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Stability of Self-Similar Flow: 6. Uniform Implosion of an Ablatively Driven Shell

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The linear stability of a uniformly imploding shell, modeled as an ideal polytropic fluid, is investigated. Two types of unstable modes are found: incompressible irrotational perturbations localized at the outer surface, ascribable to Rayleigh-Taylor instability, and compressible modes, associated with convective instability. KIDDER'S (1976) result for the Rayleigh-Taylor modes is shown to hold independently of the form of the shell density profile. By means of a variational principle it is shown that the criterion for convective instability is the existence of a region within

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CONTENTS

0. Introduction

KIDDER (1976) studied the homogeneous (i.e., uniform) isentropic compression of hollow shells imploded by the conversion of absorbed laser energy into mechanical work. (The characteristic feature of uniform motion is that the radial velocity at a given instant of time is everywhere proportional to the radial distance from the center of symmetry.) Using this model he was able to estimate the optical power required to achieve a given measure of inertial confinement for a hollow laser-driven pellet. The calculations showed that using a large-aspect-ratio target (radius much greater than thickness) reduced the power requirements compared with those for a solid spherical target by as much as a factor of five. The results were strongly influenced by the possibility of disruption caused by Rayleigh-Taylor instabilities, which imposed an upper limit on the aspect ratios which could be employed.

The time-dependent basic state in the model, a type of self-similar motion, is of considerable theoretical interest. Although simple, it subsumes a number of the principal physical features of the problem, and represents a nontrivial solution of the (nonlinear) equations of motion. Such solutions appear to have a range of applicability well beyond what one might have expected, apparently because the change of scales implicit in strong compression is large enough so that the dependence on variables other than the

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similarity variable becomes weak. Similarity solutions of this and other types have accordingly been found useful in a wide range of studies of laser-driven target behavior [see, e.g., ANISIMOV and INOGAMOV (1974)].

It is noteworthy that KIDDER (1976) found it possible to carry out in closed form the analysis of the linear stability of incompressible irrotational perturbations about the self-similar basic state. Thus he was able to determine the evolution of the Rayleigh-Taylor instability exactly, without recourse to numerical approximation, even though the perturbed equations of motion contain coefficients which depend on position and time, and the perturbations are therefore not the familiar exponential functions of time.

Somewhat later, the present authors independently developed in a different application (BOOK and BERNSTEIN, 1978) a technique for determining the evolution of all the modes (including compressible and rotational ones). Now we wish to apply this technique in order to generalize KIDDER'S (1976) results in three respects: (i) arbitary choice of γ, the ratio of specific heats: (ii) nonuniform entropy; and (iii) general perturbations.

As discussed below, we obtain the following principal results: (i) The Rayleigh-Taylor modes become more unstable as γ decreases (this is true of all modes); (ii) they are unaffected by changes in the shell density profile;

(iii) compressible modes, which are convective in nature, are destabilized by the occurrence of regions of "super-adiabaticity" $[d(p\rho^{-\gamma})/dr > 0]$, but grow less rapidly than incompressible modes, particularly at infinitely large mode number, where the Rayleigh-Taylor growth rates diverge.

The paper is organized as follows: in §1, we write down and solve the equations describing the unperturbed motion. In §2 we derive the equations describing linear perturbations about the basic state. The time and space dependence separate. The time-dependent part is solved analytically in §3 for general values of the separation constant. In §4 we solve the spatial part of the problem for incompressible irrotational modes. In §5, utilizing a variational principle, we solve the spatial equations for the case of compressible, rotational modes. In §6 we illustrate the results so obtained with a number of examples which admit of analytic solutions of the spatial problem. (The variational principle of §5 is, however, well-suited for numerical determination of the separation constant). We conclude in §7 with a discussion of our results and some applications to targets having typical parameters.

1. The model

We start with the equations of ideal hydrodynamics in the form

$$\dot{\rho} + \underset{\sim}{v} \cdot \nabla\rho = 0, \qquad (1.1a)$$

$$\frac{dp_0}{dr} = \rho_0 r \tau^{-2}. \tag{1.6}$$

Here a constant τ with units of time results from separating the spatial behavior (1.6) and the equation determining $f(t)$:

$$f^{3\gamma-2}\ddot{f} = -\tau^{-2} \tag{1.7}$$

The sign in (1.7) is chosen to yield negative acceleration (implosion). As a consequence of this choice, $p_0(r)$ increases monotonically outward. From (1.5) it follows that the pressure applied to the outer boundary of the shell at $r = r_0+d$ must grow in time proportionally to $f^{-3\gamma}$. In general a quadrature can be carried out on (1.7), yielding

$$\tau^2 \dot{f}^2 = -2\ln f \tag{1.8a}$$

for $\gamma=1$, and

$$\tau^2 \dot{f}^2 = \frac{2}{\alpha}\left(f^{-\alpha}-1\right) \tag{1.8b}$$

otherwise, where $\alpha = 3(\gamma-1)$. For $\gamma = 5/3$, (1.8b) can be solved to give

$$f(t) = (1-t^2/\tau^2)^{1/2}. \tag{1.9}$$

For other values of γ the solution is easily found numerically. In all cases $f(t)$ is approximately parabolic near $t = 0$, and vanishes with infinite slope at some time t_0 of order τ.

Although the form of the time-dependent function $f(t)$

is determined by specifying γ, there is complete freedom in the choice of nonnegative $\rho_0(r)$ [or equivalently, $p_0(r)$, provided the latter satisfies $p_0(r) > 0$]. For example, if

$$\rho_0(r) = \bar{\rho}_0(r^2/r_0^{\,2}-1)^{\varkappa}, \tag{1.10a}$$

where $\varkappa$ is an arbitary constant, then

$$p_0(r) = \bar{p}_0(r^2/r_0^{\,2}-1)^{\varkappa+1}, \tag{1.10b}$$

with $\bar{p}_0 = \bar{\rho}_0 r_0^{\,2}/2(\varkappa+1)\tau^2$. The case $\varkappa = 0$ corresponds to a shell of uniform density and quadratic pressure. When $\varkappa = \frac{1}{\gamma-1}$, the entropy $s(r)$ becomes constant. This is the model studied by KIDDER (1976). A particularly simple limiting form of (1.10), which will be employed in § 6, is

$$\rho_0(r) = \bar{\rho}(r^2/r_0^{\,2})^{\varkappa}; \tag{1.11a}$$

$$p_0(r) = \bar{p}(r^2/r_0^{\,2})^{\varkappa+1}. \tag{1.11b}$$

Note that in systems described by this model, all motion is inward — there is no ablation region or blowoff, nor does heat conduction play a role. Hence application of the model is restricted to the cold inner regions of a shell or pellet.

2. Linear perturbations

We follow BERNSTEIN and BOOK (1978) and BOOK (1978, 1979) in deriving linearized equations written in Lagrangian variables for the development of a small perturbation about the solutions of the basic equations. As a result of the perturbation, an element of fluid whose unperturbed motion was given by $\underset{\sim}{R}(\underset{\sim}{r},t)$ is located at $\underset{\sim}{R}(\underset{\sim}{r},t) + \underset{\sim}{\xi}(\underset{\sim}{r},t)$. The first-order displacement $\underset{\sim}{\xi}$ satisfies the linearized form of (1.1b),

$$\rho\ddot{\underset{\sim}{\xi}} + \rho_1\ddot{\underset{\sim}{R}} = -\nabla_{\underset{\sim}{R}} p_1 + \nabla_{\underset{\sim}{R}}\underset{\sim}{\xi}\cdot\nabla_{\underset{\sim}{R}} p, \tag{2.1}$$

where the subscript 1 denotes first-order quantities. Substituting for ρ_1 and p_1 from

$$\rho_1 = -\rho\nabla_{\underset{\sim}{R}}\cdot\underset{\sim}{\xi} \tag{2.2}$$

and

$$p_1 = \frac{\partial p}{\partial\rho}\rho_1 = -\gamma\, p\nabla_{\underset{\sim}{R}}\cdot\underset{\sim}{\xi}, \tag{2.3}$$

and using (1.3) to replace the gradient with respect to $\underset{\sim}{R}$ by the gradient with respect to $\underset{\sim}{r}$ according to $\nabla_{\underset{\sim}{R}} = f^{-1}\nabla_{\underset{\sim}{r}}$, we find

$$\tau^2 f^{3\gamma-1}\ddot{\underset{\sim}{\xi}} + \underset{\sim}{r}\,\nabla\cdot\underset{\sim}{\xi} = \tau^2\gamma\rho_0^{-1}\nabla(p_0\nabla\cdot\underset{\sim}{\xi}) + \nabla\underset{\sim}{\xi}\cdot\underset{\sim}{r}. \tag{2.4}$$

In (2.4) we have omitted the subscript $\underset{\sim}{r}$ on ∇. For convenience in what follows we will also omit the zero subscript from ρ_0 and p_0.

Like the unperturbed equation, (2.4) is separable. Writing

$$\underset{\sim}{\xi}(\underset{\sim}{r},t) = \underset{\sim}{\xi}(\underset{\sim}{r})\ T(t), \qquad (2.5)$$

we have

$$\mu\underset{\sim}{\xi} - \underset{\sim}{r}\nabla\cdot\underset{\sim}{\xi} + \tau^2\gamma\rho^{-1}\ \nabla(p\nabla\cdot\underset{\sim}{\xi}) + (\nabla\underset{\sim}{\xi})\cdot\underset{\sim}{r} = 0 \qquad (2.6)$$

and

$$\tau^2 f^{3\gamma-1}\ddot{T} = -\mu T, \qquad (2.7)$$

respectively, for the spatial and temporal dependence of the perturbations. Here the separation constant μ is an eigenvalue to be obtained through solution of (2.6), subject to the requirement that (2.3) vanish at $r = r_0$ and $r = r_0+d$.

Once μ is known, the behavior of $T(t)$ is completely determined from (2.6). In the next section we show how to solve this equation.

3. **Time dependence of the perturbations**

At early times, when $f(t) \approx 1$, the solutions of (2.7) are approximately given by $\exp[\pm i\mu^{\frac{1}{2}}t/\tau)]$. For $\mu<0$, one root is growing and one is damping, while for $\mu>0$, both are oscillatory. Consequently we can expect that in some sense positive μ will be associated with stable behavior and negative μ with instability.

It is convenient to make use of the concepts of relative amplification and relative stability, introduced by BERNSTEIN and BOOK (1978). Suppose that, as the imploding pellet or shell contracts, the size of the pertubations decreases also, but at a slower rate. That is, $f(t)$ and $T(t)$ both approach zero, but the ratio $T(t)/f(t)$ grows, thus making the perturbations more desymmetrizing or disruptive. When this happens, we say the mode is relatively unstable. Its amplitude can eventually exceed the thickness of the pellet (by which point the linear approximation is no longer valid).

The idea of "relative stability" should be applied with a degree of caution. If an imploding shell is "perturbed" by shifting its position infinitesimally, say in the direction of the z axis, the contraction takes place just as before. A linear treatment, however, predicts that the $\ell = 1$, $m = 0$ mode is overstable (i.e., it grows in time without oscillation), evidently a spurious conclusion. On the other hand, a growing oscillatory mode poses a genuine threat to the integrity of the implosion, even if the growth is only relative to the unperturbed dimensions.

When $\mu = 1$, $T = f$ is a solution of (2.7). The other solution is found by the method of variation of constants to be

$$g(t) = \tau^{-1} f \int_0^t dt\ f^{-2}. \tag{3.1}$$

We will see shortly that $\mu=1$ is the critical value for determining marginal relative stability.

When $\gamma=1$, we rewrite (2.6) in terms of the new independent variable $x = \ell n f$.

$$x \frac{d^2T}{dx^2} + \left(\frac{1}{2} - x\right) \frac{dT}{dx} - \frac{\mu}{2} T = 0, \tag{3.2}$$

a confluent hypergeometric equation. The solutions can be expressed as a linear combination of

$$T_1 = \Phi\left(\frac{1}{2}\mu;\ \frac{1}{2};\ \ell n f\right) \tag{3.3a}$$

and

$$T_2 = -(-2\ell n f)^{\frac{1}{2}}\ \Phi\left(\frac{1}{2} + \frac{1}{2}\mu;\ \frac{3}{2};\ \ell n f\right), \tag{3.3b}$$

where $\Phi(a;b;x)$ is the Kummer function (see e.g., ABRAMOWITZ and STEGUN, 1964). As functions of t, the solutions (3.3) satisfy the initial condition

$$T_1(0) = 1,\ \tau\dot{T}_1(0) = 0; \tag{3.4a}$$

$$T_2(0) = 0,\ \tau\dot{T}_2(0) = 1. \tag{3.4b}$$

for large negative values of the argument $\ell n f$ both behave as $(\ell n f)^{-\mu/2}$

If $\gamma = 1$, we write $x = 1 - f^{-\alpha}$, where $\alpha = 3(\gamma-1)$. Then (2.6) becomes the hypergeometric equation

$$x(1-x)\frac{d^2T}{dx^2} + \left[c - (a+b+1)x\right]\frac{dT}{dx} - abT = 0. \tag{3.5}$$

Here

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = (\Gamma \pm \Delta)/4\alpha, \tag{3.6a,b}$$

where $\Gamma = \alpha + 2$ and $\Delta = (\Gamma^2 - 8\alpha\mu)^{\frac{1}{2}}$, and $c = \frac{1}{2}$. The solutions of (3.5) can be chosen as

$$T_1 = F(a,b;\tfrac{1}{2};\ 1-f^{-\alpha}); \tag{3.7a}$$

$$T_2 = -\left[\frac{2}{\alpha}(f^{-\alpha} - 1)\right]^{\frac{1}{2}} F(a+\tfrac{1}{2}, b+\tfrac{1}{2}; \tfrac{3}{2};\ 1-f^{-\alpha}), \tag{3.7b}$$

where $F(a,b;c;x)$ is Gauss' hypergeometric function. The solutions (3.7) again satisfy (3.4). As $f \to 0$, both approach asymptotic forms (ABRAMOWITZ and STEGUN, 1964) containing terms proportional to $f^{(\Gamma \pm \Delta)/4}$. With the lower sign, this expression diverges as $f \to 0$ whenever $\Delta > \Gamma$, i.e., $\mu < 0$. Since the condition for Δ to be real is that μ be no greater than $6(\alpha+2)^2/8\alpha$, whose minimum value as a function of α is unity, we see that $\mu < 1$ is always sufficient to make T/f diverge. If Δ is imaginary, T/f still diverges when $\Gamma < 4$ $(\gamma < 5/3)$. BOOK (1978) has presented a simple argument involving conservation of wave action to explain this result in terms of the geometrical properties of the implosion.

4. Rayleigh-Taylor instability

We turn now to the problem of calculating μ. We dot (2.5) with ξ and use (1.6) to rewrite the coefficients (except in the first term) in terms of p instead of ρ. Introducing the notation $\sigma = \nabla \cdot \xi$ and $\omega = \nabla \times \xi$, and using the identities

$$\sigma\xi \cdot \nabla p = \nabla \cdot (p\sigma\xi) - p\sigma^2 - p\xi \cdot \nabla\sigma, \tag{4.1}$$

$$\xi \cdot \nabla(p\sigma) = \nabla \cdot (p\sigma\xi) - p\sigma^2, \tag{4.2}$$

$$\xi \cdot (\nabla\xi) \cdot \nabla p = \nabla \cdot (p\xi \cdot \nabla\xi) + p\nabla\xi : \nabla\xi^{\dagger} - p\omega^2 \tag{4.3}$$

where the transpose is defined in Cartesian coordinates by $(\nabla\xi)^{\dagger}_{ij} = (\nabla\xi)_{ji}$, we find

$$\tau^{-2}\mu\rho\xi^2 = p\left[(\gamma - 1)\sigma^2 + \nabla\xi : \nabla\xi^{\dagger} - \omega^2\right] + \gamma\nabla \cdot (p\sigma\xi). \tag{4.4}$$

Integration over the volume occupied by the shell yields

$$\tau^{-2}\mu \int_V dV\rho\xi^2 = \int_V dVp \left[(\gamma-1)\sigma^2 + \nabla\xi : \nabla\xi^{\dagger} - \omega^2\right] + \gamma \int_S dS\, p\sigma n \cdot \xi, \tag{4.5}$$

where in the last term, the unit vector n is so defined that it points away from the shell on both inner and outer surfaces. The left hand member and the first terms in the volume integral on the right hand side of (4.5) are manifestly nonnegative. From this we see that relative instability ($\mu < 1$) can result in only two ways: if $\nabla \times \xi \neq 0$, or if the integrated terms are nonvanishing. The latter is the case whenever a perturbation exists at a point where the density changes discontinuously. For the model we have assumed in § 2, one such point is always located at the shell outer radius, $r = r_0+d$.

The external pressure, which enters the model as a boundary condition, produces an inward acceleration. There is thus an effective gravity in the outward direction. Hence we anticipate that a Rayleigh-Taylor instability should occur, localized at the outer surface.

In slab geometry under uniform destabilizing gravitational acceleration G, the Rayleigh-Taylor growth time τ_g for incompressible modes is given by

$$\tau_g^{-2} = kG \frac{\rho_2-\rho_1}{\rho_2+\rho_1}, \tag{4.6}$$

where k is the wavenumber and ρ_2 and ρ_1 are the densities above and below the interface. In the present problem, $\rho_1 = 0$ and we can write $k = 2\pi/\lambda$, where the wavelength satisfies $\lambda = 2\pi r/\ell$. A reasonable approximation to the effective gravity is $G \approx r\tau^{-2}$. Substituting in (4.6), we thus estimate that the growth time will be given by

$$\tau_g^2 = \tau^2/\ell. \tag{4.7}$$

To study the analogous modes in our model of a uniformly imploding shell, we begin by operating on (2.5) with the divergence and with the curl. The results are given by

$$\mu\sigma + \gamma\{\tau^2[\rho^{-1}\nabla^2(p\sigma) + \nabla(\rho^{-1})\cdot\nabla(p\sigma)]$$

$$-2\sigma - \underline{r}\cdot\nabla\times\underline{\omega}\} = 0 \tag{4.8}$$

and

$$(\mu-1)\,\underline{\omega} + \underline{r}\times\nabla\sigma + \tau^2\gamma\nabla(\rho^{-1})\times\nabla(p\sigma) = 0. \tag{4.9}$$

From (4.8) - (4.9) it follows that $\underline{\omega}$ vanishes if σ does. If that happens,

$$\underline{\xi} = \nabla\phi, \tag{4.10}$$

where the potential ϕ satisfies Laplace's equation,

$$\nabla^2\phi = 0. \tag{4.11}$$

The general solution of (4.11) in spherical coordinates is a sum of terms of the form

$$\phi(r,\Theta,\varphi) = (\phi_+ r^{\ell} + \phi_- r^{-\ell-1})Y_{\ell m}(\Theta,\varphi), \tag{4.12}$$

where $\phi_\pm$ are constants. Now from (4.10) and (4.11), (2.6) becomes

$$(\mu-1)\underline{\xi} + \nabla(\underline{r}\cdot\underline{\xi}) = \nabla\left[(\mu-1)\ \phi + \underline{r}\cdot\nabla\phi\right] = 0. \tag{4.13}$$

Hence either $\mu = \ell + 2$ or $\mu = -\ell+1$, corresponding to the two terms in (4.12). The latter mode, localized (not surprisingly) at the outer surface of the shell, is unstable. From (2.7) we see that the time scale for this mode is

$$\tau_g^2 = \tau^2/(\ell+1), \tag{4.14}$$

essentially the same as the estimate given in (4.7).

Note that this result, identical with that obtained by KIDDER (1976) for the case of isentropic implosions with $\gamma = 5/3$, is completely independent of any details of the density, pressure or entropy profile, the magnitude of r_0 or d, and the value of γ. The unstable mode is purely incompressible, even though the unperturbed fluid is experiencing compression under the influence of the implosive force.

5. Convective instability

We consider perturbations for which σ does not vanish. In that case we see from (4.9) that $\underset{\sim}{\omega}$ is also nonzero, unless σ depends only on r or

$$\underset{\sim}{r} + \tau^2 \gamma p \nabla(\rho^{-1}) = 0, \tag{5.1}$$

which is equivalent to

$$\frac{ds}{dr} = \frac{d}{dr}(p\rho^{-\gamma}) = 0. \tag{5.2}$$

The former possibility, which amounts to looking just at $\ell=0$ modes, and the latter, which was discussed by BOOK (1978), are both included in the general treatment to be presented here.

We begin by moting that $\underset{\sim}{\omega}$ enters (4.8) only in the form $\nu \equiv \underset{\sim}{r}\cdot\nabla\times\underset{\sim}{\omega}$. An expression for this quantity can be found by taking the curl of (4.9), then dotting with $\underset{\sim}{r}$. Eliminating ν in favor of σ and writing $\sigma(r) = \sigma(r)\ Y_{\ell m}(\Theta,\varphi)$, we obtain an ordinary differential equation

$$\tau^{-2}(\mu-2)\sigma + \gamma\rho^{-1}\left[(p\sigma)'' + 2r^{-1}(p\sigma) - \ell(\ell+1)r^{-2}p\sigma\right]$$

$$+ \gamma(\rho^{-1})'\ (p\sigma)' - \frac{\ell(\ell+1)}{\mu-1}\left(1 - \frac{\gamma p\rho'}{p'\rho}\right), \tag{5.3}$$

where the prime denotes differentiation with respect to r. Equation (5.3) can be rewritten in the form

$$(\rho^{-1}r^2w')' + \left[\ell(\ell+1)\left(\frac{1}{\mu-1}\frac{\rho'}{r\rho} - \frac{1}{r^2} + \frac{Kp'}{2rp}\right]\rho^{-1}\ r^2 w = 0 \tag{5.4}$$

where $w = -p\sigma$ is the radial part of p_1 and

$$K = \frac{2}{\gamma}\left[\mu-2 - \frac{\ell(\ell+1)}{\mu-1}\right]. \qquad (5.5)$$

After multiplying (5.5) by w and integrating with respect to r, we find

$$\int_{r_0}^{r_0+d} dr\ r^2\rho^{-1}w'^2 - \int_{r_0}^{r_0+d} dr\ r^2\rho^{-1}[\ell(\ell+1)$$

$$\cdot\left(\frac{1}{\mu-1}\frac{\rho'}{r\rho} - \frac{1}{r^2}\right) + \frac{K\rho'}{2r\rho}]\ w^2 = 0. \qquad (5.6)$$

Here the surface terms left after the integration by parts vanish because the perturbed pressure must be zero at the boundaries. This equation is self-adjoint and almost in Sturm-Liouville form. Extremization subject to the requirement that μ be stationary recovers the differential equation (5.4), so that (5.6) is the basis for a variational principle.

Defining

$$A = -\int dr\ r\ w^2 p'/p\rho; \qquad (5.7a)$$

$$B = -A + \gamma\int dr\ r^2w'^2/\rho + \ell(\ell+1)\int dr\ w^2/\rho; \qquad (5.7b)$$

$$C = -\ell(\ell+1)\ (A + \gamma\int dr\ r\ w^2\rho'/\rho^2) \qquad (5.7c)$$

where all integrals run from r_0 to r_0+d, we can express (5.6) as

$$A(\mu-1)^2 + B(\mu-1) + C = 0. \tag{5.8}$$

The eigenvalue μ accordingly satisfies

$$\mu-1 = \frac{-B \pm (B^2-4AC)^{\frac{1}{2}}}{2A} . \tag{5.9}$$

From the definitions, $B > 0 > A$. Thus the lower root always satisfies $\mu > 1$, ruling out all but the weak "geometric" type of amplification (§3). We can make B arbitrarily large without changing A or C significantly by superposing a rapid infinitesimal "wiggle" on w. As B increases without bound, the upper root approaches $\mu = 1$. Hence we see that the extremal μ corresponding to the exact eigenfunction w is a minimum. It follows that there is an unstable mode ($\mu < 1$) if any test function w can be found for which (5.9) yields $\mu < 1$. The condition for this to occur is $C > 0$. Since the latter can be written

$$C = \ell(\ell+1)\int dr\ r\ w^2\ s'/s\rho, \tag{5.10}$$

a necessary and sufficient condition that $\mu < 1$ is $s'(r) > 0$ somewhere.

The mechanism for this instability was described by BOOK (1979). Because the unperturbed system is nonsteady ($\ddot{f} \neq 0$), individual elements of fluid are subject to an effective gravitational acceleration $G = -r\ddot{f}$, so that elements of differing density experience different buoyancy forces. If the result of interchanging two differential elements with different initial radii is to reduce the sum of the kinetic, compressional, and effective gravitational energies associated with these elements, the system is unstable. It can easily be shown, by modifying slightly the argument of BOOK (1979), that the condition for this to occur is just that the profiles be "superadiabatic," i.e., that $s'(r) > 0$ somewhere in the shell. We thus anticipate that the instability associated with this condition will be characterized by overturning of profiles, such as is typically seen in convective instabilities of static media (LANDAU and LIFSHITZ, 1959). There also the condition for instability is the occurrence of superadiabaticity.

Note that internal Rayleigh-Taylor modes, i.e, modes associated with the occurrence of opposing pressure and density gradients ($p'\rho' < 0$) at some point r satisfying $r_0 < r < r_0 + d$, are described by this formalism, since now both terms in

$$s'/s = p'/p - \gamma\rho'/\rho \tag{5.11}$$

are positive.

6. Examples

The variational principle developed in the previous section gives us an effective technique for approximately evaluating μ. As is usually the case,the eigenvalue calculated using a first-order approximation to w is accurate to second order.

For sufficiently simple basic state profiles, however, analytic solutions are possible. Consider the system whose initial density and pressure are given by (1.11). The corresponding entropy function is

$$s(r) = \bar{p}\,\bar{\rho}^{-\gamma}\,(r^2/r_o^{\,2})^{\varkappa+1-\varkappa\gamma}. \tag{6.1}$$

Evidently s(r) is increasing if $\varkappa < 1/(\gamma-1)$ and decreasing if $\varkappa > 1/(\gamma-1)$. Equation (5.4) becomes

$$w'' + \frac{2(1-\varkappa)}{r}w' + \frac{1}{r^2}\left[\ell(\ell+1)\left(\frac{2\varkappa}{\mu-1}-1\right) + K(\varkappa+1)\right]w = 0. \tag{6.2}$$

Writing $w = r^q$, we find two solutions for q:

$$2q = 2\varkappa - 1 \pm \left\{(2\varkappa-1)^2 - 4\left[\ell(\ell+1)\left(\frac{2\varkappa}{\mu-1}-1\right)\right.\right.$$

$$\left.\left.+K(\varkappa+1)\right]\right\}^{\frac{1}{2}} \equiv 2q_{\pm}. \tag{6.3}$$

We require w, expressed as a linear combination of the two solutions,

$$w = C_+\,r^{q_+} + C_-\,r^{q_-}, \tag{6.4}$$

to vanish at $r = r_0$ and $r = r_0+d$. Therefore $q_\pm$ must satisfy

$$r_0^{\,q_+}\,(r_0+d)^{q_-} = r_0^{\,q_-}\,(r_0+d)^{q_+}. \tag{6.5}$$

Hence

$$(2\kappa-1^2) - 4\left[\ell(\ell+1)\left(\frac{2\kappa}{\mu-1}-1\right) + K(\kappa+1)\right] = -\frac{4\pi^2 n^2}{\ell n^2(1+d/r_0)}, \tag{6.6}$$

where n is an integer.

Equation (6.6) is equivalent to (5.6), so the solution for μ is again given by (5.9), where

$$A = -2(\kappa+1); \tag{6.7a}$$

$$B = 2(\kappa+1) + \gamma\left[\frac{1}{4}(2\kappa-1)^2 + \ell(\ell+1) + \frac{\pi^2 n^2}{\ell n^2(1+d/r_0)}\right]; \tag{6.7b}$$

$$C = 2\ell(\ell+1)(\kappa+1)(\kappa+1-\gamma\kappa). \tag{6.7c}$$

The condition for instability, namely $C > 0$, reduces to

$$\kappa < 1/(\gamma-1), \tag{6.8}$$

as expected. The magnitude of μ decreases with n, so the most unstable mode is that with $n = 1$ (the boundary conditions cannot be satisfied for $n = 0$). For $\ell = 0$, $\mu = 1$. As $\ell \to \infty$, μ approaches the limiting value

$$\mu_\infty = 1 + \frac{\kappa\gamma-\kappa-1}{\gamma} = \frac{(\kappa+1)(\gamma-1)}{\gamma}. \tag{6.9}$$

A second example which can be treated analytically results if we let $r_0 \to 0$, $d \to \infty$, and choose

$$\rho(r) = \bar{\rho}(1+r^2/L^2)^{-2}, \tag{6.10}$$

where $\bar{\rho}$ is constant and L is a characteristic scale size. If $p(0) = 0$ it follows that

$$\bar{p}(r) = \bar{p}r^2/(r^2+L^2) \tag{6.11}$$

and

$$s(r) = \bar{p}\,\bar{\rho}^{\,-\gamma}(r^2/L^2)(1+r^2/L^2)^{\gamma-1}, \tag{6.12}$$

so that again $s'(r) > 0$. The logarithmic derivatives of ρ and p are

$$\frac{\rho'}{\rho} = \frac{-4r/L^2}{1+r^2/L^2} \tag{6.13}$$

and

$$\frac{p'}{p} = \frac{2}{r(1+r^2/L^2)}. \tag{6.14}$$

If we let $w(r) = r^q y(r)$, where

$$q = \frac{1}{2}\left\{(-1 + [(2\ell+1)^2 - 4K]^{\frac{1}{2}}\right\}, \tag{6.15}$$

then y satisfies the hypergeometric equation (3.5) with

$$\left.\begin{matrix} 2a \\ 2b \end{matrix}\right\} = q + \frac{5}{2} \pm \left[(q + \frac{5}{2})^2 + K + \frac{4\ell(\ell+1)}{\;} - 4q\right]^{\frac{1}{2}}, \tag{6.16}$$

$c = q + 3/2$, and $x = -r^2/L^2$. The condition that w(r) vanish at infinity reduces to $q - 2b < 0$, or $\mu > -3$, independent of ℓ. The discreet set of unstable modes has thus merged into a continum, the result of moving the boundary to infinity.

Finally, letting $r_0 = 0$ but keeping d finite, choose

$$\rho = \bar{\rho} \exp(r^2/L^2). \tag{6.17}$$

Then

$$p = \bar{p} \exp(r^2/L^2), \tag{6.18}$$

and

$$\frac{\rho'}{\rho} = \frac{p'}{p} = \frac{2r}{L^2} \ . \tag{6.19}$$

For this choice, s(r) is monotone <u>decreasing</u>:

$$s(r) = \bar{p}\bar{\rho}^{-\gamma} \exp\left[(1-\gamma)r^2/L^2\right] . \tag{6.20}$$

Writing $w(r) = r^{\ell}z(r)$, we find that z satisfies the confluent hypergeometric equation

$$xz'' + (c-x)\ z' - az = 0 \ , \tag{6.21}$$

where

$$a = -\frac{1}{4}K + \frac{\ell}{2}\,\frac{\mu-\ell}{\mu-1} \ , \tag{6.22}$$

$c = \ell+3/2$, and $x = r^2/L^2$. From the series expansion, it is clear that the Kummer function $\Phi(a;c;x)$ can vanish at $x = d^2/L^2$ only if $a < 0$. It follows that

$$\mu-1 > \tfrac{1}{2}\ \{\gamma\ell+1 \pm \left[(\gamma\ell+1)^2 - 4\ell(\ell+1)\ (\gamma-1)\right]^{\frac{1}{2}}\ \} > 0, \tag{6.23}$$

so that all modes are stable, as may be inferred from the fact that $s'(r) < 0$.

7. Discussion

We have seen that for a broad class of pressure and density profiles it is possible to calculate the evolution of linear perturbations on a uniformly imploding pellet or shell. The calculations presented above show that the time development of the perturbations depends only on the eigenvalue μ and γ, the ratio of specific heats. If $\mu > 1$, the perturbation amplitude grows weakly relative to the radius of the unperturbed pellet during implosion if and only if $\gamma < 5/3$. If $\mu < 1$, perturbation amplitudes are nonoscillatory and are always relatively amplified, the degree of amplification decreasing with γ. When $\mu < 0$ the mode amplitudes grow absolutely, not just relatively. They grow rapidly if the eigenvalue μ is large and negative. These results are summarized in Table 1, which presents the result of integrating (2.7) for various choices of γ and μ until $f = 0.1$ (tenfold radial compression), and in Table 2, which shows the corresponding results for $f = 0.05$ (twentyfold compression).

Incompressible irrotational modes, which are physically of Rayleigh-Taylor type, have $\mu = 1 - \ell$, irrespective of the shape of density and pressure profiles. For compressible (convective) modes, $\mu < 1$ or $\mu < 0$ can also occur, but the limiting value as $\ell \to \infty$ is finite. These modes are consequently less significant unless (i) the compressible disturbances are initiated with larger amplitude; or (ii) some change in the physical situation introduces a stabilizing influence at the outer boundary, where the Rayleigh-Taylor modes are localized, but does not affect the convectively unstable region where $s'(r) > 0$.

For real shells, which are affected by ablation, ionization, thermal conduction and other processes absent from our model, these results can only be applied in a rough approximation. The model is most nearly valid in the interior of the shell, far from the deposition zone. In this region it should predict correctly the stability of the target as implosion and compression take place.

A reasonable approximation is to take $\gamma = 4/3$ and inquire regarding the effect of a radial compression of 10 to 20, corresponding to a density increase of three to four orders of magnitude. As μ decreases, the oscillation period and growth rate both increase. From numerical integration of (2.6) for various choices of μ between -3 and 3 (Tables 1 and 2) it is clear that relative amplification by a factor of 100 is quite possible. This imposes a limitation on acceptable asymmetries or irregularities,restricting them to relative magnitudes of order 1% or less. Even the convective modes, for which $-\mu$ is bounded above, can be amplified significantly relative to the imploding shell raduis. Of course the Rayleigh-Taylor modes at large values of ℓ are amplified much more.

Since only the simplest properties of the target material were invoked in analyzing stability of the implosion, the predicted behavior should be observed universally. That is, it does not depend on details of the deposition process or on subtle plasma mechanisms. (Of course, if the target becomes ionized in the course of implosion, the model must be altered to reflect the ensuing complications.) Viewed in this light, the present calculation provides a zeroth order prediction of the destabilizing and desymmetrizing effects of implosion of a compressible target.

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REFERENCES

ABRAMOWITZ, M. and STEGUN, I.A., Editors (1964) Handbook of Mathematical Functions, U.S. Govt. Printing Office, Washington.

ANISIMOV, S.A. and INOGAMOV, N.A. (1974) Sov. Phys.-JETP Letters **20** , 74.

BERNSTEIN, I.B. and BOOK, D.L. (1978), Astrophys. J. **225**, 633.

BOOK, D.L. (1978) Phys. Rev. Lett. **41**, 1552.

BOOK, D.L. (1979) J. Fluid Mech. (in press).

KIDDER, R.E. (1976) Nuclear Fusion **16**, 3.

LANDAU, L.D. and LIFSHITZ, E.M. (1959) Fluid Mechanics, Addison-Wesley, Reading, MA, p.8.

SEDOV, L.I. (1959) Similarity and Dimesional Methods in Mechanics, Academic Press, New York, pp. 271-281.

Table 1.

Relative amplification $|T/f|$ at $f = 0.1$

μ	γ			
	1	4/3	5/3	2
3	-3.60	-1.58	-.475	-.0385
2	-2.85	-1.76	-.988	-.537
1	1.00	1.00	1.00	1.00
0	10.0	10.0	10.0	10.0
-1	27.0	30.4	33.9	37.7
-2	56.1	70.4	87.4	107.7
-3	102.3	142.2	194.0	260.8

Table 2.

Relative amplification T/f at f = 0.05

μ	γ			
	1	4/3	5/3	2
3	-4.99	-.677	.473	.416
2	-5.23	-2.35	-.854	-.290
1	1.00	1.00	1.00	1.00
0	20.0	20.0	20.0	20.0
-1	61.5	75.4	92.1	108.5
-2	140.0	205.2	298.0	406.0
-3	275.8	473.7	799.0	1240