

AD-A081 318

DAYTON UNIV OH RESEARCH INST
EIGENVALUE AND NEAR EIGENVALUE PROBLEMS SOLVED BY BRANDT'S MULT--ETC(U)
SEP 79 K G GUDERLEY, D S CLEMM

F/G 12/1

AFOSR-78-3524

UNCLASSIFIED

AFFDL-TR-79-3147

NL

1 of 1
AD
A081318

END
DATE
FILMED
4-80
DTIC

AD A081318

AFFDL-TR-79-3147

[Handwritten scribbles and initials]

EIGENVALUE AND NEAR EIGENVALUE PROBLEMS SOLVED BY BRANDT'S MULTIGRID METHOD

K. G. GUDERLEY
UNIVERSITY OF DAYTON RESEARCH INSTITUTE
300 COLLEGE PARK AVENUE
DAYTON, OHIO 45469

DONALD S. CLEMM
APPLIED MATHEMATICS GROUP
AFFDL/FBRD

SEPTEMBER 1979

TECHNICAL REPORT AFFDL-TR-79-3147
Interim Report for period May - June 1979

Approved for public release; distribution unlimited.

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

A

80 3 - 3 - 051

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

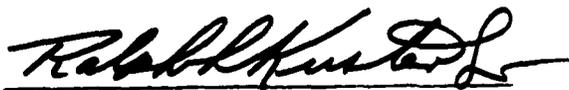


CHARLES L. KELLER
Project Engineer
Applied Mathematics Group
Analysis and Optimization Branch



FREDERICK A. PICCHIONI, Lt Col, USAF
Chief, Analysis & Optimization Br
Structures & Dynamics Division

FOR THE COMMANDER



RALPH L. KUSTER, Jr, Col, USAF
Chief, Structures and Dynamics Division

"If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify AFFDL/FBRDW-PAFB, OH 45433 to help us maintain a current mailing list".

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER AFFDL-TR-79-3147	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
6. TITLE (and Subtitle) EIGENVALUE AND NEAR EIGENVALUE PROBLEMS SOLVED BY BRANDT'S MULTIGRID METHOD.		7. Interim Report, [redacted] May - June 1979	
7. AUTHOR(s) K. G./Guderley Donald S./Clemm		8. CONTRACT OR GRANT NUMBER(s) AFOSR-78-3524	
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Dayton Research Institute 300 College Park Avenue Dayton, Ohio 45469		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2304/N102	
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Flight Dynamics Laboratory/FBRD Wright-Patterson Air Force Base, Ohio 45433		12. REPORT DATE Sep 1979	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 28		13. NUMBER OF PAGES 21	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 16 2304 17 N1		15. SECURITY CLASS. of this report: Unclassified	
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20; if different from Report)		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Transonic Flows Oscillating Flows Far Field Conditions Helmholtz Equation Radiation Condition			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) To illustrate Brandt's multigrid method this report considers the Helmholtz equation in cases where the frequency is close to an eigenfrequency. In a second part the eigenvalue problem is discussed. The basic idea for the treatment of the inhomogeneous problem is contained in Brandt's publications. Difficulties are encountered because the coarse grid eigenvalues differ from those of the finer grid. As a consequence, approximations to a (continued)			

105400 [handwritten signature]

20. (continued)

solution in the coarse grid may differ widely from the corresponding approximation in a finer grid, if the frequency defined by the inhomogeneous problem is close to one of the eigenfrequencies. This may be detrimental to the aim of the coarse grid step; namely to remove from an approximation in a fine grid long waves which cause divergence of fine grid iterations. This is remedied by eliminating long wave contributions to the residual directly in the next finer grid. The success of this procedure still depends upon the choice of the subspace from which the long wave corrections in the finer grid are taken. Even this modified method fails if the frequency pertaining to the inhomogeneous term is too close to an eigenvalue, unless the subspace contains the pertinent eigenfunction.

In dealing with the eigenvalue problem one must update the eigenfunctions and eigenvalues in each iteration step and in addition remove long wave perturbations pertaining to eigenfunctions with frequencies lower than that currently under evaluation. Multiple eigenvalues and eigenvalues that lie close together are treated simultaneously.

FOREWORD

The work was performed as a joint project between K.G. Guderley working under Grant AFOSR-78-3524 and Donald S. Clemm, US Air Force, Project 2304N102.

The authors express their appreciation for the excellent typing of Miss Norma Harting and the art work of the staff of the University of Dayton.

Dist.	
A	
Avail and/or special	

TABLE OF CONTENTS

SECTION		PAGE
I	INTRODUCTION	1
II	DESCRIPTION OF THE SAMPLE PROBLEM	2
III	EIGENVALUES AND EIGENFUNCTIONS	5
IV	COMPUTATION IF μ IS NOT AN EIGENVALUE	7
V	EIGENVALUE PROBLEMS	14
VI	NUMERICAL EXPERIENCES IN THE TREATMENT OF EIGENVALUE PROBLEM	20
REFERENCE		21

PRECEDING PAGE BLANK - NOT FILMED

LIST OF ILLUSTRATIONS

FIGURE		PAGE
1	Region for which the Computations are carried out.	3
2	Number of Iterations to Convergence for the Inhomogeneous Problem with Different Functions for the Interpolation from the Coarse to the Fine Grid.	13

SECTION I
INTRODUCTION

The present report describes some examples for Brandt's multigrid method (Ref. 1). The primary purpose is to gain some direct working experience. We have not used ready-made programs which, presumably, are available by now, and we have even disregarded some of the finer points of Ref. 1. The examples chosen are simple but not entirely trivial; we seek solutions to the Helmholtz equation for frequencies which exceed the lowest eigenfrequency of the problem. Cases of this kind are included in Brandt's work, at least in principle; but additional considerations are needed if the frequency is above that pertaining to the lowest eigenvalue and in particular if the frequency is close to one of the eigenvalues. In this respect, there is an element of novelty in the present discussions. To obtain results which are completely satisfactory in the latter case, one needs the eigenfunctions which pertain to the eigenvalues in the vicinity of the driving frequency. A modification of Brandt's procedure by which such eigenvalues and eigenfunctions can be computed is therefore included.

The work was initiated because of its possible usefulness for the flutter problem. One must realize, however, that for this problem the boundary conditions at a large distance are of a nature which precludes the occurrence of standing waves. The eigenvalue problem as such has no direct bearing on the flutter problem. In other respects Brandt's multigrid approach is likely to be very useful for problems of this kind, especially if the frequencies are not low.

SECTION II
DESCRIPTION OF THE SAMPLE PROBLEM

The region for which the numerical solution of the Helmholtz problem is carried out is square (Fig. 1). Along the boundary of the region, Dirichlet conditions are prescribed. The fine grid is obtained by dividing each side of the square into 12 intervals. The grid size of the coarse grid is four times that of the fine grid. This choice is not optimal; Brandt suggests a ratio of 2:1 for the grid parameters. Brandt envisages the use of more than two grids, while our examples use only two grids. If there are more than two grids, then one must decide when one should go to a finer or to a coarser grid. In this regard the present example fails to cover all aspects of Brandt's procedure.

The Helmholtz equation reads

$$\varphi_{xx} + \varphi_{yy} + \mu \varphi = 0 \quad (1)$$

where μ (the square of the frequency) has a fixed value. The difference form of this equation is

$$\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1} - 4\varphi_{ij} + \mu h^2 \varphi_{ij} = 0, \quad \begin{matrix} i=1,\dots,12 \\ j=1,\dots,12 \end{matrix} \quad (2)$$

where i and j give the numbering of the grid in the x and y directions, respectively, and h is the distance between adjacent grid points. The boundary conditions are

$$\varphi_{i,j} = 0, \quad i=0, i=12, j=0, j=12 \quad (3)$$

In the fine grid, one has 121 unknowns (the values of φ_{ij} at the inner grid points). In the coarse grid, one has a system of four equations.

Because of the lack of an inhomogeneous term, this system has only the trivial solution $\varphi_{ij} = 0$ unless μ is an eigenvalue. The values of φ_{ij} which arise in the course of the present computations therefore give the errors immediately. The

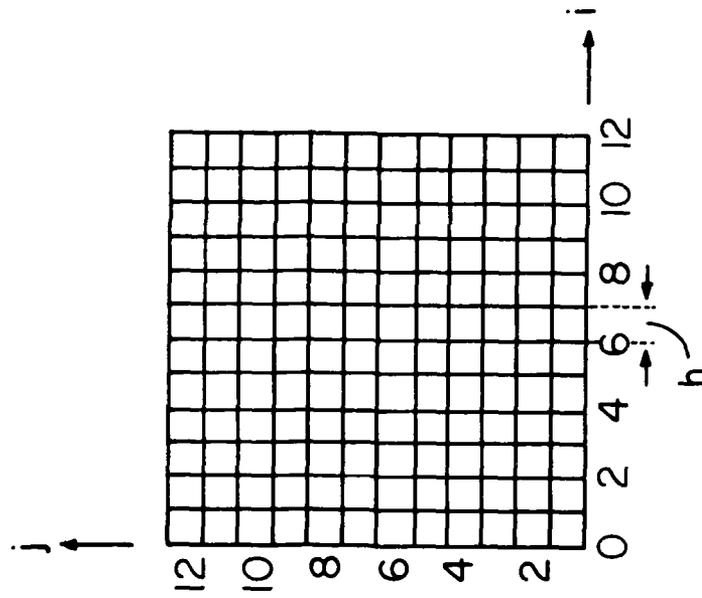


Figure 1. Region for which the computations are carried out.

solution process ought to reduce the errors to zero. In a more realistic problem the error never appears separately; one encounters approximate expressions, ϕ_{ij} , which are the sum of the exact solution and the current error. The presence of errors manifests itself in terms of "residuals" obtained by substituting the current approximation into the difference form (Eq. (2)) of the differential equation. In our discussions we are interested only in solutions to the difference equations, the question of truncation error is of a different nature.

SECTION III
EIGENVALUES AND EIGENFUNCTIONS

In the present example one can determine the eigenvalues and eigenfunctions for the difference form of the Laplace operator in a closed form. Of course this information should not be used in the numerical work; it is used to show for which values of μ difficulties may be expected.

The (nonnormalized) eigenfunctions are given by

$$\varphi_{i,k} = \sin\left(\frac{\pi}{12} ni\right) \sin\left(\frac{\tilde{x}}{12} mk\right); \quad n=1, \dots, 11; \quad m=1, \dots, 11 \quad (4)$$

Substituting this into Eq. (2) one obtains

$$\mu_{nm} h^2 = 4 \left[\sin^2\left(\frac{n\tilde{x}}{24}\right) + \sin^2\left(\frac{m\tilde{x}}{24}\right) \right]; \quad n=1, \dots, 11; \quad m=1, \dots, 11 \quad (5)$$

The eigenvalues pertaining to Eq. (1) are

$$\mu_{nm} h^2 = \left(\frac{\tilde{x}}{12}\right)^2 [n^2 + m^2]; \quad n=1, 2, \dots; \quad m=1, 2, \dots \quad (6)$$

The two expressions approach each other if n and m are sufficiently small.

For the coarse grid the eigenfunctions are again given by Eq. (4) with $i = 4$ and 8 , and $k = 4$ and 8 . The pertinent eigenvalues are

$$\mu_{nm} h^2 = \frac{1}{4} \left[\sin^2\left(\frac{n\tilde{x}}{6}\right) + \sin^2\left(\frac{m\tilde{x}}{6}\right) \right]; \quad n=1, 2; \quad m=1, 2 \quad (7)$$

The eigenfunctions for the coarse grid can be found simply by inspection

$$\begin{aligned} n=1, \quad m=1 : \quad \varphi_{44} &= \varphi_{48} = \varphi_{84} = \varphi_{88} = 1 \\ n=1, \quad m=2 : \quad \varphi_{44} &= \varphi_{84} = 1; \quad \varphi_{48} = \varphi_{88} = -1 \\ n=2, \quad m=1 : \quad \varphi_{44} &= \varphi_{48} = 1; \quad \varphi_{84} = \varphi_{88} = -1 \\ n=2, \quad m=2 : \quad \varphi_{44} &= \varphi_{88} = 1; \quad \varphi_{48} = \varphi_{84} = -1 \end{aligned}$$

The first four values of μh^2

	coarse grid	fine grid	differential equation
n=1, m=1	0.125	0.13630	0.13708
n=1, m=2 n=2, m=1	0.25	0.33610	0.34269
n=2, m=2	0.375	0.53590	0.54831

SECTION IV
COMPUTATION IF μ IS NOT AN EIGENVALUE

The computation proceeds in the following steps:

1. Introduction of a starting solution for the fine grid.
2. Fine grid iterations (in our examples carried out by the alternating direction method). Terminate the procedure if it converges.
3. If the fine grid iterations converge slowly or diverge, then one computes the residuals for the fine grid and transfers them to the coarse grid. (According to Brandt it is most economical if one simply evaluates the fine grid residuals for the points of the coarse grid.)
4. Computation of corrections in the coarse grid, either by iteration or by direct solution of the system of equations for the coarse grid.
5. One transfers these corrections to the fine grid by means of an interpolation and adds these corrections to the fine grid values of ϕ_{ik} obtained at the end of step 2. Return to step 2.

We make the following observation: Steps 3, 4 and 5 are carried out in order to remove, as well as possible, long wave contributions which appear in the results of the fine grid iterations. This purpose may not be achieved by the procedure just described if μ is close to one of the eigenvalues of the problem.

This difficulty can be ascribed to the fact that the eigenvalues are not the same in the coarse and in the fine grid. The following example shows that the corrections obtained in the coarse grid may be quite unrelated to the corrections needed in the fine grid. Assume that μ coincides with a coarse grid eigenvalue. The corrections in the coarse grid are obtained

by solving an inhomogeneous system of simultaneous linear equations. If μ is a coarse grid eigenvalue, then the solution will be infinite although the contribution of long wave errors in the fine grid is definitely finite.

The elimination of long wave contributions can actually be accomplished without solving the problem in the coarse grid. One notices that the coarse grid corrections depend upon a number of parameters which is equal to the number of grid points in the coarse grid. By an interpolation one obtains a fine grid correction for each of these possible coarse grid corrections. The elimination of long wave contributions in the fine grid can therefore be accomplished by adding to the current fine grid approximations a linear combination of the interpolated coarse grid corrections. The coefficients with which these functions are multiplied are determined in such a manner that, as far as possible, the long waves are eliminated from the fine grid approximation. The criteria by which these coefficients are determined must, of course, be derived from the fine grid residuals (see the remark at the end of Section II). Accordingly, one computes in advance the residuals in the fine grid belonging to the interpolated coarse grid corrections. In the present case there are four linearly independent residual functions of this kind. Assume that one has carried out a number of alternating direction steps. One then possesses a fine grid approximation and pertaining to it a fine grid residual. From this residual, and subsequently from the approximation for ϕ , one wants to remove long wave contributions. Let $R(\phi)$ be the residual for the approximation obtained at the end of the alternating direction iteration. Let $R(\phi_{\text{coarse}, i})$ be the residual for the i^{th} coarse grid expression after interpolation to the fine grid. In the present example $i = 1, \dots, 4$. According to the idea just described above, we form

$$R = R(\phi) + \sum_i \alpha_i R(\phi_{\text{coarse}, i}) \quad (8)$$

It is our goal to determine the coefficients α_i so that long wave contributions are (approximately) removed from R .

This task could be done perfectly if exact long wave eigenfunctions were known. The long wave eigenfunctions are approximated by linear combinations of the interpolated coarse grid expressions $\phi_{\text{coarse},i}$. (If one would compute the coarse grid eigenfunctions and then proceed to the fine grid by interpolation, then one would obtain just such expressions.) This leads to the following procedure. Let the scalar product of two functions u and v defined at the grid points be given by

$$[u, v] = \sum_{m,n} u_{m,n} v_{m,n} \quad (9)$$

Now we impose the requirement that

$$[\phi_{\text{coarse},i}, R] = 0, \quad i = 1, \dots, 4 \quad (10)$$

This leads to the system of equations

$$M \alpha + r = 0 \quad (11)$$

where M is a matrix with components

$$M_{m,n} = [\phi_{\text{coarse},m}, R(\phi_{\text{coarse},n})] \quad (12)$$

α is a vector whose n^{th} component is given by α_n , r is a vector whose m^{th} component is given by

$$r_m = [\phi_{\text{coarse},m}, R(\phi)] \quad (13)$$

The matrix $M_{m,n}$ is symmetric. Let L be the difference operator defined by Eq. (2). Then one can define an eigenvalue problem

$$(L - \lambda_k I) \phi_k = 0$$

The eigenfunctions ϕ_k are orthogonal to each other

$$[\phi_{k_1}, \phi_{k_2}] = 0, \quad k_1 \neq k_2 \quad (14)$$

The residual pertaining to ϕ_k is given by

$$R(\phi_k) = L(\phi_k) - \lambda_k \phi_k \quad (15)$$

Now assume that $\phi_{\text{coarse},i}$ is represented in the form

$$\phi_{\text{coarse},i} = \sum_k \delta_{ik} \phi_k$$

Then one has

$$\begin{aligned}
 M_{m,n} &= [\phi_{\text{coarse},m}, R(\phi_{\text{coarse},n})] = [\sum_{k_1} f_{m,k_1} \phi_{k_1}, R(\sum_{k_2} f_{n,k_2} \phi_{k_2})] \\
 &= \sum_{k_1} \sum_{k_2} f_{m,k_1} f_{n,k_2} [\phi_{k_1}, R(\phi_{k_2})] = \sum_{k_1} \sum_{k_2} f_{m,k_1} f_{n,k_2} \lambda_{k_2} [\phi_{k_1}, \phi_{k_2}]
 \end{aligned}$$

Finally, because of the orthogonality relations Eq. (14)

$$M_{m,n} = \sum_{k_2} f_{m,k_2} f_{n,k_2} \lambda_{k_2} [\phi_{k_2}, \phi_{k_2}]$$

Hence

$$M_{m,n} = M_{m,m} \tag{16}$$

The system Eq. (11) arises by forming scalar products of the expression Eq. (8) with the functions $\phi_{\text{coarse},m}$. In a first version we had introduced the condition that the scalar product of the residual with itself should be a minimum. This leads to a condition similar to Eq. (10), except that now the weight functions are given by $R(\phi_{\text{coarse},m})$, rather than $\phi_{\text{coarse},m}$ itself. If μ is close to the square of an eigenfrequency, then one of the eigenvalues λ is close to zero, and the contribution of the pertinent eigenfunction is nearly suppressed in the weight function $R(\phi_{\text{coarse},m})$. This is undesirable for the purpose of the computation is to remove the contributions of these eigenfunctions.

According to these considerations, we replace steps 3, 4 and 5 by the following procedure. In preparing for the computations, one determines the elements of the matrix M (a 4 by 4 matrix in the present example). At the end of step 2, that is, after one has decided that the convergence is too slow one evaluates the residual $R(\phi)$ pertaining to the current approximation, forms the components of the vector r, Eq. (13), and then solves the system Eq. (11) for the vector α .

With the values of α_i so obtained, one forms a corrected function ϕ

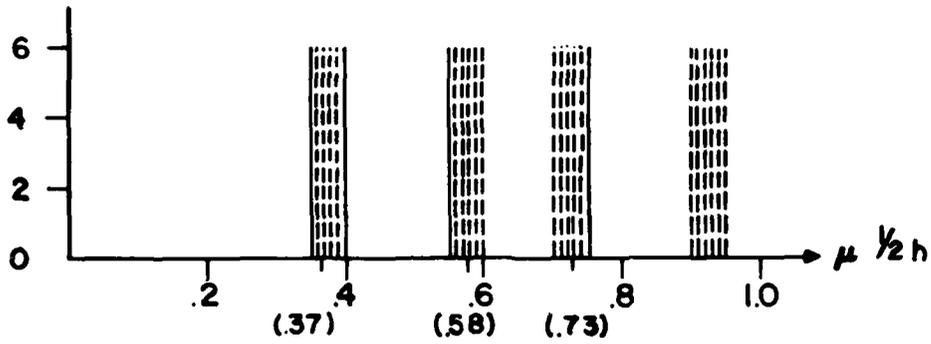
$$\phi_{\text{corrected}} = \phi + \sum_i \alpha_i \phi_{\text{coarse},i} \tag{17}$$

and returns to the beginning of step 2. Notice that the system for the α_i has the same dimension as the system of equations which one would obtain in a direct treatment of the coarse mesh (step 4 of the procedure outline above).

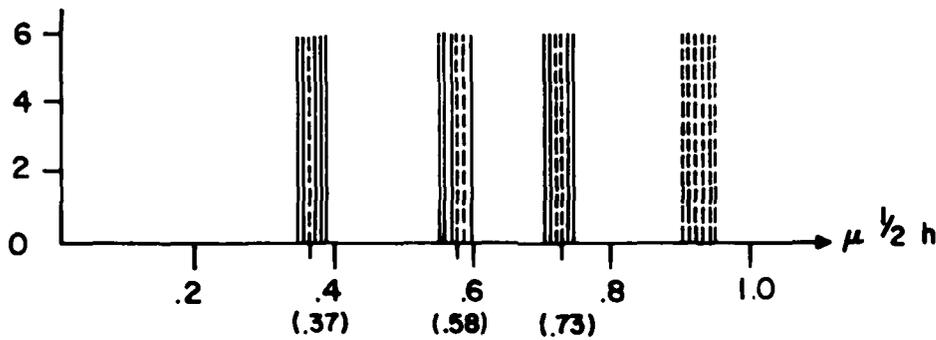
One notices that if μ is close to the square of the j^{th} eigenfrequency, say n , then the contribution of the j^{th} eigenfunction to the expressions $R(\phi_{\text{coarse},n})$ and consequently to the elements of the matrix M is small. (Of course the contribution of this eigenfunction to the inhomogeneous term $R(\phi)$ is also small.) Consequently, the effect of this eigenfunction on the determination of the coefficients α_i becomes small and the goal of the computation, namely the approximate removal of the eigenfunctions ϕ_i from the current approximation ϕ (see Eq. 17) is not achieved. The interval of values μ in which this failure occurs depends upon the character of the functions $\phi_{\text{coarse},n}$'s, for this determines how strongly eigenfunctions other than the ϕ_k (here $k = 1, \dots, 4$) occur in the $\phi_{\text{coarse},n}$'s. These other eigenfunctions will then play the dominant role in the matrix M and lead to faulty values of α . The width of the interval μ in which the approach fails depends upon the interpolation formula by which one proceeds from the coarse to the fine mesh. Best results would be obtained if the functions $\phi_{\text{coarse},n}$ span the same subspace as the eigenfunctions ϕ_n (in our case $n = 1, \dots, 4$).

These observations are borne out by our computations, which have been carried out for different interpolation routines from the coarse to the fine grid. In routine 1 linear interpolation has been used. In routine 2, we have used a third degree interpolation formula for both the x and y direction. (This is possible in the present example because the eigenfunctions arise from a product hypothesis and because, with the boundary points included, the coarse grid has only 4 points in the x , and 4 points in the y direction.) In routine 3, exact eigenfunctions are used.

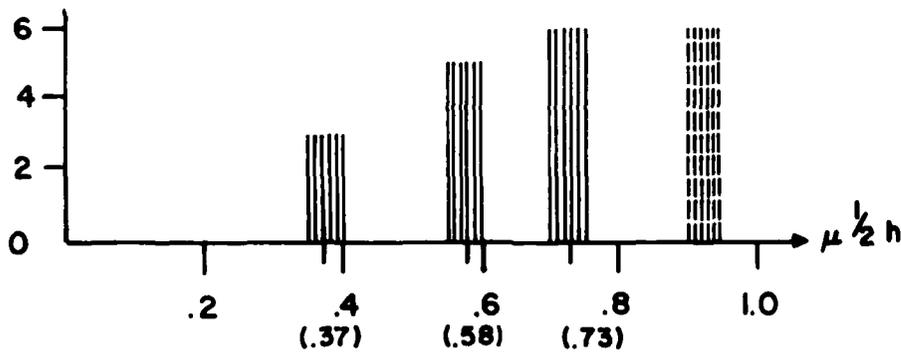
Figure 2 gives a survey of the results. The horizontal axis shows the values of $\mu^{1/2}$ (one might just as well use μ as independent variable). The eigenfrequencies for which we expect difficulties are shown by short heavy lines. The vertical directions show the number of iteration steps to convergence. If convergence is not attained by five iterations but might have been attained by more iterations, then the solid lines are drawn up to the upper end of the graph. Divergence is shown by dashed lines. As expected, there is an interval around the eigenfrequencies where the method fails to converge. This interval becomes smaller as one uses a better interpolation formula. If one uses the exact eigenfunctions to remove long wave perturbations from the residual, then one has convergence even very close to the eigenvalues. In this case, the size of the interval depends only on the precision with which the computations are carried out. In all cases the convergence is slower for larger values of μ . No convergence is obtained if μ is too large.



LINEAR INTERPOLATION



THIRD DEGREE INTERPOLATION



CORRECT EIGEN FUNCTIONS

Figure 2. Number of Iterations to Convergence for the Inhomogeneous Problem with Different Functions for the Interpolation from the Coarse to the Fine Grid.

SECTION V
EIGENVALUE PROBLEMS

The eigenvalue problems for the Helmholtz equation and the Laplace equation are the same except for a shift of the eigenvalues by a constant amount. The following discussion is therefore restricted to the eigenvalue problem for the Laplace operator. Let

$$[L(u)]_{ik} = u_{i,j,k+1} + u_{i,j,k-1} + u_{i+1,j,k} + u_{i-1,j,k} - 4u_{ik} \quad (18)$$

Then we consider

$$[L(\phi)]_{ik} + \mu \phi_{ik} = 0 \quad (19)$$

with the Dirichlet boundary conditions formulated above.

The computations start with approximations for a limited number of eigenfunctions. Such approximations can be obtained from a coarse grid formulation with a subsequent interpolation to the fine grid. In our example, approximations to the first four eigenfunctions have been determined. If one wants to compute the lowest eigenfunction only, then it will probably suffice if one derives from the coarse grid an approximation for only the lowest eigenfunction.

The discussion includes cases where a number of eigenvalues lie close together, for this happens frequently in multidimensional problems. In the present example, for instance, one has coincidence of the second and third eigenvalue. The coarse grid approximation will show for which eigenfunctions this is the case.

The number of eigenfunctions for which approximations are introduced is restricted; it cannot exceed the number of grid points in the coarse mesh, but one might use even a smaller number. The subscripts i and j which will appear subsequently refer to these eigenfunctions. Let S be the subset of subscripts for those eigenfunctions which are treated simultaneously, because the

corresponding eigenvalues lie close together or coincide. The approximations to the eigenfunctions designated by the subset S are updated in each iteration step, those not pertaining to S remain unchanged. For the latter one always uses the best approximations available. In particular, one uses the exact eigenfunctions (usually obtained in previous phases of the procedure) if they are available.

We shall denote eigenfunctions and their approximations by ψ_i , if $i \in S$ and by ϕ_j if $j \notin S$. In our example we have for the first eigenvalue $i = 1$, $j = 2, 3, 4$; for the second and third eigenvalues $i = 2$ and 3 ; $j = 1$ and 4 ; and for the fourth eigenvalue $i = 4$, $j = 1, 2, 3$.

Let a superscript p denote the iteration number. One iteration step leads from $\psi_i^{(p)}$ to $\psi_i^{(p+1)}$, $i \in S$. The iteration starts with approximations $\psi_i^{(0)}$ obtained by the interpolation from the coarse to the fine grid. We describe immediately the general case where S contains more than 1 element. Single eigenvalues are obtained by an obvious specialization.

The following steps are carried out.

1. Choose the subset S , provide approximations for the ϕ_j , $j \notin S$, and starting approximations $\psi_i^{(0)}$, $i \in S$.
2. Update the values of $\mu_i^{(p)}$ and form an intermediate update to the approximations $\psi_i^{(p)}$, $i \in S$.
3. Provide, separately for each i , $i \in S$, a second update by approximately eliminating long wave contributions due to the ϕ_j 's, $j \notin S$.
4. Smooth out by iteration, separately for each $i \in S$ the twice updated functions ψ_i . After a number of these steps this process gives the final approximation $\psi_i^{(p+1)}$. If one has convergence, then one updates the eigenvalues μ_i and terminates the computation. Otherwise one returns to step 2.

We discuss these steps in detail and provide the necessary equations. In step 2 we set

$$\bar{\psi} = \sum_{i \in S} \beta_i \psi_i^{(p)} \quad (20)$$

with coefficients β_i so far unknown. In general, such functions $\bar{\psi}$ will not satisfy Eq. (18),

$$L \bar{\psi} + \mu \bar{\psi} \stackrel{!}{=} 0$$

even if one considers μ and the β_i 's as arbitrary. To obtain conditions for μ and the β_i 's we apply to the last expression weight function $\psi_{i_1}^{(p)}$, $i_1 \in S$. Then one obtains the eigenvalue problem

$$(M^{(1)} + \mu M^{(2)}) \vec{\beta} = 0 \quad (21)$$

where $\vec{\beta}$ is a vector with components β_i , $i \in S$. The elements of $M^{(1)}$ and $M^{(2)}$ are given by

$$M_{i_1, i_2}^{(1)} = [\psi_{i_1}^{(p)}, L(\psi_{i_2}^{(p)})] \quad (22)$$

$$M_{i_1, i_2}^{(2)} = [\psi_{i_1}^{(p)}, \psi_{i_2}^{(p)}]$$

(For the solution of this problem, one will, of course, change the subscripts.) The solution of the eigenvalue problem gives the updated eigenvalues $\mu_i^{(p+1)}$ and eigenvectors $\vec{\beta}_i$ and subsequently (from Eq. (20)) updated approximations $\bar{\psi}_i$.

In the present setting the matrices $M^{(1)}$ and $M^{(2)}$ are symmetric (see the discussion in Section IV). Because of lack of precision, there may arise deviations from symmetry if one evaluates the matrix elements separately. It is then desirable to make the matrices symmetric. If the eigenvalues $\mu_i^{(p+1)}$ are different, then one obtains eigenvectors $\vec{\beta}_i$ which are orthogonal to each other. If some of the $\mu_i^{(p+1)}$'s coincide, then one can construct vectors $\vec{\beta}_i$ which are orthogonal to each other. Critical are cases where eigenvalues $\mu_i^{(p+1)}$ are different but lie very close together. The eigenvectors obtained in such a case will be different and orthogonal, but small errors in the matrices will lead to eigenvectors which are quite different (although taken together they will always span the same or approximately the same subspace of the β space). It is then possible that the

eigenvectors change considerably from iteration to iteration depending upon small changes in the matrices. This will introduce a numerical instability. For eigenvalues which nearly coincide one must therefore override the automatic determination of the eigenvectors and define eigenvectors within the pertinent subspace which change very little from iteration to iteration.

We illustrate by example another disturbing phenomenon which may occur. We consider for this purpose the computation of the second and third eigenfunctions. In this case, one encounters 2 by 2 matrices which have nearly diagonal form. Let us idealize them as diagonal matrices.

$$\left\{ \begin{matrix} M''_{11} & 0 \\ 0 & M''_{22} \end{matrix} \right\} + \mu \left\{ \begin{matrix} M^{(2)}_{11} & 0 \\ 0 & M^{(2)}_{22} \end{matrix} \right\} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0$$

The determinant vanishes if

$$(M''_{11} + \mu M^{(2)}_{11})(M''_{22} + \mu M^{(2)}_{22}) = 0$$

Hence

$$\mu = -\frac{M''_{11}}{M^{(2)}_{11}} \quad \text{or} \quad \mu = -\frac{M''_{22}}{M^{(2)}_{22}}$$

To obtain the eigenvectors one substitutes these values of μ .

One obtains in the first case

$$\begin{pmatrix} 0 & 0 \\ 0 & M''_{22} - (M''_{11}/M^{(2)}_{11})M^{(2)}_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0$$

Hence, from the second of these equations

$$\beta_2 = 0, \beta_1 = 1$$

With inaccurate numbers the zeros which appear here will be small numbers, which are uncertain because of the lack of precision. If one inadvertently determines the ratio of β_1 to β_2 from the first equation, then one will obtain a nonsensical result, while the second equation will give the correct result. Of course,

a good eigenvalue routine will guard against this occurrence. A problem may arise, however, if one uses an ad hoc program. In the present example, the difficulty is easily avoided if one determines the ratio of β_1 to β_2 from the sum of the two equations which one obtains for the different choices of $\mu^{(p+1)}$.

If two eigenvalues coincide, then one obtains for the determination of eigenvectors

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0$$

This system of equations is satisfied by any linear combination of β_1 and β_2 . With inaccurate information about the matrix elements, one will obtain specific vectors β_1 and β_2 , but the form of these vectors will depend upon the inaccuracies and may vary from iteration to iteration. To guard against this occurrence we have chosen

$$\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

if we found by a test that the eigenvalues are very close together.

In the following steps the functions $\bar{\psi}_i$ with approximate eigenvalues $\mu^{(p+1)}$ $i \in S$, are treated separately for each i .

In step 3 the long wave contributions due to the approximate eigenfunctions ϕ_j , $j \notin S$ are approximately removed. We set

$$\bar{\psi}_i = \bar{\psi}_i + \sum_{\substack{j_1 \notin S \\ j_2 \notin S}} \alpha_j \phi_j \quad (23)$$

The α_j 's are determined by the requirement that this expression be orthogonal to the functions ϕ_{j_1} , ($j_1 \in S$). One then obtains

$$M_{\alpha}^{(2)} + r^{(3)} = 0 \quad (24)$$

Where α is a vector with j^{th} component α_{j_2} , $j_2 \notin S$ and the matrix elements are given by

$$M_{j_1, j_2}^{(3)} = [\phi_{j_1}, \phi_{j_2}] \quad ; \quad j_1 \notin S; \quad j_2 \notin S \quad (25)$$

and

$$r_{j_1} = [\phi_{j_1}, \bar{\psi}_i] \quad (26)$$

For the actual computation a renumbering of the subscripts will be carried out.

In our program we have actually proceeded in a different manner because of a somewhat uncritical analogy to the procedure for the inhomogeneous problem. We believe, however, that the method described above is preferable. We have formed the residual pertaining to Eq. (23)

$$(L + \mu_i^{(p+1)}) \bar{\psi}_i + \sum_{j \notin S} \alpha_j (L + \mu_j^{(p+1)}) \phi_j$$

and obtained conditions for the α_j by postulating that this expression be orthogonal to the ϕ_{j_1} 's, $j_1 \notin S$. With the values of α_j so found one then computes $\bar{\psi}_i$ from Eq. (23).

In step 4, one eliminates short wave perturbations by an iterative procedure (in our case by the alternating direction method). This is done separately for all values of i , $i \in S$. If these iterations converge, then one terminates the procedure and proceeds to another set S of eigenvalues. Before the termination, one may update the eigenvalues μ by the method of step 2 for a last time. If the convergence is slow, then one goes back to step 2 using the expression $\bar{\psi}_i$ obtained by the alternating direction iteration as starting approximation $\bar{\psi}_i^{(p+1)}$.

SECTION VI
NUMERICAL EXPERIENCES IN THE TREATMENT OF
EIGENVALUE PROBLEM

The procedure described in Section V has been implemented using a third degree interpolation formula to proceed from the coarse to the fine grid. We found that it converges rather well. With linear interpolation, the procedure has been tried only for the single eigenvalues $i = 1$ and $i = 4$. The convergence for $i = 1$ is satisfactory, but dubious for $i = 4$. The question whether the procedure recommended in the preceding section for step 4 would be beneficial has not been explored.

REFERENCE

Brandt, Achi, Multi-Level Adaptive Solutions to Boundary Value Problems, Mathematics of Computation, Vol. 31, (April 1977) pp. 333-390.