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# Optimum Detection Of a Randomly Frequency-Modulated Carrier

Albert H. Nuttall  
Special Projects Department

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**Preface**

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. A. H. Nuttall (Code 313), Program Manager J. H. Probus (MAT 08T1), Naval Material Command. The technical reviewer for this report was Dr. G. Clifford Carter, Code 313.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The logarithm of the likelihood ratio for detection of a frequency-modulated signal tone, received in the presence of white Gaussian noise, is derived and expressed in the form of an infinite Volterra series. The amplitude of the received signal tone and the received noise power density level are presumed known; the additive phase shift is uniformly distributed; and the frequency-modulating process is Gaussian, with a spectrum such that no carrier tone remains in the frequency-modulated signal spectrum. This is called a fully-random signal herein. <i>next page</i> (over)		

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When the time-bandwidth product (of observation time and signal bandwidth) is large enough, and there is at least a moderate amount of frequency modulation, the optimum processor is well-approximated by a filter followed by an energy detector; the filter passband is that of the spectrum of the received signal. The time-bandwidth product is large enough, approximately, when its square-root is greater than the maximum of 10 and  $3d_2$ , where  $d_2$  is the voltage deflection criterion of the filter-energy-detector processor. Equivalently, the ratio of the received signal energy per independent component to the received noise power density must be small, and the time-bandwidth product large, in order for the filter-energy-detector term of the log-likelihood ratio series to dominate decisions. The frequency modulation is termed moderate when the ratio of the RMS frequency deviation to the equivalent bandwidth of the frequency-modulating process is of the order of 2-3.

Numerous approximations have been necessary to facilitate evaluation of some of the multiple integrals; to what degree the sufficient conditions cited above can be relaxed, without violating the conclusions, is unknown.

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## INTRODUCTION

Detection of a tone of known frequency and limited duration, in the presence of noise, is often accomplished by passing the received waveform through a matched filter and comparing a sample of the filter output with a threshold. (For unknown phase of the tone, it is the filter output envelope that is compared with a threshold.) The bandwidth of this receiving filter is very narrow, in fact approximately equal to the inverse of the signal duration; in this manner, the noise passed by the receiving filter is greatly attenuated, and decent decisions about signal presence or absence can be made. Typically, the ratio of received signal energy to noise power density level, of the order of 10, is required for low false alarm probabilities and acceptable detection probabilities.

However if the tone is phase- or frequency-modulated in a random fashion, the spectrum of the received signal is spread significantly beyond the inverse signal duration, and any receiving filter must be broadened to accept the received signal. As a result, considerably more noise is passed to the receiving filter output, adversely affecting decision capability. The fact that the modulation is random, that is, unknown to the receiver, prevents much coherent processing of the received signal, if the modulation is significant. If the deviation and bandwidth of the frequency-modulating process were known (or better, if the spectrum of the frequency-modulating process were known), the question arises as to just how much could be achieved in detection capability by taking advantage of this information.

We will address the problem of optimum detection of such signals in the following, with particular emphasis on the case where the product of observation time (signal duration) and signal bandwidth is much larger than unity. For long duration signals, this situation can easily arise with fairly small amounts of phase- or frequency-modulation. We will also consider simpler sub-optimum processors and attempt to deduce their performance. The severe analytic problems preclude complete solution, and some considerable simulation effort will be necessary in the future in order to give quantitative comparisons. Some related work is given in Ref. 1 for the first-order term in a series expansion of the optimum processor, useful for low input signal-to-noise ratios.

## LIKELIHOOD RATIOS

### General Case

In an observation of  $M$  samples, either the condition noise-alone or signal-plus-noise prevails. On the basis of these  $M$  samples, an optimum decision about signal presence or absence is to be made. The signal samples are denoted by

$$S = (s_1, s_2, \dots, s_M), \quad (1)$$

and the observation is denoted by

$$X = (x_1, x_2, \dots, x_M). \quad (2)$$

If we let  $H_0$  denote the signal-absent hypothesis, and  $H_1$  denote the signal-present hypothesis, the optimum decision rule is based upon a comparison of the likelihood ratio (LR) with a threshold (Refs. 2, 3):

$$LR(X) = \frac{p_1(X)}{p_0(X)}, \quad (3)$$

where  $p_0(X)$  is the noise-alone probability density function (PDF) of observation  $X$ , and  $p_1(X)$  is the PDF of  $X$  with signal present. We can express (3), for any signal statistics, as\*

$$LR(X) = \frac{\int dS p_1(X|S) p_s(S)}{p_0(X)}, \quad (4)$$

where  $p_1(X|S)$  is the conditional PDF of  $X$ , given signal values  $S$ , and  $p_s(S)$  is the a priori PDF of  $S$ .

A modified notation for (4) is adopted in the following, namely

$$LR(X) = \frac{\langle p_1(X|S) \rangle}{p_0(X)} = \left\langle \frac{p_1(X|S)}{p_0(X)} \right\rangle = \langle LR(X|S) \rangle, \quad (5)$$

where the ensemble average,  $\langle \rangle$ , is over the signal statistics, whatever they are, considering observation  $X$  as a fixed quantity. (See also Ref. 3, page 132, eq. (1.4)). (5) is an average over the conditional LR.

For continuous observation of waveform  $x(t)$  over a time interval  $(t_a, t_b)$  where the signal exists (if present), the LR in (5) becomes

---

\*Integrals without limits are over the range of non-zero integrand.

$$LR(x(t): t_a < t < t_b) = \left\langle \frac{p_i(x(t)|s(t): t_a < t < t_b)}{p_o(x(t): t_a < t < t_b)} \right\rangle, \quad (6)$$

where the ensemble average,  $\langle \rangle$ , is over the random signal process  $s(t)$ , considering observation  $x(t)$  as a fixed waveform. This interpretation of the bracket  $\langle \rangle$  will be used throughout the following report.

#### Additive Signal and White Gaussian Noise

The results above pertain to any signal and noise model. We now specialize to additive signal and noise. In particular, for flat\* low-pass Gaussian noise of bandwidth  $W_n$  ( $-W_n, W_n$  Hz double-sided), noise samples taken every  $(2W_n)^{-1}$  seconds apart are independent; then (for real processes)

$$p_o(X) = \prod_{m=1}^M \left\{ \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{X_m^2}{2\sigma_n^2}\right) \right\} \quad (7)$$

and

$$p_i(X|S) = \prod_{m=1}^M \left\{ \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{(X_m - S_m)^2}{2\sigma_n^2}\right) \right\}, \quad (8)$$

where  $\sigma_n^2$  is the noise variance, presumed known. Then (5) yields

$$LR(X) = \left\langle \exp\left[\frac{1}{\sigma_n^2} \sum_{m=1}^M X_m S_m - \frac{1}{2\sigma_n^2} \sum_{m=1}^M S_m^2\right] \right\rangle, \quad (9)$$

where the ensemble average is over  $\{S_m\}_1^M$  only.

Since  $\sigma_n^2 = N_0 W_n$ , where  $N_0$  is the single-sided noise density level in watts/Hz (presumed known), then in the limit as the noise bandwidth  $W_n \rightarrow \infty$ , (9) (and (6)) becomes

$$LR(x(t): t_a < t < t_b) = \left\langle \exp\left[\frac{2}{N_0} \int_{t_a}^{t_b} dt x(t) s(t) - \frac{1}{N_0} \int_{t_a}^{t_b} dt s^2(t)\right] \right\rangle. \quad (10)$$

(See also Ref. 3, pages 108-9.)

If the signal  $s(t)$  and observation  $x(t)$  are narrowband about center

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\*For narrowband signals, the assumption that the noise spectrum is flat in this narrow band is not very restrictive.

frequency  $f_0$ , and if  $Tf_0 \gg 1$ , where observation time (signal duration)  $T = t_b - t_a$ , then (10) can be expressed as (see also Ref. 3, page 160, eq. (5.2))

$$LR(x(t): t_a < t < t_b) = \left\langle \exp \left[ \frac{1}{N_0} \operatorname{Re} \int_{t_a}^{t_b} dt x(t) z^*(t) - \frac{1}{2N_0} \int_{t_a}^{t_b} dt |z(t)|^2 \right] \right\rangle, \quad (11)$$

where  $\operatorname{Re}$  denotes the real part, and  $z(t)$  and  $x(t)$  are the complex envelopes of signal and observation respectively:

$$s(t) = \operatorname{Re} \{ z(t) \exp(i2\pi f_0 t) \}, \quad x(t) = \operatorname{Re} \{ x(t) \exp(i2\pi f_0 t) \}. \quad (12)$$

It should be noted that although we have presumed  $Tf_0 \gg 1$ , we have not fixed  $TW_s$ , the product of observation time (signal duration) and signal bandwidth. Also we have not yet specified the signal statistics in any way.

#### Fixed Received Signal Energy

A special case of the above is afforded when the received signal energy in the observation interval,

$$E = \int_{t_a}^{t_b} dt s^2(t) = \frac{1}{2} \int_{t_a}^{t_b} dt |z(t)|^2, \quad (13)$$

is constant, even though the received signal  $s(t)$  is a random process. (An example is the narrowband phase-modulated signal

$$s(t) = A \cos [2\pi f_0 t + \theta(t) + \phi], \quad t_a < t < t_b, \quad (14)$$

where  $Tf_0 \gg 1$ ,  $A$  is a constant (but perhaps unknown),  $\theta(t)$  is a random process, and  $\phi$  is a random variable; the product  $TW_s$  is arbitrary. Then,  $E = A^2 T / 2$ ).

In this special case of (13), (11) yields

$$LR(x(t): t_a < t < t_b) = \exp\left(-\frac{E}{N_0}\right) \left\langle \exp \left[ \frac{1}{N_0} \operatorname{Re} \int_{t_a}^{t_b} dt x(t) z^*(t) \right] \right\rangle. \quad (15)$$

This simplified form for the LR (for fixed received signal energy) points out the magnitude of the analytical difficulty of the problem we are addressing. Namely, the ensemble average on the right-hand side of (15) is the characteristic function\* of the random variable

$$\operatorname{Re} \int_{t_a}^{t_b} dt x(t) z^*(t), \quad (16)$$

---

\*Actually, it is the moment-generating function.

at argument  $i\mathcal{S} = 1/N_0$ . But (16) is a linearly-filtered version of random process  $s(t)$ , with fixed waveform  $x(t)$ ; recall that  $x(t)$  is considered fixed in the ensemble average in (15). And since calculation of the PDF of the output of a linear filter (for other than Gaussian input processes) has remained an unsolved problem in the literature for many years now, there is no hope of exact evaluation of (15) for general random processes  $s(t)$ . Rather, all we can hope for is good approximations for some signal processes under certain conditions. The situation for the more general case, (11), is even worse, due to the additional randomness of the received signal energy. Notice that the only restriction on the signal statistics in (15) is that stated in (13); we are not restricted to the model of example (14).

In the following, we shall generally make the assumption about the received signal energy being constant; notice that this disallows  $s(t)$  or  $x(t)$  from being a Gaussian random process. We shall also concentrate on narrowband signals, for which  $Tf_b \gg 1$ ; however,  $TW_b$  is arbitrary initially. Hence the pertinent expression for the LR is (15).

It is perhaps worthwhile at this point to fix the ideas presented thus far with a couple of examples. In particular, averaging with respect to signal statistics, while holding the received waveform (observation) fixed, will be illustrated. The received signal energy is not constant for these two examples.

#### Low Pass Example

Let the received signal be

$$s(t) = A m(t), \quad t_a < t < t_b, \quad (17)$$

where  $A$  is random but positive, and  $m(t)$  is a known deterministic real function. Since the only randomness in the received signal is through the scaling parameter  $A$ , (10) becomes

$$\begin{aligned} LR &= \left\langle \exp \left[ \frac{2A}{N_0} \int_{t_a}^{t_b} dt x(t) m(t) - \frac{A^2}{N_0} \int_{t_a}^{t_b} dt m^2(t) \right] \right\rangle_A \\ &= \left\langle \exp [Aa - A^2 b] \right\rangle_A, \end{aligned} \quad (18)$$

where

$$a \equiv \frac{2}{N_0} \int_{t_a}^{t_b} dt x(t) m(t), \quad b \equiv \frac{1}{N_0} \int_{t_a}^{t_b} dt m^2(t). \quad (19)$$

Notice that the observation  $x(t)$  appears in the LR only through the coherent integration yielding parameter  $a$ . Now we can express (18) as

$$LR = \int_0^{\infty} dA p(A) \exp[AA - A^2 b]. \quad (20)$$

But since

$$\frac{\partial}{\partial a} LR = \int_0^{\infty} dA p(A) A \exp[AA - A^2 b] \quad (21)$$

is positive for any PDF  $p(A)$ , when  $A$  is limited to positive values, then LR is monotonically increasing with the parameter  $a$ . Hence comparison of LR with a threshold is synonymous with comparison of parameter  $a$  with a threshold.

Thus the optimum processor is

$$\int_{t_a}^{t_b} dt x(t) m(t) \underset{H_0}{\overset{H_1}{\geq}} \text{threshold}, \quad (22)$$

regardless of the PDF  $p(A)$ , when random variable  $A$  is limited to positive values. (22) is recognized as the standard coherent correlation detector operating on observation  $x(t)$ .

#### Narrowband Example

Let the received signal be

$$s(t) = A \operatorname{Re} \{ \xi(t) \exp(i2\pi f_0 t + i\phi) \}, \quad t_a < t < t_b, \quad (23)$$

where  $A$  is random and positive, and  $\phi$  is random and uniformly distributed over  $2\pi$ . Variables  $A$  and  $\phi$  are independent; there is no need to consider negative  $A$ , since this effect can be absorbed by values of  $\phi$ . The complex envelope is then

$$\underline{s}(t) = A \exp(i\phi) \xi(t), \quad (24)$$

where  $\xi(t)$  is a known deterministic low-frequency complex function. Since the randomness of  $\underline{s}(t)$  arises only through  $A$  and  $\phi$ , (11) yields

$$\begin{aligned} LR &= \left\langle \exp \left[ \frac{A}{N_0} \operatorname{Re} \int_{t_a}^{t_b} dt x(t) \xi^*(t) e^{-i\phi} - \frac{A^2}{2N_0} \int_{t_a}^{t_b} dt |\xi(t)|^2 \right] \right\rangle_{A, \phi} \\ &= \left\langle I_0 \left( \frac{A}{N_0} \left| \int_{t_a}^{t_b} dt x(t) \xi^*(t) \right| \right) \exp \left( - \frac{A^2}{2N_0} \int_{t_a}^{t_b} dt |\xi(t)|^2 \right) \right\rangle_A \\ &= \left\langle I_0(Aa) \exp(-A^2 b) \right\rangle_A = \int_0^{\infty} dA p(A) I_0(Aa) \exp(-A^2 b), \quad (25) \end{aligned}$$

where

$$a = \frac{1}{N_0} \left| \int_{t_a}^{t_b} x(t) s^*(t) dt \right|, \quad b = \frac{1}{2N_0} \int_{t_a}^{t_b} |s(t)|^2 dt. \quad (26)$$

Notice that observation  $x(t)$  appears in the LR only through the coherent integration yielding parameter  $a$ . But since (using (25))

$$\frac{\partial}{\partial a} \text{LR} = \int_0^\infty dA p(A) A I_1(Aa) \exp(-A^2 b) \quad (27)$$

is positive for any PDF  $p(A)$ , then LR is monotonically increasing with the parameter  $a$ . Hence comparison of LR with a threshold is synonymous with comparison of  $a$  with a threshold. Thus the optimum processor is

$$\left| \int_{t_a}^{t_b} dt x(t) s^*(t) \right| \underset{H_0}{\overset{H_1}{\geq}} \text{threshold}, \quad (28)$$

regardless of the PDF  $p(A)$ . (28) is recognized as a threshold comparison of the envelope of the output of a matched filter operating on observation  $x(t)$ .

## SERIES EXPANSION FOR LOGARITHM OF LIKELIHOOD RATIO

The starting point for this development is (15), which applies to arbitrary narrowband signal statistics, subject to fixed received signal energy. When we define random variable

$$v = \frac{1}{N_0} \operatorname{Re} \int dt x(t) s^*(t), \quad (29)$$

(using the fact that received signal  $s(t)$  and observation  $x(t)$  are non-zero only in  $(t_a, t_b)$ ), then (15) can be expressed as

$$LR = \exp\left(-\frac{E}{N_0}\right) \langle \exp[v] \rangle, \quad (30)$$

where the ensemble average is over the signal statistics. Now from (29), we find

$$\langle v \rangle = \frac{1}{N_0} \operatorname{Re} \int dt x(t) \langle s^*(t) \rangle, \quad (31)$$

where  $\langle s(t) \rangle$  will be called the deterministic signal component. Then define the residual of random variable  $v$  as

$$r = v - \langle v \rangle = \frac{1}{N_0} \operatorname{Re} \int dt x(t) a^*(t), \quad (32)$$

where we have also defined the "ac" component of the signal complex envelope as

$$a(t) = s(t) - \langle s(t) \rangle. \quad (33)$$

(Later we shall generally consider signal models for which the deterministic signal component is absent, i.e.,

$$\langle s(t) \rangle = 0, \quad (34)$$

which is the usual case for phase-random narrowband waveforms. For example, (24) was just such a case.)

Substitution of (32) in (30) yields

$$\begin{aligned} LR &= \exp\left(-\frac{E}{N_0}\right) \langle \exp[\langle v \rangle + r] \rangle \\ &= \exp\left(-\frac{E}{N_0} + \langle v \rangle\right) \langle \exp[r] \rangle. \end{aligned} \quad (35)$$

At this point, we define the characteristic function of random variable  $r$  in (32) to be the ensemble average

$$f_r(i\mathcal{E}) = \langle \exp(i\mathcal{E}r) \rangle. \quad (36)$$

Once again, we note, with reference to the definition of  $r$  in (32), that the randomness of  $r$  is to be considered due to that of ac signal  $g(t)$  (through (33)), and that observation  $\mathcal{X}(t)$  must be considered fixed in (32) for this ensemble average in (36).

The natural logarithm of the characteristic function of  $r$  can be expanded as

$$\ln f_r(i\mathcal{E}) = \sum_{k=2}^{\infty} \frac{(i\mathcal{E})^k}{k!} \chi_k^{(r)}, \quad (37)$$

where  $\chi_k^{(r)}$  is the  $k$ -th cumulant of random variable  $r$ . (See, for example, Ref. 4, Chapter 3). The sum starts at  $k = 2$  since (see (32))

$$\chi_1^{(r)} = \langle r \rangle = \langle v - \langle v \rangle \rangle = 0. \quad (38)$$

For example, we have (Ref. 4, page 71, eq. 3.43)

$$\begin{aligned} \chi_2^{(r)} &= \langle r^2 \rangle \\ \chi_3^{(r)} &= \langle r^3 \rangle \\ \chi_4^{(r)} &= \langle r^4 \rangle - 3\langle r^2 \rangle^2. \end{aligned} \quad (39)$$

Since the natural logarithm is a monotonic transformation, comparison of LR in (35) with a threshold is synonymous with comparison of  $\ln$  LR with a (different) threshold. Then using (35), (36), and (37), we find

$$\begin{aligned} \ln LR &= -\frac{E}{N_0} + \langle v \rangle + \ln f_r(i) \\ &= -\frac{E}{N_0} + \langle v \rangle + \sum_{k=2}^{\infty} \frac{1}{k!} \chi_k^{(r)} = -\frac{E}{N_0} + \sum_{k=1}^{\infty} \frac{1}{k!} \chi_k^{(v)}, \end{aligned} \quad (40)$$

where  $\chi_k^{(v)}$  is the  $k$ -th cumulant of random variable  $v$ ; see (32). The first term in  $\ln LR$ , namely  $-E/N_0$ , is a constant, independent of the observation  $\mathcal{X}(t)$ , and can therefore be ignored (i.e., absorbed in a modified threshold in the decision comparison).

The second term,  $\langle v \rangle$ , in  $\ln LR$  is proportional to a coherent correlation of the received waveform with the deterministic component of the signal

waveform; consider the following development (where  $\omega_0 = 2\pi f_0$ ):

$$\begin{aligned} \frac{2}{N_0} \int dt x(t) \langle s(t) \rangle &= \frac{2}{N_0} \int dt \operatorname{Re}\{x(t) e^{i\omega_0 t}\} \langle \operatorname{Re}\{s(t) e^{i\omega_0 t}\} \rangle \\ &= \frac{2}{N_0} \int dt \operatorname{Re}\{x(t) e^{i\omega_0 t}\} \operatorname{Re}\{\langle s(t) \rangle e^{i\omega_0 t}\} \\ &= \frac{2}{N_0} \int dt \frac{1}{2} \{x(t) e^{i\omega_0 t} + x^*(t) e^{-i\omega_0 t}\} \frac{1}{2} \{\langle s(t) \rangle e^{i\omega_0 t} + \langle s^*(t) \rangle e^{-i\omega_0 t}\} \\ &= \frac{1}{N_0} \operatorname{Re} \int dt x(t) \langle s^*(t) \rangle = \langle v \rangle, \end{aligned} \quad (41)$$

where we have used (12), the fact that  $Tf_0 \gg 1$ , and (31). (As noted in (34), this term will generally be zero for our signal models of interest; that is, there will be no deterministic signal component in the received waveform, and hence the  $\langle v \rangle = \chi_1^{(v)}$  term will be absent from the ln LR expansion in (40).)

All the other terms in ln LR in (40) involve  $r$  defined in (32) (or  $v$  defined in (29)). In particular, since (32) can be written as

$$r = \frac{1}{2N_0} \int dt [x(t) a^*(t) + x^*(t) a(t)], \quad (42)$$

then we have, using (39), the second cumulant

$$\begin{aligned} \chi_2^{(r)} = \langle r^2 \rangle &= \frac{1}{4N_0^2} \iint dt_1 dt_2 \langle [x(t_1) a^*(t_1) + x^*(t_1) a(t_1)] [x(t_2) a^*(t_2) + x^*(t_2) a(t_2)] \rangle \\ &= \frac{1}{4N_0^2} \iint dt_1 dt_2 x(t_1) x^*(t_2) \langle a(t_2) a^*(t_1) \rangle + \text{complex conjugate}, \end{aligned} \quad (43)$$

where we have used the fact that

$$\langle a(t_2) a(t_1) \rangle = 0, \quad \text{all } t_1, t_2 \quad (44)$$

for complex envelopes of narrowband random processes; see for example, Ref. 3, page 53, eq. (5.12). Now it may easily be shown that each of the two terms in (43) is real; therefore

$$\frac{1}{2} \chi_2^{(r)} = \frac{1}{4N_0^2} \iint dt_1 dt_2 x(t_1) x^*(t_2) R_a(t_2 - t_1), \quad (45)$$

where  $R_a$  is the autocorrelation of ac component  $a(t)$ , assumed stationary in the observation interval; that is,

$$\langle a(t_2) a^*(t_1) \rangle = R_a(t_2 - t_1) \quad \text{for } t_1, t_2 \in (t_0, t_0). \quad (46)$$

An alternative expression to (45) will be developed below.

The remaining cumulants in (40) require higher-order moments of  $a(t)$ , and must await evaluation until we specify the exact signal model of interest, which we have not yet done. However it is very important to observe that if

$$r = \frac{1}{N_0} \operatorname{Re} \int dt \underline{x}(t) \underline{a}^*(t) \quad (47)$$

were a Gaussian random variable, then all its cumulants  $\chi_k^{(r)}$ , for  $k \geq 3$ , would be identically zero, and the series for  $\ln LR$  in (40) would terminate with the term (45). Since  $\underline{x}(t)$  in (47) is to be considered fixed in this particular statistical consideration of  $r$ , it is seen from (47) and (33) that  $r$  is the sum of a large number of statistically independent components if the time-bandwidth product of the received signal is large, that is, if

$$TW_s \gg 1, \quad (48)$$

and if  $\underline{a}(t)$  is fully random (to be explained). This is due to the fact that the integral in (47) is over a time duration of  $T$  seconds and that  $\underline{a}(t)$  has a statistically-independent "wiggle" approximately every  $1/W_s$  seconds. Then by appeal to the Central Limit Theorem,  $r$  will be nearly Gaussian when (48) obtains, and the cumulants for  $k \geq 3$  in (40) can then be well approximated by zero. We will attempt to quantitatively justify this claim later, for a particular signal model.

The expression for the  $\ln LR$  is, from (40),

$$\ln LR = -\frac{E}{N_0} + \langle v \rangle + \frac{1}{2} \chi_2^{(r)} + \frac{1}{6} \chi_3^{(r)} + \frac{1}{24} \chi_4^{(r)} + \dots \quad (49)$$

Now  $\chi_k^{(r)}$  is actually a random variable, governed by the statistics of the observation  $\underline{x}(t)$ ,  $t_a < t < t_b$ ; remember we have already averaged over the statistics of  $\underline{s}(t)$ , in (36) and (37). Furthermore  $\chi_k^{(r)}$  has different statistics for signal-present ( $H_1$ ) versus signal-absent ( $H_0$ ). Although the sum of the random variables in (49) is to be compared with a threshold, it is not the absolute level of the mean values of  $\chi_k^{(r)}/k!$ , under  $H_0$  and  $H_1$ , that is important; rather, it is the difference in these means under  $H_0$  and  $H_1$ , that is important. After all, means can be absorbed in a modified threshold. Therefore we will attempt to determine the means and standard deviations of some of the random variables  $\{\chi_k^{(r)}/k!\}$ , and make some quantitative statements as to when and where (49) can be terminated.

Another important point should be made here regarding (49). One sub-optimum processor is afforded by simply terminating (49) with the term  $\chi^2/2$ . Now even though the rest of the terms in the series may not be small, and therefore cannot be dropped from the optimum processor, it is conceivable that the decisions yielded by this sub-optimum processor often agree with those yielded by the optimum processor (each processor with its own threshold). This will be the case when (48) is true; an important question is: how small can  $TW_s$  get before the optimum processor significantly outperforms this sub-optimum processor? An example in appendix A shows no difference in performance for very small  $TW_s$ , namely  $1/f_0 \ll T < 1/W_s$ . Thus at the two extremes of very large and small  $TW_s$ , this sub-optimum processor performs just as well as the optimum processor.

It is worthwhile to note that if we were to expand the LR in (35) in a series, we would obtain a far less useful result; namely

$$LR = \exp\left(-\frac{E}{N_0} + \langle v \rangle\right) \sum_{k=0}^{\infty} \frac{1}{k!} \langle r^k \rangle. \quad (50)$$

Now this series would not terminate, even for  $r$  equal to a Gaussian random variable. And for  $r$  near-Gaussian, it is not obvious how many terms of (50) need to be retained for a good approximation. Thus (49) is expected to be more useful, especially when (48) is true.

#### Example of Exact LR for Several Narrowband Pulses

By way of illustration of some of the techniques above, consider the signal model

$$s(t) = \text{Re} \sum_{\ell} A_{\ell} \xi_{\ell}(t) \exp[i\omega_{\ell} t + i\phi_{\ell}], \quad t_a < t < t_b, \quad (51)$$

where  $\{\phi_{\ell}\}$  are independent random variables uniformly distributed over  $2\pi$ ,  $\{\xi_{\ell}(t)\}$  are known complex deterministic low-frequency waveforms,  $\{A_{\ell}\}$  are known amplitudes, and  $\{f_{\ell}\}$  are known center frequencies. We assume that the complex waveforms  $\{\xi_{\ell}(t) \exp(i\omega_{\ell} t)\}$  are orthogonal, perhaps through time or frequency separation, and that  $T_{\ell} f_{\ell} \gg 1$ , where  $T_{\ell}$  is the duration of  $\xi_{\ell}(t)$ . Then it follows that the received signal energy,  $\int dt s^2(t)$ , is virtually independent of  $\{\phi_{\ell}\}$ , and hence is non-random.

The signal complex envelope is

$$\underline{s}(t) = \sum_{\ell} A_{\ell} \xi_{\ell}(t) \exp[i\phi_{\ell} + i(\omega_{\ell} - \omega_0)t], \quad (52)$$

for which, obviously, the deterministic component  $\langle \varepsilon(t) \rangle = 0$ . Then (32) yields

$$r = \frac{1}{N_0} \operatorname{Re} \int dt \varepsilon(t) \sum_l A_l \varepsilon_l^*(t) \exp[-i\phi_l - i(\omega_l - \omega)t]. \quad (53)$$

The ensemble average in (35) is then

$$\begin{aligned} \langle e^r \rangle &= \prod_l \left\langle \exp \left[ \frac{A_l}{N_0} \operatorname{Re} \int dt \varepsilon(t) \varepsilon_l^*(t) \exp[-i(\omega_l - \omega)t] \exp[-i\phi_l] \right] \right\rangle \\ &= \prod_l I_0 \left( \frac{A_l}{N_0} \left| \int dt \varepsilon(t) \varepsilon_l^*(t) \exp[-i(\omega_l - \omega)t] \right| \right). \end{aligned} \quad (54)$$

Hence (35) yields (with the aid of (31))

$$\ln LR = -\frac{E}{N_0} + \sum_l \ln I_0 \left( \frac{A_l}{N_0} \left| \int dt \varepsilon(t) \varepsilon_l^*(t) \exp(i\omega_l t - i\omega t) \right| \right). \quad (55)$$

A block diagram of the optimum processor for this signal model is shown in figure 1 below. The non-linearity  $\ln I_0(\cdot)$  is approximately a squarer for small arguments (inputs), and is approximately linear for large arguments. Thus coherent processing of each component is accomplished, and then followed by non-linear envelope detection and summation.

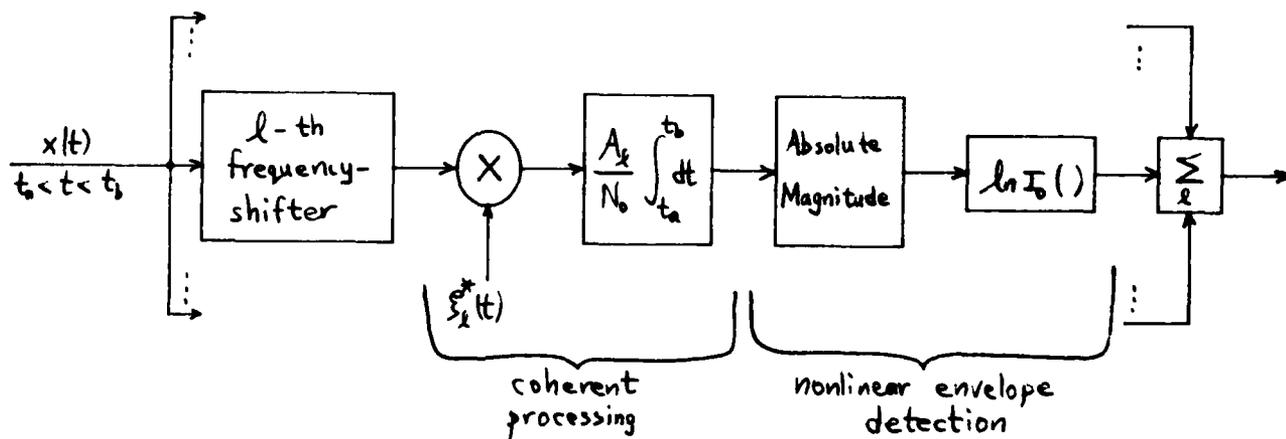


Figure 1. Optimum Processor for Several Narrowband Pulses

## BLOCK DIAGRAM FOR PROCESSOR YIELDING $\chi_2^{(r)}/2$

In this section we will give a block diagram of a processor which accepts complex envelope waveform  $x(t)$ ,  $t_a < t < t_b$ , at its input, and emits the quantity  $\chi_2^{(r)}/2$ . To do this, we first define correlation

$$R_o(\tau) \equiv R_a(\tau) - R_a(\infty) ; R_o(\infty) = 0. \quad (56)$$

An example of a signal process for which  $R_a(\infty) \neq 0$  (even though the deterministic signal component  $\langle \varepsilon(t) \rangle = 0$ ) is given in appendix B. Furthermore, it is shown there that the deterministic signal component corresponds to pure tones in  $s(t)$  of known phase, whereas the ac component with  $R_a(\infty) \neq 0$  corresponds to pure tones in  $s(t)$  of random phase. The quantity  $R_a(\infty)$  must be real, since it gives rise to an impulse in real power spectrum  $G_a(f)$ , at  $f=0$ .

Substitution of (56) in (45) yields

$$\frac{1}{2} \chi_2^{(r)} = \frac{R_a(\infty)}{(2N_b)^2} \left| \int_{t_a}^{t_b} dt x(t) \right|^2 + \frac{1}{(2N_b)^2} \iint dt_1 dt_2 x(t_1) x^*(t_2) R_o(t_2 - t_1). \quad (57)$$

This processor is indicated in block diagram form in figure 2. The power

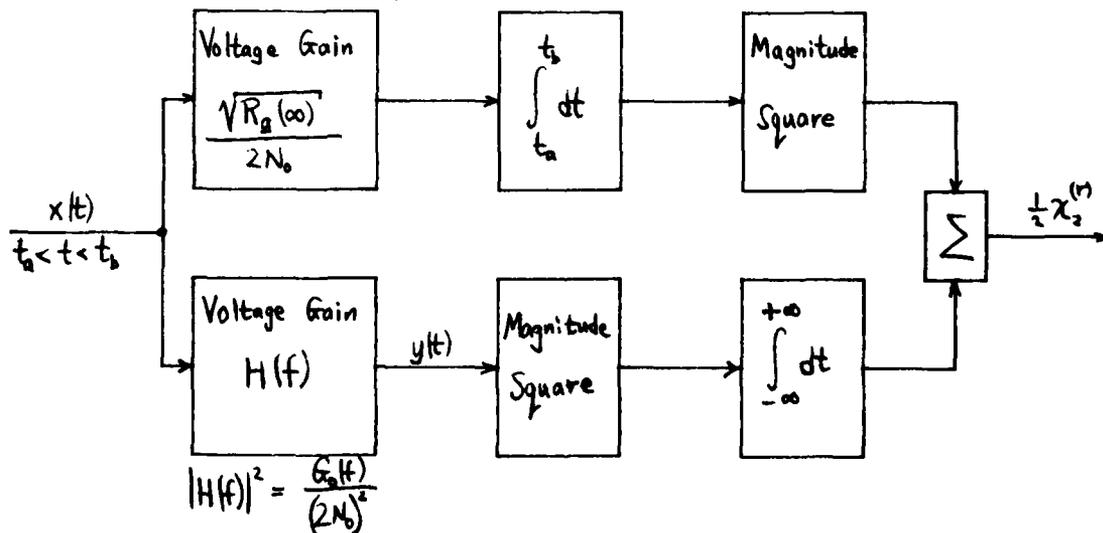


Figure 2. Processor to Generate  $\frac{1}{2} \chi_2^{(r)}$  from  $x(t)$

spectrum\*  $G_o(f)$  is defined as the usual Fourier transform of correlation  $R_o(\tau)$ ,

$$G_o(f) = \int d\tau \exp(-i2\pi f\tau) R_o(\tau), \quad (58)$$

and  $H(f)$  is a low-pass Eckart filter.

The way to show that the lower branch of figure 2 generates the second term in (57) is as follows: the output of the bottom integrator in figure 2 is

$$\int dt \left| \int du h(t-u) x(u) \right|^2 = \int \int du dv x(u) x^*(v) \int dt h(t-u) h^*(t-v). \quad (59)$$

But

$$\begin{aligned} \int dt h(t-u) h^*(t-v) &= \int df |H(f)|^2 \exp[i2\pi f(v-u)] \\ &= \frac{1}{(2N_o)^2} \int df G_o(f) \exp[i2\pi f(v-u)] = \frac{R_o(v-u)}{(2N_o)^2}, \end{aligned} \quad (60)$$

where  $h$  and  $H$  are a Fourier transform pair (impulse response and transfer function), and we used (58). Thus the last quantity in (59) becomes precisely the last term in (57).

\* Since  $G_o(f) = G_a(f) - R_a \delta(f)$ ,  $G_o(f)$  must be non-negative, since  $G_a(f)$  is a legal spectrum. Therefore the choice of  $|H(f)|^2$  in figure 2 is always legal.

The upper branch of the processor in figure 2 depends on the statistic  $|\int_{t_a}^{t_b} x(t)|^2$  (see (57)), which corresponds to a coherent cross-correlation, over the entire observation interval, of the received waveform with a constant carrier at  $f_0$ , followed by square-law envelope detection at the end of the observation interval. Whether this coherent component is significant depends on the relative sizes of  $R_s(\omega)$  and  $R_s(\phi)$ ; see appendix B in this regard. It should be made clear that the upper branch of figure 2 is present even when the received signal has no deterministic component, i.e., when  $\langle x(t) \rangle = 0$ . The distinction between the two cases is well illustrated by the special case of a pure tone signal of random phase,  $\cos(2\pi f_0 t + \phi)$ . This signal has no deterministic component because the average over  $\phi$  gives zero. However, cross-correlation of this signal with a local carrier at  $f_0$ , with in-phase and quadrature components, followed by squaring and summing outputs, affords a significant amount of coherent signal processing gain that cannot be disregarded; in fact, the performance is but slightly poorer than detection of a signal with known phase,  $\phi$  (see Ref. 3, pages 88 and 155). When the signal satisfies both of the properties

$$\langle x(t) \rangle = 0 \quad \text{and} \quad R_s(\omega) = 0, \quad (61)$$

we then say that the signal is fully-random; this case will occupy most of our attention in the following.

The lower branch of the processor in figure 2 is a linear Eckart filter followed by an energy detector. Since the linear filter impulse response is non-zero in duration (being the Fourier transform of  $H(f)$ ), the finite-duration input  $x(t)$  is smeared by this filtering operation; hence the integration in this lower branch is over all time, not just  $(t_a, t_b)$  as in the upper branch. This Eckart filter followed by an energy detector is exactly the filter characteristic which maximizes the output deflection; see appendix C.

Furthermore, the linear filtering action of  $H(f)$  in figure 2 is such as to perform coherent integration on the random input signal over as long a time as possible. To see this, we notice that the output of filter  $H(f)$  can be written as

$$y(t) = \int d\tau x(\tau) h(t-\tau). \quad (62)$$

But since the passband of  $H(f)$  is the same as the bandwidth of  $G_o(f)$ , the impulse response duration of  $h(t)$  is approximately the correlation time of  $R_o(\tau)$ . That is,

$$|H(f)|^2 = \frac{G_o(f)}{(2N_o)^2} \quad (63)$$

transforms into

$$\int du h(u) h^*(u-\tau) = \frac{R_o(\tau)}{(2N_o)^2} \quad (64)$$

Thus the effective integration time in (62) is approximately the correlation time of the random signal process, and is as long a time as coherent integration could reasonably be anticipated. The squarer and integrator following  $H(f)$  in figure 2 can be looked upon as a sum of squared envelopes, for the continuum of possible relative time delays that the filter impulse response and input waveform can take on (see (62)). For a very narrowband spectrum  $G_o(f)$ , the correlation time of  $R_o(\tau)$  will be large, and considerable coherent processing is achievable via (62), prior to energy detection. This processing of the random signal is achievable even if the upper branch of figure 2 is absent; that is, even if there are no pure tones of random phase in the signal.

An alternative interpretation of the quantity  $\frac{1}{2} \chi_2^{(n)}$  in (57) is possible. Let the voltage density spectrum of the received waveform complex envelope be defined as

$$X(f) = \int dt x(t) \exp(-i2\pi ft) \quad (65)$$

This is fine-grained frequency analysis, i.e., resolution  $1/\tau$  in frequency. Then it is easily shown that (57) can be written as

$$\frac{1}{2} \chi_2^{(n)} = \frac{R_g(f_0)}{(2N_o)^2} |X(f_0)|^2 + \int df \frac{G_o(f)}{(2N_o)^2} |X(f)|^2 \quad (66)$$

Realization of this form requires a fine-grained frequency analysis over  $(t_a, t_b)$ , weighting of the energy-density function  $|X(f)|^2$  with an Eckart filter function, and summing; however the weighting at zero-frequency is distinctly different

from that at non-zero frequency. Namely the weighting is (see (45) and (56))

$$G_g(f) = R_g(\infty) \delta(f) + G_o(f). \quad (67)$$

Since the expression for  $\frac{1}{2} \chi_2^{(n)}$  in (45) involves  $R_g(t_2 - t_1)$  only for the range of arguments  $t_1 < t_1, t_2 < t_2$ , it is seen that only the values of  $R_g(u)$  for  $|u| < T$  are relevant. The question then arises as to whether the procedure in (56) et seq. is necessary. This problem is addressed in appendix D, where it is shown that other approaches may not yield legal choices for  $|H(f)|^2$ ; that is,  $|H(f)|^2$  may be required to be negative for some values of  $f$ , which is impossible. At least, (56) guarantees a legal choice for  $|H(f)|^2$ .

## MEAN VALUES OF $\chi_2^{(n)}$

It was noted in (49) et seq. that cumulants  $\{\chi_k^{(n)}\}$  are actually random variables, since they are functionals of the observation  $x(t)$ . In an attempt to determine what terms in (49) are most important, we begin by first evaluating the mean values of  $\chi_2^{(n)}$ , both with and without signal present. For hypothesis  $H_0$  (noise-only), the mean value of  $\frac{1}{2}\chi_2^{(n)}$  is available from (45) as

$$\frac{1}{2} \overline{\chi_2^{(n)}(N)} = \frac{1}{4N_b^2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 \overline{n(t_1) n^*(t_2)} R_n(t_2 - t_1), \quad (68)$$

where an overbar denotes an ensemble average\* with respect to the statistics of the received waveform. For the complex envelope noise, we have (Ref. 3, ch. 2, (4.10) and (5.15))

$$\overline{n(t_1) n^*(t_2)} = 2N_b \delta(t_1 - t_2). \quad (69)$$

Substitution in (68) yields

$$\begin{aligned} \frac{1}{2} \overline{\chi_2^{(n)}(N)} &= \frac{1}{2N_b} \int_{t_a}^{t_b} dt R_n(0) = \frac{1}{2N_b} \int_{t_a}^{t_b} dt \langle |a(t)|^2 \rangle \\ &= \frac{1}{2N_b} \left\langle \int_{t_a}^{t_b} dt |a(t)|^2 \right\rangle. \end{aligned} \quad (70)$$

Now if  $\langle s(t) \rangle = 0$ , no deterministic signal component, which is the usual case for phase-random narrowband signals, and the only case that we will consider henceforth, then  $a(t) = s(t)$ , and we have, using (13),

$$\frac{1}{2} \overline{\chi_2^{(n)}(N)} = \frac{1}{2N_b} \left\langle \int_{t_a}^{t_b} dt |s(t)|^2 \right\rangle = \frac{1}{2N_b} \langle 2E \rangle = \frac{E}{N_b}. \quad (71)$$

---

\* This average is over an ensemble of received signal and noise realizations which are both totally independent of the earlier signal ensemble average, although the same signal statistics are valid.

(From (70) and (71), we extract the relation (to be used often later)

$$\text{TR}_s |0\rangle = \text{TR}_s |0\rangle = 2E. \quad (72)$$

Since the quantity  $E/N_b$  must be large compared to 1 in order to get decent detection performance, even for coherent processing of a known deterministic signal, it is seen that, in this more-random signal case, the mean value  $\frac{1}{2} \chi_2^{(s+N)}$  is very large, even when no signal is present. However, of more importance is the difference of means of  $\frac{1}{2} \chi_2^{(s)}$ , with and without signal.

For signal present, (45) yields (since signal and noise are independent)

$$\frac{1}{2} \overline{\chi_2^{(s+N)}} = \frac{1}{4N_b^2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 \left[ \overline{s(t_1) s^*(t_2)} + \overline{n(t_1) n^*(t_2)} \right] R_s(t_2 - t_1). \quad (73)$$

Now recalling  $\langle s(t) \rangle = 0$ , we have\*

$$\overline{s(t_1) s^*(t_2)} = \langle s(t_1) s^*(t_2) \rangle = R_s(t_1 - t_2) = R_s(t_2 - t_1), \quad (74)$$

and (73) becomes, upon use of (69) and (72),

$$\begin{aligned} \frac{1}{2} \overline{\chi_2^{(s+N)}} &= \frac{1}{4N_b^2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 |R_s(t_1 - t_2)|^2 + \frac{E}{N_b} \\ &= \left(\frac{E}{N_b}\right)^2 I_2 + \frac{E}{N_b}, \end{aligned} \quad (75)$$

where we have defined

$$I_2 = \frac{1}{T^2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 |\rho(t_1 - t_2)|^2 = \frac{1}{T} \int_{-T}^T d\tau \left(1 - \frac{|\tau|}{T}\right) |\rho(\tau)|^2 \quad (76)$$

and

$$\rho(\tau) = \frac{R_s(\tau)}{R_s(0)}. \quad (77)$$

\* See footnote on previous page.

The difference of mean outputs, with and without signal present, is available from (71) and (75):

$$\frac{1}{2} \overline{\chi_2^{(r)}(S+N)} - \frac{1}{2} \overline{\chi_2^{(r)}(N)} = \left(\frac{E}{N_0}\right)^2 I_2. \quad (78)$$

The size of this difference depends critically on the value of  $I_2$ . The quantity  $I_2^{-1}$  is a measure of the effective number of statistically independent wiggles of  $s(t)$  in  $(t_a, t_b)$ ; see Ref. 5, pages 5-10. Its value is strongly dependent on the size of  $R_s(t_0)$ . For example, if  $R_s(t)$  does not decay with  $\tau$ , but remains constant at its origin value, then  $\rho(\tau)=1$  and  $I_2=1$ ; this is consistent with the physical observation that an unmodulated signal has only one statistically independent value. On the other hand, if (61) is true, then for  $T > 3/W_s$ , (76) yields

$$\begin{aligned} I_2 &\cong \frac{1}{T} \int_{-\infty}^{+\infty} dt |\rho(\tau)|^2 = \frac{1}{T} \frac{\int dt |R_s(t)|^2}{R_s^2(0)} = \frac{1}{T} \frac{\int dt |R_s(t)|^2}{R_s^2(0)} \\ &= \frac{1}{T} \frac{\int df G_s^2(f)}{[\int df G_s(f)]^2} = \frac{1}{TW_s}, \end{aligned} \quad (79)$$

where we have employed (77), (74), Parseval's Theorem, and defined the statistical bandwidth of fully-random received signal  $s(t)$  as\*

$$W_s = \frac{[\int df G_s(f)]^2}{\int df G_s^2(f)}. \quad (80)$$

In this case,  $I_2 < \frac{1}{3}$ ; generally  $I_2$  can be much smaller than unity if  $TW_s \gg 1$ . Notice that  $TW_s$  only has to be moderately larger than 1 in order for (79) to be a rather good approximation.

In general, (76) and (77) show that  $I_2 \leq 1$ . The nature of the dependence of  $I_2$  on  $R_s(t_0)$  and  $TW_s$  can perhaps best be illustrated by an

\*-----  
Since  $G_s(f)$  is the power density spectrum of complex envelope  $s(t)$ , it is a low-pass spectrum centered about  $f=0$ . Thus  $W_s$  measures the "width" of  $G_s(f)$  on both the positive and negative frequency scales together. Alternatively,  $W_s$  measures the width of the narrowband spectrum of  $s(t)$  about its carrier frequency.

example. Let the signal correlation be exponential:

$$R_g(\tau) = [R_g(0) - R_g(\infty)] \exp(-W_s |\tau|) + R_g(\infty); \quad (81)$$

equivalently, using (77),

$$\rho(\tau) = (1-c) \exp(-W_s |\tau|) + c, \quad c \equiv R_g(\infty)/R_g(0). \quad (82)$$

Then from (76),

$$I_2 = c^2 + \frac{(1-c)(1+3c)}{x} - \frac{1-c}{2x^2} [1+7c-8ce^{-x} - (1-c)e^{-2x}], \quad x \equiv TW_s. \quad (83)$$

As particular cases,

$$\begin{aligned} \lim_{x \rightarrow 0} I_2 &= 1 \quad \text{for all } c; \\ \lim_{x \rightarrow +\infty} I_2 &= c^2 \quad \text{if } c \neq 0; \\ I_2 &\sim \frac{1}{x} \left(1 - \frac{1}{2x}\right) \quad \text{as } x \rightarrow +\infty \quad \text{if } c = 0; \\ I_2 &= 1 \quad \text{for all } x \quad \text{if } c = 1; \\ I_2 &= \frac{2x-1+e^{-2x}}{2x^2} \quad \text{if } c = 0. \end{aligned} \quad (84)$$

These relations agree with those stated above. A plot of (83) is given in figure 3. The smallest value of  $I_2$ , and hence the smallest value of the difference of means, (78), is attained when  $c = 0$ , i.e.,  $R_g(\infty) = 0$ . This situation will yield the most difficulty in signal detection, and is the one we concentrate on henceforth; that is, we consider only the fully-random signal, characterized by (61). Notice from figure 3, that for  $c = 0$ ,  $I_2$  is very well approximated by  $\frac{1}{TW_s} \left(1 - \frac{1}{2TW_s}\right)$ , for  $TW_s$  as small as 1.5. In fact,  $(TW_s)^{-1}$  is a good approximation to  $I_2$  for  $TW_s > 10$ .

When the signal is obtained via Gaussian frequency-modulation of a carrier, the pertinent requirements on the spectrum  $G_m(f)$  of the modulating process, to yield a fully-random signal, are derived in appendix E and are related to the behavior of  $G_m(f)$  near  $f = 0$ . This case of Gaussian frequency-modulation with property (61) will occupy the remainder of our attention.

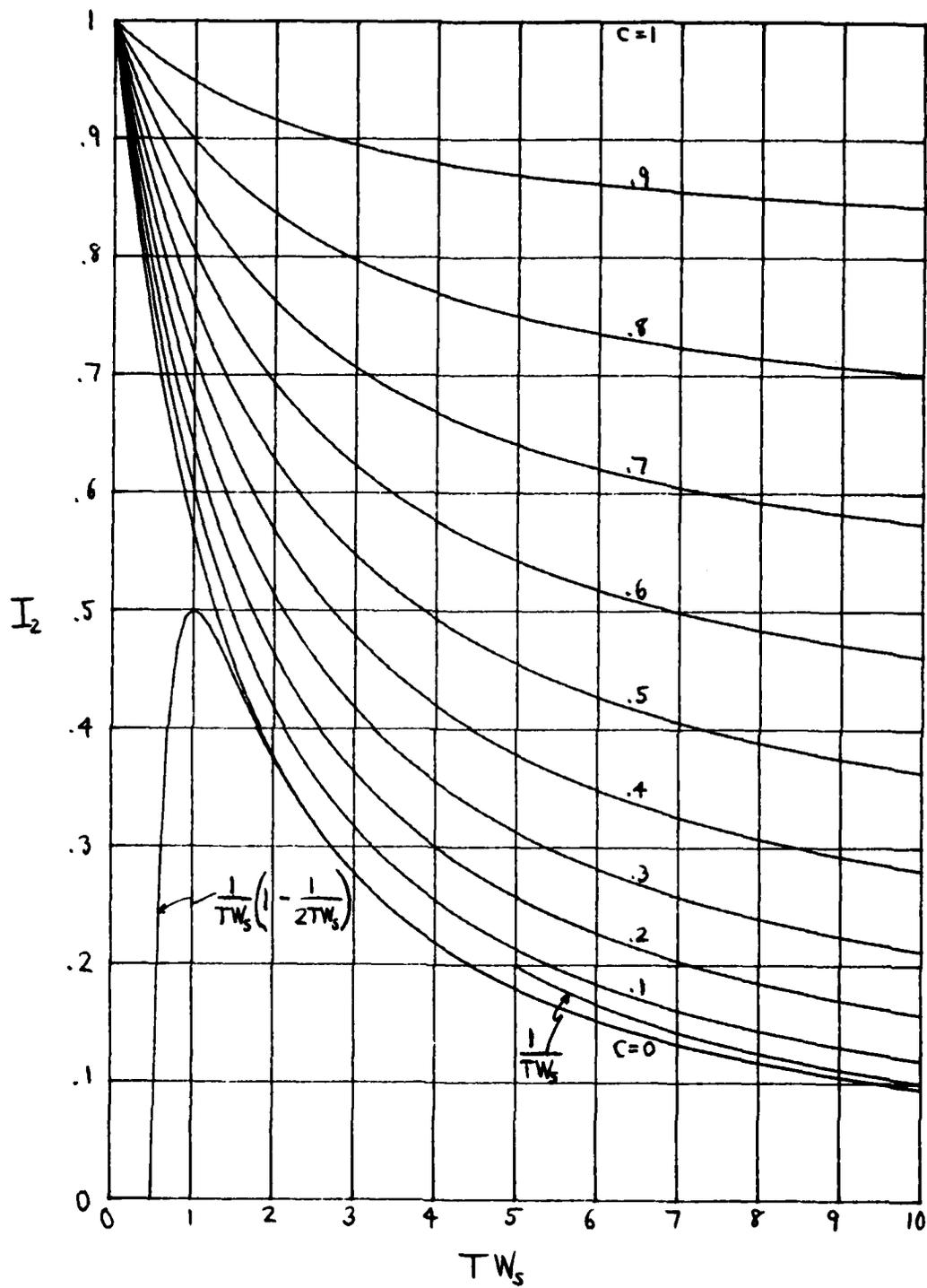


Figure 3.  $I_2$  for Exponential Signal Correlation

## VARIANCES OF $\chi_2^{(n)}$

For noise-alone, the mean value of  $\chi_2^{(n)}$  is available in (71). To evaluate the variance, we start with (45) and (74) and form the mean square quantity

$$MS\left\{\frac{1}{2}\chi_2^{(n)}(N)\right\} = \frac{1}{16N_0^4} \int_{t_a}^{t_b} \int_{t_a}^{t_b} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 dt_3 dt_4 \overline{\underline{n}(t_1) \underline{n}^*(t_2) \underline{n}(t_3) \underline{n}^*(t_4)} R_{\underline{z}}(t_2-t_1) R_{\underline{z}}(t_4-t_3)}. \quad (85)$$

Now since  $\underline{n}(t)$  is a complex-envelope Gaussian process, we can employ Ref. 6 and (69), and obtain for the fourth-order average in (85), the value

$$(2N_0)^2 \left[ \delta(t_1-t_2) \delta(t_3-t_4) + \delta(t_1-t_4) \delta(t_2-t_3) \right]. \quad (86)$$

Substitution of (86) into (85) yields

$$MS\left\{\frac{1}{2}\chi_2^{(n)}(N)\right\} = \frac{1}{4N_0^2} \int_{t_a}^{t_b} \int_{t_a}^{t_b} dt_1 dt_2 \left[ R_{\underline{z}}^2(0) + |R_{\underline{z}}(t_2-t_1)|^2 \right]. \quad (87)$$

Employment of (72) and (75) yields

$$MS\left\{\frac{1}{2}\chi_2^{(n)}(N)\right\} = \left(\frac{E}{N_0}\right)^2 (1 + I_2). \quad (88)$$

Combining this result with (71), the standard deviation of interest is

$$SD\left\{\frac{1}{2}\chi_2^{(n)}(N)\right\} = \frac{E}{N_0} \sqrt{I_2}. \quad (89)$$

Notice that this quantity can be significantly smaller than (71), the mean value, if  $I_2 \ll 1$ . This latter situation will occur if  $TW_s \gg 1$ ; see (79).

We are now able to form a deflection statistic for the random quantity  $\frac{1}{2}\chi_2^{(n)}$  in the ln LR. Using (78) and (89), it is defined as

$$d_2 \equiv \frac{\overline{\frac{1}{2}\chi_2^{(n)}(S+N)} - \overline{\frac{1}{2}\chi_2^{(n)}(N)}}{SD\left\{\frac{1}{2}\chi_2^{(n)}(N)\right\}} = \frac{E}{N_0} \sqrt{I_2}. \quad (90)$$

This relation has been derived without any specific assumptions about the detailed signal statistics, except for the randomness assumption (61). If our processor is limited to just the  $\frac{1}{2}\chi_2^{(r)}$  term of the  $\ln$  LR processor in (40) (recall that the  $\langle v \rangle$  term is zero by (61)), we would require that (90) be somewhat larger than unity, in order that good performance be attainable. A reasonable ballpark figure for low false alarm probability ( $10^{-3}$ ) and decent detection probability (.5) is to have deflection

$$d_2 \sim \sqrt{10}; \quad (91)$$

higher-quality performance will require still larger values of  $d_2$ . Then (90) indicates that we will have to require

$$\frac{E}{N_0} = \frac{d_2}{\sqrt{I_2}} \sim \sqrt{\frac{10}{I_2}}. \quad (92)$$

But if  $TW_b \gg 1$ ,  $E/N_0$  is going to have to be rather large to get good performance from  $\frac{1}{2}\chi_2^{(r)}$  alone, for this fully-random signal. The factor  $I_2^{-1/2}$  is a measure of the penalty of having to detect a fully-random signal rather than a completely-known deterministic signal, by employing just the second cumulant,  $\frac{1}{2}\chi_2^{(r)}$ , of the infinite series (49).

For the signal-plus-noise hypothesis, the mean value of  $\frac{1}{2}\chi_2^{(r)}$  is available in (75). To evaluate the variance, we start with (45) and (74) and form the mean square quantity

$$MS\left\{\frac{1}{2}\chi_2^{(r)}(S+N)\right\} = \frac{1}{16N_0^4} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_{t_0}^{t_1} dt_1 dt_2 dt_3 dt_4 R_s(t_2-t_1)R_s(t_4-t_3) F(t_1, t_2, t_3, t_4), \quad (93)$$

where fourth-order average

$$F(t_1, t_2, t_3, t_4) \equiv \overline{[s(t_1) + n(t_1)][s^*(t_2) + n^*(t_2)][s(t_3) + n(t_3)][s^*(t_4) + n^*(t_4)]}. \quad (94)$$

Using the statistical independence of signal and noise, their zero means, and property (44), we find (using an obvious shorthand notation)

$$F = \overline{s_1 s_2^* s_3 s_4^*} + \overline{s_1 s_2^* n_3 n_4^*} + \overline{s_1 s_4^* n_2^* n_3} \\ + \overline{s_2^* s_3 n_1 n_4^*} + \overline{s_3 s_4^* n_1 n_2^*} + \overline{n_1 n_2^* n_3 n_4^*} \quad (95)$$

All the terms in (95) have been encountered previously, with the exception of the leading term. In order to evaluate it, we will need to be specific about the signal statistics. As noted below (84), our interest is in Gaussian frequency-modulation of a carrier, with property (61). The pertinent second-order characteristics of this signal model are presented in appendix E. In order to evaluate the fourth-order moment required in (95), we employ (E-6) in the form

$$\underline{s}(t) = A \exp\left[i 2\pi \int_{t_a}^t du m(u) + i\phi\right], \quad t_a < t < t_b, \quad (96)$$

where  $m(t)$  is the Gaussian frequency-modulating process, to obtain

$$\overline{s_1 s_2^* s_3 s_4^*} = A^4 \overline{\exp\left[i 2\pi \int_{t_1}^{t_2} du m(u) + i 2\pi \int_{t_3}^{t_4} du m(u)\right]} \\ = A^4 \overline{\exp\left[i 2\pi \int du m(u) w(u; \underline{t})\right]} \quad (97)$$

where

$$w(u; \underline{t}) \equiv U(u - t_2) - U(u - t_1) + U(u - t_4) - U(u - t_3) \quad (98)$$

and  $U(x)$  is the unit step function:

$$U(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \quad (99)$$

Equations (97) and (98) hold for any values of  $t_1, t_2, t_3, t_4$  in  $(t_a, t_b)$ , irrespective of their location and ordering in size.

Now since frequency-modulating process  $m(u)$  in (97) is zero-mean Gaussian, the integral on  $u$ , denoted by  $g$ , is also zero-mean Gaussian, and we can immediately write (97) as

$$\begin{aligned} \overline{s_1 s_2^* s_3 s_4^*} &= A^4 \overline{\exp[i2\pi g]} = A^4 \exp\left[-\frac{1}{2}(2\pi)^2 \overline{g^2}\right] \\ &= A^4 \exp\left[-2\pi^2 \iint du dv R_m(u-v) w(u; \pm) w(v; \pm)\right] \\ &= A^4 \exp\left[-2\pi^2 \int df G_m(f) |W(f; \pm)|^2\right], \end{aligned} \quad (100)$$

where

$$\begin{aligned} W(f; \pm) &\equiv \int du \exp(-i2\pi fu) w(u; \pm) \\ &= \frac{1}{i2\pi f} \left[ \exp(-i2\pi f t_2) - \exp(-i2\pi f t_1) + \exp(-i2\pi f t_4) - \exp(-i2\pi f t_3) \right]. \end{aligned} \quad (101)$$

Here we have employed the Fourier transform relation between frequency-modulating spectrum  $G_m(f)$  and its correlation  $R_m(t)$  (see appendix E), and used (98).

Equation (100) is a compact expression for the fourth-order moment, and can be numerically evaluated by one integral when values for  $t_1, t_2, t_3, t_4$  are specified. However, it is not directly suitable for our use in (95) and (93). Rather, we expand  $|W|^2$  by means of (101), and obtain

$$\begin{aligned} \overline{s_1 s_2^* s_3 s_4^*} &= A^4 \exp\left[\frac{1}{2} \left\{ \sigma^2(t_2 - t_4) + \sigma^2(t_1 - t_3) \right. \right. \\ &\quad \left. \left. - \sigma^2(t_1 - t_2) - \sigma^2(t_2 - t_3) - \sigma^2(t_3 - t_4) - \sigma^2(t_4 - t_1) \right\}\right], \end{aligned} \quad (102)$$

where

$$\sigma^2(\tau) \equiv 2 \int df \frac{G_m(f)}{f^2} [1 - \cos(2\pi f \tau)]. \quad (103)$$

In appendix E, this real even function  $\sigma^2(t)$  is interpreted as the variance of the "random walk" process  $2\pi \int du m(u)$  during an interval of  $|T|$  seconds. Thus we have a closed form expression for the fourth-order moment in (102), provided that (103) can be evaluated in closed form.

Now also, for the frequency-modulated signal process in (96), we have

$$\begin{aligned} R_s(t_1, -t_2) &= \overline{s(t_1) s^*(t_2)} = A^2 \overline{\exp \left[ i 2\pi \int_{t_2}^{t_1} du m(u) \right]} \\ &= A^2 \exp \left[ -\frac{1}{2} \sigma^2(t_1, -t_2) \right], \end{aligned} \quad (104)$$

which is real. The last step follows from appendix E, or by setting  $t_3 = t_4$  in (97) and (102), and noting from (103) that  $\sigma^2(t) = 0$ . But now (104) enables us to write (102) as

$$\overline{s(t_1) s^*(t_2) s(t_3) s^*(t_4)} = \frac{R_s(t_1 - t_2) R_s(t_2 - t_3) R_s(t_3 - t_4) R_s(t_4 - t_1)}{R_s(t_2 - t_3) R_s(t_1 - t_3)}, \quad (105)$$

in terms of second-order statistics. This equation is exact for signal model (96). For comparison, if  $s(t)$  were a Gaussian process, the right-hand side of (105) would have been expressible in terms of second-order statistics (according to Ref. 6) as

$$R_s(t_1 - t_2) R_s(t_3 - t_4) + R_s(t_1 - t_4) R_s(t_2 - t_3). \quad (106)$$

(The approximation afforded by the replacement of (105) by (106) will be of interest later; see especially appendix F, (F-42) et seq.)

Thus the leading term in (95) is given by (105); the remaining terms in (95) are developed below, by using (74) and (69):

$$\begin{aligned}
\overline{s_1 s_2^*} \overline{n_3 n_4^*} &= R_z(t_1 - t_2) 2N_b \delta(t_3 - t_4) \\
\overline{s_1 s_4^*} \overline{n_2 n_3^*} &= R_z(t_1 - t_4) 2N_b \delta(t_2 - t_3) \\
\overline{s_2^* s_3} \overline{n_1 n_4^*} &= R_z(t_3 - t_2) 2N_b \delta(t_1 - t_4) \\
\overline{s_3^* s_4} \overline{n_1 n_2^*} &= R_z(t_3 - t_4) 2N_b \delta(t_1 - t_2),
\end{aligned} \tag{107}$$

and the last term in (95) is given by (86). Thus F in (95) is given by the sum of (105), (107), and (86). Employing these results in (93), there follows, by use of (74), (77), (72), and (76),

$$MS\left\{\frac{1}{2}\chi_2^{(r)}(S+N)\right\} = \left(\frac{E}{N_b}\right)^2 (1 + I_2) + 2\left(\frac{E}{N_b}\right)^3 (I_3 + I_2) + \left(\frac{E}{N_b}\right)^4 I_{62} \tag{108}$$

where we have defined\*

$$I_3 = \frac{1}{T^3} \iiint_{t_a}^{t_b} dt_1 dt_2 dt_3 \rho(t_2 - t_1) \rho(t_1 - t_3) \rho(t_3 - t_2) \tag{109}$$

and

$$I_{62} = \frac{1}{T^4} \iiint_{t_a}^{t_b} dt_1 dt_2 dt_3 dt_4 \frac{|\rho(t_1 - t_2)|^2 |\rho(t_3 - t_4)|^2 \rho(t_2 - t_3) \rho(t_4 - t_1)}{|\rho(t_2 - t_4)| \rho(t_1 - t_3)} \tag{110}$$

Both of these quantities are smaller than unity; in fact, they can be much smaller than unity. For signal absent, only the first term in (108) remains (see (95)), and (108) reduces to (88).

When we combine (75) with (108), there follows

\*  $I_{pq}$  denotes a  $p$ - $q$  fold integral, with  $p$  numerator  $\rho$ -functions and  $q$  denominator  $\rho$ -functions. If  $q$  is zero, the notation is simply  $I_p$ .

$$\text{Var}\left\{\frac{1}{2}\chi_2^{(k)}(S+N)\right\} = \left(\frac{E}{N_0}\right)^2 I_2 + 2\left(\frac{E}{N_0}\right)^3 I_3 + \left(\frac{E}{N_0}\right)^4 (I_{62} - I_2^2). \quad (111)$$

The first term in (111) is identical to the square of (89); the other two terms are due to the randomness of the signal.

#### Approximation to $I_2$

In order to evaluate  $I_3$  and  $I_{62}$  of (109) and (110), we need to make some approximations. These approximations will be developed by re-consideration of  $I_2$ , and carried over to the cases for  $I_3$  and  $I_{62}$ , where exact evaluation is very difficult. First, in (76), let  $u_k = t_k - (t_a + t_b)/2$  for  $k = 1, 2$ . Then

$$I_2 = \frac{1}{T^2} \iint_{-T/2}^{T/2} du_1 du_2 |\rho(u_1 - u_2)|^2 = \iint du_1 du_2 r(u_1) r(u_2) |\rho(u_1 - u_2)|^2, \quad (112)$$

where rectangular weighting

$$r(u) \equiv \begin{cases} 1/T, & |u| < T/2 \\ 0, & \text{otherwise} \end{cases}. \quad (113)$$

Let  $x = u_1 - u_2$  in (112); then

$$I_2 = \iint du_1 dx r(u_1) r(u_1 - x) |\rho(x)|^2 = \int dx \phi_2(x) |\rho(x)|^2, \quad (114)$$

where

$$\phi_2(x) \equiv \int du r(u) r(u-x) \quad (115)$$

is the autocorrelation function of rectangular weighting (113). This autocorrelation is triangular and extends over the range  $(-T, T)$ . On the other hand, the effective width of  $|\rho(x)|^2$  is approximately  $W_s^{-1}$ . Thus if  $TW_s \gg 1$ ,  $|\rho(x)|^2$  has decayed to zero long before  $\phi_2(x)$  has changed significantly from its origin value. This allows (114) to be expressed approximately as

$$I_2 \approx \phi_2(0) \int dx |\rho(x)|^2 = \frac{1}{T} \int dx |\rho(x)|^2 \quad \text{if } TW_s \gg 1. \quad (116)$$

In order to evaluate (116), we will approximate signal correlation  $\rho$  by a Gaussian function. We choose

$$\rho(\tau) \approx \exp\left(-\frac{\pi}{2} W_s^2 \tau^2\right) \quad \text{for all } \tau. \quad (117)$$

The choice of scale factor has been made to satisfy condition (80); thus  $W_s$  is the statistical bandwidth of the received signal. In appendix E, it is shown in (E-28)-(E-33) that (117) is in fact a good approximation to the true signal correlation when the rms frequency deviation is somewhat larger than the equivalent bandwidth of the frequency-modulating process. Furthermore, even when this condition is not true,  $W_s$  in (117) could be chosen so that (117) fits (as well as possible in some sense) to the exact signal correlation, given by (E-11) and (E-9). Several examples for modulating and signal spectra are given at the end of appendix E, and the approximation (117) is demonstrated.

Substituting (117) in (116), we find

$$I_2 \cong \frac{1}{TW_s} \quad \text{for } TW_s \gg 1; \quad (118)$$

this result is consistent with (79) et seq.  $TW_s$  does not have to be too large for (118) to be a reasonable approximation.

Equation (118) is a fairly good approximation to the exact curve for  $c=0$  in figure 3, where the signal correlation was exponential; see (81). When the signal correlation  $\rho$  is Gaussian, as given by (117), the exact value of  $I_2$  is available from (76) according to

$$I_2 = \frac{2\sqrt{2\pi}}{\alpha} \left[ \Phi(\alpha) - \frac{1}{2} - \frac{1 - \exp(-\alpha^2/2)}{\sqrt{2\pi} \alpha} \right] \quad ; \quad \alpha = \sqrt{2\pi} TW_s. \quad (119)$$

Error function  $\Phi$  is defined in appendix E, example 2. Comparison of the approximate and exact values of  $I_2$ , given by (118) and (119), is presented in figure 4. It is seen that (118) is a rather good approximation, especially for large  $TW_s$ . Thus the approximation technique developed in (112)-(116) will be used to approximately evaluate  $I_3$  and  $I_{cs}$ , given in (109) and (110). Actually, all the above has been done merely to set the stage for the approximate evaluation

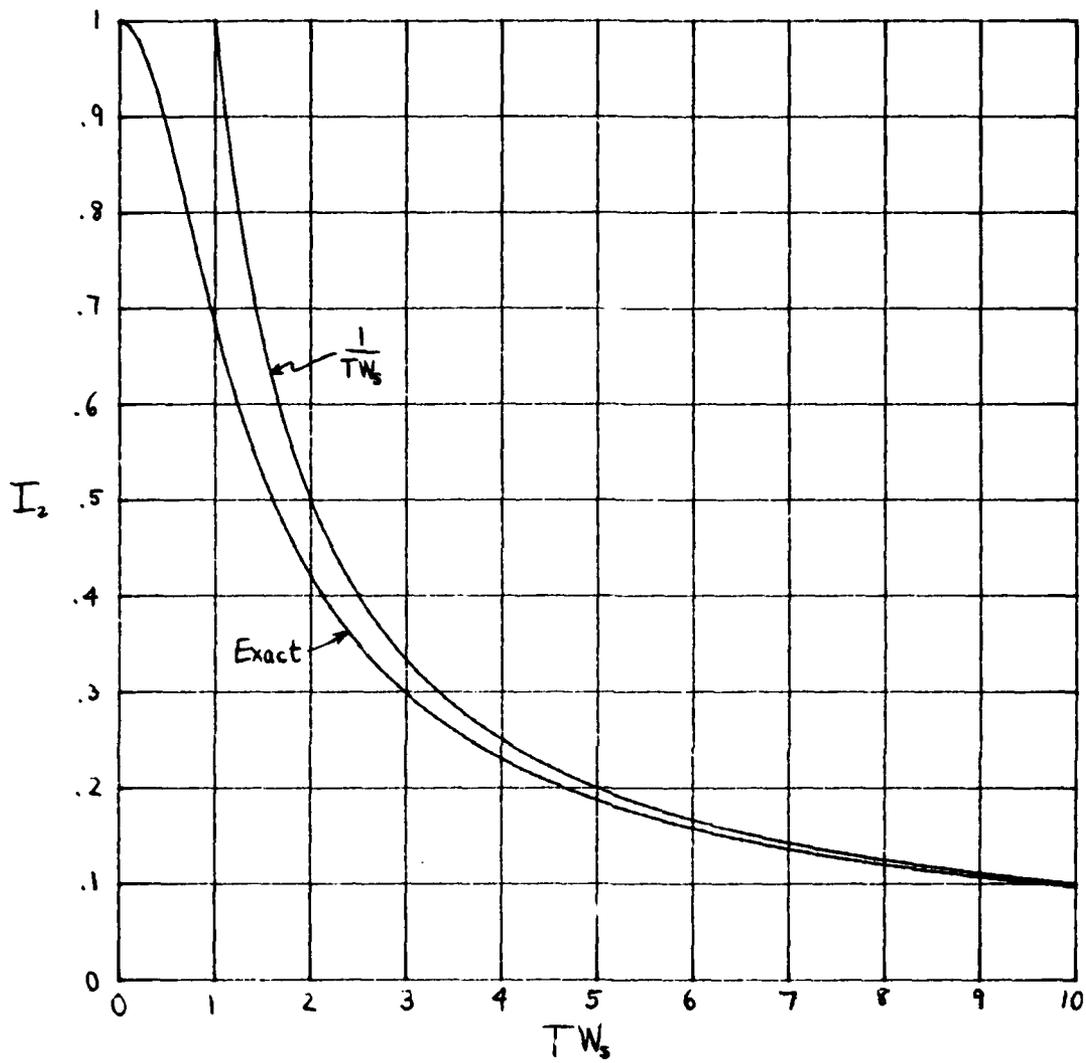


Figure 4.  $I_2$  for Gaussian Signal Correlation

of  $I_3$  and  $I_{62}$ , since exact evaluation of  $I_2$  via (76) is rather straightforward in most cases; see, e.g., (83) and (119). However, exact evaluation of (109) and (110) can be extremely difficult.

Approximation to  $I_3$

Use of (113) allows us to express (109) as

$$I_3 = \iiint du_1 du_2 du_3 r(u_1) r(u_2) r(u_3) \rho(u_2 - u_1) \rho(u_1 - u_3) \rho(u_3 - u_2). \quad (120)$$

Letting  $x = u_2 - u_1$ ,  $y = u_3 - u_2$ , there follows

$$\begin{aligned} I_3 &= \iiint dx du_2 dy r(u_2 - x) r(u_2) r(u_2 + y) \rho(x) \rho(-x - y) \rho(y) \\ &= \iint dx dy \phi_3(x, y) \rho(x) \rho(-x - y) \rho(y), \end{aligned} \quad (121)$$

where

$$\phi_3(x, y) \equiv \int du r(u - x) r(u) r(u + y). \quad (122)$$

Now if  $\tau W_s \gg 1$ , (121), (122), and (113) yield

$$I_3 \approx \phi_3(0, 0) \iint dx dy \rho(x) \rho(-x - y) \rho(y) = \frac{1}{\tau^2} \iint dx dy \rho(x) \rho(-x - y) \rho(y), \quad (123)$$

since the integrand of (123)  $\rightarrow 0$  no matter how  $x, y \rightarrow \pm \infty$ . Next we appeal to approximation (117) in order to evaluate (123); we have then

$$I_3 = \frac{1}{\tau^2} \iint dx dy \exp[-\pi W_s^2 (x^2 + y^2 + xy)] = \frac{2/\sqrt{3}}{(\tau W_s)^2}. \quad (124)$$

The last step was made by use of K-fold multiple integral relation

$$\int dX \exp[-X^T M X] = \left( \frac{\pi^K}{\det M} \right)^{1/2}, \quad \det M > 0, \quad (125)$$

where column matrix  $X = [x_1, \dots, x_k]^T$ , and  $M$  is a symmetric  $K \times K$  matrix. Equation (124) constitutes our desired approximation to  $I_1$ .

Approximation to  $I_{62} - I_2^2$

If we try to approximate  $I_{62}$  in (110) directly, as above, we encounter the difficulty that the appropriate correlation does not decay to zero for large arguments. Specifically, if  $t_1 = t_2$  and  $t_3 = t_4$  in (110), the integrand remains at value 1. Furthermore the variance expression of interest, (111), involves the combination  $I_{62} - I_2^2$ . Evaluation of this latter quantity is undertaken in appendix F; the end result is

$$I_{62} - I_2^2 = \frac{I_D^{(0)}}{(TW_s)^3}, \quad (126)$$

where  $I_D^{(0)}$  is a dimensionless quantity of the order of 1. A short list of values of  $I_D^{(0)}$  is given in table 1, for Gaussian frequency-modulating process  $m(t)$  with an exponential correlation  $R_m(\tau)$ ; the frequency-modulation index

$$D = \frac{\sigma_m}{W_m} \quad (127)$$

is the ratio of the RMS frequency-deviation to the equivalent bandwidth of the frequency-modulating process  $m(t)$ . A larger table and plot are given in appendix F.

D	.5	1	2	3	4	5	6
$I_D^{(0)}$	.356	.418	.642	.881	1.124	1.368	1.612

Table 1. Values of  $I_D^{(0)}$  for Exponential Correlation of Frequency Modulating Process  $m(t)$

### Approximation to Variance

When we combine (118), (124), and (126) in (111), we obtain the approximation

$$\begin{aligned} \text{Var}\left\{\frac{1}{2}\chi_2^M(S+N)\right\} &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} + \left(\frac{E}{N_0}\right)^3 \frac{4/\sqrt{3}}{(TW_s)^2} + \left(\frac{E}{N_0}\right)^4 \frac{I_D^{(4)}}{(TW_s)^3} \\ &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} \left[1 + \frac{4}{\sqrt{3}}R_s + I_D^{(4)}R_s^2\right] \quad \text{for large } TW_s, \end{aligned} \quad (128A)$$

where

$$R_s \equiv \frac{E/N_0}{TW_s} = \frac{A^2/2}{N_0W_s} = \frac{\text{received signal power}}{\text{received noise power in signal band}} \quad (128B)$$

is the input signal-to-noise power ratio (SNR) in the signal bandwidth\*. For very large  $TW_s$ , keeping  $E/N_0$  fixed for the moment, the first term in (128A) dominates; that is, for low input SNR, the first term is the same as the square of (89) with approximation (118). In terms of deflection  $d_2$  defined in (90), we can employ (92) and (118) to write the requirement on  $E/N_0$  as

$$\frac{E}{N_0} = \frac{d_2}{\sqrt{I_2}} \approx d_2 \sqrt{TW_s} \quad \text{for large } TW_s, \quad (129)$$

in which case (128A) and (128B) become

$$\text{Var}\left\{\frac{1}{2}\chi_2^M(S+N)\right\} = d_2^2 \left[1 + \frac{4}{\sqrt{3}} \frac{d_2}{\sqrt{TW_s}} + I_D^{(4)} \left(\frac{d_2}{\sqrt{TW_s}}\right)^2\right] \quad (130A)$$

and

$$R_s = \frac{E/N_0}{TW_s} = \frac{d_2}{\sqrt{TW_s}} \quad (130B)$$

\* Another interpretation of  $R_s$  is furnished by expressing  $R_s = E_i/N_0$ , where  $E_i = \frac{E}{TW_s}$  is the signal energy per independent component. That is, in observation time  $T$ , there are  $TW_s$  effectively independent components of the received signal (see the footnote to equation (80)). Then small  $R_s$  is synonymous with a small ratio  $E_i/N_0$  of the component signal energy to noise density ratio. This interpretation will be of use later.

The first term in (130A) is the only one that would remain for hypothesis  $H_0$ , signal absent; see (89). Thus the increase caused by signal randomness is indicated by the last two terms in (128A) or (130A), in terms of input measure  $\mathcal{R}_s$  or output measure  $d_2$ , respectively. If  $\sqrt{TW_s}$  is somewhat larger than the performance quality measure  $d_2$  (see (90)-(91)), then (130A) is only slightly larger than the variance with signal absent.

Although the variability of  $\frac{1}{2}\chi_2^{(p)}$  is somewhat greater with signal-present than with signal-absent, this may not matter much if we are asking for detection probabilities in the neighborhood of 0.5. For this case, the threshold will lie approximately in the center of the probability density function of the decision variable when signal is present, and the increased variability is not relevant. And for false alarm probabilities in the  $10^{-3}$  range, with  $TW_s \gg 1$ , the deflection criterion of (90) is the appropriate quantity to focus on. Thus, if we confine attention to the  $\frac{1}{2}\chi_2^{(p)}$  term of the  $\ln LR$  development in (49), requirement (129) is the one of major significance; whether higher-order terms are important has not yet been ascertained.

### DERIVATION OF $\chi_3^{(r)}$

We are now prepared and interested in determining some of the higher-order terms in the series expansion of  $\ln LR$  in (49). We recall that  $r$  is given by (32) or (42), and that  $\chi_3^{(r)}$  and  $\chi_k^{(r)}$  are given by (39). Substituting (96) into (32), we have

$$r = \text{Re} \{ C \exp(-i\phi) \} = \frac{1}{2} (C e^{-i\phi} + C^* e^{i\phi}), \quad (131)$$

where we have defined the complex quantity

$$C = \frac{A}{N_0} \int dt x(t) \exp\left[-i 2\pi \int_0^t du m(u)\right]. \quad (132)$$

Now, according to (39), we must average  $r^3$  over the signal statistics, i.e., over  $\phi$  and  $m(t)$ . Performing the average over  $\phi$  first, we obtain from (131)

$$\langle r^3 \rangle_\phi = \frac{1}{8} \langle (C e^{-i\phi} + C^* e^{i\phi})^3 \rangle_\phi = 0, \quad (133)$$

since  $\phi$  is uniformly distributed. Therefore

$$\chi_3^{(r)} = 0, \quad (134)$$

meaning that the term  $\frac{1}{6} \chi_3^{(r)}$  in  $\ln LR$  of (49) is absent. In fact, all cumulants,  $\chi_k^{(r)}$ , for  $k$  odd, are zero, by an argument similar to (131)-(133), for the signal model (96).

DERIVATION OF  $\chi_4^{(n)}$

We now have to average  $r^+$  over the signal statistics. From (131), we find

$$\langle r^+ \rangle_\phi = \frac{1}{16} \langle (C e^{-i\phi} + C^* e^{i\phi})^4 \rangle_\phi = \frac{3}{8} |C|^4. \quad (135)$$

Hence, from (39),

$$\chi_4^{(n)} = \frac{3}{8} \langle |C|^4 \rangle_m - 3 \chi_2^{(n)2}; \quad (136)$$

here we must still average  $|C|^4$  over the random signal frequency-modulating process  $m(t)$ . We have, from (132),

$$\langle |C|^4 \rangle_m = \left(\frac{A}{N_0}\right)^4 \iiint \int dt_1 dt_2 dt_3 dt_4 \chi(t_1) \chi^*(t_2) \chi(t_3) \chi^*(t_4) \cdot \left\langle \exp \left[ -i2\pi \int_{t_2}^{t_1} du m(u) - i2\pi \int_{t_4}^{t_3} du m(u) \right] \right\rangle_m. \quad (137)$$

But we have already evaluated this statistical average, in (97)-(105). Therefore we find

$$\langle |C|^4 \rangle_m = \left(\frac{A}{N_0}\right)^4 \iiint \int dt_1 dt_2 dt_3 dt_4 \chi(t_1) \chi^*(t_2) \chi(t_3) \chi^*(t_4) \cdot \frac{\rho(t_1-t_2) \rho(t_2-t_3) \rho(t_3-t_4) \rho(t_4-t_1)}{\rho(t_2-t_4) \rho(t_1-t_3)}, \quad (138)$$

where we have also used (77).

Combining (138), (45), (74), and (75) in (136), we obtain

$$\frac{1}{4!} \chi_4^{(n)} = \frac{1}{64} \left(\frac{A}{N_0}\right)^4 \iiint \int dt_1 dt_2 dt_3 dt_4 \chi(t_1) \chi^*(t_2) \chi(t_3) \chi^*(t_4) \cdot \rho(t_1-t_2) \rho(t_3-t_4) \left[ \frac{\rho(t_2-t_3) \rho(t_4-t_1)}{\rho(t_2-t_4) \rho(t_1-t_3)} - 2 \right]. \quad (139)$$

This is the next non-zero term\* in expansion (49). Although it indicates explicitly what has to be done on the received waveform  $x(t)$ , no simple way of realizing this processing has been discovered; in general, it appears that the difficult processing indicated by the fourth-order integral must be evaluated. Whether there is a special property of the combination of correlations in (139), that enables a simpler or approximate realization, is unknown. The magnitude of the difficulty of calculating (139) directly is pointed out by noting that since samples must be taken in time at least as often as  $1/N_s$  seconds, and  $x(t)$  is of duration  $T$ , a sum of terms of order  $(TN_s)^4$  must be effected. For large  $TN_s$ , this is not going to be possible, in which case approximations must be found. For now, however, we are interested in the average size of the term (139) in the  $\ln LR$  series.

\*-----  
 A partial check on (139) is afforded by using the example of  $x(t) = \delta(t-t_0)$ , which leads to  $\chi_4^{(n)}/24 = -(A/N_0)^4/64$ . At the same time, (132) then yields  $|C|^2 = A^2/N_0^2$ , while (131) yields  $\langle r^2 \rangle = \frac{1}{2} A^2/N_0^2$  and  $\langle r^4 \rangle = \frac{3}{2} (A/N_0)^4$ , and (39) yields the same value as above for  $\chi_4^{(n)}/24$ . Alternatively, use of the replacement-approximation of (105)-(106) results in zero for (139); this is consistent with a Gaussian assumption for  $x(t)$ . More precisely, use of (F-47) in (139) yields

$$\frac{1}{4!} \chi_4^{(n)} = \frac{1}{64} \left( \frac{A}{N_0} \right)^4 \iiint \int dt_1 dt_2 dt_3 dt_4 x(t_1) x^*(t_2) x(t_3) x^*(t_4) R_3(t_2-t_1, t_3-t_1, t_4-t_1),$$

where remainder  $R_3(x, y, z)$  tends to zero no matter how  $x, y, z$  increase to infinity.

MEAN VALUES OF  $\chi_4^{(n)}$

Signal Absent

Let the combination of correlation functions in (139) be denoted by kernel  $K(t_1, t_2, t_3, t_4)$  (which equals  $R_3(t_2-t_1, t_3-t_1, t_4-t_1)$ ). Then for noise-alone, using (85) and (86), along with (76), we have mean value

$$\begin{aligned} \frac{1}{24} \overline{\chi_4^{(n)}(N)} &= \frac{1}{16} \frac{A^4}{N_0^2} \iiint_{t_0}^{t_b} dt_1 dt_2 dt_3 dt_4 \cdot \\ & \left[ \delta(t_1-t_2) \delta(t_3-t_4) + \delta(t_1-t_3) \delta(t_2-t_4) \right] K(t_1, t_2, t_3, t_4) \\ &= \frac{1}{16} \frac{A^4}{N_0^2} \iint_{t_0}^{t_b} dt_1 dt_3 \left[ K(t_1, t_1, t_3, t_3) + K(t_1, t_3, t_3, t_1) \right] \\ &= -\frac{1}{8} \frac{A^4}{N_0^2} \iint_{t_0}^{t_b} dt_1 dt_3 \left| \rho(t_1-t_3) \right|^2 = -\frac{1}{2} \left( \frac{E}{N_0} \right)^2 I_2. \end{aligned} \quad (140)$$

For comparison, (71) yields the mean value of the previous term in the  $\ln LR$  expansion (49):

$$\frac{1}{2} \overline{\chi_2^{(n)}(N)} = \frac{E}{N_0}. \quad (141)$$

Since  $I_2 \approx (TW_b)^{-1}$  for  $TW_b \gg 1$  via (118), (140) becomes small relative to (141), as  $TW_b$  increases, as was anticipated, provided that  $E/N_0$  is held constant; however, large values of  $TW_b$  would be required, due to the additional power of  $E/N_0$  involved in (140). Furthermore, the situation of fixed  $E/N_0$  is not the one of interest; rather, if we attempt to keep the deflection criterion  $d_2$  of (90) constant, we have from (92),

$$\frac{1}{24} \overline{\chi_4^{(n)}(N)} = -\frac{1}{2} d_2^2, \quad (142)$$

which is constant (and large), while

$$\frac{1}{2} \overline{\chi_2^{(r)}(N)} = \frac{d_2}{\sqrt{I_2}} \approx d_2 \sqrt{TW_s} \quad \text{for } TW_s \gg 1. \quad (143)$$

In order that (142) be much smaller (in magnitude) than (143), we would require  $\frac{1}{2} d_2 \sqrt{TW_s} = \frac{1}{2} R_s \ll 1$ , which would indicate large values for  $TW_s$ , since values of  $d_2 \geq \sqrt{10}$  are typical requirements for good detection performance; see the discussion surrounding (90)-(91). However, these mean values are not the final or most meaningful measures of performance; additional important parameters are evaluated below.

#### Signal Present

The derivation of the mean value of  $\frac{1}{24} \chi_4^{(r)}$  in (139), for signal present, is rather involved and is presented in appendix G; the result is

$$\frac{1}{24} \overline{\chi_4^{(r)}(S+N)} = -\frac{1}{2} \left(\frac{E}{N_0}\right)^2 I_2 - \left(\frac{E}{N_0}\right)^3 I_3 + \frac{1}{4} \left(\frac{E}{N_0}\right)^4 (I_{64} - 2I_{62}), \quad (144)$$

where

$$I_{64} = \frac{1}{T^4} \int_{t_1}^b \int_{t_2}^b \int_{t_3}^b \int_{t_4}^b dt_1 dt_2 dt_3 dt_4 \left[ \frac{\rho(t_1-t_2)\rho(t_2-t_3)\rho(t_3-t_4)\rho(t_4-t_1)}{\rho(t_1-t_3)\rho(t_2-t_4)} \right]^2, \quad (145)$$

and  $I_2, I_3$ , and  $I_{62}$  are given by (76), (109), and (110). Evaluation of  $I_{64} - 2I_{62}$  in appendix G, and use of (118) and (124), yields

$$\frac{1}{24} \overline{\chi_4^{(r)}(S+N)} = -\left(\frac{E}{N_0}\right)^2 \frac{\sqrt{2}}{TW_s} - \left(\frac{E}{N_0}\right)^3 \frac{2\sqrt{3}}{(TW_s)^2} + \left(\frac{E}{N_0}\right)^4 \frac{I_D^{(4)}/4}{(TW_s)^3} \quad (146)$$

$$= -\left(\frac{E}{N_0}\right)^2 \frac{\sqrt{2}}{TW_s} \left[ 1 + \frac{4}{\sqrt{3}} R_s - \frac{1}{2} I_D^{(4)} R_s^2 \right] \quad \text{for large } TW_s, \quad (147)$$

where  $I_D^{(4)}$  is a dimensionless quantity of the order of unity. In terms of deflection  $d_2$ , we use (129) to express (146) as

$$\frac{1}{24} \overline{\chi_4^{(4)}(S+N)} = -\frac{1}{2} d_2^2 \left[ 1 + \frac{4}{\sqrt{3}} \frac{d_2}{\sqrt{TW_5}} - \frac{1}{2} I_D^{(4)} \left( \frac{d_2}{\sqrt{TW_5}} \right)^2 \right]. \quad (148)$$

Comparison of (142) and (148) reveals that the difference of mean fourth-order outputs is

$$\frac{1}{24} \overline{\chi_4^{(4)}(S+N)} - \frac{1}{24} \overline{\chi_4^{(4)}(N)} = -\frac{2}{\sqrt{3}} d_2^2 \frac{d_2}{\sqrt{TW_5}} \left[ 1 - \frac{\sqrt{3} I_D^{(4)}}{8} \frac{d_2}{\sqrt{TW_5}} \right], \quad (149)$$

which can be large for  $d_2 \sim \sqrt{10}$ . However, when we recall that the difference of mean second-order outputs is

$$\frac{1}{2} \overline{\chi_2^{(2)}(S+N)} - \frac{1}{2} \overline{\chi_2^{(2)}(N)} = \left( \frac{E}{N_0} \right)^2 I_2 = d_2^2, \quad (150)$$

where we used (78) and (92), then we see that the latter difference is much larger if  $R_S = d_2/\sqrt{TW_5}$  is small. But this is still not the final statement, for we still have to address the deflection criterion of random variable  $\chi_4^{(4)}$ , which will involve the variance of  $\chi_4^{(4)}(N)$ , in addition to the difference of means considered above.

## VARIANCE OF $\chi_4^{(N)}$

Since  $\chi_4^{(N)}$  is given by the four-fold integral in (139), evaluation of its variance will involve an eight-fold integral. To simplify this task, we will consider the variance only for noise-alone. The derivation is given in appendix H, with the result that

$$\text{Var}\left\{\frac{1}{24}\chi_4^{(N)}\right\} = \frac{1}{4}\left(\frac{E}{N_0}\right)^4 \left(I_{84} - 4I_{62} + 6I_4 + 2I_2^2\right), \quad (151)$$

where the various quantities have been defined in (76), (110), (145), and also

$$I_4 \equiv \frac{1}{T^4} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_{t_0}^{t_1} dt_1 dt_2 dt_3 dt_4 \rho(t_1-t_2) \rho(t_2-t_3) \rho(t_3-t_4) \rho(t_4-t_1). \quad (152)$$

The expression (151) is simplified in appendix H, to

$$\text{Var}\left\{\frac{1}{24}\chi_4^{(N)}\right\} = \frac{I_D^{(4)}}{4} \left(\frac{E}{N_0}\right)^4 \frac{1}{(TW_b)^3}, \quad (153)$$

where  $I_D^{(4)}$  is a dimensionless quantity of the order of unity, and is available from  $I_D^{(1)}$  and  $I_D^{(2)}$  according to (H-8). In terms of deflection  $d_2$ , (129) yields

$$\text{Var}\left\{\frac{1}{24}\chi_4^{(N)}\right\} = \frac{I_D^{(4)}}{4} \frac{d_2^4}{(TW_b)^3}. \quad (154)$$

For comparison, the variance of the first term in the  $\ln LR$  series is given by (89) and (118) as

$$\text{Var}\left\{\frac{1}{2}\chi_2^{(N)}\right\} = \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_b} = d_2^2. \quad (155)$$

The ratio of (154) to (155) is of the order of  $(d_2/\sqrt{TW_b})^2 = R_s^2$ , which will be

rather small if  $\sqrt{TW_5}$  is somewhat larger than  $d_2$ .

We can now form a deflection statistic for the second term alone,  $\chi_4^{(r)}/24$ , in the ln LR series. Using (140), (146), and (153), it is

$$d_4 = \frac{\frac{1}{24} \overline{\chi_4^{(r)}(S+N)} - \frac{1}{24} \overline{\chi_4^{(r)}(N)}}{SD\left\{\frac{1}{24} \chi_4^{(r)}(N)\right\}} = \frac{-\frac{2}{\sqrt{3}} \frac{E}{N_6} \frac{1}{(TW_6)^{1/2}} + \frac{I_D^{(r)}}{4} \left(\frac{E}{N_6}\right)^2 \frac{1}{(TW_6)^{3/2}}}{\left(\frac{I_D^{(r)}}{4}\right)^{1/2}} \quad (156)$$

In terms of deflection  $d_2$ , we use (129) to find

$$d_4 = d_2 \frac{-\frac{2}{\sqrt{3}} + \frac{I_D^{(r)}}{4} \frac{d_2}{\sqrt{TW_6}}}{\left(\frac{I_D^{(r)}}{4}\right)^{1/2}} \quad (157)$$

Thus the second term in the ln LR series has about the same deflection criterion as the first term, when  $\sqrt{TW_6}$  is large relative to deflection  $d_2$ . At first, this result would seem to suggest that the ln LR series ought to be carried to at least two terms; however, a better measure for this conclusion is based upon the deflection criterion of the combined statistic

$$l_{24} \equiv l_2 + l_4 \equiv \frac{1}{2} \chi_2^{(r)} + \frac{1}{24} \chi_4^{(r)}, \quad (158)$$

rather than upon each term alone. We now undertake the evaluation of the deflection of statistic (158).

## DEFLECTION OF $l_{24}$

The difference of means of  $l_{24}$  for signal present versus noise-alone, can be obtained from (78), (118), (140), and (146); we find, with the help of (130B),

$$\begin{aligned} \overline{l_{24}(S+N)} - \overline{l_{24}(N)} &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} - \frac{2}{\sqrt{3}} \left(\frac{E}{N_0}\right)^3 \frac{1}{(TW_s)^2} + \frac{I_D^{(b)}}{4} \left(\frac{E}{N_0}\right)^4 \frac{1}{(TW_s)^3} \\ &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} \left[ 1 - \frac{2}{\sqrt{3}} R_s + \frac{I_D^{(b)}}{4} R_s^2 \right]. \end{aligned} \quad (159)$$

The variance of  $l_{24}$  for noise-alone is given by

$$\begin{aligned} \text{Var}\{l_{24}(N)\} &= \text{Var}\{l_2(N)\} + \text{Var}\{l_4(N)\} \\ &+ 2 \left[ \overline{l_2(N)l_4(N)} - \overline{l_2(N)}\overline{l_4(N)} \right]. \end{aligned} \quad (160)$$

With the exception of the third term, these quantities are available from (89), (153), (71), and (140). The third term is evaluated in appendix I, with the result

$$\overline{l_2(N)l_4(N)} - \overline{l_2(N)}\overline{l_4(N)} = -\left(\frac{E}{N_0}\right)^3 I_3. \quad (161)$$

Combining this result with those cited above, and employing (118), (124), and (130B), there follows

$$\begin{aligned} \text{Var}\{l_{24}(N)\} &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} - \frac{4}{\sqrt{3}} \left(\frac{E}{N_0}\right)^3 \frac{1}{(TW_s)^2} + \frac{I_D^{(b)}}{4} \left(\frac{E}{N_0}\right)^4 \frac{1}{(TW_s)^3} \\ &= \left(\frac{E}{N_0}\right)^2 \frac{1}{TW_s} \left[ 1 - \frac{4}{\sqrt{3}} R_s + \frac{I_D^{(b)}}{4} R_s^2 \right]. \end{aligned} \quad (162)$$

The deflection of  $l_{2+}$  is

$$d_{2+} = \frac{\overline{l_{2+}(S+N)} - \overline{l_{2+}(N)}}{SD\{l_{2+}(N)\}} = \frac{E}{N_0} \frac{1}{\sqrt{TW_s}} \frac{1 - \frac{2}{\sqrt{3}} R_s + O(R_s^2)}{\sqrt{1 - \frac{4}{3} R_s + O(R_s^2)}}, \quad (163)$$

where we employed (159) and (162), and recall (130B),

$$R_s = \frac{E}{N_0} \frac{1}{TW_s} = \frac{d_2}{\sqrt{TW_s}}. \quad (164)$$

Expanding (163) in terms of powers of  $R_s$ , we find

$$d_{2+} = \frac{E}{N_0} \frac{1}{\sqrt{TW_s}} \left[ 1 + O(R_s^2) \right] = d_2 \left[ 1 + O\left(\frac{d_2^2}{TW_s}\right) \right]. \quad (165)$$

Notice that the perturbation from  $d_2$  is an order of magnitude smaller, depending on  $(d_2/\sqrt{TW_s})^2$  now. Thus the addition of  $l_4$  to  $l_2$  does not significantly affect the deflection criterion; this strongly suggests that the first non-zero data-dependent term in the  $\ln$  LR series,  $l_2 = \frac{1}{2} \chi_2^{(r)}$ , is sufficient to base optimum decisions on, without regard for the higher-order terms, when  $\sqrt{TW_s} \geq 3d_2$  and  $d_2 > \sqrt{10}$ . This criterion is significantly weaker than the earlier ones which were based on preliminary statistics, and not a final performance measure.

#### Correlation Coefficient

One additional statistic relating  $l_2$  and  $l_4$  is of interest, namely their correlation coefficient; for noise-alone, it is given by\*

\* Calculation of the correlation coefficient for signal-plus-noise would require knowledge of the mean of  $l_2(S+N)l_4(S+N)$  and the variance of  $l_4(S+N)$ , in addition to the other quantities that have been evaluated already. This would mean evaluation of sixth-order and eighth-order signal and noise correlations and subsequent simplification and approximation; this task has not been undertaken.

$$C_{24}(N) = \frac{\overline{l_2(N)l_4(N)} - \overline{l_2(N)}\overline{l_4(N)}}{SD\{l_2(N)\} SD\{l_4(N)\}} \quad (166)$$

Substitution of (161), (89), (153), (118), and (124) yields

$$C_{24}(N) = -\frac{4}{\sqrt{3I_D^{(D)}}} \text{ for large } TW_s. \quad (167)$$

Evaluation of  $I_D^{(D)}$  is undertaken in appendix J; we find via (J-13)

$$C_{24}(N) = -0.671 \text{ for } D = 3. \quad (168)$$

This rather large correlation coefficient indicates that the statistics  $l_2(N)$  and  $l_4(N)$  tend to vary together; that is, when one gets large, so does the other (in magnitude), and vice versa. This result, combined with (165) above, is further confirmation that the addition of  $l_4$  to  $l_2$  does not significantly alter the decisions yielded by  $l_2$  alone. This conclusion is drawn for large  $TW_s$  only. As an example, if we desire  $d_2 \sim \sqrt{10}$  (see (90)-(91)), then (165) suggests  $TW_s > 100$  as the condition for neglect of  $l_4$ . Consideration of (130B) reveals that we are then talking about input SNRs less than  $1/\sqrt{10} = -5 \text{ dB}$ .

To summarize, if we want decent performance in terms of detection and false alarm probabilities ( $d_2 \geq \sqrt{10}$ ), and if  $\sqrt{TW_s}$  is large relative to  $d_2$  ( $\sqrt{TW_s} \geq 3d_2$ ), then the input SNR is small ( $R_s < 1/3$ ), the time-bandwidth product is large ( $TW_s > 100$ ), and the  $l_4$  term in the  $\ln LR$  series can be dropped (use  $l_2$  alone). In fact, if  $\sqrt{TW_s} > \max(10, 3d_2)$ , but  $d_2$  is small, then  $R_s < 1/3$ , and the  $l_4$  term can still be dropped without degradation in performance; however the performance level, in terms of false-alarm and detection probabilities, will be poor.

If we attempt to state the condition for neglect of  $l_4$  in terms of input SNR  $R_s$ , we can say that if  $R_s < 1/3$ , then from (164),  $\sqrt{TW_s} > 3d_2$ . However,

this does not necessarily mean that  $TW_s$  is large unless  $d_2 \geq 3$ . So small input SNR is not a sufficient condition by itself; it must be augmented with the requirement of good performance by the processor. Then we can state that  $l_4$  can be dropped in the  $\ln LR$  series.

ON THE APPROACH OF  $r$  TO GAUSSIAN

It was noted in (47) et seq. that the random variable  $r$  should approach Gaussian as  $TW_s$  increases; we can now make some quantitative statements about the rate of approach. If  $r$  were equal to a sum of  $N$  independent identically-distributed random variables  $\{z_n\}_1^N$ , then we would find that the measure of non-Gaussianness afforded by

$$\frac{\chi_4^{(r)}}{\chi_2^{(r)^2}} = \frac{1}{N} \frac{\chi_4^{(z)}}{\chi_2^{(z)^2}} \quad (169)$$

indicates a  $1/N$  decay. For our problem, however,  $r$  is given by (47) as a sum of approximately  $TW_s$  unequally-distributed independent components; the unequal distribution follows from the weighting by the received waveform  $x(t)$ , which varies with time  $t$ . Furthermore, this results in  $\{\chi_k^{(r)}\}$  themselves being random variables. Nevertheless, we can still derive a meaningful measure, similar to (169), for our problem.

We observe first that the random variables  $\chi_2^{(r)}(N)$ ,  $\chi_2^{(r)}(S+N)$ ,  $\chi_4^{(r)}(N)$  are clustered if  $TW_s \gg 1$ ; that is, their standard deviations are much smaller than their means, if  $TW_s \gg 1$ . From (71), (89), and (79),

$$\frac{SD\{\chi_2^{(r)}(N)\}}{\chi_2^{(r)}(N)} = \frac{1}{\sqrt{TW_s}} \quad (170)$$

From (75), (79), and (128),

$$\frac{SD\{\chi_2^{(r)}(S+N)\}}{\chi_2^{(r)}(S+N)} = \frac{1}{\sqrt{TW_s}} \frac{\left(1 + \frac{4}{\sqrt{3}}R_s + I_D^{(r)} R_s^2\right)^{1/2}}{1 + R_s}, \quad (171)$$

where  $R_s$  is defined in (164); the function of  $R_s$  is approximately unity. And from (140), (79), and (153),

$$\frac{SD\{\chi_4^{(n)}(N)\}}{\overline{\chi_4^{(n)}(N)}} = - \frac{\sqrt{I_D^{(n)}}}{\sqrt{TW_s}}; \quad (172)$$

the numerator of (172) is approximately 3.44 for  $D = 3$ ; see (J-13). For large  $TW_s$ , these last three quantities are small, and our conclusion about clustered behavior is drawn.

This leads us to replace (169) by the measures

$$\frac{\overline{\chi_4^{(n)}(N)}}{\overline{\chi_2^{(n)}(N)}^2} = - \frac{3}{TW_s} \quad (173)$$

and

$$\frac{\overline{\chi_4^{(n)}(S+N)}}{\overline{\chi_2^{(n)}(S+N)}^2} = - \frac{3}{TW_s} \frac{1 + \frac{4}{\sqrt{3}}R_s - \frac{I_D^{(n)}}{2}R_s^2}{(1 + R_s)^2}, \quad (174)$$

which are obtainable respectively from (140), (79), (71), and (146), (75), (79), (164). Since the function of  $R_s$  in (174) is approximately unity, both measures (173) and (174) indicate a rate of decay proportional to  $(TW_s)^{-1}$ ; this inverse dependence on the effective number of independent components in  $r$  is similar to (169). The scale factor in (173) and (174) is somewhat larger and is probably due to the unequal weighting in (47), as mentioned above.

The cumulants  $\chi_3^{(n)}$  and  $\chi_5^{(n)}$  are zero, whether signal is present or not, as are all odd cumulants; thus  $\chi_4^{(n)}$  is the first non-Gaussian contributor to the  $\ln LR$  series:

$$\ln LR = - \frac{E}{N_0} + \frac{1}{2} \chi_2^{(n)} + \frac{1}{24} \chi_4^{(n)} + \frac{1}{720} \chi_6^{(n)} + \dots \quad (175)$$

where we employed (49), (41) et seq., and (134). All of the above measures lead to the conclusion that the most important term in (175) is the first data-dependent term  $\chi_2^m$ , at least for large  $\tau W_s$ .

## POWER AND ENERGY RELATIONS

It has been noted above that, for large  $TW_s$ , the dominant term of the  $\ln LR$  series is the  $\frac{1}{2}\chi_2^2$  term, and that in order for this term to realize deflection  $d_2$ , we must have the ratio of received signal energy to noise-density

$$\frac{E}{N_0} = d_2 \sqrt{TW_s}, \quad (176)$$

where  $d_2 \sim \sqrt{10}$  for good performance; see (129). This relation requires the received signal energy  $E$  to increase according to the square root of the signal bandwidth,  $W_s^{1/2}$ , for fixed deflection  $d_2$ , noise density  $N_0$ , and observation time  $T$ .

The ratio of the received signal power to the noise power in the signal band is then

$$R_s = \frac{A^2/2}{N_s} = \frac{E/T}{N_0 W_s} = \frac{d_2}{\sqrt{TW_s}}, \quad (177)$$

which allows  $R_s$  to decrease as  $W_s^{-1/2}$  for fixed  $d_2$  and  $T$ . The reason for this decrease is that the received noise power in the signal band is increasing according to  $W_s$ , as  $W_s$  increases. A better measure may be the ratio of received signal power to the noise power in a 1 Hz band:

$$\frac{A^2/2}{N_0 \cdot 1} = d_2 \sqrt{\frac{W_s}{T}}. \quad (178)$$

Like  $E/N_0$ , this must also increase as  $W_s^{1/2}$ , for fixed  $d_2$  and  $T$ . Some examples of these relations are presented in table 2.

$d_2$	$W_s$ (Hz)	T (sec)	$TW_s$	$E/N_b$ (dB)	$R_s$ (dB)	$\frac{A^2/2}{N_b}$ (dB)
$\sqrt{10}$	$3\frac{1}{3}$	300	1000	20	-10	-4.8
$\sqrt{10}$	$\frac{1}{3}$	300	100	15	-5	-9.8
$\sqrt{10}$	$\frac{1}{30}$	300	$10^*$	10	0	-14.8

Table 2. Power and Energy Relations

The input signal-to-noise ratio, (177), can range below 0 dB if large values of  $TW_s$  are attainable; but rather large values for  $E/N_b$  are required in this case. For comparison and partial verification of the last example in table 2, Ref. 8, page 14 gives, for  $M = 1$  alternative,  $D = 10$ -fold diversity,  $P_F = 10^{-3}$ ,  $P_D = .5$ , the value  $\bar{E}_r/N_b = 11.3$  dB; thus, high orders of "diversity" ( $TW_s \gg 1$ ) require large values for the received signal energy to noise density ratio, in order to attain decent performance.

\* This is not a very large value of  $TW_s$ ; thus the conclusion about  $d_2 = \frac{1}{2} \chi^2_m$  being a sufficient statistic, and the adequacy of  $d_2$  as a performance measure, are suspect for this last example.

### THIRD CUMULANT OF $\frac{1}{2}\chi_2^{(r)}(N)$

The first two cumulants of  $\frac{1}{2}\chi_2^{(r)}(N)$  have been derived in (71) and (89); they are, using (90),

$$C_1 = \frac{E}{N_0}, \quad C_2 = \left(\frac{E}{N_0}\right)^2 I_2 = d_2^2. \quad (179)$$

For large  $TW_s$ , these relations were the basis in (170) for showing that  $\frac{1}{2}\chi_2^{(r)}(N)$  is clustered around its mean value. Now we wish to show that  $\frac{1}{2}\chi_2^{(r)}(N)$  is tending to a Gaussian random variable as  $TW_s$  increases. (This is true even though  $\frac{1}{2}\chi_2^{(r)}(N)$  is a nonlinear double integral of the noise waveform.) We do this by considering the normalized third cumulant  $C_3/C_2^{3/2}$ , which is a standard measure of non-Gaussianness of a random variable (see, e.g. Ref. 4). We have

$$C_3 = \overline{\left[ \frac{1}{2}\chi_2^{(r)}(N) - \frac{1}{2}\overline{\chi_2^{(r)}(N)} \right]^3}. \quad (180)$$

We expand (180) out and use (71) and (88) to evaluate the first- and second-power moments. The third moment can be evaluated by using the properties of complex white Gaussian noise in (45), with the result that

$$C_3 = 2 \left(\frac{E}{N_0}\right)^3 I_3, \quad (181)$$

where  $I_3$  is defined in (109). Therefore (179), in conjunction with (118) and (124), yields

$$\frac{C_3}{C_2^{3/2}} = 2 \frac{I_3}{I_2^{3/2}} = \frac{4/\sqrt{3}}{\sqrt{TW_s}} \quad \text{for large } TW_s. \quad (182)$$

The square-root decay with the number of independent contributors is again standard for this particular normalized cumulant; thus  $\frac{1}{2}\chi_2^{(r)}(N)$  is tending towards Gaussian. This result (182) could also be useful in setting thresholds for specified false alarm probabilities of the processor employing only  $\frac{1}{2}\chi_2^{(r)}$  in its decision-making.

## SIXTH CUMULANT $\chi_6^{(p)}$ AND ITS MEAN

The conclusions drawn above have been based on consideration of  $\chi_2^{(p)}$  and  $\chi_4^{(p)}$  and their statistics. We now consider the sixth cumulant  $\chi_6^{(p)}$  and its mean  $\overline{\chi_6^{(p)}(N)}$  with noise-only; any other statistics require a prohibitive amount of manipulations. The derivation is presented in appendix K, with the result

$$\frac{1}{6!} \chi_6^{(p)} = \frac{1}{2304} \left(\frac{A}{N_0}\right)^6 \iiint \int \int \int \int dt_1 \dots dt_6 x(t_1) \dots x^*(t_6) \cdot$$

$$\left[ K_6(t_1, \dots, t_6) - 9 K_4(t_1, \dots, t_4) K_2(t_5, t_6) + 12 K_2(t_1, t_2) K_2(t_3, t_4) K_2(t_5, t_6) \right], \quad (183)$$

where

$$K_6(t_1, \dots, t_6) \equiv \langle p^*(t_1) p(t_2) p^*(t_3) p(t_4) p^*(t_5) p(t_6) \rangle, \quad (184)$$

$p(t)$  is the complex phasor process defined in (F-43),  $K_4$  is available in (F-44), and  $K_2 = \rho$ . The complex processing required by the kernel of (183) has not been evaluated, because it is too difficult to realize physically and sheds no light on the processor.

The mean of (183) for noise-alone is derived in appendix K:

$$\frac{1}{6!} \overline{\chi_6^{(p)}(N)} = \frac{2}{3} \left(\frac{E}{N_0}\right)^3 I_3 \cong \frac{4}{3\sqrt{3}} \frac{(E/N_0)^3}{(TW_s)^2} \text{ for large } TW_s. \quad (185)$$

For comparison, the means of the first three non-zero data-dependent terms in the ln LR series are summarized:

$$\frac{1}{2} \overline{\chi_2^{(p)}(N)} = \frac{E}{N_0}, \quad \frac{1}{4!} \overline{\chi_4^{(p)}(N)} = \frac{E}{N_0} \left(-\frac{1}{2} R_s\right), \quad \frac{1}{6!} \overline{\chi_6^{(p)}(N)} = \frac{E}{N_0} \frac{4}{3\sqrt{3}} R_s^2. \quad (186)$$

Thus the mean values of the higher-order terms are dropping off by additional powers of the input SNR  $R_s$ , which is small. This leads to the conjecture that the higher-order terms in the ln LR series are progressively less important when  $\sqrt{TW_s} > 3 d_2$ , i.e., small  $R_s$ .

## SUMMARY

The logarithm of the likelihood ratio has been developed in an infinite Volterra series, and the leading terms, from zeroth-order through sixth-order, have been investigated in terms of their size and statistical significance. It has been found for the fully-random signal, which is characterized by no deterministic component and no carrier, that if false alarm probabilities in the order of  $10^{-3}$  and detection probabilities in the order of .5 are desired, if the product,  $TW_s$ , of observation time and received signal bandwidth is large, and if there is at least a moderate frequency modulation index  $D$ , then the ratio of total received signal energy to noise power density level must be rather large, of the order of  $3\sqrt{TW_s}$ , and the dominant term in the log likelihood series is given by the filter-energy-detector term,  $\chi_2^{(n)}$ . The requirement on the frequency modulation index  $D$  being at least of the order of 2-3 was introduced in the analysis in order to facilitate approximate evaluation of some of the multiple integrals by means of Gaussian functions. Thus this is a sufficient condition employed for tractability; how small the modulation index can become, without violating the conclusions above, is unknown.

Alternatively, if  $\sqrt{TW_s}$  is larger than the maximum of 10 and  $3d_2$ , where  $d_2$  is the deflection criterion of the filter-energy-detector term  $\chi_2^{(n)}$ , then the dominant term in the log likelihood series is  $\chi_2^{(n)}$ ; this holds regardless of the size of  $d_2$ , i.e., whether good or poor performance is to be obtained from the filter-energy-detector. In this case, the input signal-to-noise power ratio in the signal band is small. It is not sufficient to say that small input signal-to-noise power ratio alone yields the filter-energy-detector as near-optimum. Rather this condition must be augmented with a requirement for large  $TW_s$ , i.e.,  $\sqrt{TW_s} > (10, 3d_2)$ , which means that larger time-bandwidth products are required for better performance.

The input signal-to-noise ratio,  $R_s$ , can be put in the form  $R_s = E_i/N_0$ , where  $E_i$  is the received signal energy per independent signal component and  $N_0$  is the noise power density level; see the footnote to (128B). When this ratio of component signal energy to noise density is small, and the number of components,  $TW_s$ , is large, the optimum processor is well approximated by the filter-energy-detector. This same conclusion has been reached by the author in some as yet unpublished work (Ref. 9) on the exact performance of a related processor, the  $\ln I_0$  combiner; see (51) et seq. Specifically, for a signal with  $K$  orthogonal signal components, deterministic except for phase, the optimum processor

is a sum of  $\ln \mathcal{I}_0$  of the sampled envelopes of the outputs of  $K$  filters, each one matched to a different signal component. When the number of independent components,  $K$ , tends to infinity, the optimum processor does not approach a sum of envelopes-squared (the filter-energy-detector) unless the signal energy to noise density ratio of each component tends to zero as  $K$  increases. Also, in this case, the performance of the optimum processor and the sum of envelopes-squared processor are close to each other in the region of reasonably-good false alarm and detection probabilities. This exact back-up analysis of a related processor tends to confirm the results achieved in the current study.

We have assumed here that the signal amplitude,  $A$ , is a constant over the observation interval. This led to a more tractable analysis and did not require any additional assumptions about amplitude statistics and their dependence on frequency- or phase-modulation statistics. It is this author's conjecture that in the case of random amplitude-modulation in addition, the filter-energy-detector will again turn out to be near-optimum under the conditions of small component signal energy to noise density and a large number of independent components. This is based, in part, on the observation that the received signal would be even more random than in this study, and that the filter-energy-detector is a robust processor for the more-noise-like signals. Of course, the pre-filter in the energy detector (see figure 2) would have to be broadened to cover the total bandwidth of the received signal (with both AM and FM), thereby lowering the input signal-to-noise ratio in the received signal band. This would cause a loss in performance, but is unavoidable as the randomness of the received signal is increased. Alternatively, to maintain a desired level of performance, the observation time would have to be increased; the result is a larger number of independent signal components and a still-better approximation to the optimum processor.

Due to mathematical difficulty, it has not been possible to develop anything but an infinite series for the log likelihood ratio. We then had to analyze the low-order terms (through order seven which is zero) to determine which were significant, and under what conditions the leading data-dependent term was dominant. Every statistic (we could reasonably evaluate) pointed to the filter-energy-detector as being the dominant term, under the conditions cited above. Evaluation of higher-order statistics of  $\chi_k^{(n)}$ , such as their higher-order cumulants (see (182), for example) or higher-order terms (i.e., larger  $k$ ,

as in (183) and (186), for example) is possible, but is extremely tedious. It also requires approximations to still higher-order multiple integrals, which have proven very time-consuming to evaluate exactly. Furthermore, the difficulty of realizing Volterra kernels such as (139) and (183) appears to preclude this series as a practical solution to the problem of optimum detection of frequency-modulated tones. Rather it appears to this author that for moderate  $TW_s$  products, approximations to the likelihood ratio directly should be attempted. Analysis of their performance will probably require extensive simulation effort.

APPENDIX A. OPTIMUM PROCESSOR FOR SMALL  $TW_s$

If  $TW_s < 1$ , we can give an explicit expression for the LR.  
From (32),

$$r \cong \frac{T}{N_b} \operatorname{Re}\{x a^*\} \quad \text{for } \frac{1}{f_0} \ll T < \frac{1}{W_s}, \quad (\text{A-1})$$

where  $x$  and  $a$  are the values of  $x(t)$  and  $a(t)$  at the endpoint  $t_0$ . Now let us consider  $\underline{z} = |z| \exp(i\phi)$ , where  $\phi$  is uniformly distributed over  $2\pi$ . From (13), since  $E = \frac{1}{2}|z|^2 T$ , then  $|z|$  can not be random, since  $E$  is fixed. Now we have deterministic component  $\langle \underline{z} \rangle = 0$ , and therefore, from (33),

$$\underline{a} = \underline{z} = |z| \exp(i\phi). \quad (\text{A-2})$$

Also  $\langle v \rangle = 0$  from (31). Then (35) yields (since  $\phi$  is uniformly distributed)

$$\begin{aligned} LR &= \exp\left(-\frac{E}{N_b}\right) \left\langle \exp\left[\frac{T}{N_b} \operatorname{Re}\{x a^*\}\right] \right\rangle \\ &= \exp\left(-\frac{E}{N_b}\right) I_0\left(\frac{T}{N_b} |z| |x|\right). \end{aligned} \quad (\text{A-3})$$

Hence comparison of  $|x|$  with a threshold is optimum.

On the other hand, the quantity

$$\frac{1}{2} \chi_2^2 = \frac{1}{2} \langle r^2 \rangle = \frac{1}{2} \left\langle \left\{ \frac{T}{N_b} |z| |x| \cos(\arg x - \phi) \right\}^2 \right\rangle = \frac{1}{4} \left( \frac{T}{N_b} |z| |x| \right)^2. \quad (\text{A-4})$$

This again yields the rule: compare  $|x|$  with a threshold.

This rule for decision-making is physically reasonable, since a single sample, of a complex envelope with uniform phase, contains only one item of information on which to base decisions, namely the magnitude of the complex envelope.

APPENDIX B. SOME SIGNAL MODELS AND THEIR INTERPRETATION

Consider the phase-modulated signal process (also given in (14))

$$s(t) = A \cos [2\pi f_c t + \theta(t) + \phi], \quad t_a < t < t_b, \quad (B-1)$$

where phase-modulating process  $\theta(t)$  is a zero-mean real stationary Gaussian random process, and  $\phi$  is uniformly distributed over  $2\pi$ . Then complex envelope

$$\underline{s}(t) = A \exp[i\theta(t) + i\phi], \quad t_a < t < t_b. \quad (B-2)$$

Obviously  $\langle \underline{s}(t) \rangle = 0$ ; hence there is no deterministic signal component. The ac signal component is, from (33),

$$\underline{a}(t) = \underline{s}(t) = A \exp[i\theta(t) + i\phi], \quad t_a < t < t_b. \quad (B-3)$$

Then the averages over  $\phi$ ,

$$\langle \underline{a}(t) \rangle_{\phi} = 0, \quad \langle \underline{a}(t_1) \underline{a}(t_2) \rangle = 0, \quad (B-4)$$

as expected. Also

$$\begin{aligned} \langle \underline{a}(t_2) \underline{a}^*(t_1) \rangle &= A^2 \langle \exp[i\theta(t_2) - i\theta(t_1)] \rangle_{\theta} \\ &= A^2 \exp[-R_{\theta}(0) + R_{\theta}(t_2 - t_1)] = R_{\underline{a}}(t_2 - t_1), \end{aligned} \quad (B-5)$$

where we have used the Gaussian character of the difference variable  $\theta(t_2) - \theta(t_1)$ , and defined the autocorrelation of stationary process  $\theta(t)$  as

$$\langle \theta(t_2) \theta(t_1) \rangle_{\theta} = R_{\theta}(t_2 - t_1). \quad (B-6)$$

So we have, from (B-5)

$$R_{\underline{a}}(0) = A^2, \quad R_{\underline{a}}(\infty) = A^2 \exp[-R_{\theta}(0)], \quad (B-7)$$

since  $R_\theta(\infty) = 0$ . The latter quantity in (B-7) is not zero unless  $R_\theta(0) = \langle \theta^2(t) \rangle_0$  is infinite. However if the mean-square phase-modulation is large, i.e.,

$$\langle \theta^2(t) \rangle_0 = R_\theta(0) \gg 1, \quad (\text{B-8})$$

then (B-7) yields

$$\frac{R_g(\infty)}{R_g(0)} = \exp[-R_\theta(0)] \approx 0. \quad (\text{B-9})$$

But, in general, a small mean-square phase-modulation will give rise to a non-zero value for  $R_g(\infty)$ . This holds even though there is no deterministic signal component, i.e.,  $\langle s(t) \rangle = 0$ .

For the case of frequency-modulation rather than phase-modulation, we have

$$\theta(t) = 2\pi \int_{t_0}^t du m(u), \quad t_a < t < t_b, \quad (\text{B-10})$$

where  $m(t)$  is the instantaneous frequency-modulating process, assumed stationary. This form allows  $\theta(t)$  to be a non-stationary process. The properties of Gaussian frequency-modulation are taken up in appendix E; it is shown that the exponential quantity in (B-5) is replaced according to

$$R_\theta(0) - R_\theta(\tau) \Rightarrow \int df G_m(f) \frac{1 - \cos(2\pi f\tau)}{f^2}, \quad (\text{B-11})$$

where  $G_m(f)$  is the power density spectrum of stationary frequency-modulating process  $m(t)$ . Hence (B-5) yields

$$R_g(\tau) = A^2 \exp\left[-\int df G_m(f) \frac{1 - \cos(2\pi f\tau)}{f^2}\right]. \quad (\text{B-12})$$

The behavior of (B-11), as  $\tau \rightarrow \infty$ , depends on the behavior of  $G_m(f)$  near  $f=0$ ; numerous examples are given in appendix E. Suffice it to say, for now, that examples exist where (B-11) tends to infinity, and other examples where (B-11) tends to a finite constant. Thus (B-12) can tend to zero or non-zero values, respectively, as  $\tau \rightarrow \infty$ . Furthermore, this holds even though the mean-square frequency modulation,

$$\sigma_m^2 \equiv \langle m^2(t) \rangle = \int df G_m(f), \quad (B-13)$$

is finite. Thus detection of frequency-modulated processes will be different for the two cases of (B-11) being finite or infinite in the limit as  $T \rightarrow \infty$ ; see figure 2 in the main text.

When the signal has no deterministic component, i.e.,

$$\langle s(t) \rangle = 0, \quad (B-14)$$

and in addition

$$R_s(\infty) = 0, \quad (B-15)$$

we call the signal fully-random; this is the case of major interest here. However, below, we delineate the components of the signal in the general case, so that we can properly interpret our case of interest.

#### Interpretation of Signal Components

The narrowband signal is represented in terms of its complex envelope according to

$$s(t) = \text{Re} \{ \underline{s}(t) \exp(i 2\pi f_0 t) \}, \quad t_0 < t < t_1. \quad (B-16)$$

The ac component of the complex envelope is

$$\underline{a}(t) = \underline{s}(t) - \langle \underline{s}(t) \rangle. \quad (B-17)$$

Hence the signal can be represented as

$$\begin{aligned} s(t) &= \text{Re} \{ \langle \underline{s}(t) \rangle \exp(i 2\pi f_0 t) \} + \text{Re} \{ \underline{a}(t) \exp(i 2\pi f_0 t) \} \\ &\equiv d(t) + a(t), \end{aligned} \quad (B-18)$$

where  $d(t)$  is a deterministic narrowband waveform with known phase-modulation, centered at frequency  $f_0$ . This component could include pure-tone components of known phase, including as a special case, a carrier at  $f_0$ , and/or sidebands in the neighborhood of  $f_0$ .

The correlation of signal  $s(t)$  is, using (B-18),

$$R_s(t_1, t_2) \equiv \langle s(t_1) s(t_2) \rangle = d(t_1) d(t_2) + R_a(t_1 - t_2), \quad t_a < t_1, t_2 < t_b, \quad (\text{B-19})$$

where ac component  $a(t)$  is assumed wide-sense stationary in the observation interval. The latter term can be developed as follows:

$$R_a(\tau) = \langle a(t) a(t-\tau) \rangle = \frac{1}{4} \langle \{ a(t) \exp(i2\pi f_0 t) + a^*(t) \exp(-i2\pi f_0 t) \} \cdot \{ a(t-\tau) \exp(i2\pi f_0 (t-\tau)) + a^*(t-\tau) \exp(-i2\pi f_0 (t-\tau)) \} \rangle = \frac{1}{2} \text{Re} \{ R_a(\tau) \exp(i2\pi f_0 \tau) \}, \quad (\text{B-20})$$

where  $R_a(\tau)$  is the correlation of the ac component of the complex envelope. Now if

$$R_a(\tau) \sim \sum_k c_k \exp(i2\pi f_k \tau) \quad \text{as } \tau \rightarrow +\infty, \quad (\text{B-21})$$

each term represents a pure-tone component of random phase, at frequency  $f_0 + f_k$ , of magnitude  $\frac{1}{2}|c_k|$ . If  $f_k = 0$  for some value of  $k$ , this term indicates a pure-tone component of random phase at carrier frequency  $f_0$ .

Thus the deterministic component of the signal can contain pure-tone components of known phase, while the ac component can contain pure-tone components of random phase. Coherent processing on both of these components is possible and will be so indicated by the optimum processor.

## APPENDIX C. DEFLECTION CRITERION

In this appendix, we consider maximization of the deflection at the output of the filter-energy-detector in figure C-1.

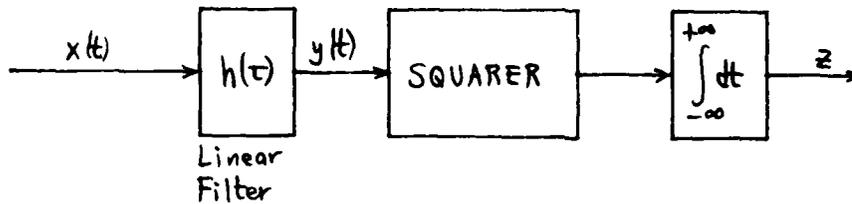


Figure C-1. Filter Energy-Detector

The input  $x(t)$  is composed of either signal-plus-noise or noise-alone:

$$x(t) = \left\{ \begin{array}{l} s(t) + n(t) \\ \text{OR} \\ n(t) \end{array} \right\} \text{ for } t_a < t < t_b, \quad T = t_b - t_a, \quad (\text{C-1})$$

where the signal and noise are stationary processes during the gated interval of time, independent of each other. The only assumption on the signal statistics is that ensemble average

$$\overline{s(t_1)s(t_2)} = \left\{ \begin{array}{l} R_s(t_1 - t_2), \quad t_a < t_1, t_2 < t_b \\ 0, \text{ otherwise} \end{array} \right\}. \quad (\text{C-2})$$

That is,  $s(t)$  is stationary in the observation interval. A similar relationship holds for the noise  $n(t)$ . Since input  $x(t)$  is a gated process,  $y(t)$  is

non-stationary. We allow filter  $h(t)$  to be unrealizable, and we integrate the squarer output over all time to ensure we collect all the signal output.

The output of figure C-1 is

$$\begin{aligned} z &= \int_{-\infty}^{+\infty} dt y^2(t) = \int_{-\infty}^{+\infty} dt \left[ \int_{t_a}^{t_b} du x(u) h(t-u) \right]^2 \\ &= \int_{-\infty}^{+\infty} dt \iint_{t_a}^{t_b} du dv x(u) x(v) h(t-u) h(t-v) \\ &= \iint_{t_a}^{t_b} du dv x(u) x(v) A_h(u-v), \end{aligned} \quad (C-3)$$

where  $A_h$  is the autocorrelation function of the filter impulse response:

$$A_h(u-v) = \int_{-\infty}^{+\infty} dt h(t-u) h(t-v). \quad (C-4)$$

Equation (C-3) may be expressed alternatively as

$$z = \int_{-\infty}^{+\infty} df |Y(f)|^2 = \int_{-\infty}^{+\infty} df |H(f)|^2 |X(f)|^2, \quad (C-5)$$

using Parseval's theorem, where

$$\begin{aligned} H(f) &= \int_{-\infty}^{+\infty} dt \exp(-i2\pi ft) h(t), \\ X(f) &= \int_{t_a}^{t_b} dt \exp(-i2\pi ft) x(t). \end{aligned} \quad (C-6)$$

Equations (C-3) and (C-5) indicate the generic form of processing attainable by the block diagram of figure C-1, for any time-bandwidth values and any signal and noise statistics.

The mean value of the output of figure C-1, for signal and noise present, is available from (C-3) and (C-2) as

$$E\{z | S+N\} = \iint_{t_a}^{t_b} du dv [R_S(u-v) + R_n(u-v)] A_h(u-v). \quad (C-7)$$

Then obviously, for noise-alone, the mean output is

$$E\{z|N\} = \int_{t_a}^{t_b} du dv R_n(u-v) A_h(u-v). \quad (C-8)$$

The difference in these mean outputs is equal to the mean output for signal-alone at the input, and is

$$\begin{aligned} E\{z|S+N\} - E\{z|N\} &= E\{z|S\} = \int_{t_a}^{t_b} du dv R_s(u-v) A_h(u-v) \\ &= \int_{-T}^T d\tau (T-|\tau|) R_s(\tau) A_h(\tau) \\ &= \int_{-\infty}^{+\infty} df |H(f)|^2 \left[ G_s(f) \otimes \frac{\sin^2(\pi f T)}{\pi^2 f^2} \right], \end{aligned} \quad (C-9)$$

where  $T = t_b - t_a$  and  $G_s(f)$  is the signal power density spectrum. (We think of underlying process  $s(t)$  as being stationary over all time, but being gated and observable only during  $(t_a, t_b)$ . If we let  $g(t)$  be the gating function, then we have (more generally than (C-2)) the input observable signal correlation

$$\overline{[g(t_1) s(t_1)] [g(t_2) s(t_2)]} = g(t_1) g(t_2) R_s(t_1 - t_2) \text{ for all } t_1, t_2. \quad (C-10)$$

Exact relationship (C-9) holds for any interval  $T$  and any signal spectrum  $G_s(f)$  (including a possible dc component, corresponding to a deterministic component of  $s(t)$ ).

Now we make a simplifying approximation: let

$$T \gg W_s^{-1} \text{ and } R_s(\infty) = 0. \quad (C-11)$$

Thus,  $G_s(f)$  will have no impulse at the origin. Then (C-9) yields the useful approximation

$$E\{z|S\} \approx T \int_{-\infty}^{+\infty} d\tau R_s(\tau) A_h(\tau) = T \int_{-\infty}^{+\infty} df G_s(f) |H(f)|^2. \quad (C-12)$$

Now we address the noise output component. For noise-alone at the input to figure C-1, we have

$$z_n = \int_{-\infty}^{+\infty} dt y_n^2(t). \quad (C-13)$$

Its mean-square value is

$$\overline{z_n^2} = \iint_{-\infty}^{+\infty} dt_1 dt_2 \overline{y_n^2(t_1) y_n^2(t_2)}. \quad (C-14)$$

We now assume that  $y_n(t)$  is a zero-mean Gaussian process; a sufficient condition for this to be true is for noise input  $n(t)$  to be zero-mean Gaussian. Then (C-14) becomes

$$\overline{z_n^2} = \iint_{-\infty}^{+\infty} dt_1 dt_2 \left[ \overline{y_n^2(t_1) y_n^2(t_2)} + 2 \overline{y_n(t_1) y_n(t_2)}^2 \right], \quad (C-15)$$

giving

$$\text{Var} \{z|N\} = 2 \iint_{-\infty}^{+\infty} dt_1 dt_2 \overline{y_n(t_1) y_n(t_2)}^2. \quad (C-16)$$

But consideration of figure C-1 and (C-1) gives

$$\begin{aligned} \overline{y_n(t_1) y_n(t_2)} &= \iint_{t_a}^{t_b} du dv \overline{n(u) n(v)} h(t_1 - u) h(t_2 - v) \\ &= \iint_{t_a}^{t_b} du dv R_n(u - v) h(t_1 - u) h(t_2 - v). \end{aligned} \quad (C-17)$$

Hence (C-16), (C-17), and (C-4) yield

$$\begin{aligned} \text{Var} \{z|N\} &= 2 \iint_{-\infty}^{+\infty} dt_1 dt_2 \left[ \iint_{t_a}^{t_b} du dv R_n(u - v) h(t_1 - u) h(t_2 - v) \right]^2 \\ &= 2 \iiint_{t_a}^{t_b} du dv \mu \nu R_n(u - v) R_n(\mu - \nu) A_h(u - \mu) A_h(v - \nu), \end{aligned} \quad (C-18)$$

which is exact for any noise spectrum.

An alternative exact expression to (C-18) is available by employing (C-6); it is

$$\text{Var}\{z|N\} = 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} df_a df_b |H(f_a)|^2 |H(f_b)|^2 |R_n(f_a, f_b)|^2, \quad (\text{C-19})$$

where

$$R_n(f_a, f_b) \equiv \int_{t_a}^{t_b} \int_{t_a}^{t_b} du dv R_n(u-v) \exp[-i2\pi(f_a u + f_b v)] \quad (\text{C-20})$$

is a two-dimensional noise spectrum.

For the special case of white noise, we have

$$R_n(t) = \frac{N_0}{2} \delta(t), \quad G_n(f) = \frac{N_0}{2}, \quad \text{all } f, \quad (\text{C-21})$$

and (C-18) yields

$$\text{Var}\{z|N\} = \frac{1}{2} N_0^2 \int_{t_a}^{t_b} \int_{t_a}^{t_b} du d\mu A_h^2(u-\mu) = \frac{1}{2} N_0^2 \int_{-T}^T d\tau (T-|\tau|) A_h^2(\tau). \quad (\text{C-22})$$

This expression is exact for white noise. The alternative expression (C-19) is unchanged, but (C-20) specializes to

$$|R_n(f_a, f_b)|^2 = \frac{1}{4} N_0^2 T^2 \text{sinc}^2[(f_a + f_b)T], \quad (\text{C-23})$$

where  $\text{sinc}(x) \equiv \sin(\pi x)/(\pi x)$ .

If we make the reasonable assumption that the frequency width of  $H(f)$  in figure C-1 is comparable to that of the input signal, then for  $T \gg W_s^{-1}$  and  $A_h(\omega) = 0$ , we have, from (C-22)

$$\text{Var}\{z|N\} \cong \frac{1}{2} N_0^2 T \int_{-\infty}^{+\infty} d\tau A_h^2(\tau) = \frac{1}{2} N_0^2 T \int_{-\infty}^{+\infty} df |H(f)|^4, \quad (\text{C-24})$$

the latter step via (C-4) and (C-6). This approximation is valid for large  $TW_s$ .

Finally, we define the deflection at the output of figure C-1 as

$$d^2 = \frac{[E\{z|S+N\} - E\{z|N\}]^2}{\text{Var}\{z|N\}} \quad (\text{C-25})$$

Now calling on (C-9), (C-12), and (C-24), we obtain the deflection as

$$d^2 = \frac{2T \left[ \int_{-\infty}^{+\infty} df G_s(f) |H(f)|^2 \right]^2}{N_0^2 \int_{-\infty}^{+\infty} df |H(f)|^4} \quad (\text{C-26})$$

This expression is valid for white noise and  $TW_s \gg 1$ . Its sensitivity to different filters  $H(f)$  in figure C-1 is easily investigated.

The deflection in (C-26) is maximized by the choice of the Eckart filter

$$|H_0(f)|^2 = k \frac{G_s(f)}{N_0^2} \quad \text{for large } TW_s. \quad (\text{C-27})$$

The maximum value of the deflection (C-26) is then

$$d_0^2 = \frac{2T}{N_0^2} \int_{-\infty}^{+\infty} df G_s^2(f) \quad \text{for large } TW_s. \quad (\text{C-28})$$

Alternative expressions to (C-28) are

$$d_0^2 = \frac{T}{W_s} \left( \frac{S}{N_0} \right)^2 = TW_s \left( \frac{S}{N_0 W_s} \right)^2, \quad (\text{C-29})$$

where the single-sided signal statistical bandwidth  $W_s$  is defined as

$$W_s = \frac{\left[ \int_0^{\infty} df G_s(f) \right]^2}{\int_0^{\infty} df G_s^2(f)}, \quad (\text{C-30})$$

and the received signal power is

$$S = \int_{-\infty}^{+\infty} df G_s(f). \quad (C-31)$$

The quantity  $S/N_b$  in (C-29) is the ratio of total received signal power to the noise power in a 1 Hz band; the latter quantity  $S/(N_b W_s)$  in (C-29) is the ratio of total signal power to the noise power in the signal band.

In summary, for maximum deflection at the output of figure C-1, the optimum filter-power-transfer function is proportional to the signal power density spectrum (for white noise and no deterministic signal component); see (C-27). This result obtains regardless of the size of the input signal-to-noise ratio. Also then, from (C-4), (C-6), and (C-27),

$$A_{h_0}(\tau) = \int_{-\infty}^{+\infty} df \exp(i2\pi f\tau) |H_0(f)|^2 = k R_s(\tau), \quad (C-32)$$

which is proportional to the signal autocorrelation function.

APPENDIX D. DIFFERENT FILTER CHOICES

We have, from (45),

$$\frac{1}{2} \chi_2^{(r)} = \frac{1}{4N_0^2} \iint dt_1 dt_2 x(t_1) x^*(t_2) R_g(t_2 - t_1). \quad (D-1)$$

Let weighting

$$w(u) = \begin{cases} 1, & |u| < T \\ \text{arbitrary,} & |u| > T \end{cases} \quad (D-2)$$

and define

$$\int_{-\infty}^{+\infty} du w(u) R_g(u) \exp(-i2\pi fu) = G_w(f), \quad \text{all } f. \quad (D-3)$$

Then inverting (D-3), and appealing to (D-2), we have

$$R_g(u) = \int_{-\infty}^{+\infty} df \exp(i2\pi fu) G_w(f) \quad \text{for } |u| < T. \quad (D-4)$$

Now we employ (D-4) in (D-1), recalling that  $x(t) = 0$  for  $t < t_0, t > t_1$ , to get

$$\begin{aligned} \frac{1}{2} \chi_2^{(r)} &= \frac{1}{4N_0^2} \iint dt_1 dt_2 x(t_1) x^*(t_2) \int df \exp[i2\pi f(t_2 - t_1)] G_w(f) \\ &= \frac{1}{4N_0^2} \int df G_w(f) |X(f)|^2. \end{aligned} \quad (D-5)$$

This relation holds true for any weighting  $w$  satisfying (D-2).

Since the output of the bottom channel of figure 2 can be expressed as

$$\int_{-\infty}^{+\infty} dt |y(t)|^2 = \int df |Y(f)|^2 = \int df |H(f)|^2 |X(f)|^2, \quad (D-6)$$

it would appear that the choice

$$|H(f)|^2 = \frac{G_w(f)}{4N_0^2} \quad (D-7)$$

yields the desired quantity (D-5). However, use of (D-3) shows that

$$G_w(f) = G_a(f) \otimes W(f), \quad (D-8)$$

where  $W(f)$  is the Fourier transform of weighting (D-2). But (D-8) might be negative for some  $f$ ; for example, if  $w(u)$  is chosen to be zero for  $|u| > T$  in (D-2), then  $W(f) = \sin(2\pi fT)/(\pi f)$ , which goes negative repeatedly, and could cause (D-8) to become negative. Of course the selection of  $w(u) = 1$  for all  $u$  yields  $W(f) = \delta(f)$ , and (D-8) is guaranteed non-negative; this is the case considered in the main body of the report.

#### Realization of Second-Order Kernel via Bush's Technique

Here we will instead employ the technique given in Ref. 7, pp. 4-6, to derive the filtering block diagram necessary to realize  $\chi_2^{(n)}$ . From (D-1), the second-order kernel is  $R_a(t_2 - t_1)$ , all  $t_1, t_2$ , with kernel transform

$$\iint dt_1 dt_2 \exp(-i\omega_1 t_1 - i\omega_2 t_2) R_a(t_2 - t_1) = \delta(\omega_1 + \omega_2) G_a(\omega_1). \quad (D-9)$$

Now the most general second-order kernel is (Ref. 7, eq. 7)

$$K_a(\omega_1) K_b(\omega_2) K_c(\omega_1 + \omega_2). \quad (D-10)$$

So if we choose

$$K_c(\omega) = \delta(\omega), \quad k_c(\tau) = 1 \text{ for all } \tau, \quad (D-11)$$

we have to require

$$\delta(\omega_1 + \omega_2) G_a(\omega_1) = \delta(\omega_1 + \omega_2) K_a(\omega_1) K_b(-\omega_1) = \delta(\omega_1 + \omega_2) K_a(\omega_1) K_b^*(\omega_1), \quad (D-12)$$

where we have used the real character of impulse response  $k_b(\tau)$ . That is,

$$K_a(\omega)K_b^*(\omega) = G_g(\omega). \quad (D-13)$$

The block diagram of the second-order processor is then indicated in figure D-1; some arbitrariness is present in (D-13).

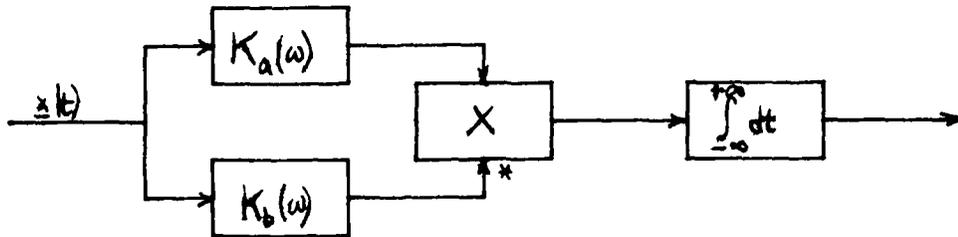


Figure D-1. Second-Order Processor

For the particular special case of  $K_b(\omega) = K_a(\omega)$ , (D-13) becomes

$$|K_a(\omega)|^2 = G_g(\omega), \quad (D-14)$$

and we have the block diagram presented in the main body of this report, when we recall that  $G_g(\omega)$  could contain an impulse at the origin if  $R_g(\infty) \neq 0$ ; see (67). Thus this alternative procedure yields the same second-order processor.

## APPENDIX E. GAUSSIAN FREQUENCY-MODULATION

The signal of interest here is the frequency-modulated process

$$s(t) = A \cos \left[ 2\pi f_0 t + 2\pi \int_{t_a}^t du m(u) + \phi \right], \quad t_a < t < t_b. \quad (\text{E-1})$$

The phase shift  $\phi$  is a random variable and is uniformly distributed over  $2\pi$ . The instantaneous frequency of the signal in (E-1) is

$$f_0 + m(t) \quad \text{Hz}, \quad (\text{E-2})$$

where frequency-modulating process  $m(t)$  is stationary, Gaussian, zero mean (no loss of generality), and correlation

$$\overline{m(t_1)m(t_2)} = R_m(t_1 - t_2), \quad t_a < t_1, t_2 < t_b. \quad (\text{E-3})$$

The (double-sided) power density spectrum of the low-pass frequency-modulating process  $m(t)$  is

$$G_m(f) = \int d\tau \exp(-i2\pi f\tau) R_m(\tau), \quad (\text{E-4})$$

and the mean-square frequency deviation is

$$\sigma_m^2 \equiv \overline{m^2(t)} = R_m(0) = \int df G_m(f). \quad (\text{E-5})$$

The complex-envelope signal is, from (E-1),

$$s(t) = A \exp \left[ i 2\pi \int_{t_a}^t du m(u) + i\phi \right], \quad t_a < t < t_b. \quad (\text{E-6})$$

The mean of this signal is zero, since  $\phi$  is uniformly distributed. Thus there is no deterministic signal component.

The autocorrelation of  $\underline{s}(t)$  is, for  $t_a < t_1, t_2 < t_b$ ,

$$R_s(t_1, t_2) \equiv \overline{\underline{s}(t_2) \underline{s}^*(t_1)} = A^2 \overline{\exp\left[i2\pi \int_{t_1}^{t_2} du m(u)\right]}, \quad (\text{E-7})$$

where we have assumed  $A$  non-random. Since  $m(t)$  is a Gaussian random process, the dimensionless real random variable

$$y \equiv 2\pi \int_{t_1}^{t_2} du m(u) \quad (\text{E-8})$$

is Gaussian, with mean zero, and mean-square value

$$\begin{aligned} \overline{y^2} &= 4\pi^2 \iint_{t_1}^{t_2} du dv R_m(u-v) = 8\pi^2 \int_0^\tau dx \int_0^x dy R_m(y) = 4\pi^2 \int_{-\tau}^\tau du (\tau - |u|) R_m(u) \\ &= 4 \int df G_m(f) \frac{\sin^2(\pi f \tau)}{f^2} = 2 \int df \frac{G_m(f)}{f^2} [1 - \cos(2\pi f \tau)] \equiv \sigma^2(\tau), \end{aligned} \quad (\text{E-9})$$

where we have employed (E-3), (E-4), and defined  $\tau = |t_2 - t_1|$ . Therefore (E-7) is expressible as

$$R_s(t_1, t_2) = A^2 \overline{\exp(iy)} = A^2 \exp\left(-\frac{1}{2} \overline{y^2}\right) = A^2 \exp\left[-\frac{1}{2} \sigma^2(\tau)\right], \quad (\text{E-10})$$

using the Gaussian character of  $y$  and (E-9). Since the right-hand side of (E-10) is a function only of  $\tau = |t_2 - t_1|$ ,  $\underline{s}(t)$  is wide-sense stationary for  $t_a < t < t_b$ , and we say that

$$R_s(\tau) = A^2 \exp\left[-\frac{1}{2} \sigma^2(\tau)\right] \text{ in the observation interval.} \quad (\text{E-11})$$

According to (E-9),  $\sigma^2(\tau)$  can be interpreted as the variance, after  $\tau$  seconds, of the "random walk" process  $y$  in (E-8).

Since  $\sigma^2(0) = 0$  from (E-9), we find

$$R_s(0) = \overline{|\underline{s}(t)|^2} = A^2, \quad (\text{E-12})$$

using (E-7) and (E-11); that is, the power in the frequency-modulated signal  $s(t)$  is independent of the frequency modulating process  $m(t)$  or its spectrum  $G_m(f)$ .

Limiting Behavior

From (E-9) and (E-5), we find

$$\sigma^2(\tau) \leq 4\pi^2 \int_{-\tau}^{\tau} du (\tau - |u|) R_m(u) = 4\pi^2 \sigma_m^2 \tau^2, \text{ all } \tau \geq 0. \quad (\text{E-13})$$

Furthermore (E-9) yields directly

$$\sigma^2(\tau) \sim 4\pi^2 \sigma_m^2 \tau^2 \text{ as } \tau \rightarrow 0+. \quad (\text{E-14})$$

In order to determine the behavior of  $R_s(\tau)$  as  $\tau \rightarrow +\infty$ , we need to know the behavior of  $\sigma^2(\tau)$ , which in turn depends on  $G_m(f)$  via (E-9). In particular, suppose that

$$G_m(f) \sim \alpha |f|^\nu \text{ as } f \rightarrow 0; \nu > -1. \quad (\text{E-15})$$

We require  $\nu > -1$  so that the mean-square frequency deviation (E-5) remain finite. Then we find, from (E-9) and (E-15), the proportionality behaviors

$$\sigma^2(\tau) \propto \begin{cases} \tau^{1-\nu} & \text{for } -1 < \nu < 1 \\ \ln \tau & \text{for } \nu = 1 \\ c & \text{for } \nu > 1 \end{cases} \text{ as } \tau \rightarrow +\infty. \quad (\text{E-16})$$

Thus for  $-1 < \nu \leq 1$ ,  $\sigma^2(\tau)$  tends to infinity as  $\tau$  does. This in turn dictates, via (E-11), that

$$R_s(\tau) \sim \begin{cases} 0, & -1 < \nu \leq 1 \\ A^2 \exp\left[-\frac{1}{2} \sigma^2(\tau)\right], & \nu > 1 \end{cases} \text{ as } \tau \rightarrow +\infty. \quad (\text{E-17})$$

Thus, according to appendix B, there is no carrier term present if  $-1 < \nu \leq 1$ ; but

there is a carrier term of random phase if  $\nu > 1$ , that is, if  $G_m(f) \rightarrow 0$  faster than  $f$  at the origin. In either case, there is no carrier term of known phase present, because the uniform phase of  $\phi$  in (E-1) precludes any deterministic component.

If we assume that  $\sigma^2(\tau)$  approaches the constant value  $\sigma^2(\infty)$  as  $\tau \rightarrow +\infty$ , then we can express (E-11) as

$$R_s(\tau) = A^2 \left\{ \exp\left[-\frac{1}{2}\sigma^2(\tau)\right] - \exp\left[-\frac{1}{2}\sigma^2(\infty)\right] + \exp\left[-\frac{1}{2}\sigma^2(\infty)\right] \right\}. \quad (\text{E-18})$$

Then the spectrum of the complex envelope  $\underline{s}(t)$  is

$$G_s(f) = A^2 \exp\left[-\frac{1}{2}\sigma^2(\infty)\right] \delta(f) + B(f), \quad (\text{E-19})$$

where

$$B(f) = A^2 \int d\tau \exp(-i2\pi f\tau) \left\{ \exp\left[-\frac{1}{2}\sigma^2(\tau)\right] - \exp\left[-\frac{1}{2}\sigma^2(\infty)\right] \right\}. \quad (\text{E-20})$$

The former component in (E-19) represents a pure-tone carrier term of random phase at frequency  $f_0$ , while the latter term represents a distributed spectrum about the carrier; recall we are dealing with complex envelopes here. The relations (E-19) and (E-20) actually hold whether  $\sigma^2(\infty)$  is finite or infinite; in the latter case, there is no carrier.

If  $G_m(0) \neq 0$ , then  $\nu = 0$  in (E-15). Then (E-9) yields

$$\sigma^2(\tau) \sim 4\pi^2 \tau \int du R_m(u) = 4\pi^2 G_m(0) \tau \quad \text{as } \tau \rightarrow +\infty, \quad (\text{E-21})$$

in agreement with the first line of (E-16). Thus in this special case, there is no carrier term, and the signal correlation (E-11) or (E-18) decays exponentially with  $\tau$ .

From (E-9), there also follows

$$\frac{d}{d\tau} \sigma^2(\tau) = 4\pi \int df G_m(f) \frac{\sin(2\pi f\tau)}{f} = 4\pi^2 \int_{-\tau}^{\tau} du R_m(u), \quad (\text{E-22})$$

and

$$\frac{d^2}{d\tau^2} \sigma^2(\tau) = 8\pi^2 R_m(\tau). \quad (\text{E-23})$$

Since the integral on  $f$  in (E-22) can be negative for some values of  $\tau$ ,  $\sigma^2(\tau)$  is not always monotonically increasing with  $\tau$ , despite its interpretation below (E-9) as the variance of the random walk process  $y$  in (E-8). An example would occur for spectrum  $G_m(f)$  narrowband about a non-zero frequency.

#### Approximations

Equations (E-14) and (E-21) suggest an approximation for  $\sigma^2(\tau)$  when  $G_m(0) \neq 0$ , namely

$$\sigma^2(\tau) \approx \frac{4\pi^2 \sigma_m^2 \tau^2}{1 + \sigma_m^2 \tau / G_m(0)}, \quad \tau \geq 0. \quad (\text{E-24})$$

Substitution in (E-11) then yields the approximation

$$R_s(\tau) \approx A^2 \exp\left[-\frac{2\pi^2 \sigma_m^2 \tau^2}{1 + \sigma_m^2 \tau / G_m(0)}\right], \quad \tau \geq 0, \quad G_m(0) \neq 0. \quad (\text{E-25})$$

We now give several examples of spectrum  $G_m(f)$  for the frequency-modulating process  $m(t)$ , and illustrate the properties derived above. In these examples,  $W_m$  is a characteristic frequency of spectrum  $G_m(f)$ , but it is not necessarily the bandwidth of  $m(t)$ . However, in the first three examples, where  $G_m(0) \neq 0$ ,  $W_m$  can be interpreted as an equivalent bandwidth, in the following sense:

$$W_m = \frac{\int_{-\infty}^{+\infty} df G_m(f)}{G_m(0)} = \frac{R_m(0)}{G_m(0)} = \frac{\sigma_m^2}{G_m(0)}, \quad G_m(0) \neq 0. \quad (\text{E-26})$$

That is,  $W_m$  is the width of a rectangle with amplitude equal to the origin value of the spectrum  $G_m(f)$ , such that the rectangle and true spectrum have the same power. In this case, (E-24) yields the approximation

$$\sigma^2(\tau) \approx \frac{4\pi^2 \sigma_m^2 \tau^2}{1 + W_m \tau} = 4\pi^2 \frac{\sigma_m^2}{W_m^2} \frac{(W_m \tau)^2}{1 + W_m \tau}, \quad \tau \geq 0, \quad G_m(0) \neq 0. \quad (\text{E-27})$$

This approximation is plotted in figure E-1 as the bottom curve; the remaining curves are the exact results for the first three examples given below, and are seen to be well approximated by (E-27).

Equation (E-9) points out another approximation that could be extremely useful; we have

$$\sigma^2(\tau) = 4\pi^2\tau^2 \int df G_m(f) \left[ \frac{\sin(\pi f \tau)}{\pi f \tau} \right]^2. \quad (\text{E-28})$$

Now the equivalent width of  $G_m(f)$  is  $W_m$ ; see (E-26). On the other hand, the  $\text{sinc}^2$  function in (E-28) reaches a null at  $f = 1/\tau$ . Then if

$$\frac{1}{\tau} > 4W_m, \quad (\text{E-29})$$

(E-28) yields

$$\sigma^2(\tau) \cong 4\pi^2\tau^2 \int df G_m(f) = 4\pi^2\sigma_m^2\tau^2. \quad (\text{E-30})$$

That is,

$$\frac{1}{2}\sigma^2(\tau) \cong 2\pi^2 \frac{\sigma_m^2}{W_m^2} (W_m\tau)^2 \quad \text{if } W_m\tau < \frac{1}{4}. \quad (\text{E-31})$$

But (E-31) is equal to

$$\frac{\pi^2}{8} \frac{\sigma_m^2}{W_m^2} \quad \text{at } W_m\tau = \frac{1}{4}, \quad (\text{E-32})$$

which is large compared with unity if  $\frac{\sigma_m}{W_m} > 3$ . Therefore  $R_s(\tau)$  is substantially zero for  $\tau > (4W_m)^{-1}$ , and we have

$$R_s(\tau) = A^2 \exp\left[-\frac{1}{2}\sigma^2(\tau)\right] \cong A^2 \exp\left[-2\pi^2\sigma_m^2\tau^2\right] \quad \text{for all } \tau, \quad \text{if } D \equiv \frac{\sigma_m}{W_m} > 3. \quad (\text{E-33})$$

That is, the frequency-modulated signal correlation is approximately Gaussian if the RMS frequency deviation is somewhat larger than the equivalent bandwidth

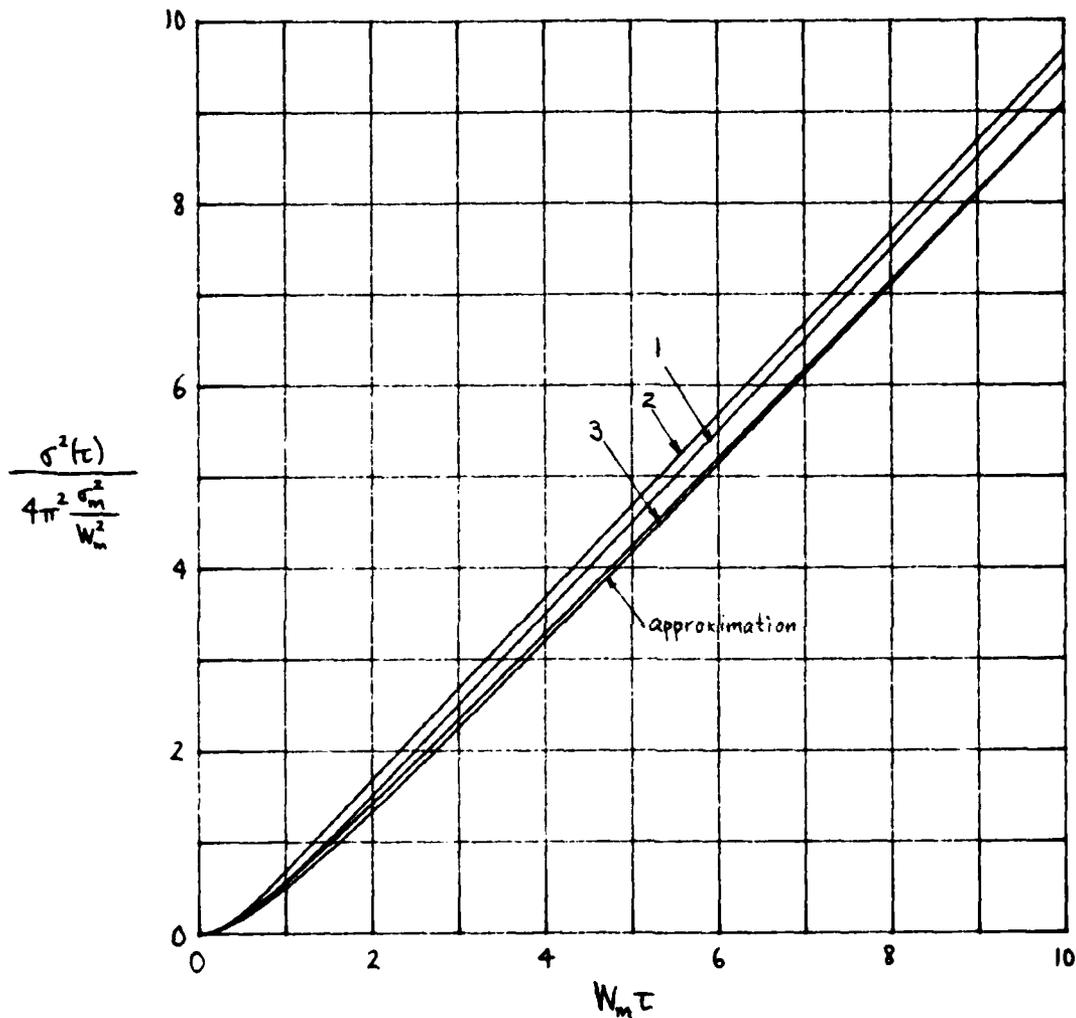


Figure E-1.  $\sigma^2(\tau)$  for Three Examples

of the frequency-modulating process  $m(t)$ . Some examples of this comparison are given below.

The spectrum corresponding to (E-33) is

$$G_s(f) \approx \frac{A^2}{\sqrt{2\pi} \sigma_m} \exp\left[-\frac{f^2}{2\sigma_m^2}\right] \quad \text{if } D = \frac{\sigma_m}{W_m} > 3. \quad (\text{E-34})$$

This is a Gaussian spectrum with standard deviation equal to  $\sigma_m$ , which is the RMS frequency deviation of the frequency-modulating process  $m(t)$ .

#### Examples

Example 1.

$$R_m(t) = \sigma_m^2 \exp(-2W_m |t|)$$

$$G_m(f) = \frac{\sigma_m^2 / W_m}{1 + (\pi f / W_m)^2}$$

$$G_m(W_m/2) = .29 G_m(0), \quad G_m(W_m) = .092 G_m(0)$$

$$\sigma^2(t) = 4\pi^2 \frac{\sigma_m^2}{W_m^2} \frac{2W_m t - 1 + \exp(-2W_m t)}{2}, \quad t \geq 0$$

This curve is plotted in figure E-1. Also

$$\sigma^2(t) \sim 4\pi^2 \frac{\sigma_m^2}{W_m^2} t \quad \text{as } t \rightarrow +\infty.$$

Since  $\nu = 0$  (see (E-15) and (E-16)),  $\sigma^2(t) = \infty$ , and there is no carrier. The distributed spectral component of the frequency-modulated signal is given by (E-20) as

$$\begin{aligned} B(f) &= A^2 \int_0^\infty dt \exp(-i2\pi f t) \exp\left[-\frac{1}{2}\sigma^2(t)\right] \\ &= \frac{A^2}{W_m} \int_0^\infty dx \cos\left(\pi \frac{f}{W_m} x\right) \exp\left[-\pi^2 D^2 (x-1+e^{-x})\right]. \end{aligned}$$

The parameter  $D = \sigma_m / W_m$  is a measure of the frequency modulation index, being the ratio of the RMS frequency deviation,  $\sigma_m$ , to the equivalent bandwidth,  $W_m$ , (characteristic frequency) of the frequency modulating process  $m(t)$ . It is also a measure of the RMS phase deviation of the signal process, since  $D^2$  appears as a factor in the expression for  $\sigma^2(t)$  above.

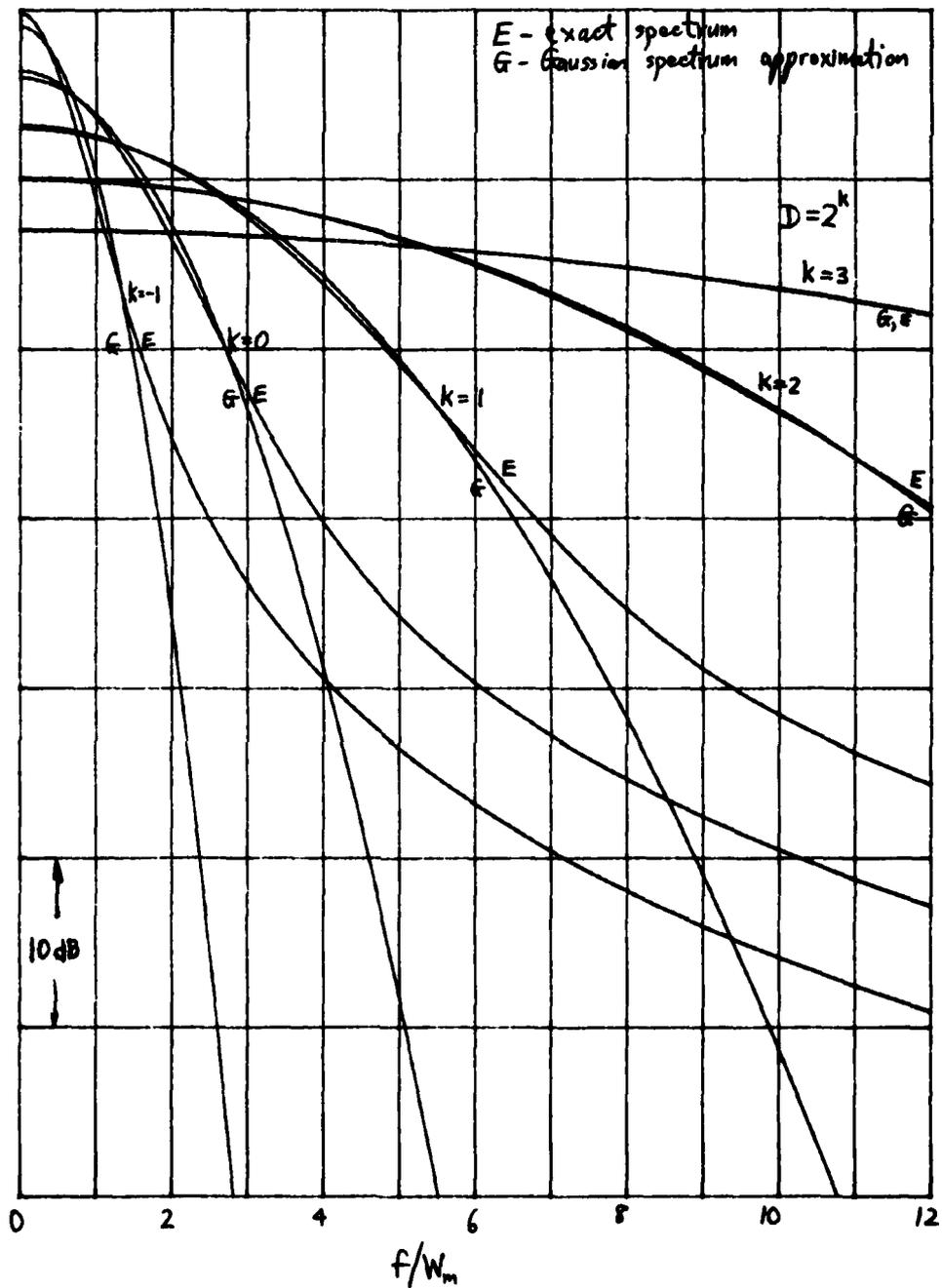
Plots of  $B(f)$  are given in ref. 10, page 609, figure 14.4; however, they are plotted in linear units (watts/Hz) rather than in dB, and the deep skirts of the modulated-signal spectrum are not observable. We plot the (positive-frequency part of the even) signal power density spectrum for this example in figure E-2 in dB, where the curves are all normalized to unit area (over  $-\infty, +\infty$ ). The three parts of the figure present the same information on different abscissas. Figure E-2(a) is most easily interpreted as considering  $W_m$  fixed, and varying  $\sigma_m$  through the frequency-modulation index  $D$ . The curves labelled with an "E" are the exact signal spectra obtained from the above equation for  $B(f)$ , for  $D=2^k$ ,  $k=-(1)3$ . Corresponding to each value of  $D$  (or  $k$ ) is also plotted the Gaussian approximation, labelled with "G", afforded by saying that

$$\sigma^2(\tau) \approx 4\pi^2 \sigma_m^2 \tau^2 \quad \text{for all } \tau$$

although it is recognized to be poor for large  $\tau$  (see figure E-1); it can be seen that the Gaussian spectral approximation is very poor in the deep skirt region for small values of index  $D$ , but is better for larger values of index  $D$ . As  $D$  increases, the signal bandwidth  $W_s$  increases progressively.

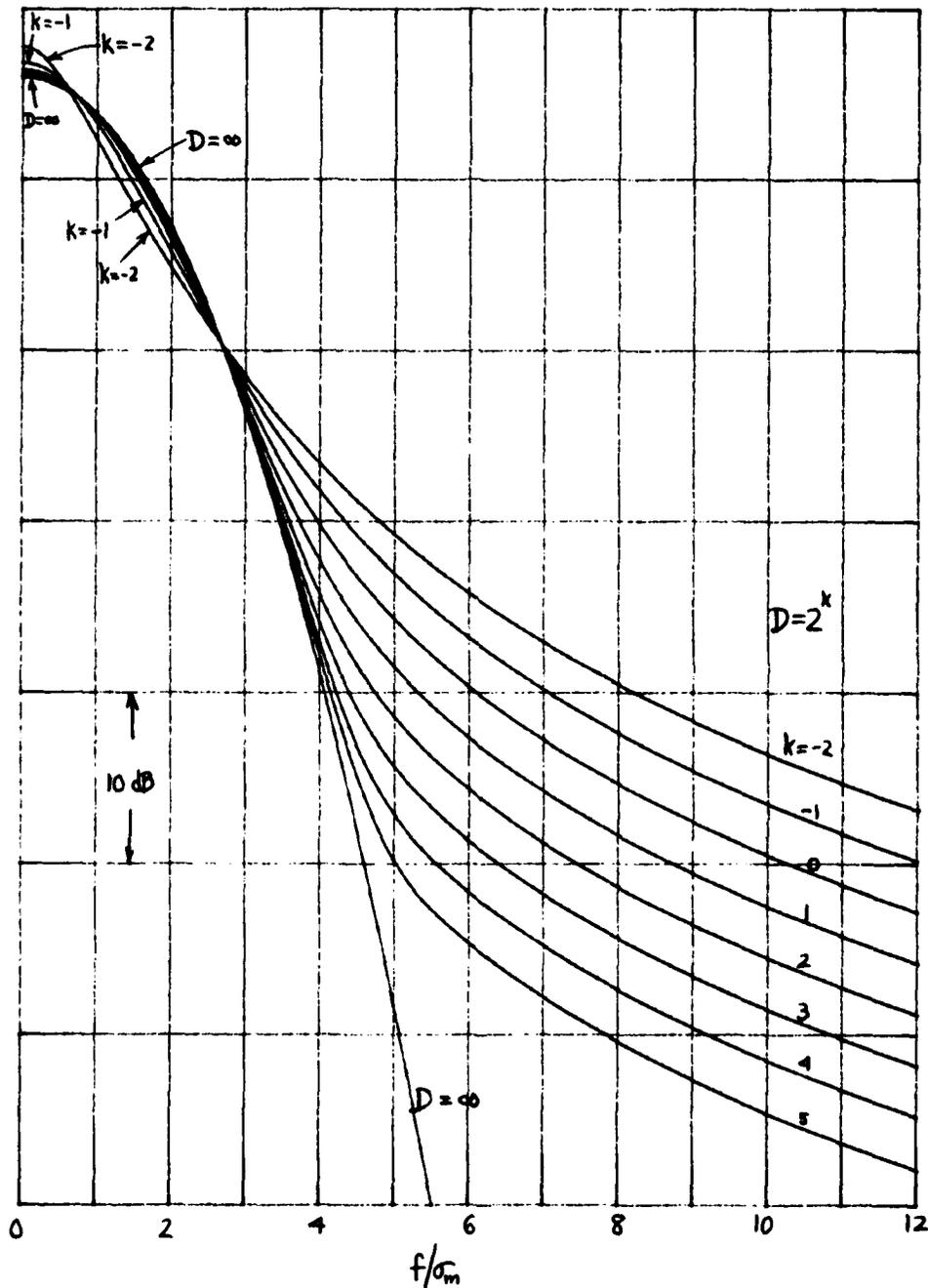
Figures E-2(b) and E-2(c) are most easily interpreted as considering RMS frequency deviation  $\sigma_m$  fixed, and varying modulating bandwidth  $W_m$  through the index  $D$ . As  $D$  increases (i.e., modulating bandwidth  $W_m$  decreases), the deep skirts become narrower and drop by 3 dB per doubling of  $D$ ; however, the skirts are very slow in approaching the limiting Gaussian spectrum indicated by the  $D = \infty$  curve. For  $D > 2$ , the -3 dB bandwidth of the signal spectrum is virtually independent of  $D$ ; in fact, this is true of any bandwidth measure above -20 dB.

However, it should be noted from figures E-2(b) and E-2(c) that the -3 dB bandwidth decreases as  $D$  decreases below 1 ( $W_m$  increases above  $\sigma_m$ ). That is, if RMS deviation  $\sigma_m$  is held fixed, and the modulating bandwidth  $W_m$  is increased, the -3 dB signal bandwidth decreases; this result is consistent with Ref. 10, page 609, figure 14.4b. It is also consistent with the observation above that the RMS phase deviation is smaller, since it is proportional to  $D = \sigma_m/W_m$ ; hence the frequency-modulated signal is more nearly a carrier of small phase deviation and a narrowband spectrum. Physically, the frequency-modulating bandwidth  $W_m$  can get so large that the accumulated phase perturbation during a time coherence interval  $W_m^{-1}$  seconds is tending to zero, and the waveform is tending to be progressively more narrowband.



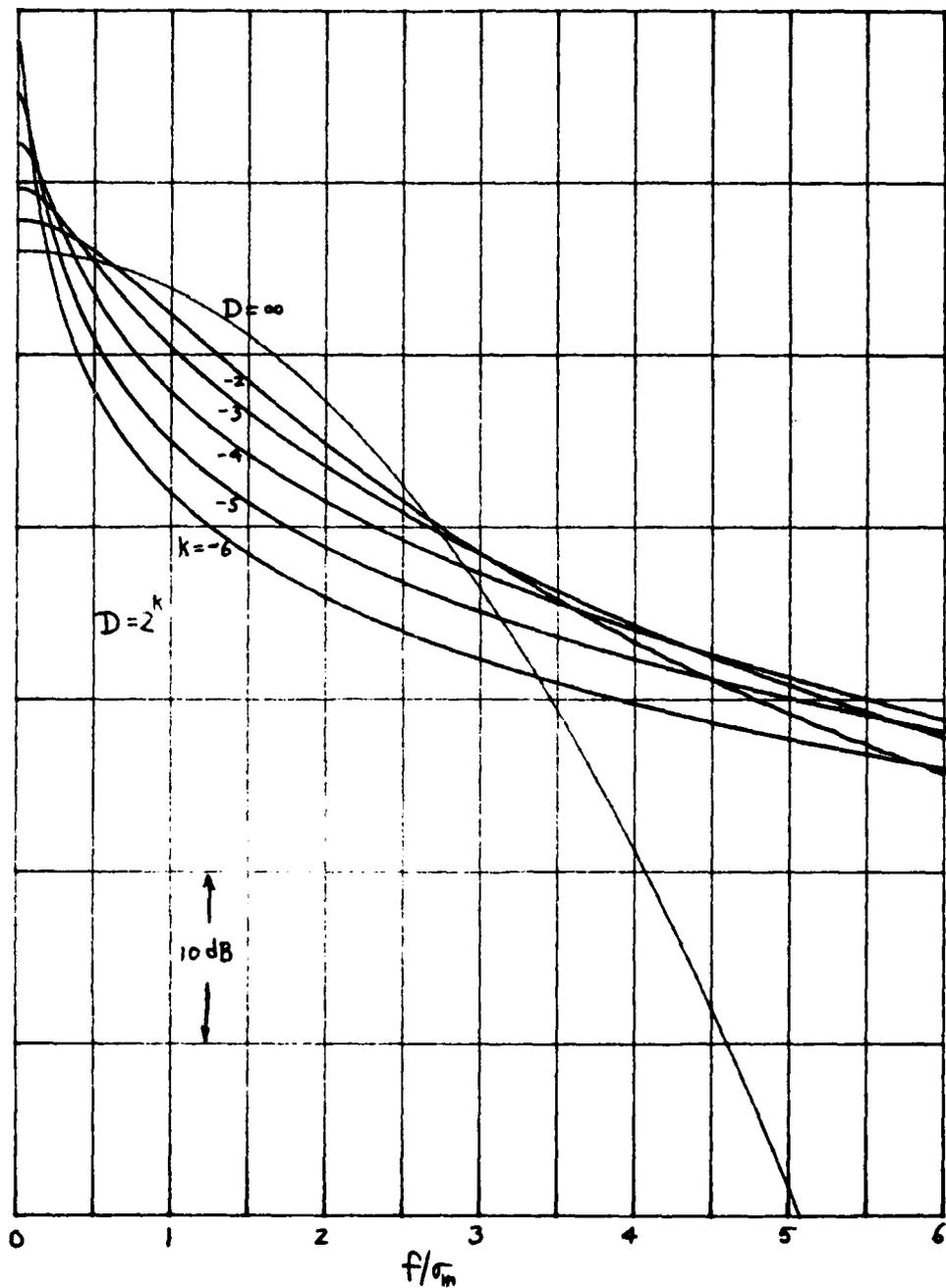
(a) Abscissa normalized with respect to equivalent modulating bandwidth  $W_m$

Figure E-2. Normalized Signal Spectrum; Exponential Correlation for Frequency-Modulating Process  $m(t)$ .



(b) Abscissa normalized with respect to RMS frequency deviation  $\sigma_m$

Figure E-2. Normalized Signal Spectrum; Exponential Correlation for Frequency-Modulating Process  $m(t)$



(c) Abscissa normalized with respect to RMS frequency deviation  $\sigma_m$ ; small  $D$

Figure E-2. Normalized Signal Spectrum; Exponential Correlation for Frequency-Modulating Process  $m(t)$ .

However, the figures clarify that the approach to the narrowband limit is not monotonic at all frequencies. Thus at  $f/\sigma_m = 6$ , for example, the spectral density initially increases as  $D$  decreases, until  $D = 2^4 = 1/16$ , at which point the spectral density decreases with further decreases in  $D$ . In fact, by this time, the normalized spectral density in figure E-2(c) is well approximated by the spectrum

$$\frac{D}{(f/\sigma_m)^2 + \pi^2 D^2}$$

For  $f=0$ , this function increases without limit as  $D$  decreases; for any other  $f$ , it eventually decreases as  $D$  decreases. Hence the bandwidth  $W_s$  of the signal spectrum tends to zero as  $D$  decreases, and the assumption of large  $TW_s$  will be violated.

We will limit consideration to  $D > 2$ , for which the signal spectrum is approximately Gaussian, and the bandwidth on the  $f/\sigma_m$  scale is virtually independent of  $D$  (see figure E-2(b)). That is,  $W_s$  is directly proportional to  $\sigma_m$  for  $D > 2$ . This case also leads to a significant RMS phase deviation of the frequency-modulated signal in the observation interval  $T$ . Once again, this is in keeping with our interest of detecting the most random signal.

Some analytic expressions are available for the origin value and the statistical bandwidth of the signal spectrum  $B(f)$ . We have, from above,

$$\begin{aligned} B(0) &= \frac{A^2}{W_m} \int_0^{\infty} dx \exp[-\pi^2 D^2 (x-1+e^{-x})] \\ &= \frac{A^2}{W_m} \int_0^1 dt t^{\pi^2 D^2 - 1} \exp[-\pi^2 D^2 (t-1)] \\ &= \frac{A^2}{W_m} \frac{1}{\pi^2 D} {}_1F_1(1; 1 + \pi^2 D^2; \pi^2 D^2) \equiv \frac{A^2}{W_m} \mathcal{N}(D), \end{aligned}$$

where we have used Ref. 11, 3.383 1 and 9.212 1. For the normalized plots in figure E-2, the quantity  $\mathcal{N}(D)$  represents the origin value. A short table follows. Notice that the origin value increases without limit as  $D$  approaches zero, but saturates as  $D \rightarrow \infty$ .

$D$	$\mathcal{N}(D)$	$10 \log \mathcal{N}(D)$
small	$(\pi^2 D)^{-1}$	$-10 \log D - 9.943$
$2^{-6}$	6.500	8.129
$2^{-5}$	3.273	5.150
$2^{-4}$	1.682	2.259
$2^{-3}$	.9270	-0.329
$2^{-2}$	.6035	-2.193
$2^{-1}$	.4824	-3.166
1	.4364	-3.601
2	.4167	-3.802
4	.4076	-3.898
8	.4032	-3.945
16	.4011	-3.968
32	.4000	-3.979
$\infty$	$\frac{1}{\sqrt{2\pi}} = .3989$	$-3.991 = -5 \log(2\pi)$

Table E-1. Origin Value of Spectrum

The statistical bandwidth of the signal is

$$\begin{aligned}
 W_s &= \frac{[\int df G_s(f)]^2}{\int df G_s^2(f)} = \frac{R_s^2(0)}{\int dt R_s^2(t)} = \frac{1}{\int dt \rho^2(t)} \\
 &= \left( \int dt \exp[-2\pi^2 D^2 \{2W_m |t| - 1 + \exp(-2W_m |t|)\}] \right)^{-1} \\
 &= \sigma_m \frac{2\pi^2 D}{F_1(1; 1+2\pi^2 D^2; 2\pi^2 D^2)},
 \end{aligned}$$

by a technique similar to that given above. Notice that the statistical bandwidth  $W_s$  decreases to zero as  $D$  decreases (i.e.,  $W_m$  increases for  $\sigma_m$  fixed), as noted above. Also, the bandwidth on the  $f/\sigma_m$  scale saturates as  $D \rightarrow \infty$ . A table of  $W_s/\sigma_m$  follows.

<u>D</u>	<u><math>W_s/\sigma_m</math></u>	<u>D</u>	<u><math>W_s/\sigma_m</math></u>
small	$2\pi^2 D$	1	3.331
$2^{-6}$	.3070	2	3.438
$2^{-5}$	.6553	6	3.509
$2^{-4}$	1.148	10	3.524
$2^{-3}$	1.942	20	3.534
$2^{-2}$	2.679	50	3.541
$2^{-1}$	3.113	$\infty$	$3.545 = 2\sqrt{\pi}$

Table E-2. Bandwidth of Signal Spectrum

A cautionary note is in order here regarding the physical significance of the statistical bandwidth. Consider a unit-power spectrum

$$a_1 \text{rect}\left(\frac{f}{\Delta_1}\right) + a_2 \text{rect}\left(\frac{f}{\Delta_2}\right)$$

where

$$\text{rect}(u) \equiv \begin{cases} 1, & |u| < \frac{1}{2} \\ 0, & |u| > \frac{1}{2} \end{cases}$$

The requirement of unit power means that  $a_1 \Delta_1 + a_2 \Delta_2 = 1$ . Now suppose that

$$a_2 = \Delta_2 = 1 - \exp(-x), \quad a_1 = \exp(-x^2).$$

Then

$$\Delta_1 = \exp(-x^2) [2 \exp(-x) - \exp(-2x)].$$

We then find that

$$\left. \begin{array}{l} a_1 \rightarrow \infty, \Delta_1 \rightarrow 0, a_1 \Delta_1 \rightarrow 0 \\ a_2 \rightarrow 1, \Delta_2 \rightarrow 1, a_2 \Delta_2 \rightarrow 1 \end{array} \right\} \text{as } x \rightarrow \infty.$$

That is, the spectrum tends to a flat one over  $(\frac{1}{2}, \frac{1}{2})$ , with power 1; the power in the narrow component of width  $\Delta_1$  tends to zero. However, the statistical bandwidth is

$$W_s = \frac{(a_1 \Delta_1 + a_2 \Delta_2)^2}{a_1^2 \Delta_1 + a_2^2 \Delta_2} = \left[ e^{x^2} (2e^{-x} - e^{-2x}) + (1 - e^{-x})^3 \right]^{-1} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Thus, this measure of bandwidth tends to zero even though the spectrum approaches a flat one of unit power over  $(-\frac{1}{2}, \frac{1}{2})$ . The situation is no better for the -3 dB bandwidth, for it too approaches zero for this spectrum, in the limit.

Calculation of Spectrum:

The normalized spectrum is

$$\begin{aligned} G(f) &= \int dt \exp(-i2\pi ft) \rho(t) \\ &= \frac{1}{\sigma_m} \int_0^\infty dy \cos(\pi \frac{f}{\sigma_m} y) \exp\left[-\pi^2 D^2 \left\{ \frac{y}{D} - 1 + e^{-y/b} \right\}\right]. \end{aligned}$$

If we add and subtract the asymptotic value,  $\exp[-\pi^2 D^2 \{ \frac{y}{D} - 1 \}]$ , from the integrand, we obtain

$$\sigma_m G(\sigma_m v) = \frac{D e^{\pi^2 D^2}}{v^2 + \pi^2 D^2} + \int_0^\infty dy \cos(\pi v y) g(y)$$

where

$$g(y) = \exp\left[-\pi^2 D^2 \left\{ \frac{y}{D} - 1 \right\}\right] \left\{ \exp\left[-\pi^2 D^2 e^{-y/b}\right] - 1 \right\}.$$

Using Simpson's rule with end correction, and sampling at intervals conducive to FFT processing, we obtain, for the integral above,

$$\Delta \text{Re} \sum_{k=0}^{\infty} w_k \exp(-i2\pi kn/N) \tilde{g}(k\Delta)$$

where

$$\tilde{g}(k\Delta) = g(k\Delta) + \frac{1}{7} \Delta g'(0) \delta_{k0},$$

$$w_k = \frac{1}{15} \{ 7, 16, 14, 16, 14, \dots \}.$$

For large  $D$ , this addition and subtraction procedure is not recommended, because it requires the cancellation of two large quantities to realize the result. Programs for the spectrum calculation are presented in table E-3.

Example 2.

$$R_m(\tau) = \sigma_m^2 \exp(-\pi W_m^2 \tau^2), \quad G_m(f) = \frac{\sigma_m^2}{W_m} \exp(-\pi f^2 / W_m^2)$$

$$G_m\left(\frac{W_m}{2}\right) = .46 G_m(0), \quad G_m(W_m) = .043 G_m(0),$$

$$\sigma^2(\tau) = 4\pi^2 \sqrt{\frac{2}{\pi}} \frac{\sigma_m^2}{W_m^2} \left[ \alpha \left\{ \Phi(\alpha) - \frac{1}{2} \right\} - \frac{1 - \exp(-\alpha^2/2)}{\sqrt{2\pi}} \right]$$

where

$$\alpha \equiv \sqrt{2\pi} W_m \tau, \quad \Phi(x) \equiv \int_{-\infty}^x du \frac{\exp(-u^2/2)}{\sqrt{2\pi}}.$$

The curve for  $\sigma^2(\tau)$  is plotted in figure E-1. We also have

$$\sigma^2(\tau) \sim 4\pi^2 \frac{\sigma_m^2}{W_m} \tau \quad \text{as } \tau \rightarrow +\infty.$$

Since  $\nu = 0$ , there is no carrier.  $B(f)$  would have to be evaluated from (E-20), perhaps via an FFT.

Example 3.

$$R_m(\tau) = \frac{\sigma_m^2}{1 + (\pi W_m \tau)^2}$$

$$G_m(f) = \frac{\sigma_m^2}{W_m} \exp(-2|f|/W_m), \quad G_m\left(\frac{W_m}{2}\right) = .37 G_m(0), \quad G_m(W_m) = .14 G_m(0)$$

$$\sigma^2(\tau) = 8 \frac{\sigma_m^2}{W_m^2} \left[ \alpha \arctan(\alpha) - \frac{1}{2} \ln(1 + \alpha^2) \right],$$

where  $\alpha \equiv \pi W_m \tau$ . The curve for  $\sigma^2(\tau)$  is plotted in figure E-1. Also

$$\sigma^2(\tau) \sim 4\pi^2 \frac{\sigma_m^2}{W_m} \tau \quad \text{as } \tau \rightarrow +\infty.$$

There is no carrier.

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```
10 DIM X(1:2048),Y(1:2048),C(1:513)
20 N=2048      NK=2048      N=2^INTEGER
30 Del=.01
40 PLOTTER IS "9872A"
50 PLOTTER IS "GRAPHICS"
60 GRAPHICS
70 SCALE 0,12,-7,0
80 GRID 1,1
90 PENUP
100 S=2*(N*Del)
110 FOR J=-1 TO 3
120 D=2^J
130 F=LGT(SQR(2+PI)*D)
140 T=.5/D^2
150 FOR I=0 TO N/2
160 U=I*S
170 Y=LGT(EXP(-T*U^2))-F
180 PLOT U,Y
190 IF Y<-7 THEN 220
200 IF U>12 THEN 220
210 NEXT I
220 PENUP
230 NEXT J
240 FOR J=-1 TO 3
250 D=2^J
260 MAT X=ZER
270 MAT Y=ZER
280 S=(PI+D)^2
290 X(1)=.5
300 FOR K=1 TO N-1
310 P=K*Del
320 X(K+1)=EXP(-S*(P-1+EXP(-P)))
330 IF X(K+1)<1E-20 THEN 350
340 NEXT K
```

FFT Procedure in here.

```
1050 F=LGT(Del)
1060 S=2*(N*Del)
1070 FOR I4=1 TO N/2+1
1080 U=S*(I4-1)
1090 PLOT U,LGT(X(I4))+F
1100 IF U>12 THEN 1120
1110 NEXT I4
1120 PENUP
1130 NEXT J
1140 END
```

(a) Program for Figure E-2(a)  
Table E-3. Programs for Spectrum Calculation

```

10 GRAPHICS
20 PLOTTER IS "9872A"
30 Vm=6
40 SCALE 0,Vm,-6,1
50 GRID .5,1
60 PENUP
70 DIM X(1:2048),Y(1:2048),C(1:513)
80 N=2048 ! N<=2048 N=2^INTEGER
90 COM S
100 F=.5*LG(2*PI)
110 FOR V=0 TO Vm STEP .05
120 Y=LG(EXP(-.5*V^2))-F
130 PLOT V,Y
140 NEXT V
150 PENUP
160 FOR Id=-2 TO -6 STEP -1
170 D=2^Id
180 Del=.02*D
190 MAT X=ZER
200 MAT Y=ZER
210 S=(PI*D)^2
220 F=Del/D
230 FOR K=0 TO N-1
240 X(K+1)=FNG(K*F)
250 NEXT K
260 OUTPUT 0;"D =";D;"Del =";Del;X(1);X(2);X(3)
270 OUTPUT 0;K;X(K-2);X(K-1);X(K)
280 X(1)=(7*X(1)+Del*PI+2*D*EXP(S))*.15
290 Q=16*.15
300 FOR K=2 TO N STEP 2
310 X(K)=X(K)*Q
320 NEXT K
330 Q=14*.15
340 FOR K=3 TO N-1 STEP 2
350 X(K)=X(K)*Q
360 NEXT K

```

FFT Procedure in here.

```

1070 T=D*EXP(S)
1080 Del=2^Id*Del
1090 FOR V=0 TO Vm STEP .01
1100 V1=T*(V+2+3)
1110 F1=V1*Del
1120 T=INT(F1)
1130 F1=F1-T
1140 V2=X(1+1)*.1-F1+(X(1+2)+2)*F1
1150 PLOT V, LGT(V1+Del*V2)
1160 NEXT V
1170 PENUP
1180 NEXT Id
1190 END
1200 DEF FNG(P)
1210 COM S
1220 X=S*EXP(-P)
1230 T=1
1240 G=0
1250 FOR N=1 TO 1000
1260 T=-T*X/N
1270 G=G+T
1280 IF ABS(T)<(ABS(G)+1E-12) THEN 1300
1290 NEXT N
1300 OUTPUT 0;"1000 TERMS AT";P
1310 PAUSE
1320 RETURN EXP(-S*(P-1))*G
1330 FNGND

```

(b) Program for Figure E-2(c)  
 Table E-3. Programs for Spectrum Calculation

Example 4.

$$R_m(\tau) = \sigma_m^2 \frac{1 - (\pi W_m \tau)^2}{[1 + (\pi W_m \tau)^2]^2}$$

$$G_m(f) = 2 \frac{\sigma_m^2}{W_m} \frac{|f|}{W_m} \exp(-2|f|/W_m).$$

This spectrum goes to zero linearly at  $f=0$ , i.e.  $\nu=1$  in (E-15). It peaks at  $f=.5W_m$ ; we have

$$G_m(W_m) = .74 G_m(.5W_m), \quad G_m(1.5W_m) = .41 G_m(.5W_m),$$

$$\sigma^2(\tau) = 4 \frac{\sigma_m^2}{W_m^2} \ln [1 + (\pi W_m \tau)^2],$$

by use of Ref. 11, 3.943. Also

$$\sigma^2(\tau) \sim 8 \frac{\sigma_m^2}{W_m^2} \ln \tau \quad \text{as } \tau \rightarrow +\infty,$$

in keeping with (E-16). Thus there is no carrier.  $B(f)$  follows from (E-20) as

$$B(f) = \frac{2}{\pi} \frac{A^2}{W_m} \int_0^{\infty} dx \cos\left(2 \frac{f}{W_m} x\right) \exp\left[-2 \frac{\sigma_m^2}{W_m^2} \ln(1+x^2)\right].$$

The previous examples all had  $\sigma^2(f) = \infty$ , and hence no carrier. The next two examples have a carrier, since  $G_m(f) \propto f^2$  as  $f \rightarrow 0$ .

Example 5.

$$R_m(\tau) = \sigma_m^2 (1 - 2\pi W_m^2 \tau^2) \exp(-\pi W_m^2 \tau^2)$$

$$G_m(f) = 2\pi \frac{\sigma_m^2}{W_m} \frac{f^2}{W_m^2} \exp(-\pi f^2/W_m^2).$$

The peak of  $G_m(f)$  occurs at  $f = W_m/\sqrt{\pi} = .56 W_m$ ; then we have

$$G_m(W_m) = .37 G_m(.56 W_m), \quad G_m(1.5 W_m) = .016 G_m(.56 W_m),$$

$$\sigma^2(\tau) = 4\pi \frac{\sigma_m^2}{W_m^2} \left[ 1 - \exp(-\pi W_m^2 \tau^2) \right]$$

$$\sigma^2(\infty) = 4\pi \frac{\sigma_m^2}{W_m^2} \neq \infty.$$

Thus there is a carrier. Define  $\alpha = 2\pi \frac{\sigma_m^2}{W_m^2}$ . Then

$$\begin{aligned} B(f) &= 2 \frac{A^2}{W_m} \int_0^\infty dx \cos(2\pi \frac{f}{W_m} x) \left[ \exp\{-\alpha[1 - e^{-\pi x^2}]\} - \exp\{-\alpha\} \right] \\ &= \frac{A^2}{W_m} e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n! \sqrt{n!}} \exp\left(-\frac{\pi f^2}{n W_m^2}\right). \end{aligned}$$

The total signal spectrum is available from (E-19) as

$$G_s(f) = A^2 e^{-\alpha} \delta(f) + B(f).$$

Evaluation of  $B(f)$  directly via an FFT appears fruitful; we have rapid convergence of the integrand since

$$\exp\{\alpha e^{-\pi x^2}\} - 1 \sim \alpha e^{-\pi x^2} \quad \text{as } x \rightarrow \infty.$$

Example 6.

$$R_m(\tau) = \sigma_m^2 \frac{1 - 3(\pi W_m \tau)^2}{[1 + (\pi W_m \tau)^2]^3}$$

$$G_m(f) = 2 \frac{\sigma_m^2}{W_m} \frac{f^2}{W_m^2} \exp(-2|f|/W_m).$$

This spectrum peaks at  $f = W_m$ . We find

$$G_m(2W_m) = .54 G_m(W_m), \quad G_m(3W_m) = .16 G_m(W_m)$$

$$\sigma^2(\tau) = 4\pi^2 \sigma_m^2 \frac{\tau^2}{1 + (\pi W_m \tau)^2}$$

$$\sigma^2(\infty) = 4 \frac{\sigma_m^2}{W_m^2} \neq \infty.$$

Thus there is a carrier. Define  $\alpha = 2 \frac{\sigma_m^2}{W_m}$ . Then

$$B(f) = \frac{2}{\pi} \frac{A^2}{W_m} \int_0^{\infty} dx \cos\left(2 \frac{f}{W_m} x\right) \left[ \exp\left\{-\alpha \frac{x^2}{1+x^2}\right\} - \exp\{-\alpha\} \right]$$

$$= \frac{2}{\pi} \frac{A^2}{W_m} e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \int_0^{\infty} dx \cos\left(2 \frac{f}{W_m} x\right) \frac{1}{(1+x^2)^k}.$$

Direct numerical evaluation of  $B(f)$  via an FFT could take advantage of the following:

$$\exp\left\{-\alpha \frac{x^2}{1+x^2}\right\} - \exp\{-\alpha\} = e^{-\alpha} \left[ \frac{\alpha}{1+x^2} + \frac{\alpha^2/2}{(1+x^2)^2} \right. \\ \left. + \left\{ \exp\left(\frac{\alpha}{1+x^2}\right) - 1 - \frac{\alpha}{1+x^2} - \frac{\alpha^2/2}{(1+x^2)^2} \right\} \right],$$

and the bracketed term in the last line is  $O(x^{-6})$  as  $x \rightarrow +\infty$ .

#### Validity of Approximation (E-33)

In order to determine the accuracy of (E-33), we define the two dimensionless parameters

$$x = 2\pi\sigma_m\tau, \quad D = \frac{\sigma_m}{W_m}. \quad (\text{E-35})$$

Then we wish to compare correlations

$$\exp\left[-\frac{1}{2}\sigma^2\left(\frac{x}{2\pi\sigma_m}\right)\right] \quad \text{and} \quad \exp\left[-\frac{1}{2}x^2\right], \quad (\text{E-36})$$

for different values of  $D$ . In this normalized form, examples 1-3 take the following respective forms for  $\sigma^2\left(\frac{x}{2\pi\sigma_m}\right)$ :

$$4\pi^2 D^2 \frac{\beta x - 1 + e^{-\beta x}}{2} \quad \text{where} \quad \beta = \frac{1}{\pi D}; \quad (\text{E-37A})$$

$$4\pi^2 \sqrt{\frac{2}{\pi}} D^2 \left[ \beta x \left\{ \Phi(\beta x) - \frac{1}{2} \right\} - \frac{1 - e^{-\beta^2 x^2/2}}{\sqrt{2\pi}} \right] \quad \text{where} \quad \beta = \frac{1}{\sqrt{2\pi} D}; \quad (\text{E-37B})$$

$$8 D^2 \left[ \beta x \arctan(\beta x) - \frac{1}{2} \ln(1 + \beta^2 x^2) \right] \quad \text{where} \quad \beta = \frac{1}{2D}. \quad (\text{E-37C})$$

Asymptotically, these results take the forms

$$\left. \begin{array}{l} x^2 \left(1 - \frac{x}{3\pi D}\right) \text{ as } x \rightarrow 0+ \\ 2\pi D x - 2\pi^2 D^2 \text{ as } x \rightarrow +\infty \end{array} \right\}; \quad (\text{E-38A})$$

$$\left. \begin{array}{l} x^2 \left(1 - \frac{x^2}{24\pi D^2}\right) \text{ as } x \rightarrow 0+ \\ 2\pi D x - 4\pi D^2 \text{ as } x \rightarrow +\infty \end{array} \right\}; \quad (\text{E-38B})$$

$$\left. \begin{array}{l} x^2 \left(1 - \frac{x^2}{24D^2}\right) \text{ as } x \rightarrow 0+ \\ 2\pi D x - 8D^2 \ln\left(\frac{x}{2D}\right) - 8D^2 \text{ as } x \rightarrow +\infty \end{array} \right\}. \quad (\text{E-38C})$$

Equation (E-36), with each of the examples in (E-37) substituted, is plotted in figures E-3 through E-5, along with a good-fitting Gaussian approximation. We notice that as  $D$  increases, the good-fitting Gaussian curve approaches  $\exp(-x^2/2)$ . Also the approximation of (E-33) is excellent for  $D > 2$ .

Thus when the RMS frequency deviation of the modulating process is somewhat greater than the equivalent bandwidth of the modulating process, the correlation of the FM signal is approximately Gaussian. See also Ref. 10, Ch. 14.

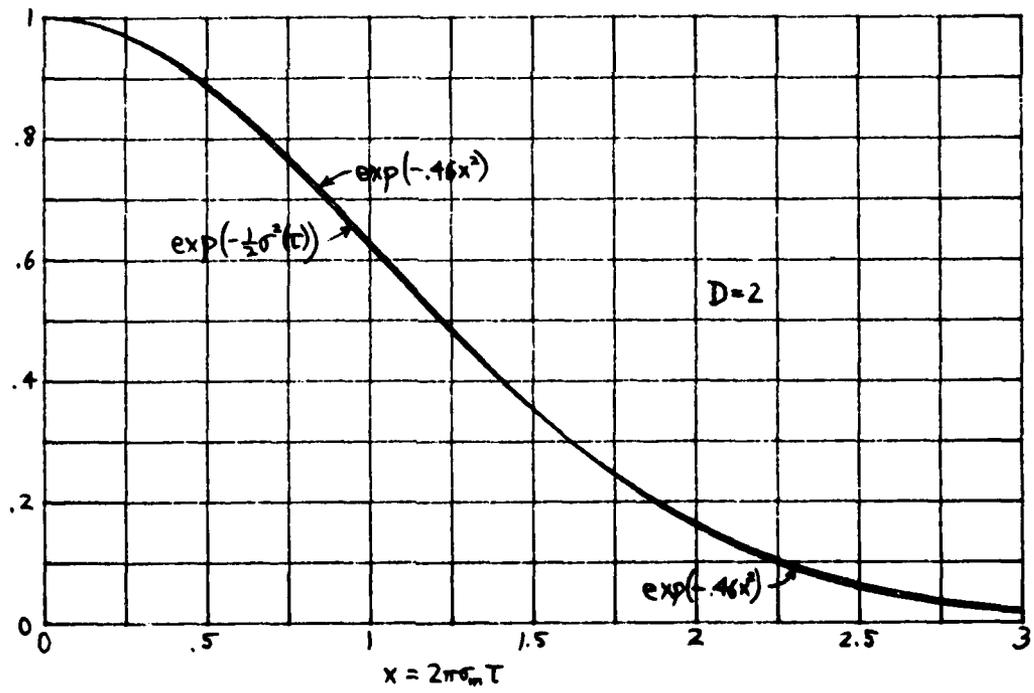
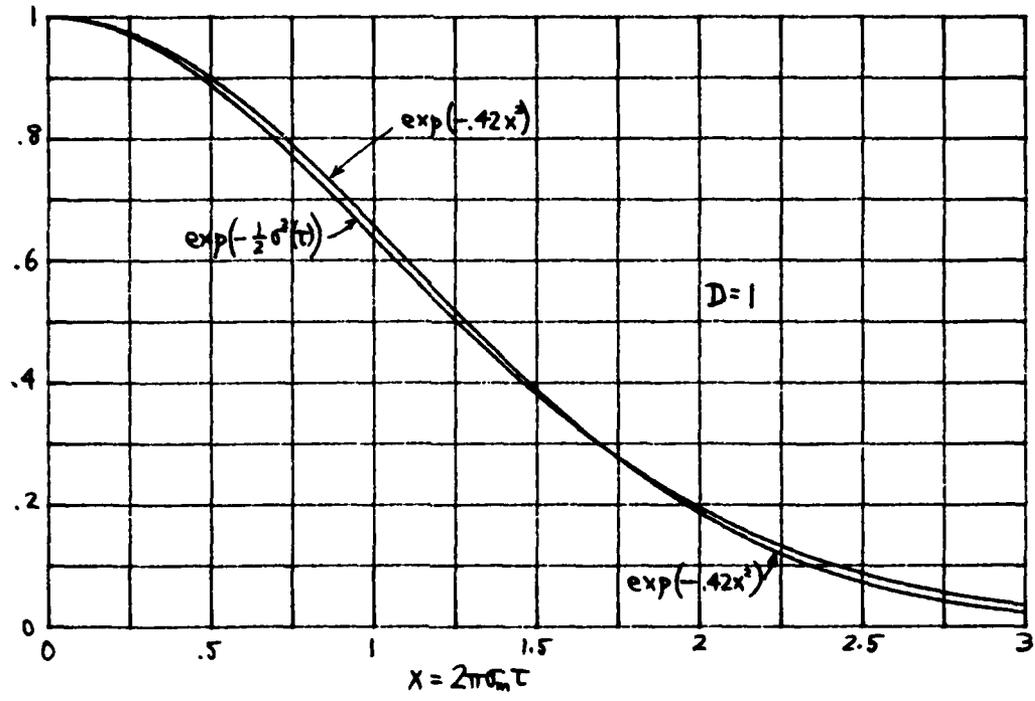


Figure E-3. Correlations for Example 1

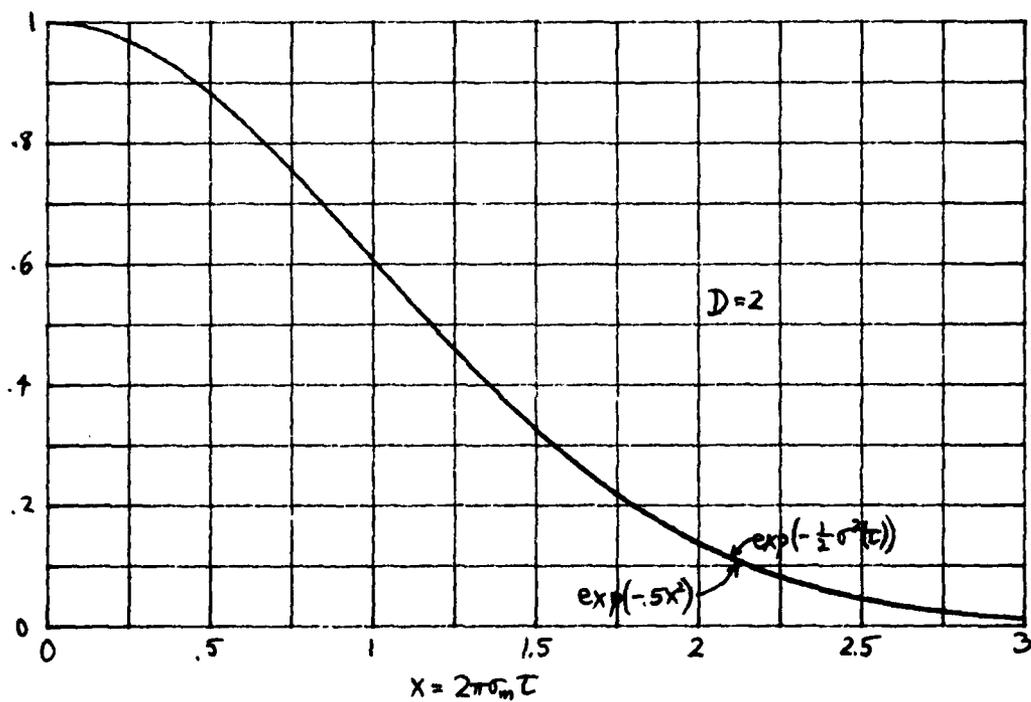
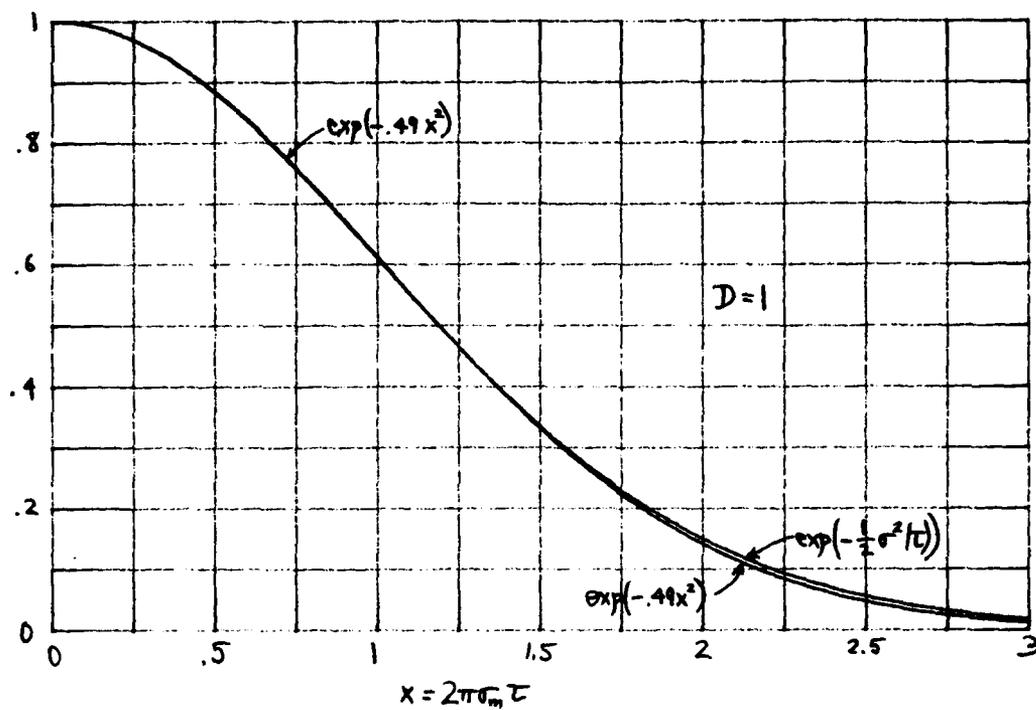


Figure E-4. Correlations for Example 2

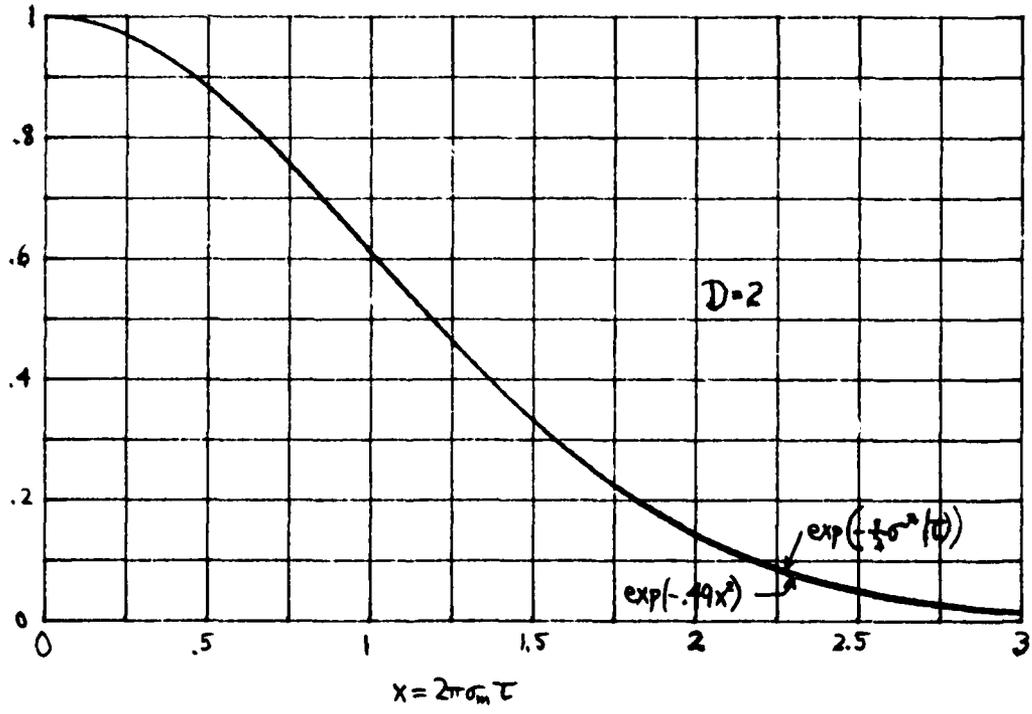
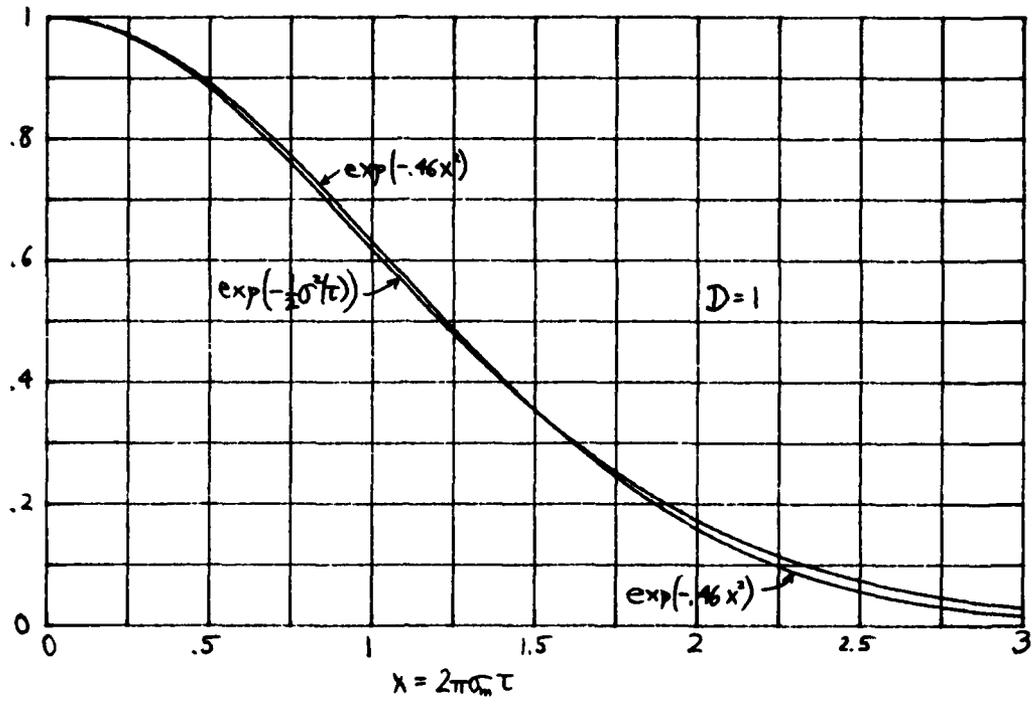


Figure E-5. Correlations for Example 3

APPENDIX F. EVALUATION OF  $I_{62}$

We wish to evaluate  $I_{62}$ , as given by (110). To this aim, we let  $u_k = t_k - (t_a + t_b)/2$ , and employ (113) to express, for real  $\rho$ ,

$$I_{62} = \iiint \int du_1 du_2 du_3 du_4 r(u_1) r(u_2) r(u_3) r(u_4) \rho^2(u_1 - u_2) \rho^2(u_3 - u_4) \frac{\rho(u_1 - u_4) \rho(u_2 - u_3)}{\rho(u_1 - u_3) \rho(u_2 - u_4)}. \quad (F-1)$$

Approximate Evaluation

Before we evaluate this quantity, we first approximate it by using the Gaussian replacement-approximation mentioned in (105)-(106). (A more thorough treatment is given in (F-42) et seq.) Then  $I_{62}$  is replaced by

$$\begin{aligned} I_{62} &\rightarrow \iiint \int du_1 du_2 du_3 du_4 r(u_1) r(u_2) r(u_3) r(u_4) \rho(u_1 - u_2) \rho(u_3 - u_4) [\rho(u_1 - u_4) \rho(u_2 - u_3) + \rho(u_1 - u_3) \rho(u_2 - u_4)] \\ &= \left[ \iint du_1 du_2 r(u_1) r(u_2) \rho^2(u_1 - u_2) \right]^2 + \iiint \int du_1 du_2 du_3 du_4 r(u_1) r(u_2) r(u_3) r(u_4) \rho(u_1 - u_2) \rho(u_2 - u_3) \rho(u_3 - u_4) \rho(u_4 - u_1) \\ &= I_2^2 + I_4. \end{aligned} \quad (F-2)$$

Letting  $u_k = u_4 + v_k$  for  $k=1,2,3$ , the second term in (F-2) becomes

$$I_4 = \iiint \int dv_1 dv_2 dv_3 \phi_4(v_1, v_2, v_3) \rho(v_1 - v_2) \rho(v_2 - v_3) \rho(v_3) \rho(v_1), \quad (F-3)$$

where

$$\phi_4(v_1, v_2, v_3) \equiv \int du_4 r(u_4) r(u_4 + v_1) r(u_4 + v_2) r(u_4 + v_3). \quad (F-4)$$

For large  $TW_s$ , (F-3) becomes

$$I_4 \cong \phi_4(0,0,0) \iiint \int dv_1 dv_2 dv_3 \rho(v_1 - v_2) \rho(v_2 - v_3) \rho(v_3) \rho(v_1). \quad (F-5)$$

And we have, from (F-4) and (113),  $\phi_4(0,0,0) = 1/T^3$ . Furthermore, making the Gaussian assumption (117) for the signal correlation  $\rho$ , (F-5) becomes,

with the aid of (125)\*,

$$I_4 = \frac{1}{T^3} \iiint dv_1 dv_2 dv_3 \exp\left[-\frac{\pi}{2} W_s^2 \{2v_1^2 + 2v_2^2 + 2v_3^2 - 2v_1 v_2 - 2v_2 v_3\}\right] = \frac{\sqrt{2}}{(TW_s)^3} \quad (F-6)$$

Employing this result in (F-2), we have

$$I_{62} \rightarrow I_2^2 + \frac{\sqrt{2}}{(TW_s)^3}; \quad (F-7)$$

i.e., the replacement approximation leads to

$$I_{62} - I_2^2 \rightarrow I_4 = \frac{\sqrt{2}}{(TW_s)^3} \quad (F-8)$$

How good this replacement is, quantitatively, will be indicated below.

#### Exact Evaluation for Large $TW_s$

Returning to the exact expression (F-1), let  $x = u_1 - u_2$ ,  $y = \frac{1}{2}(u_1 + u_2)$ ,  $v = u_3 - u_4$ ,  $w = \frac{1}{2}(u_3 + u_4)$ , and then eliminate  $y$  via  $y = w + \tau$ ; there results

$$I_{62} = \iint dx dv \rho^2(x) \rho^2(v) \int d\tau \hat{\phi}_4(\tau, x, v) \frac{\rho(\tau - \frac{x+v}{2}) \rho(\tau + \frac{x+v}{2})}{\rho(\tau - \frac{x-v}{2}) \rho(\tau + \frac{x-v}{2})}, \quad (F-9)$$

where  $\hat{\phi}_4$  is a shifted version of  $\phi_4$  given in (F-4). Now as  $\tau \rightarrow \pm\infty$ , the ratio of four  $\rho$ -functions in (F-9) tends to 1. Therefore we add and subtract 1 from this ratio; the added 1 leads to the quantity

$$\iint dx dv \rho^2(x) \rho^2(v) \int d\tau \hat{\phi}_4(\tau, x, v) = \left[ \int dx \rho^2(x) \phi_2(x) \right]^2 = I_2^2, \quad (F-10)$$

where we employed (115) and (114). The remaining quantity is

\* More generally, we find, by use of (117) and (125), that  $I_p = (2^{r/p}/p)^{1/p} (TW_s)^{1-p}$ ; this includes (118), (124), and (F-6) as special cases. When the signal correlation is not assumed Gaussian,  $I_p$  depends on the exact frequency-modulation statistics; an example is presented at the end of this appendix.

$$\begin{aligned}
I_{62} - I_2^2 &= \iint dx dv \rho^2(x) \rho^2(v) \int dt \hat{\phi}_4(t, x, v) [g(t, x, v) - 1] \\
&\cong \hat{\phi}_4(0, 0, 0) \iint dx dv \rho^2(x) \rho^2(v) \int dt [g(t, x, v) - 1] \\
&= \frac{1}{T^3} \iint dx dv \rho^2(x) \rho^2(v) \int dt [g(t, x, v) - 1], \tag{F-11}
\end{aligned}$$

where

$$g(t, x, v) \equiv \frac{\rho\left(\tau - \frac{x+v}{2}\right) \rho\left(\tau + \frac{x+v}{2}\right)}{\rho\left(\tau - \frac{x-v}{2}\right) \rho\left(\tau + \frac{x-v}{2}\right)}. \tag{F-12}$$

The particular value of (F-11) depends on the exact signal correlation  $\rho$  considered. We will consider Gaussian frequency modulation with an exponential correlation, as an example. Then from appendix E, example 1, we have

$$\rho(\tau) = \exp\left[-\pi^2 D^2 \left\{ \frac{2\sigma_m |\tau|}{D} - 1 + \exp\left(-\frac{2\sigma_m |\tau|}{D}\right) \right\}\right], \tag{F-13}$$

where  $D = \sigma_m/W_m$  is the ratio of RMS frequency deviation to the equivalent bandwidth of the modulating process. For large  $D$ , we have the approximation to (F-13)

$$\rho_a(\tau) \equiv \exp\left[-2\pi^2 \sigma_m^2 \tau^2\right]. \tag{F-14}$$

In order to make this limit agree with the Gaussian approximation (117), we set  $2\sqrt{\pi} \sigma_m = W_s$ ; we then eliminate  $\sigma_m$  from (F-13) to obtain

$$\rho(\tau) = \exp\left[-\pi^2 D^2 \left\{ \frac{W_s |\tau|}{\sqrt{\pi} D} - 1 + \exp\left(-\frac{W_s |\tau|}{\sqrt{\pi} D}\right) \right\}\right]. \tag{F-15}$$

(Strictly, this expression, in terms of  $W_s$ , holds only for large  $D$ ; however, the dependence on  $D$  is rather weak, as will be seen below, so we use (F-15) for moderate  $D$  also.)

Now we can say that

$$\rho\left(\frac{x}{\sqrt{\pi}W_s}\right) = \exp\left[-\pi^2 D^2 \left\{\frac{|x|}{\pi D} - 1 + \exp\left(-\frac{|x|}{\pi D}\right)\right\}\right] \equiv \rho_D(x). \quad (\text{F-16})$$

The reason for this scale change in (F-16) is that we can now write the convenient normalization

$$\rho_D(x) \approx \exp\left(-\frac{x^2}{2}\right) \text{ for large } D. \quad (\text{F-17})$$

Plots of  $\rho_D(x)$  are given in figure F-1. It is seen to be weakly dependent on  $D$ . Also, when we make the substitutions  $x \rightarrow \frac{x}{\sqrt{\pi}W_s}$ ,  $v \rightarrow \frac{v}{\sqrt{\pi}W_s}$ ,  $\tau \rightarrow \frac{\tau}{\sqrt{\pi}W_s}$ , in (F-11), we have the clear dependence on  $\tau W_s$ :

$$I_{62} - I_2^2 = \frac{I_D^{(b)}}{(\tau W_s)^3}, \quad (\text{F-18})$$

where

$$I_D^{(b)} = \frac{1}{\pi^{3/2}} \iint dx dv \rho_D^2(x) \rho_D^2(v) K(x, v) \quad (\text{F-19})$$

is a function of only the one parameter  $D$ , and

$$K(x, v) \equiv \int d\tau \left[ g_D(\tau, x, v) - 1 \right], \quad (\text{F-20})$$

$$g_D(\tau, x, v) \equiv \frac{\rho_D\left(\tau - \frac{x+v}{2}\right) \rho_D\left(\tau + \frac{x+v}{2}\right)}{\rho_D\left(\tau - \frac{x-v}{2}\right) \rho_D\left(\tau + \frac{x-v}{2}\right)}. \quad (\text{F-21})$$

We would like to determine the exact dependence of  $I_D^{(b)}$  on  $D$  and see if it is of the order of 1, as suggested by (F-8). It should be observed that substitution of the limiting form (F-17) directly in (F-21) yields  $g_D(\tau, x, v) = \exp(xv)$  and hence a divergent integral on  $\tau$  in (F-20). Thus, the exact dependence of (F-16) (or whatever example is being considered) must be preserved for accurate evaluation of (F-19); this is in contrast to the evaluation

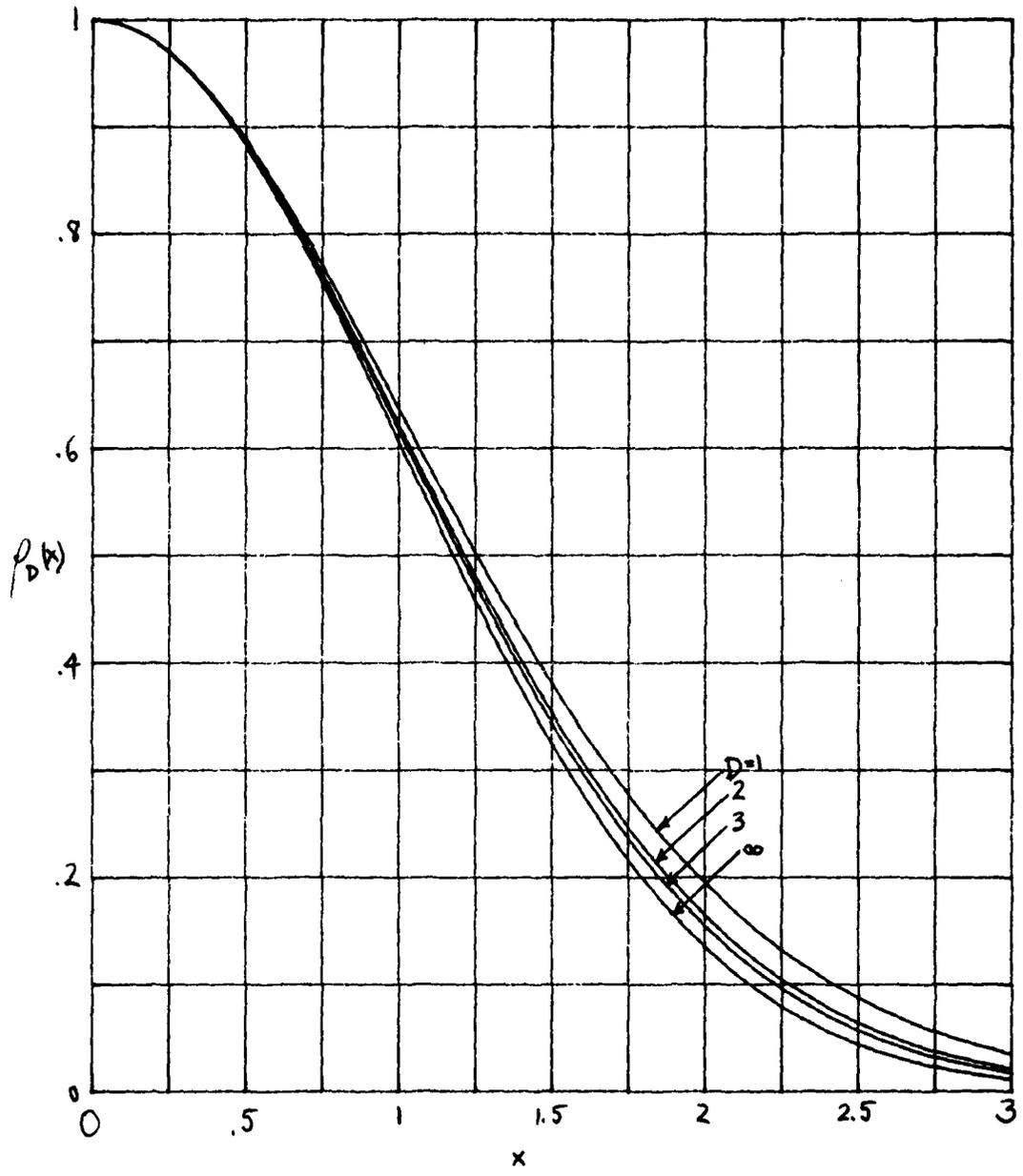


Figure F-1. Normalized Signal Correlation  $\rho_D(x)$ , for Exponential Correlation of Frequency-Modulation

of  $I_2$  and  $I_3$ , where the Gaussian approximation (F-17) was applicable; see the end of this appendix. The key reason for the dichotomy is the presence of  $\rho$ -functions in the denominator of (F-1), instead of only in the numerator. When a  $\rho$ -function appears in the denominator, its exact rate of approach to zero is of paramount importance, whereas if it is in the numerator, its rate of approach to zero is far less critical. Then when we evaluate any  $I_n$  for  $q > 0$ , it is very example-dependent and must be handled very carefully.

#### Numerical Considerations

Since (F-19) must be evaluated numerically, it behooves us to utilize any symmetries involved; these symmetries will be derived for a general signal correlation. First, since  $\rho_D$  is even,

$$g_D(-\tau, x, v) = g_D(\tau, x, v); \quad K(x, v) = 2 \int_0^{\infty} dt [g_D(\tau, x, v) - 1]. \quad (F-22)$$

Second,

$$\begin{aligned} g_D(\tau, -x, -v) &= g_D(\tau, x, v); \quad K(-x, -v) = K(x, v); \\ I_D^{(1)} &= \frac{2}{\pi^{3/2}} \int_0^{\infty} dx \int_{-\infty}^{+\infty} dv \rho_D^2(x) \rho_D^2(v) K(x, v) \\ &= \frac{2}{\pi^{3/2}} \int_0^{\infty} dx \int_0^{\infty} dv \rho_D^2(x) \rho_D^2(v) [K(x, v) + K(x, -v)]. \end{aligned} \quad (F-23)$$

Third,

$$\begin{aligned} g_D(\tau, x, -v) &= \frac{1}{g_D(\tau, x, v)}; \quad K(x, -v) = 2 \int_0^{\infty} dt \left[ \frac{1}{g_D(\tau, x, v)} - 1 \right]; \\ I_D^{(1)} &= \frac{4}{\pi^{3/2}} \int_0^{\infty} dx \int_0^{\infty} dv \rho_D^2(x) \rho_D^2(v) \int_0^{\infty} dt \left[ g_D(\tau, x, v) + \frac{1}{g_D(\tau, x, v)} - 2 \right]. \end{aligned} \quad (F-24)$$

Fourth,

$$\begin{aligned} g_D(\tau, x, v) &= g_D(\tau, v, x); \quad K(x, v) = K(v, x); \\ I_D^{(1)} &= \frac{8}{\pi^{3/2}} \int_0^{\infty} dx \rho_D^2(x) \int_0^x dv \rho_D^2(v) \int_0^{\infty} dt \left[ g_D(\tau, x, v) + \frac{1}{g_D(\tau, x, v)} - 2 \right]. \end{aligned} \quad (F-25)$$

Now let  $x = 2u, v = 2y$  in (F-25). There follows

$$I_D^{(1)} = \frac{32}{\pi^{3/2}} \int_0^{\infty} du \rho_D^2(2u) \int_0^u dy \rho_D^2(2y) \int_0^{\infty} dt [R + 1/R - 2], \quad (F-26)$$

where

$$R \equiv \frac{\rho_D(\tau-u-y) \rho_D(\tau+u+y)}{\rho_D(\tau-u+y) \rho_D(\tau+u-y)} \quad (F-27)$$

(F-26) is a general result, applicable to any signal correlation example, where  $\rho_D(x) \equiv \rho\left(\frac{x}{\sqrt{W_x W_y}}\right)$ .

Let the result of the integration on  $\tau$  in (F-26) be denoted by  $A(u, y)$ , and let the result of the integration on  $y$  be denoted by  $B(u)$ . Then

$$I_D^{(1)} = \frac{32}{\pi^{3/2}} \int_0^\infty du \rho_D^2(2u) B(u) \equiv \frac{32}{\pi^{3/2}} \Delta \sum_{k=0}^K w_k^{(1)} \rho_D^2(2k\Delta) B(k\Delta), \quad (F-28)$$

where  $\{w_k^{(1)}\}$  is a set of integration weights (e.g., Trapezoidal or Simpson), and  $K\Delta$  must be taken large enough to include all relevant contributions to the integral. Similarly,

$$B(k\Delta) = \int_0^{k\Delta} dy \rho_D^2(2y) A(k\Delta, y) \equiv \Delta \sum_{l=0}^k w_l^{(2)} \rho_D^2(2l\Delta) A(k\Delta, l\Delta), \quad (F-29)$$

and

$$A(k\Delta, l\Delta) \equiv \Delta \sum_{m=0}^M w_m^{(3)} [R_\Delta + 1/R_\Delta - 2], \quad (F-30)$$

where

$$R_\Delta = \frac{\rho_D((m-k-l)\Delta) \rho_D((m+k+l)\Delta)}{\rho_D((m-k+l)\Delta) \rho_D((m+k-l)\Delta)}. \quad (F-31)$$

In practice, evaluation of (F-30) for the example of (F-16) turned out to be exceedingly slow, due to the slow rate of decay of the integrand. Accordingly, for this particular example, a special modification was employed; we express

$$A(u, y) = \left\{ \int_0^{u+y} dt + \int_{u+y}^{\infty} dt \right\} [R + 1/R - 2] \equiv A_1 + A_2. \quad (\text{F-32})$$

Now for  $\tau \geq u+y$ , the example of (F-16) yields

$$R = \exp\left[-F \exp\left(-\frac{\tau}{\pi D}\right)\right], \quad F = 4\pi^2 D^2 \sinh\left(\frac{u}{\pi D}\right) \sinh\left(\frac{y}{\pi D}\right). \quad (\text{F-33})$$

In this case, then

$$A_2 = \int_{u+y}^{\infty} dt \left\{ \exp\left[-F \exp\left(-\frac{\tau}{\pi D}\right)\right] + \exp\left[F \exp\left(-\frac{\tau}{\pi D}\right)\right] - 2 \right\}. \quad (\text{F-34})$$

Letting  $x = F \exp\left(-\frac{\tau}{\pi D}\right)$ , and employing Ref. 12, (5.2.4),

$$A_2 = \pi D \int_0^G \frac{dx}{x} (e^x + e^{-x} - 2) = 2\pi D \left\{ \text{Chi}(G) - \ln G - \gamma \right\}, \quad (\text{F-35})$$

where

$$G = \pi^2 D^2 \left(1 - \exp\left(-\frac{2u}{\pi D}\right)\right) \left(1 - \exp\left(-\frac{2y}{\pi D}\right)\right). \quad (\text{F-36})$$

By use of Ref. 12, (5.2.18) and (5.1.11), an alternative form of  $A_2$  is given in terms of the exponential integral:

$$\begin{aligned} A_2 &= -\pi D \left\{ E_1(G) + E_1(-G) + i\pi + 2 \ln G + 2\gamma \right\} \\ &= -\pi D \left\{ E_1(G) + \text{Re } E_1(-G) + 2 \ln G + 2\gamma \right\}. \end{aligned} \quad (\text{F-37})$$

This latter form is the one utilized here.

Hence (F-32) becomes

$$A(u, y) = \int_0^{u+y} dt [R+1/R-2] - \pi D \{E_1(G) + \operatorname{Re} E_1(-G) + 2 \ln G + 2\gamma\}, \quad (\text{F-38})$$

and (F30) is replaced by

$$A(k\Delta, l\Delta) = \Delta \sum_{m=0}^{k+l} w_m^{(b)} [R_0 + 1/R_0 - 2] - \pi D \{E_1(G_0) + \operatorname{Re} E_1(-G_0) + 2 \ln G_0 + 2\gamma\}, \quad (\text{F-39})$$

where

$$G_0 = \pi^2 D^2 \left(1 - \exp\left(-\frac{2k\Delta}{\pi D}\right)\right) \left(1 - \exp\left(-\frac{2l\Delta}{\pi D}\right)\right). \quad (\text{F-40})$$

Utilizing  $A(0, y) = 0$ , and  $A(u, 0) = 0$ , we can combine (F-28), (F-29), and (F-39), to get

$$I_D^{(1)} = \frac{32}{\pi^2} \Delta^3 \sum_{k=1}^K w_k^{(b)} \rho_D^2(2k\Delta) \sum_{l=1}^k w_l^{(b)} \rho_D^2(2l\Delta) \left( \sum_{m=0}^{k+l} w_m^{(b)} [R_0 + 1/R_0 - 2] - \frac{\pi D}{\Delta} \{E_1(G_0) + \operatorname{Re} E_1(-G_0) + 2 \ln G_0 + 2\gamma\} \right). \quad (\text{F-41})$$

There are only two parameters to choose in (F-41),  $\Delta$  and  $K$ . For a given  $\Delta$ , we compute for as large  $K$  as necessary to realize an unchanging sum; then we decrease  $\Delta$  by a factor of 2 and repeat the procedure. Then for the Trapezoidal rule that we used, since the error, in approximating an integral by a sum, is proportional to  $\Delta^2$ , we extrapolated the approximate values to the limit (of  $\Delta = 0$ ). Results for this limiting procedure, using  $\Delta = .2, .1, .05$  are given in table F-1 and figure F-2; the program in BASIC for the HP 9845 follows. It is observed that  $I_D^{(1)}$  is of the order of 1, and increases linearly with  $D$  for large  $D$ .

$D$	$I_D^{(1)}$	$D$	$I_D^{(1)}$
.25	.515852	1	.418002
.32	.416039	2	.641552
.4	.372120	3	.880873
.5	.355822	4	1.12359
.6	.357359	5	1.36758
.75	.374109	6	1.61220

Table F-1. Values of  $I_D^{(1)}$  for Exponential Correlation of Frequency-Modulating Process  $m(t)$

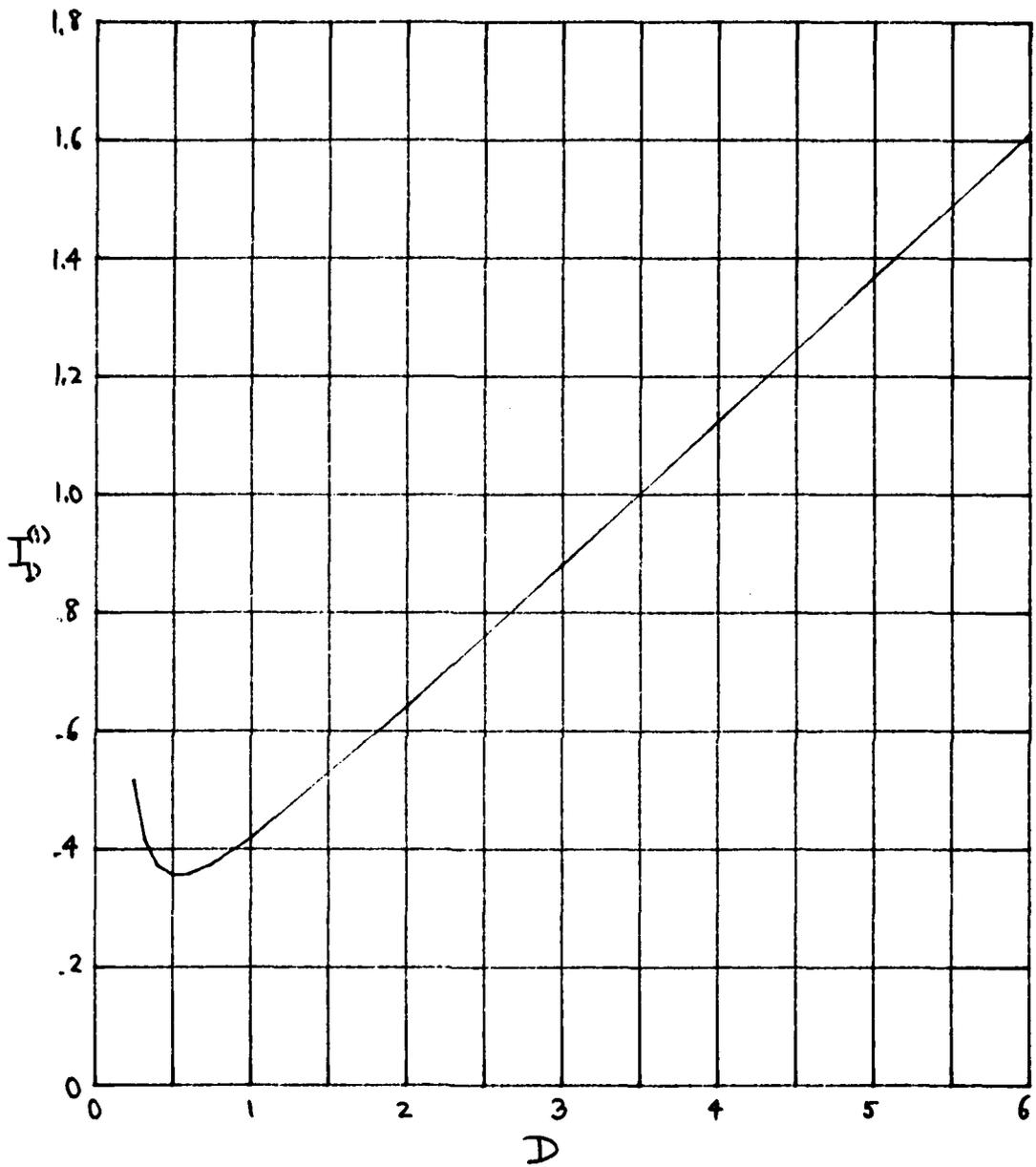


Figure F-2.  $I_D^{(0)}$  for Exponential Correlation of Frequency-Modulation

```

10  K=50
20  Del=.1
30  D=3
40  DIM R(0:400),F2(1:100),E(1:100)
50  REDIM R(0:4*K),R2(1:K),E(1:K)
60  PRINT "K Del D:",K;Del;D
70  COM T,S
80  T=1/(PI*D)
90  S=(PI*D)^2
100 FOR Ks=0 TO 4*K
110  R(Ks)=FNP(Ks*Del)
120 NEXT Ks
130 T=2*Del*T
140 FOR Ks=1 TO K
150  R2(Ks)=R(2*Ks)^2
160  E(Ks)=1-EXP(-Ks*T)
170 NEXT Ks
180 S1=0
190 FOR Ks=1 TO K
200  S2=0
210  FOR Ls=1 TO Ks
220  T=(R(Ks+Ls)/R(Ks-Ls))-2
230  S3=.5*(T+1/T-2)
240  T=(R(2*Ks+2*Ls)/(R(2*Ks)*R(2*Ls)))
250  S3=S3+.5*(T+1/T-2)
260  T=S*(E(Ks)+E(Ls))
270  T=FNE1(T)+FNE1(-T)+2+LOG(T)+.577215664902
280  S3=S3-PI*D*T*Del
290  FOR Ms=1 TO Ks+Ls-1
300  I1=Ks+Ls-Ms
310  I2=Ms+Ks+Ls
320  I3=ABS(Ms-Ks+Ls)
330  I4=Ms+Ks-Ls
340  T=R(I1)*R(I2)/(R(I3)+R(I4))
350  S3=S3+(T+1/T-2)
360 NEXT Ms
370 IF Ls=Ks THEN S3=S3*.5
380 S2=S2+R2(Ls)*S3
390 NEXT Ls
400 S1=S1+R2(Ks)*S2
410 PRINT S2/PI*1.5*Del+S1,Ks,R2(Ks)
420 NEXT Ks
430 STOP
440 DEF FNP(X)
450 COM T,S
460 P=ABS(X)*T
470 RETURN EXP(-S*(P-1+EXP(-P)))
480 FNE1
490 DEF FNE1(X) ! Exponential Integral E1(X)

```

Table F-2. Program for (F-41)

### Development of Fourth-Order Correlation

For the signal process in (96), we derived the fourth-order correlation in (105):

$$\overline{s(t_1)s^*(t_2)s(t_3)s^*(t_4)} = \frac{R_s(t_1-t_2)R_s(t_2-t_3)R_s(t_3-t_4)R_s(t_4-t_1)}{R_s(t_1-t_3)R_s(t_2-t_4)} \quad (F-42)$$

If we define the complex phase process

$$p(t) = \exp\left[i2\pi \int_{t_0}^t du m(u)\right], \quad t_0 < t < t_0, \quad (F-43)$$

then (96) and (77) enable us to write (F-42) as

$$\overline{p(t_1)p^*(t_2)p(t_3)p^*(t_4)} = \frac{p(t_1-t_2)p(t_2-t_3)p(t_3-t_4)p(t_4-t_1)}{p(t_1-t_3)p(t_2-t_4)} \equiv P_3(t_2-t_1, t_3-t_1, t_4-t_1), \quad (F-44)$$

where we have made express use of the stationarity. Furthermore, the left-hand side of (F-44) can be written in the two alternative forms

$$\overline{\exp\left[i2\pi \int_{t_1}^{t_2} du m(u) + i2\pi \int_{t_1}^{t_3} du m(u)\right]} = \overline{\exp\left[i2\pi \int_{t_1}^{t_2} du m(u) + i2\pi \int_{t_2}^{t_3} du m(u)\right]}. \quad (F-45)$$

The first form indicates that if  $t_1-t_2$  and  $t_2-t_3$  are kept fixed, but the separation between these pairs of time points is increased, then  $P_3 \rightarrow p(t_1-t_2)p(t_2-t_3)$ ; the second form indicates that if  $t_1-t_2$  and  $t_3-t_2$  are kept fixed, but the separation between these pairs of time points is increased, then  $P_3 \rightarrow p(t_1-t_2)p(t_3-t_2)$ . Furthermore, these are the only ways these time differences can increase without  $P_3$  tending to zero. Thus we can express

$$P_3(t_2-t_1, t_3-t_1, t_4-t_1) = p(t_1-t_2)p(t_2-t_3) + p(t_1-t_2)p(t_3-t_2) + R_3(t_2-t_1, t_3-t_1, t_4-t_1), \quad (F-46)$$

where the remainder function  $R_3$  goes to zero, no matter how the three difference variables  $t_2-t_1, t_3-t_1, t_4-t_1$  increase to infinity. Thus the first two terms of (F-46), (which would be yielded for a complex Gaussian process  $\rho(t)$ ), are merely those terms which do not necessarily decay to zero as the difference variables increase in an arbitrary manner.

Combining (F-44) and (F-46), we have the very useful expression

$$\frac{\rho(t_2-t_1)\rho(t_3-t_1)\rho(t_4-t_1)}{\rho(t_2-t_3)\rho(t_2-t_4)} = \rho(t_2-t_3)\rho(t_3-t_4) + \rho(t_1-t_2)\rho(t_3-t_4) + R_3(t_2-t_1, t_3-t_1, t_4-t_1), \quad (F-47)$$

which is exact, and where  $R_3(x, y, z)$  tends to zero as  $x, y, z$  increase, whatever their fashion. It should be noted that this expression and conclusion only hold for a phase process like that in (F-43), where  $\rho$  is a legal correlation function for  $p$ . Thus for example, although  $\rho(x) = \exp(-x^2/2)$  is a legal correlation function, it is not a legal correlation function for  $p$ , and  $R_3$  need not tend to zero as its arguments increase. In fact, we find for Gaussian  $\rho$ ,

$$R_3(x, y, z) = \exp\left[-\frac{1}{2}(x-y+z)^2\right] - \exp\left[-\frac{1}{2}x^2 - \frac{1}{2}(z-y)^2\right] - \exp\left[-\frac{1}{2}z^2 - \frac{1}{2}(y-x)^2\right], \quad (F-48)$$

which does not tend to zero for  $y = x + z$ .

If we employ (F-47) in (F-1), there follows immediately

$$\begin{aligned} I_{t_2} &= I_2^2 + I_4 + \iiint \rho(u_2, u_3, u_4) \rho(u_2, u_3) \rho(u_2, u_4) \rho(u_3, u_4) R_3(u_2-u_1, u_3-u_1, u_4-u_1) \\ &= I_2^2 + I_4 + \iiint dx dy dz \phi_+(x, y, z) \rho(x) \rho(y-z) R_3(x, y, z) \\ &\cong I_2^2 + \frac{\sqrt{2}}{(\Gamma W_s)^2} + \frac{1}{\Gamma^3} \iiint dx dy dz \rho(x) \rho(y-z) R_3(x, y, z), \end{aligned} \quad (F-49)$$

where we employed (F-4) et seq. Thus the first two terms of (F-49) corroborate (F-2), (F-7), and (F-8); however, we now see from (F-49) that the size of the remainder term is also proportional to  $(\Gamma W_s)^{-2}$ . And indeed, (F-18) et seq. and table F-1 are a numerical calculation of the sum of the last two terms in (F-49); in fact, the sum of the last two terms in (F-49) is exactly (F-11):

$$I_{t_2} - I_2^2 = \frac{1}{\Gamma^3} \iiint dx dy dz \rho(x) \rho(y-z) [\rho(z) \rho(y-x) + R_3(x, y, z)] = \frac{I_D^{(0)}}{(\Gamma W_s)^3} \quad (F-50)$$

Exact Evaluation of  $I_p$  for Frequency-Modulation with Exponential Correlation

The quantities  $I_2$ ,  $I_3$ , and  $I_4$  are defined in (76), (109), and (152) respectively. For large  $TW_s$ , these can be simplified to

$$\begin{aligned} I_2 &= \frac{1}{T} \int dx \rho^2(x), \\ I_3 &= \frac{1}{T^2} \iint dx dy \rho(x) \rho(y-x) \rho(y), \\ I_4 &= \frac{1}{T^3} \iiint dx dy dz \rho(x) \rho(y-x) \rho(z-y) \rho(z). \end{aligned} \tag{F-51}$$

If we assume that signal correlation  $\rho$  is Gaussian, then we get the approximate results (118), (124), and (F-6). However, it is of interest to determine the exact dependence of  $\{I_p\}$  on  $D$  for the example of (F-13), to determine the accuracy of this approximation. Making the change of variable employed in (F-16), we obtain

$$I_p = \frac{\hat{I}_p}{(TW_s)^{p-1}}, \tag{F-52}$$

where

$$\begin{aligned} \hat{I}_2 &= \frac{1}{\pi^{1/2}} \int du \rho_0^2(u), \\ \hat{I}_3 &= \frac{1}{\pi} \iint du dv \rho_0(u) \rho_0(u-v) \rho_0(v), \\ \hat{I}_4 &= \frac{1}{\pi^{3/2}} \iiint du dv dw \rho_0(u) \rho_0(v-u) \rho_0(w-v) \rho_0(w). \end{aligned} \tag{F-53}$$

Now the question is: how close are  $\hat{I}_2, \hat{I}_3, \hat{I}_4$  to  $1, 2/\sqrt{3}, \sqrt{2}$  respectively?

We employ (F-16) in (F-53) to obtain

$$\hat{I}_2 = \frac{2}{\pi^{1/2}} \int_0^\infty du \exp\left[-2\pi^2 D^2 \left\{ \frac{u}{\pi D} - 1 + \exp\left(-\frac{u}{\pi D}\right) \right\}\right]. \tag{F-54}$$

By the change of variable  $u = -\pi D \ln t$ , there follows

$$\hat{I}_2 = 2\pi^{1/2} D \int_0^1 dt t^{2\pi^2 D^2 - 1} \exp(-2\pi^2 D^2(t-1)). \tag{F-55}$$

By repeated integration by parts, this is reduced to

$$\hat{I}_2 = \frac{1}{\pi^{3/2} D} F_1(1; 1 + 2\pi^2 D^2; 2\pi^2 D^2), \quad (\text{F-56})$$

where  $F_1$  is a confluent hypergeometric function. Evaluation of (F-55) or (F-56) yields table F-3. The values of  $\hat{I}_2$  are well approximated by

$D$	$\hat{I}_2$
.25	1.32344
.5	1.13884
.75	1.08798
1	1.06436
2	1.03102
3	1.02043
4	1.01523
5	1.01214
6	1.01010
7	1.00864
8	1.00755
9	1.00670
10	1.00603
20	1.00300
30	1.00200
100	1.00060
$\infty$	1.00000

Table F-3.  $\hat{I}_2$  Dependence on  $D$

$$\hat{I}_2 \cong 1 + \frac{1}{3\pi^{3/2} D} = 1 + \frac{.06}{D}, \quad (\text{F-57})$$

which is obtainable from (F-16) by expanding in powers of  $1/D$ . Thus a value of  $D = 3$  yields only a 2% error from the Gaussian-approximation value of 1.

For the evaluation of  $\hat{I}_3$  in (F-53), we denote the kernel by  $K_3(u, v)$ , and note that

$$\begin{aligned} K_3(-u, -v) &= K_3(u, v), \\ K_3(v, u) &= K_3(u, v), \end{aligned} \tag{F-58}$$

to express

$$\begin{aligned} \hat{I}_3 &= \frac{4}{\pi} \int_0^\infty du \rho_D(u) \int_{-u}^u dv \rho_D(v) \rho_D(u-v) \\ &\cong \frac{4}{\pi} \Delta^2 \sum_{k=0}^K w_k^{(0)} \rho_D(k\Delta) \sum_{l=-k}^k w_l^{(0)} \rho_D(l\Delta) \rho_D(k\Delta - l\Delta), \end{aligned} \tag{F-59}$$

where weights  $w_k^{(0)} = \frac{1}{2}$  for  $k=0$ , and  $w_k^{(0)} = \frac{1}{2}$  for  $|k|=k$ , for Trapezoidal rule, and  $K$  must be chosen large enough for negligible change in the sum. A program and numerical values for (F-59) follow in tables F-4 and F-5 respectively. The values of  $\hat{I}_3$  are well approximated by

$$\hat{I}_3 \cong \frac{2}{\sqrt{3}} + \frac{8}{9\pi^{3/2} D} = \frac{2}{\sqrt{3}} \left( 1 + \frac{.14}{D} \right). \tag{F-60}$$

A value of  $D = 3$  yields a 4.8% error, while a value of  $D = 6$  yields a 2.3% error from the Gaussian-approximation value of  $2/\sqrt{3}$ .

For the evaluation of  $\hat{I}_4$  in (F-53), we denote the kernel by  $K_4(u, v, w)$  and notice that

$$K_4(-u, -v, -w) = K_4(u, v, w). \tag{F-61}$$

Therefore

$$\hat{I}_4 = \frac{2}{\pi^{3/2}} \int_0^\infty dv \left[ \int_{-v}^v du \rho_D(u) \rho_D(u-v) \right]^2 = \frac{8}{\pi^{3/2}} \int_0^\infty dv \left[ \int_{v/2}^\infty du \rho_D(u) \rho_D(u-v) \right]^2, \tag{F-62}$$

```

10 D=10
20 Kc=100
30 Eps=1E-6
40 Del=.2
50 PRINT "Kc Eps Del D:",Kc;Eps;Del;D
60 DIM Pd(0:200)
70 COM T,S
80 T=1-(PI*D)
90 S=(PI*D)^2
100 FOR K=0 TO 2*Kc
110 Pd(K)=FNP(K*Del)
120 NEXT K
130 S1=0
140 FOR K=0 TO Kc
150 S2=0
160 FOR L=-K TO K
170 R=ABS(L)
180 T=Pd(R)+Pd(K-L)
190 IF R=K THEN T=T+.5
200 S2=S2+T
210 NEXT L
220 IF K=0 THEN S2=S2+.5
230 Q=Pd(K)*32
240 S1=S1+Q
250 IF Q<Eps THEN 230
260 NEXT K
270 PRINT "END OF LOOP"
280 PRINT 4*(PI*Del)^2*S1
290 STOP
300 DEF FNP(X)
310 COM T,S
320 P=ABS(X)+T
330 RETURN EXP(-S*(P-1+EXP(-P)))
340 FNEND

```

Table F-4. Program for (F-59)

D	$\hat{I}_3$
1	1.33412
2	1.23923
3	1.20997
4	1.19576
5	1.18736
6	1.18181
7	1.17788
8	1.17494
9	1.17266
10	1.17085
100	1.15630
$\infty$	1.15470 = $2/3$

Table F-5.  $\hat{I}_3$  Dependence on D

using the evenness of  $\rho_D(x)$  about  $x=0$ . If we choose increments  $\Delta$  in  $u$  and  $2\Delta$  in  $v$ , we obtain

$$\hat{I}_4 \cong \frac{16}{\pi^{3/2}} \Delta^3 \sum_{k=0}^K w_k^{(0)} \left[ \sum_{l=k}^L w_l^{(0)} \rho_D(l\Delta) \rho_D(l\Delta - 2k\Delta) \right]^2, \quad (\text{F-63})$$

where weights  $w_k^{(0)} = \frac{1}{2}$  for  $k=0$ , and  $w_l^{(0)} = \frac{1}{2}$  for  $l=k$ , and  $K$  and  $L$  must be chosen large enough for negligible change in the final sum. A program and numerical values for (F-63) follow in tables F-6 and F-7 respectively. The values of  $\hat{I}_4$  are well approximated by

$$\hat{I}_4 \cong \sqrt{2} + \frac{\sqrt{3}}{\pi^{3/2} D} = \sqrt{2} \left( 1 + \frac{.22}{D} \right). \quad (\text{F-64})$$

A value of  $D = 3$  yields a 7.7% error, while a value of  $D = 6$  yields a 3.8% error from the Gaussian-approximation value of  $\sqrt{2}$ .

```

10  D=10
20  Kc=150
30  Lc=150
40  Eps=1E-9
50  Del=.1
60  PRINT "Kc Lc Eps Del D:",Kc;Lc;Eps;Del;D
70  DIM Pd(0:150)
80  COM T,S
90  T=1/(PI*D)
100 S=(PI+D)^2
110 FOR K=0 TO 150
120 Pd(K)=FNP(K*Del)
130 NEXT K
140 S1=0
150 FOR L=0 TO Kc
160 S2=0
170 FOR L=L TO Lc
180 T=Pd(L)+Pd(ABS(L-2*K))
190 IF L=L THEN T=T*.5
200 S2=S2+T
210 IF T<Eps THEN 230
220 NEXT L
230 S2=S2*2
240 IF K=0 THEN S2=S2*.5
250 S1=S1+S2
260 PRINT 16/PI*1.5*Del^3*S1
270 NEXT K
280 STOP
290 DEF FNP(X)
300 COM T,S
310 P=ABS(X)*T
320 RETURN EXP(-S*(P-1+EXP(-P)))
330 FNPEND

```

Table F-6. Program for  $\hat{I}_4$

$D$	$\hat{I}_4$
1	1.78079
2	1.58273
3	1.52355
4	1.49512
5	1.47842
6	1.46744
7	1.45966
8	1.45387
9	1.44938
10	1.44581
100	1.41733
$\infty$	$1.41421 = \sqrt{2}$

Table F-7.  $\hat{I}_4$  Dependence on  $D$

APPENDIX G. MEAN VALUE OF  $\chi_4^{(p)}(S+N)$

We wish to evaluate the mean value of (139) for signal and noise present. In this case, the fourth-order average of interest was encountered in (94) and is given by the sum of (105), (107), and (86). Substituting these results in the average of (139) yields

$$\begin{aligned} \frac{1}{24} \overline{\chi_4^{(p)}(S+N)} &= \frac{1}{64} \left(\frac{A}{N_0}\right)^4 \iiint_{t_0}^{t_3} dt_1 dt_2 dt_3 dt_4 \left[ P_3(t_2-t_1, t_3-t_1, t_4-t_1) - 2\rho(t_1-t_2)\rho(t_3-t_4) \right] \\ &\quad \left[ A^4 P_3(t_2-t_1, t_3-t_1, t_4-t_1) + 2A^2 N_0 \left\{ \rho(t_1-t_2) \delta(t_3-t_4) + \rho(t_1-t_4) \delta(t_2-t_3) + \right. \right. \\ &\quad \left. \left. \rho(t_3-t_4) \delta(t_1-t_2) + \rho(t_3-t_4) \delta(t_1-t_2) \right\} + 4N_0^2 \left\{ \delta(t_1-t_2) \delta(t_3-t_4) + \delta(t_1-t_4) \delta(t_2-t_3) \right\} \right], \quad (G-1) \end{aligned}$$

where

$$P_3(t_2-t_1, t_3-t_1, t_4-t_1) \equiv \frac{\rho(t_1-t_2)\rho(t_2-t_3)\rho(t_3-t_4)\rho(t_4-t_1)}{\rho(t_2-t_4)\rho(t_1-t_3)}, \quad (G-2)$$

as in (F-44). Evaluating the delta function integrals and employing  $E = AT/2$ , there results

$$\frac{1}{24} \overline{\chi_4^{(p)}(S+N)} = -\frac{1}{2} \left(\frac{E}{N_0}\right)^2 I_2 - \left(\frac{E}{N_0}\right)^3 I_3 + \frac{1}{4} \left(\frac{E}{N_0}\right)^4 (I_{84} - 2I_{62}), \quad (G-3)$$

where

$$I_{84} \equiv \frac{1}{T^4} \iiint_{t_0}^{t_3} dt_1 dt_2 dt_3 dt_4 P_3^2(t_2-t_1, t_3-t_1, t_4-t_1), \quad (G-4)$$

and  $I_2, I_3$ , and  $I_{62}$  are defined in (76), (109), and (110).

We have already evaluated  $I_2$  and  $I_3$  in (118) and (124). We now evaluate  $I_{84} - 2I_{62}$  by means of the correlation development in (F-47). First, from (F-49),

$$I_{62} = I_2^2 + I_4 + \frac{1}{T^3} \iiint dx dy dz \rho(x) \rho(y-z) R_3(x, y, z). \quad (G-5)$$

Next, employing (F-44) and (F-47), (G-4) becomes

$$\begin{aligned} I_{84} &= \frac{1}{T^4} \iiint_{t_0}^{t_1} dt_1 dt_2 dt_3 dt_4 \left[ \rho(t_1-t_2) \rho(t_3-t_4) + \rho(t_1-t_3) \rho(t_2-t_4) + R_3(t_1-t_1, t_3-t_1, t_4-t_1) \right]^2 \\ &= 2I_2^2 + 2I_4 + \frac{1}{T^4} \iiint_{t_0}^{t_1} dt_1 dt_2 dt_3 dt_4 Q_3(t_2-t_1, t_3-t_1, t_4-t_1), \end{aligned} \quad (G-6)$$

where we have also employed (76) and (F-2), and defined

$$Q_3 = R_3 \left[ R_3 + 2\rho(t_1-t_2) \rho(t_3-t_4) + 2\rho(t_1-t_3) \rho(t_2-t_4) \right]. \quad (G-7)$$

Since  $Q_3(x, y, z) \rightarrow 0$  as  $x, y, z \rightarrow \infty$  (see (F-46) et seq.), we can modify (G-6) (by methods now standard) to

$$I_{84} = 2I_2^2 + 2I_4 + \frac{1}{T^3} \iiint dx dy dz Q_3(x, y, z). \quad (G-8)$$

Combining (G-5), (G-7), and (G-8), there follows

$$I_{84} - 2I_{62} = \frac{1}{T^3} \iiint dx dy dz R_3(x, y, z) \left[ R_3(x, y, z) + 2\rho(z) \rho(y-x) \right], \quad (G-9)$$

in terms of remainder function  $R_3$  defined by (F-47). Since  $R_3(x, y, z)$  decays to zero within an interval of the order of  $W_s^{-1}$  in each variable, we have

$$I_{84} - 2I_{62} = \frac{I_D^{(2)}}{(TW_s)^3} \quad (G-10)$$

where  $I_D^{(2)}$  is a quantity of the order of unity, as in appendix F.

Substituting (118), (124), and (G-10) in (G-3), there follows

$$\frac{1}{24} \overline{\chi_4^{(2)}(s+N)} = -\frac{1}{2} \left( \frac{E}{N_0} \right)^2 \frac{1}{TW_s} - \frac{2}{\sqrt{3}} \left( \frac{E}{N_0} \right)^3 \frac{1}{(TW_s)^2} + \frac{I_D^{(2)}}{4} \left( \frac{E}{N_0} \right)^4 \frac{1}{(TW_s)^3} \quad (G-11)$$

for large  $TW_s$ .

APPENDIX H. VARIANCE OF  $\chi_4^{(n)}(N)$

Let the collection of correlation functions in (139) be denoted by  $K(t_1, t_2, t_3, t_4)$ . Then for noise-alone, let

$$\begin{aligned} X &= \iiint_{t_0}^{t_0+T} dt_1 dt_2 dt_3 dt_4 n(t_1) n^*(t_2) n(t_3) n^*(t_4) K(t_1, t_2, t_3, t_4) \\ &= \int_4 n_1 n_2^* n_3 n_4^* K_{1234}, \end{aligned} \quad (H-1)$$

where an obvious shorthand notation has been adopted. Then employing (86), (139), and (76), the mean is

$$\begin{aligned} \bar{X} &= 4 N_0^2 \int_4 (\delta_{12} \delta_{34} + \delta_{14} \delta_{23}) K_{1234} = 4 N_0^2 \int_2 (K_{1133} + K_{1331}) \\ &= 4 N_0^2 \int_2 \left( (-1) + (1 - 2\rho_{13}^2) \right) = -8 N_0^2 T^2 I_2. \end{aligned} \quad (H-2)$$

The derivation of  $\bar{X}^2$  will be abbreviated considerably. First, using Ref. 6 and a technique similar to (H-2),

$$\begin{aligned} \bar{X}^2 &= \int_8 \overbrace{n_1 n_2^* n_3 n_4^* n_5 n_6^* n_7 n_8^*} K_{1234} K_{5678} \\ &= 16 N_0^4 \int_8 A K_{1234} K_{5678}, \end{aligned} \quad (H-3)$$

where quantity A is composed of 24 delta function terms which involve the various combinations of one even with one odd number, out of the numbers 1-8. For example, one combination is 12, 34, 56, 78; this leads to the integral contribution in (H-3) containing  $K_{1133} K_{5577} = (-1)(-1) = 1$ . Some of the other 24 terms are far more complicated; for example, combination 16, 27, 38, 45 leads to

$$K_{1735} K_{5173} = \left( \frac{\rho_{73} \rho_{51}}{\rho_{13} \rho_{57}} - 2 \right) \rho_{17} \rho_{35} \left( \frac{\rho_{17} \rho_{35}}{\rho_{13} \rho_{57}} - 2 \right) \rho_{15} \rho_{37}. \quad (H-4)$$

Substitution of all the 24 terms in (H-3), and collection of like terms, yields, after

considerable labor,

$$\overline{X^2} = 16 N_0^4 T^4 (I_{84} - 4I_{62} + 6I_4 + 3I_2^2), \quad (\text{H-5})$$

where all the terms involving constants,  $I_2$ , and  $I_3$  cancel out. Combining (H-5) with (H-2), and recollecting the scale factor in (139), we get

$$\text{Var}\left\{\frac{1}{24}\chi_4^{(4)}(N)\right\} = \frac{1}{4}\left(\frac{E}{N_0}\right)^4 (I_{84} - 4I_{62} + 6I_4 + 2I_2^2). \quad (\text{H-6})$$

In order to evaluate (H-6), we appeal to the development of the fourth-order correlation given in (F-42) et seq. Specifically, we employ (G-8), (F-49), and (G-7), to obtain

$$\begin{aligned} I_{84} - 4I_{62} + 6I_4 + 2I_2^2 &= 4I_4 + \frac{1}{T^3} \iiint dx dy dz R_3(x, y, z) \times \\ &\quad [R_3(x, y, z) - 2\rho(x)\rho(y-z) + 2\rho(z)\rho(x-y)]. \end{aligned} \quad (\text{H-7})$$

Alternatively, by use of (G-10) and (F-50), we find

$$\begin{aligned} I_{84} - 4I_{62} + 6I_4 + 2I_2^2 &= \frac{I_D^{(2)} - 2I_D^{(1)}}{(TW_S)^3} + 6I_4 \\ &= \frac{I_D^{(2)} - 2I_D^{(1)} + 6\sqrt{2}}{(TW_S)^3} \equiv \frac{I_D^{(2)}}{(TW_S)^3}. \end{aligned} \quad (\text{H-8})$$

Finally, substitution in (H-6) yields

$$\text{Var}\left\{\frac{1}{24}\chi_4^{(4)}(N)\right\} = \frac{I_D^{(2)}}{4} \left(\frac{E}{N_0}\right)^4 \frac{1}{(TW_S)^3}. \quad (\text{H-9})$$

APPENDIX I. CORRELATION OF  $l_2$  AND  $l_4$

The quantities  $l_2$  and  $l_4$  are given by (158), (45), (77), and (139)

as

$$l_2 = \frac{1}{2} \chi_2^{(p)} = \frac{1}{4} \left( \frac{A}{N_0} \right)^2 \iint dt_1 dt_2 x(t_1) x^*(t_2) \rho(t_2 - t_1), \quad (I-1)$$

$$l_4 = \frac{1}{24} \chi_4^{(p)} = \frac{1}{64} \left( \frac{A}{N_0} \right)^4 \iiint dt_1 dt_2 dt_3 dt_4 x(t_1) x^*(t_2) x(t_3) x^*(t_4) K(t_1, t_2, t_3, t_4), \quad (I-2)$$

where kernel K is the collection of correlation functions in (139) (as employed in (140) and appendix H). For noise-alone, the average of the product is (Ref. 6 and appendix H)

$$\begin{aligned} \overline{l_2(N) l_4(N)} &= \frac{1}{256} \left( \frac{A}{N_0} \right)^6 (2N_0)^3 \int_6 \rho_{64} K_{1234} \times \\ & \left[ \delta_{12} \delta_{34} \delta_{56} + \delta_{12} \delta_{36} \delta_{45} + \delta_{14} \delta_{23} \delta_{56} + \delta_{14} \delta_{25} \delta_{36} + \delta_{16} \delta_{23} \delta_{45} + \delta_{16} \delta_{25} \delta_{34} \right] \\ &= \frac{1}{32} \left( \frac{A^2}{N_0} \right)^3 \int_3 \left[ K_{1133} + \rho_{35} K_{1135} + K_{1331} + \rho_{35} K_{1531} + \rho_{15} K_{1335} + \rho_{15} K_{1533} \right] \\ &= \frac{1}{4} \left( \frac{E}{N_0} \right)^3 \frac{1}{T^3} \int_3 \left[ -2\rho_{13}^2 - 4\rho_{15}\rho_{33}\rho_{31} \right] = -\frac{1}{4} \left( \frac{E}{N_0} \right)^3 \left[ 2 I_2 + 4 I_3 \right]. \quad (I-3) \end{aligned}$$

Combining this result with (71) and (140), there follows

$$\overline{l_2(N) l_4(N)} - \overline{l_2(N)} \overline{l_4(N)} = -\left( \frac{E}{N_0} \right)^3 I_3. \quad (I-4)$$

APPENDIX J. EVALUATION OF  $I_D^{(b)}$

$I_D^{(b)}$  is given in (H-8) in terms of  $I_D^{(1)}$  and  $I_D^{(2)}$ , which are in turn given by (F-50) and (G-9)-(G-10). Namely

$$I_D^{(b)} = I_D^{(2)} - 2 I_D^{(1)} + 6 (TW_s)^2 I_4. \quad (J-1)$$

Substitution yields

$$I_D^{(b)} = W_s^3 \iiint dx dy dz R_3^2(x, y, z) + 4 (TW_s)^3 I_4, \quad (J-2)$$

where two terms of the  $I_D^{(2)} - 2 I_D^{(1)}$  expression cancel each other by use of the property

$$R_3(x, y, z) = R_3(z, y, x); \quad (J-3)$$

see (F-47). Letting  $x = u/(\sqrt{\pi} W_s)$ ,  $y = v/(\sqrt{\pi} W_s)$ ,  $z = w/(\sqrt{\pi} W_s)$  in (J-2), and employing (F-52), there follows

$$I_D^{(b)} = \frac{1}{\pi^{3/2}} \iiint du dv dw R_{3D}^2(u, v, w) + 4 \hat{I}_4, \quad (J-4)$$

where

$$R_{3D}(u, v, w) \equiv \frac{\rho_D(u) \rho_D(w-v) \rho_D(w) \rho_D(v-u)}{\rho_D(v) \rho_D(w-u)} - \rho_D(u) \rho_D(w-v) - \rho_D(w) \rho_D(v-u). \quad (J-5)$$

An example of  $\rho_D$  is given in (F-16).

Two useful properties follow immediately from (J-5):

$$R_{3D}(u, v, w) = R_{3D}(w, v, u),$$

$$R_{3D}(-u, -v, -w) = R_{3D}(u, v, w). \quad (J-6)$$

They enable us to express (J-4) as

$$I_D^{(b)} = \frac{4}{\pi^{3/2}} \int_0^\infty dv \int_{-\infty}^\infty dw \int_{-\infty}^\infty du R_{3D}^2(u, v, w) + 4 \hat{I}_4. \quad (J-7)$$

In order to numerically evaluate the triple-integral in (J-7), it is necessary to determine the regions of significant value of  $R_{3D}^2$ , and to terminate the integrals. Detailed evaluation of  $R_{3D}^2$  for the example of (F-16) for  $D = 3$  was undertaken, with the result that the first term in (J-7) can be well-approximated by

$$\frac{4}{\pi^{3/2}} \int_0^V dv \int_{v-L(v)}^{v+L(v)} dw \int_{-L(v)}^{\min\{w, L(v)\}} du R_{3D}^2(u, v, w) \equiv Q, \quad (J-8)$$

where

$$L(v) = 6.5 - .08v, \quad V = 40. \quad (J-9)$$

Letting the integral on  $u$  be denoted as  $A(v, w)$ , and the integral on  $w$  denoted as  $B(v)$ , we have

$$Q = \frac{4}{\pi^{3/2}} \int_0^V dv B(v) \approx \frac{4}{\pi^{3/2}} \Delta_v \sum_{k=0}^K w_k^{(v)} B(k\Delta_v); \quad K\Delta_v = V, \quad w_k^{(v)} = \begin{cases} 1/2, & k=0 \\ 1, & \text{otherwise} \end{cases}. \quad (J-10)$$

Also

$$B(v) = \int_{v-L(v)}^{v+L(v)} dw A(v, w) \approx \Delta_w \sum_{l=0}^L A(v, v-L(v) + l\Delta_w); \quad L\Delta_w = 2L(v), \quad (J-11)$$

and

$$A(v, w) = \int_{-L(v)}^{\min\{w, L(v)\}} du R_{3D}^2(u, v, w) \approx \Delta_u \sum_{m=0}^M w_m^{(w)} R_{3D}^2(-L(v) + m\Delta_u, v, w);$$

$$M\Delta_u = \min\{w, L(v)\} + L(v), \quad w_m^{(w)} = \begin{cases} 1/2, & m=M \\ 1, & \text{otherwise} \end{cases}. \quad (J-12)$$

A program for the evaluation of (J-9)-(J-12) is given in table J-1. Extreme care is necessary in the evaluation of  $R_{3D}$ , due to the presence of  $\beta_D$  functions in the denominator of (J-5). Exercise of the program for sampling increments .5 and .25 yielded the value  $Q = 5.75415$  for  $D = 3$ .

Combining this result with table F-7, we find

$$I_D^{(2)} = 11.848 \quad \text{for } D = 3. \quad (J-13)$$

```

10  Vc=50
20  Del=.25
30  B=3
40  F1=1-(PI*D)
50  F2=(PI*D)^2
60  COM Del,Lv,F1,F2
70  Kc=INT(Vc/Del)+1
80  Dv=Vc/Kc
90  S=0
100 FOR K=0 TO Kc
110  T=FNB(K*Dv)
120  IF K=0 THEN T=T*.5
130  S=S+T
140  PRINT 4/PI*(1.5+Dv*S),K
150  NEXT K
160  END
170  DEF FNB(V)
180  COM Del,Lv
190  Lv=6.5-.08*V
200  Lc=INT(2*Lv/Del)+1
210  Dv=2*Lv/Lc
220  W0=V-Lv
230  S=0
240  FOR L=0 TO Lc
250  S=S+FNA(V,W0+L*Dv)
260  NEXT L
270  RETURN Dv*S
280  FNEND
290  DEF FNA(V,W)
300  COM Del,Lv
310  U0=MIN(W,Lv)
320  Mc=INT((U0+Lv)/Del)+1
330  Du=(U0+Lv)/Mc
340  S=0
350  FOR M=0 TO Mc
360  T=FNR3(-Lv+M*Du,V,W)*2
370  IF M=Mc THEN T=T*.5
380  S=S+T
390  NEXT M
400  RETURN Du*S
410  FNEND
420  DEF FNR3(U,V,W)
430  A1=FNG(U)+FNG(W-V)
440  A2=FNG(W)+FNG(V-U)
450  A3=FNG(V)+FNG(W-U)
460  RETURN EXP(-A1-A2+A3)*(EXP(-A1)-EXP(-A3))
470  FNEND
480  DEF FNG(X)
490  COM Del,Lv,F1,F2
500  P=ABS(X)*F1
510  RETURN F2*(P-1+EXP(-P))
520  FNEND

```

Table J-1. Program for (J-9)-(J-12)

In equations (153), (154), (156), (157), and (162), the quantity

$$\frac{I_D^{(b)}}{4} (= 2.962 \text{ for } D=3) \quad (\text{J-14})$$

appears; it is of the order of unity, as anticipated.

Finally, reference to (J-1), (F-52), table F-1, and table F-7 enables us to evaluate

$$I_D^{(b)} = 4.469 \text{ for } D=3. \quad (\text{J-15})$$

In equations (146), (148), (149), (156), (157), and (159), the quantity

$$\frac{I_D^{(b)}}{4} (= 1.117 \text{ for } D=3) \quad (\text{J-16})$$

appears; it is of the order of unity, as anticipated.

APPENDIX K. DERIVATION OF  $\chi_6^{(n)}$  AND ITS MEAN

The cumulants through order 4 were given in (39). The fifth (and all odd-orders) cumulant is zero, as shown in (131)-(133). The sixth cumulant is given by (Ref. 4, 3.43)

$$\chi_6^{(n)} = \langle r^6 \rangle - 15 \langle r^4 \rangle \langle r^2 \rangle + 30 \langle r^2 \rangle^3. \quad (K-1)$$

From (131) and (132),

$$\langle r^6 \rangle_\phi = \frac{20}{2^6} |C|^6 = \frac{5}{16} |C|^6, \quad (K-2)$$

where

$$C = \frac{A}{N_0} \int dt x(t) p^*(t) \equiv \frac{A}{N_0} \int x p^*. \quad (K-3)$$

Here  $p(t)$  is the complex phasor process

$$p(t) = \exp\left[i 2\pi \int_{t_a}^t du m(u)\right], \quad t_a < t < t_b. \quad (K-4)$$

Using relations similar to (K-2) for lower-orders, we find from (K-1)

$$\chi_6^{(n)} = \frac{5}{16} \left[ \langle |C|^6 \rangle - 9 \langle |C|^4 \rangle \langle |C|^2 \rangle + 12 \langle |C|^2 \rangle^3 \right], \quad (K-5)$$

where the averages are over the frequency-modulating process  $m(t)$  in (K-4), or equivalently over the phasor process  $p(t)$ . Now from (K-3),

$$\begin{aligned} \langle |C|^2 \rangle &= \left(\frac{A}{N_0}\right)^2 \iint x_1 x_2^* K_{12} \\ \langle |C|^4 \rangle &= \left(\frac{A}{N_0}\right)^4 \int \int x_1 \cdots x_4^* K_{1234} \\ \langle |C|^6 \rangle &= \left(\frac{A}{N_0}\right)^6 \int \int x_1 \cdots x_6^* K_{123456} \end{aligned} \quad (K-6)$$

where

$$\begin{aligned} K_{12} &= \langle p_1^* p_2 \rangle = \langle p^*(t_1) p(t_2) \rangle, \\ K_{1234} &= \langle p_1^* p_2 p_3^* p_4 \rangle, \quad K_{123456} = \langle p_1^* p_2 p_3^* p_4 p_5^* p_6 \rangle. \end{aligned} \quad (K-7)$$

Substitution of (K-6) in (K-5) yields the desired result

$$\chi_6^{(r)} = \frac{5}{16} \left(\frac{A}{N_0}\right)^6 \int \dots \int \chi_1 \dots \chi_6^* \left[ K_{123456} - 9 K_{1234} K_{56} + 12 K_{12} K_{34} K_{56} \right]. \quad (K-8)$$

Evaluation of the kernel of (K-8) would require evaluation of the sixth-order average in (K-7); although possible, this is very tedious and sheds no light on the processor. Also, we can evaluate the mean value of  $\chi_6^{(r)}(N)$ , without knowing all the details of the kernel of (K-8). Using the properties of complex white Gaussian noise, we have

$$\begin{aligned} \overline{\chi_6^{(r)}(N)} &= \frac{5}{16} \left(\frac{A}{N_0}\right)^6 (2N)^3 \iiint \left[ K_{113355} + K_{113353} + K_{133155} + K_{153153} + K_{133551} + K_{153351} \right. \\ &\quad \left. - 9 \{ K_{1133} K_{55} + K_{1135} K_{53} + K_{1331} K_{55} + K_{1531} K_{53} + K_{1335} K_{51} + K_{1533} K_{51} \} \right. \\ &\quad \left. + 12 \{ K_{11} K_{33} K_{55} + K_{11} K_{35} K_{53} + K_{13} K_{31} K_{55} + K_{15} K_{31} K_{53} + \right. \\ &\quad \left. + K_{13} K_{35} K_{51} + K_{15} K_{33} K_{51} \} \right] \\ &= \frac{5}{2} \left(\frac{A^2 T}{N_0}\right)^3 \left[ 6 - 9 \{ 2 + 4 I_2 \} + 12 \{ 1 + 3 I_2 + 2 I_3 \} \right] \\ &= 480 \left(\frac{E}{N_0}\right)^3 I_3. \end{aligned} \quad (K-9)$$

Therefore

$$\frac{1}{6!} \overline{\chi_6^{(r)}(N)} = \frac{2}{3} \left(\frac{E}{N_0}\right)^3 I_3 \approx \frac{4}{3\sqrt{3}} \frac{(E/N_0)^3}{(TW_s)^2} \text{ for large } TW_s. \quad (K-10)$$

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