-r,0]$ and $t \in [0,T)$. The delay r^- in (1) is chosen in $[-\infty,0]$ and f is a nonlinear, not necessarily bounded operator from $\mathrm{Dom}(f) \subset Y$ into Y. The initial datum at time 0 is a Y-valued function defined on $[-r,0]$. The existence and uniqueness problem as well as some qualitative aspects of (1) have been treated in different state spaces in a number of recent papers, some of which are mentioned in the references [5,6,7,8,15,16,17].

The objective of this investigation is to give sufficient conditions on f and g such that local semigroups can be associated with (1) - this will imply representation formulas for the solutions of (1) - and, secondly, to discuss various notions of solutions which have arisen in the study of (1). Although only the autonomous equation is considered here, many of the results remain true if f and g depend on t.

The state-space chosen for the presentation is $Y \times L^p(-r,0;Y)$, where for $(\eta,\phi) \in Y \times L^p(-r,0;Y)$ we use the norm

$$\|(\eta,\phi)\| = (|\eta|^p + \int_{-r}^{0} \exp(\rho s)|\phi(s)|^p ds)^{1/p}$$

for some $\rho \geq 0$. Thus $Y \times L^p(-r,0;Y)$ becomes a reflexive Banach space, denoted by Z. In case $0 \leq r < \infty$, one may chose $\rho = 0$. On the other hands, for $r = \infty$, the need for weighting the norm is quite obvious and our results will remain true for weighting functions different from the one used here, as long as they are bounded from above and below by an exponential function. The projection of Z onto the first and second components will be denoted by P_1 and P_2, respectively.

Now that the state-space is fixed we specify as initial data for (1) at $t = 0$

$$(2) \qquad (x(0),x_0) = (\eta,\phi) \quad \text{for} \quad (\eta,\phi) \in Z.$$

The conditions on f and g will guarantee that the solutions of (1) and (2) do not depend on a specific representative in the class $\phi \in L^p(-r,0;Y)$.

Next we reformulate (1) and (2) in Z. This abstract equation is not a consequence of calculations, but it is motivated by previous knowledge about semigroup-theory treatment of (1) and (2) in case $Y = \mathbb{R}^n$. Therefore, we consider

$$(3) \qquad \begin{cases} \frac{d}{dt} z(t) = Az(t), & \text{in } Z \\ z(0) = z_0, & \text{for } z_0 \in Z \end{cases}$$

where $\mathrm{Dom}(A) = \{(\eta,\phi) \,|\, \phi \in W^{1,p}(-r,0;Y),\ \eta = \phi(0),\ \phi(0) \in \mathrm{Dom}(f)\}$ and for $(\phi(0),\phi) \in \mathrm{Dom}(A)$

$$A(\phi(0),\phi) = (f(\phi(0)) + g(\phi(0),\phi),\dot{\phi}).$$

Here $W^{1,p}(-r,0;Y)$ stands for the Sobolev-space of absolutely continuous functions defined on $[-r,0]$ with first derivative in $L^p(-r,0;Y)$. Conditions will be given that guarantee that A generates a semigroup, and it then needs extra analysis to clarify how these "generalized" solutions are associated with (1) and (2). The conditions on f and g are motivated by the following two examples:

(i) $\quad g(\eta,\phi) = h_1(\eta) + \int_{-\infty}^{0} k(s)h_2(\phi(s))ds$

for $(\eta,\phi) \in Z$, $h_i : Y \to Y$, for $i = 1,2$ and $k:[-\infty,0] \to \mathbb{R}$. Here, for a sufficiently rich class of kernels k, the smoothness of the maps h_i determines the smoothness of $g:Z \to Y$.

$$\text{(ii)} \qquad g(\eta,\phi) = h_3(\eta,\phi(-r_1),\ldots,(-r_\ell)),$$

with $h_3:Y^{\ell+1} \to Y$, which corresponds to the case when (1) is a difference differential equation. Contrary to (i), Lipschitz continuity of h_3, for example, does not imply Lipschitz continuity of g, and the situation is even worse, since h_3 is not even well defined on Z.

For the convenience of the reader we end this section by recalling the definition of local semigroup.

Definition [4].

Assume that for each $z \in Z$ there is associated a strictly positive number $t(z)$. Let t^+ denote the suprenum of these numbers. For each $t \in [0,t^+)$ let $D(t) = \{z \in Z: t < t(z)\}$. A family of operators $\{T(t)\}:D(t) \to Z$ is called a strongly continuous local semigroup in Z if

a) $D(0) = Z$ and $T(0)$ is the identity operator on Z,

b) $D(t_2) \subset D(t_1)$ for $0 \leq t_1 < t_2 < t^+$, and $z \in D(t)$ for all $0 \leq t < t(z)$,

c) if $t, s \geq 0$ and $t + s < t^+$, then
$T(s)D(t+s) \subset D(t)$ and
$T(t)T(s)z = T(t+s)z$ for all $z \in D(t+s)$,

d) for each t, $T(t)$ is a continuous operator on $D(t)$,

e) for each $z \in Z$, the map $t \to T(t)z$ is continuous on $[0,t(z))$.

2. Local Semigroups

We begin by listing all the hypotheses that are needed in this section. Some familiarity with semigroup theory is assumed; as a reference we refer to [2].

(H1) The operator f: Dom$(f) \to Y$, Dom$(f) \subset Y$, is densely defined and $(f-\omega I)$ is m-dissipative for some $\omega \geq 0$.

(H2) $g: Z \to Y$ is locally Lipschitzian, i.e. there exists a nondecreasing real-valued function L such that

$$|g(x) - g(y)| \leq L(r) \|x-y\|$$

for all $\|x\| \leq r$ and $\|y\| \leq r$.

Condition (H3) below is a generalization of Borisovich-Turbabin type conditions previously used in case $Y = \mathbb{R}^n$. For a discussion of this condition we refer to [9], where it is also shown that a large class of maps g of the form (ii) satisfy (H3).

(H3) (a) If for some $\alpha > 0$, $x \in L^p(-\infty,\alpha;Y)$ and x is absolutely continuous on $[0,\alpha)$, then the map $G: t \to g_2(x(t),x_t)$

is defined a.e. on $[0,\alpha)$, depends on the equivalence class of x only and is in $L^1(0,\alpha;Y)$.

(b) There exists a nonnegative, nondecreasing function $\gamma: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\alpha > 0$ and $\beta > 0$ the inequality

$$\int_0^t |g_2(x(s),x_s)-g_2(y(s),y_s)|ds \le \gamma(t,\beta)\left(\int_r^t \bar{\rho}(s)|x(s)-y(s)|^p ds\right)^{1/p},$$

with $\bar{\rho}(s) = e^{\rho s}$ for $s \in [-r,0]$ and $\bar{\rho}(s) \equiv 1$ on $[0,\infty)$, holds for $t \in [0,\alpha)$ and all functions x,y in $L^p(-\infty,\alpha;Y)$ which are absolutely continuous on $[0,\infty)$ with $\|x_s\| \le \beta$, $\|y_s\| \le \beta$ for $s \in [0,\infty)$.

(H4) g is defined on $\mathscr{W}^{1,p} = \{(\phi(0),\phi) \mid \phi \in W^{1,p}(-r,0;Y)\}$ and is locally lipschtzian from $\mathscr{W}^{1,p}$, endowed with the suprenum norm, to Y.

(H5) g is positive definite with constant k_2, i.e. for all $\phi \in \mathrm{Dom}(g)$, g does not depend on the values that ϕ takes on $[-k_2,0]$.

For $(\eta,\phi) \in Z$ the function $x(\cdot;\eta,\phi)$ will be called (strong) solution of (1) and (2), if it is defined on $[-r,t_1)$ with $t_1 > 0$, if it is absolutely continuous on $[0,t_1)$, and satisfies (2) and (1) almost everywhere.

<u>Theorem 1.</u> Assume that (H1) and (H2) hold. Then

(a) A generates a local semigroup $T(t)$ in Z, given by $T(t)z = \lim_n (I-\frac{t}{n}A)^{-n}z$, for all $t \in [0,t(z))$ and $z \in Z$.

(b) For $z \in Dom(A)$, $T(t)z$ satisfies (3) and the solution $x(\cdot\,;z)$ of (1) and (2) is given by

$$x(t;z) = P_1T(t)z, \quad \text{for } t \in [0,t(z)) \quad \text{and} \quad x(t;z) = (P_2z)(t) \quad \text{for } t \in [-r,0]. \tag{4}$$

Of course, if in (H2) the Lipschitz constant can be chosen globally, then $t(z) = \infty$ for all $z \in Z$ and A generates a (global) semigroup on Z.

<u>Proof.</u> We give here an outline of the proof and refer to [12] for the details. For each $\beta \in [0,\infty)$ let Π^β denote the radial projection on Z, so that for $z \in Z$

$$\Pi^\beta z = \begin{cases} z & \text{for } \|z\| \le \beta \\ \frac{\beta}{\|z\|}z & \text{for } \|z\| > \beta \end{cases} .$$

For fixed but arbitrary $\beta > 0$ we remark that the map $z \to g(\Pi^\beta z)$ is globally Lipschitz continuous with Lipschitz constant $2L(\beta)$. Some calculations then show that the operator A_β given by

$$\mathrm{Dom}(A_\beta) = \mathrm{Dom}(A)$$

and

$$A_\beta(\phi(0),\phi) = (f(\phi(0)) + g(\Pi^\beta(\phi(0),\phi)),\dot{\phi})$$

satisfies the conditions of the Crandall-Ligett theorem [3], i.e. $A_\beta - w(\beta)I$ is dissipative for some $w(\beta) \in \mathbb{R}$ and range of $(I-\lambda(A_\beta)) = Z$ for all sufficiently small nonnegative λ. This implies that A_β generates a (global) semigroup $T_\beta(t)$, $t \geq 0$, on Z for all $\beta > 0$ given by

(5) $$T_\beta(t)z = \lim_n (I - \frac{t}{n} A_\beta)^{-n} z \quad \text{for } z \in Z.$$

Moreover $T_\beta(t)$ is Lipschitz continuous with Lipschitz constant $\exp(w(\beta)t)$, and $T_\beta(\cdot)z$ is Lipschitz continuous for each fixed $z \in Z$. For each $z \in Z$ with $\|z\| < \beta$ let

$$t_\beta(z) = \{\inf t : \|T_\beta(t)z\| \geq \beta\}.$$

We shall verify that for $t \in [0,t_\beta(z))$ we can replace A_β by A in (4), so that

(6) $$T(t)z = \lim_n (I - \frac{t}{n} A_\beta)^{-n} z = \lim_n (I - \frac{t}{n} A_\beta)^{-n} z \quad \text{on } [0,t_\beta(z)).$$

Choose $T \in (0,t_\beta(z))$ and put $\gamma = \beta - \sup_{t\in[0,T]} \|T_\beta(t)z\|$. Obviously $\gamma > 0$. **Assume** first that $z \in \mathrm{Dom}(A)$ and let $J_\lambda = (I - \lambda A_\beta)^{-1}$ for nonnegative, sufficiently small λ. Then by [18,pg.457] we have for all $m \geq n > 0$, and $j = 1,\dots,n$ and $t \in [0,T]$

$$\|J^j_{\frac{t}{n}} z - J^m_{\frac{jt}{nm}} z\| \leq (2(\frac{jt}{n} \frac{t}{n}(1-\frac{j}{m})))^{1/2} \exp(4w(\beta)\frac{jt}{n}) \|A_\beta z\|$$

$$\leq \frac{2t}{\sqrt{n}} \exp(4w(\beta)t) \|A_\beta z\| \leq \frac{2T}{\sqrt{n}} \exp(4w(\beta)\frac{jt}{n}) \|A_\beta z\| ,$$

where n and m are chosen sufficiently large, so that both $J_{\frac{T}{n}}$ and $J_{\frac{T}{m}}$ exist. Taking the limit as $m \to \infty$ in the last estimate, we get

$$\|J^j_{\frac{t}{n}} z - T_\beta(\frac{jt}{n})z\| \leq \frac{2T}{\sqrt{N_0}} \exp(4w(\beta)T) \|A_\beta z\| .$$

Choosing N_0 such that $\frac{2T}{\sqrt{N_0}} \exp(4w(\beta)T) \|A_\beta z\| < \gamma$, we get for all $n \geq N_0$, $j = 1,\dots,n$ and $t \in [0,T]$

$$(7) \quad \|J^j_{\frac{t}{n}} z\| \leq \|T_\beta(\frac{tj}{n})z - J^j_{\frac{t}{n}} z\| + \|T_\beta(\frac{tj}{n})\| < \beta.$$

Since β was arbitrary (7) implies (6) for $z \in \mathrm{Dom}(A)$. For arbitrary $z \in Z$, (6) follows from the density of Dom(A) and the Lipschitz continuity of $(I - \frac{t}{n} A_\beta)_z^{-1}$. Now take $0 < \beta_1 < \beta_2$, then $t_{\beta_1}(z) \leq t_{\beta_2}(z)$ and therefore $t(z) = \lim_{\beta\to\infty} t_\beta(z) \in (0,\infty]$ exists for every $z \in Z$. Finally, it is simple to check that $\{T(t)z: t \in [0,t(z))\}$ is a local semigroup in Z and that (6) holds for all $t \in [0,t(z))$. Assertion (b) of the theorem follows from [3, Theorem 2]. Indeed, if $z \in$ Dom(A), then (3) holds on $[0,t(z))$. Here we note that Z is reflexive and that $T(t)z$ is Lipschitz continuous in t. By a general result in [14], $T(t)$ is a local translation semigroup. Therefore for $z \in$ Dom(A) we may define

$$x(s;z) = P_2T(0)z)(s) \quad \text{for almost every } s \in [-r,0],$$

$$x(s;z) = P_1T(s)z \quad \text{for } s \in [0,t(z)),$$

and taking projection P_1 in (3) we see that $x(\cdot;z)$ is a solution of (1) and (2) on $[-r,t(z))$. This ends the proof.

To include a more general class of equations we now assume that

$$g = g_1 + g_2$$

where g_1 satisfies (H1) and (H2) and g_2 satisfies (H3) - (H5).

We shall make use of the family of operators defined by

$$S(t)(\eta,\phi) = (\eta,\psi),$$

where $$\psi(s) = \begin{cases} \phi(s+t) & \text{for } s+t < 0 \\ \eta & \text{for } s+t \geq 0 \end{cases} .$$

Replacing A by A^{ε}, $\varepsilon > 0$, given by $\mathrm{Dom}(A^{\varepsilon}) = \mathrm{Dom}(A)$ and

$$A^{\varepsilon}(\phi(0),\phi) = (f(\phi(0)) + g_1(\phi(0),\phi) + \frac{1}{\varepsilon}\int_0^{\varepsilon} g(S_{\sigma}(\phi(0),\phi))d\sigma, \ \dot{\phi})$$

one can see that for each fixed $\varepsilon > 0$ Theorem 1 is applicable, which implies the existence of local semigroups $T^{\varepsilon}(t)$ generated by A^{ε}. The problem of taking the limit as $\varepsilon \to 0$ in $T^{\varepsilon}(t)z$ can be treated with techniques as if A^{ε} would arise from a Yosida approximation and the following result can be derived.

<u>Theorem 2.</u> Assume that $g = g_1 + g_2$, where g_1 satisfies (H1) and (H2), and g_2 satisfies (H3) - (H5). Further, let Y have a uniformly convex dual Y^*. Then for each $z \in \mathrm{Dom}(A)$ there exists a unique solution $x(\cdot;z)$ of (1) and (2) on $[-r,t(z))$. Moreover for $z \in \mathrm{Dom}(A)$

(8) $T(t)z \overset{\text{def}}{=} (x(t;z), x_t(z)) = \lim_{\varepsilon \downarrow 0} T^{\varepsilon}(t)z = \lim_{\varepsilon \downarrow 0} \lim_{n} (I - \frac{t}{n} A^{\varepsilon})_{z}^{-n}$

for $t \in [0,t(z))$, and the limit is uniform on compact subintervals of $[0,t(z))$.

For the proof of this theorem under a weaker hypothesis then (H4) we refer to [12].

The following Corollary asserts that $t(z)$ in Theorem 2 is actually the best possible choice.

Corollary. (a) For $z \in \mathrm{Dom}(A)$ the alternative

$$t(z) = \infty \quad \text{or} \quad \overline{\lim_{t \uparrow t(z)}} \|T(t)z\| = \infty$$

holds.

(b) For each $z \in \mathrm{Dom}(A)$ and each $t^* \in (0,t(z))$ there exist constants $\varepsilon = \varepsilon(z,t^*)$ and $\Gamma = \Gamma(z,t^*)$ such that for all $y \in \mathrm{Dom}(A)$ with $\|z-y\| \leq \varepsilon$ $t^* \leq t(y)$ and $\|T(t)z - T(t)y\| \leq \Gamma \|z-y\|$ for all $t \in [0,t^*]$.

(c) For each $\eta > 0$ there exists a $\tau(\eta) > 0$ such that $t(z) > \tau(\eta)$ for all $z \in \mathrm{Dom}(A)$ with $\|z\| \leq \eta$.

The operators $T(t)$ given in Theorem 2 can be extended to a local semigroup in Z. For $z \in Dom(A)$ we take $t(z)$ as in Theorem 2 and for $z \in Z \setminus Dom(A)$ we let

$$M(z,\rho) = \{y : y \in Dom(A),\ z \in B(y,\varepsilon(y,t(y)-\rho)),\ t(y) > \rho\},$$

where $\rho \in \mathbb{R}$, $\rho > 0$, ε is defined in the above Corollary and $B(y,r)$ is the open ball in Z centered at y with radius r. Next we define

$$(9) \qquad t(z) = \sup_{\rho>0} \sup_{y \in M(z,\rho)} (t(y)-\rho)$$

for $z \in Z \setminus Dom(A)$. The Corollary and the fact that $Dom(A)$ is dense in Z imply that $t(z) > 0$. For $T \in (0,t(z))$ and $z \in Z \setminus Dom(A)$ there exists $\tilde{\rho} > 0$ and $\tilde{y} \in M(z,\tilde{\rho})$ such that $z \in B(\tilde{y},\varepsilon(\tilde{y},t(\tilde{y})-\rho))$. We may therefore define

$$(10) \qquad T(t)z = \lim_n T(t)z_n \quad \text{for} \quad t \in [0,T]$$

where $z_n \in Dom(A) \cap B(\tilde{y},\varepsilon(\tilde{y},t(\tilde{y})-\rho))$ and $\lim_n z_n = z$. By the Corollary $T(t)z$ is well defined via (10) and the limit is uniform in $t \in [0,T]$. Moreover, the operators $T(t)$, $t \geq 0$, being continuous extensions of continuous operators, are continuous operators on their respective domains. It is now simple to see that also (a), (b), (c) and (e) in the Definition of local semigroups are satisfied. We may therefore summarize the above discussion in a theorem.

Theorem 3. Let the assumptions of Theorem 2 hold. Then $\{T(\cdot)\}:D(\cdot) \to Z$, with $T(t)$ defined as in (8) respectively (10) and

$$D(t) = \{z : t < t(z)\}$$

with $t(z)$ as in Theorem 2 respectively (9), is a strongly continuous local semigroup in Z.

3. Mild Solutions.

In this Section we discuss further the relationship between the semigroups given by Theorems 1 and 2 and solutions of (1) and (2). For $z \in Z \setminus \mathrm{Dom}(A)$, $T(t)z$ will in general not be associated with a strong solution of (1) and (2) via (4). However, if f is linear, the local semigroup $T(t): D(t) \to Z$, gives rise to mild solutions. By definition a function $z(\cdot)$ is called mild solution of (1) and (2) if it satisfies

$$(11) \quad \begin{cases} z(t) = U(t)\eta + \int_0^t U(t-s)g(z(s),z_s)ds, & \text{in } Y, \text{ for } t \in [0,t(\eta,\phi)) \\ z(t) = \phi(t) & \text{for almost every } t \in [-r,0]. \end{cases}$$

Here we assumed that (H1) holds and denote by $U(t)$ the linear semigroup generated by f.

<u>Theorem 4</u>. Assume that f is linear and let the assumptions of Theorem 2 hold. Then for each $(\eta,\phi) \in Z$ there exists a function $v:[-r,t(\eta,\phi)) \to Y$ such that $T(t)(\eta,\phi) = (v(t),v_t)$ for $t \in [0,t(\eta,\phi))$ and v satisfies (11).

<u>Proof</u>. The existence of the map v, just as in the proof of Theorem 1, is a consequence of the fact that $T(t)$ is a translation semi-group. For $z \in \mathrm{Dom}(A)$ the claim follows from Theorem 2 and [12, Theorem 2.2]. If $z \in Z \setminus \mathrm{Dom}(A)$, let $T \in (0,t(z))$. Then by definition of $(t(z)$ in (9) there exists a sequence $z_n = (\eta_n,\phi_n) \in \mathrm{Dom}(A)$ with $\lim_n z_n = z$, $\lim_n t(z_n) > T$ and

(12) $$\lim_n T(t)z_n = T(t)z \text{ uniformly on } [0,T].$$

Notice first that $s \to U(t-s)g(T(s)z)$ is integrable on $[0,T]$ and that $T(t)(z_n) = (v_n(t),(v_n)_t)$ for a family of maps v_n. Since v_n satisfies (11) for each n and since the family $T(\cdot)z_n:[0,T] \to Z$ is uniformly bounded, (H3) together with (12) and the fact that $U(t)$ is a linear C_o-semigroup imply the result.

We close with a theorem further clarifying the relationship between mild solutions and (strong) solutions.

<u>Theorem 5</u>. Under the assumptions of Theorem 4 the map v defined there satisfies

$$(13) \qquad v(t) = \eta + f\left(\int_0^t v(s)ds\right) + \int_0^t g(v(s), v_s)ds,$$

for all $(\eta,\phi) \in Z$, and $t \in [0, t(\eta,\phi))$.

This result is a special case of [13, Theorem 2.3]. Of course, (13) is just the integrated form of (1) with integration and operator f interchanged in the second summand.

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