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SENSITIVITY COEFFICIENT OF EXTERIOR BALLISTICS WITH VELOCITY SQUARE DAMPING

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The principal equation of exterior ballistics has a drag term which, in this case, is proportional to the square of the velocity in the tangential direction of the trajectile. The sensitivity coefficient is expressed as the ratio of the initial elevation angle deviation to the initial percentage velocity deviation. The work in this paper is to find analytically the sensitivity coefficient of the exterior ballistics with velocity square damping which comes from the nonlinear air resistance for a projectile. This prin-

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cipal equation is integrated analytically in obtaining the solution for tangential velocity in terms of the elevation angle, together with all the necessary initial conditions. The horizontal range and the vertical range are also expressed as integrals of certain function of the elevation angles. In order to obtain the sensitivity coefficient it is necessary to find the perturbations of the horizontal and vertical ranges. This procedure is similar to that of evaluating differentiation under the integral sign. The perturbation of the ranges is the sum of the perturbations due to the initial velocity, the initial elevation angle, and the impact elevation angle. By setting to zeroes the range perturbations we can group the coefficients of the perturbations into two separate equations. The ratio of the perturbations for initial elevation angle to that for initial velocity is the sensitivity coefficient for exterior ballistics that we are seeking.

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SUMMARY. The principal equation of exterior ballistics has a drag term which, in this case, is proportional to the square of the velocity in the tangential direction of the projectile. The sensitivity coefficient is expressed as the ratio of the initial elevation angle deviation to the initial percentage velocity deviation. The work in this paper is to find analytically the sensitivity coefficient of the exterior ballistics with velocity square damping which comes from the nonlinear air resistance for a projectile. This principal equation is integrated analytically in obtaining the solution for tangential velocity in terms of the elevation angle, together with all the necessary initial conditions. The horizontal range and the vertical range are also expressed as integrals of certain function of the elevation angles. In order to obtain the sensitivity coefficient it is necessary to find the perturbations of the horizontal and vertical ranges. This procedure is similar to that of evaluating differentiation under the integral sign. The perturbation of the ranges is the sum of the perturbations due to the initial velocity, the initial elevation angle and the impact elevation angle. By setting to zeroes the range perturbations we can group the coefficients of the perturbations into two separate equations. The ratio of the perturbations for initial elevation angle to that for initial velocity is the sensitivity coefficient for exterior ballistics that we are seeking.

I. INTRODUCTION. The design of a gun involves numerous parameters. These parameters should be in such a combination that the best first round accuracy is given. While the shell leaves the gun it has perturbations for the muzzle elevation angle and the muzzle velocity. The ratio of the two is the sensitivity coefficient of the interior and the exterior ballistics. It is desired to compensate the errors due to uncertain changes of muzzle velocity by the automatic response of the muzzle elevation angle within the gun system. With a correct design this compensation can be made by matching the exterior ballistics to the interior ballistics through the analysis of gun dynamics. This process is called passive control since there is no external measurement involved nor instrumentation needed for control. This general problem can be formulated by first investigating the sensitivity coefficients for exterior ballistics with velocity square damping.

II. DYNAMICAL EQUATIONS FOR TRAJECTORIES. For a constant mass travelling in a vertical plane with no lift and applied thrust, but having drag and velocity vectors contained in the plane of symmetry as shown in Figure 1, the dynamical equations of motion are [1]:

$$\frac{dx}{dt} - v\cos\theta = 0 \tag{1}$$

$$\frac{dy}{dt} - v\sin\theta = 0 \tag{2}$$

$$m(g\cos\theta + v\frac{d\theta}{dt}) = 0 \tag{3}$$

$$\frac{d^2x}{dt^2} = -\frac{D\cos\theta}{m} \tag{4}$$

where m = the mass of the projectile
g = the acceleration due to gravity
D = the drag of the projectile
v = the velocity of the projectile
θ = the path inclination (elevation angle)
x = the horizontal distance of the projectile
y = the altitude or vertical distance of the projectile.

It is noticed that deviations due to anomalies in the azimuth direction arenot considered here.

By differentiating Equation (1) with respect to t one obtains

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(v\cos\theta) \tag{5}$$

Substituting into Eq. (4) we have

$$\frac{d}{dt}(v\cos\theta) = -\frac{D\cos\theta}{m} \; . \tag{6}$$

Solving or $d\theta/dt$ in Eq. (3) one obtains

$$\frac{d\theta}{dt} = \frac{-g\cos\theta}{v} \; . \tag{7}$$

Equation (7) indicates that the differential equations can be transformed from the time domain in t to the angle domain in θ. Equations (6), (1), and (2) are divided by Equation (7) in achieving this transformation as

$$\frac{d(v\cos\theta)}{d\theta} = \frac{Dv}{mg} \tag{8}$$

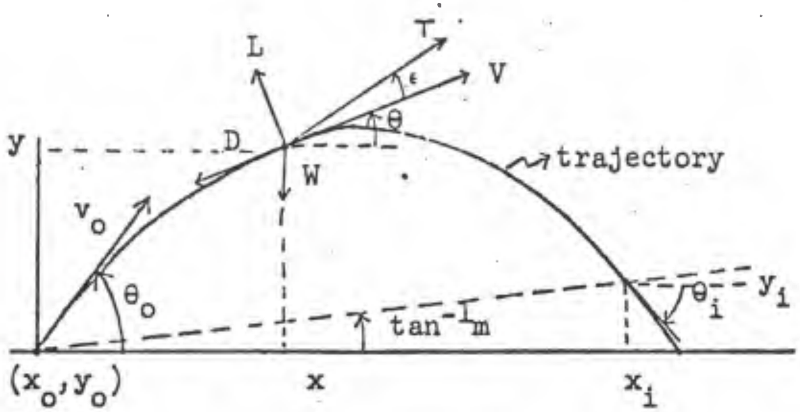

Fig. 1. Forces, Slopes, and Initial and Final Parameters for a Trajectory

$$\frac{dx}{d\theta} = -\frac{v^2}{g} \tag{9}$$

$$\frac{dy}{d\theta} = -\frac{v^2}{g}\tan\theta \tag{10}$$

Equation (8) is called the principal equation of exterior ballistics [2]. It can be integrated if the drag D is a known function of velocity v.

III. TRANSFORMATION OF VARIABLES UNDER HEAD WIND DRAG. The head wind drag D is a velocity square damping term given as

$$D = mcv^2 \tag{11}$$

where

$$c = c_W(\frac{\pi}{4}d^2)(\rho/2) \tag{12}$$

c_W = the dimensionless resistant coefficient
d = the diameter of the projectile
and ρ = the air density.

Thus the principal equation of exterior ballistics (Equation (8)) becomes

$$\frac{d}{d\theta}(v\cos\theta) = \frac{cv^3}{g}\,. \tag{13}$$

A further transformation of the dependent variable is necessary by letting

$$u = v\cos\theta \tag{14}$$

where u is the horizontal component of the projectile velocity. Then the dynamical Equations (13), (9) and (10) become

$$\frac{du}{d\theta} = \frac{c}{g}u^3\sec^3\theta \tag{15}$$

$$\frac{dx}{d\theta} = -\frac{u^2}{g}\sec^2\theta \tag{16}$$

$$\frac{dy}{d\theta} = -\frac{u^2}{g}\sec^2\theta\tan\theta\,. \tag{17}$$

To simplify further the form of the dynamical equations another transformation of the independent variable is made by letting

$$q = \tan\theta \tag{18}$$

where q is the projectile slope.

Thus

$$\frac{dq}{d\theta} = \sec^2\theta = 1+q^2 . \tag{19}$$

Equations (15), (16) and (17) are divided by Eq. (19) to give

$$du = \frac{c}{g} u^3 (1+q^2)^{1/2} dq \tag{20}$$

$$dx = - \frac{u^2}{g} dq \tag{21}$$

$$dy = - \frac{u^2}{g} qdq \tag{22}$$

Solution for u in Equation (20) can be readily integrated in closed form. Solutions for x and y can be expressed in the form of integrals once u^2 is obtained.

IV. SOLUTION FOR HORIZONTAL COMPONENT OF PROJECTILE VELOCITY AS FUNCTION OF TRAJECTORY SLOPE. The solution for horizontal component of velocity u can be obtained by integrating Equation (20)

$$- \frac{1}{2} (u^{-2}-u_o^{-2}) = \frac{1}{2} \frac{c}{g} \{q(1+q^2)^{1/2} + \ln[q+(1+q^2)^{1/2}]\}\Big|_{q_o}^{q} \tag{23}$$

where q_o equals projectile slope initially at launch and

$$u_o^2 = v_o^2\sec^{-2}\theta_o = v_o^2(1+q_o^2)^{-1} \tag{24}$$

by virtue of Equations (14) and (19).

Equation (23) can be written as

$$\frac{1}{u^2} = \frac{1}{u_o^2} \{1-u_o^2 \frac{c}{g} [p(q) - p_o(q_o)\} \tag{25}$$

where

$$p(q) = q(1+q^2)^{1/2} + \ln[q + (1+q^2)^{1/2}] \tag{26a}$$

and

$$p_o(q_o) = q_o(1+q_o^2)^{1/2} + \ln[q_o + (1+q_o^2)^{1/2}] . \tag{26b}$$

Finally, Equation (25) takes the form

$$u^2 = \frac{v_o^2}{1+q_o^2} \{1 + \frac{H(q,q_o,v_o^2,c/g)}{1 - H(q,q_o,v_o^2,c/g)}\} \tag{27}$$

where

$$H(q,q_o,v_o^2,c/g) = \frac{c}{g}\frac{v_o^2}{1+q_o^2}[p(q) - p_o(q_o)] \tag{28}$$

It is noted from the above equation that as

$$q \to q_o \qquad H_{q=q_o} = 0 \cdot \tag{29}$$

For the case with no air resistance as

$$c \to 0 \qquad H_{c=0} = 0 , \tag{30}$$

which implies that the horizontal component of projectile velocity u at any time is a constant.

<u>V. SOLUTION FOR NONDIMENSIONAL RANGE</u>. In determining the range x for the trajectory the closed form solution of u^2 in Equation (27) can be substituted into Equation (21) to obtain the solution in integral form as

$$x_i - x_o = -\frac{v_o^2}{g(1+q_o^2)}\left[(q_i-q_o) + \int_{q_o}^{q_i}\frac{H(q,q_o,v_o^2,c/g)}{1 - H}dq\right] \tag{31}$$

where x_i = range at impact point
x_o = range at initial point
and q_i = projectile slope at impact point.

To non-dimensionize the range, Equation (31) is divided by the factor v_o^2/g as

$$X(x_i,x_o,v_o)/\Lambda(q_i,q_o) = G_x(q_i,q_o,v_o^2,c/g) \tag{32}$$

where the nondimensional range is

$$X(x_i,x_o,v_o) = (x_i-x_o)g/v_o^2 , \tag{33}$$

the slope function is

$$\Lambda(q_o,q_i) = \frac{q_o-q_i}{1+q_o^2} , \tag{34}$$

and the range drag function due to air resistance is

$$G_x(q_i,q_o,v_o^2,c/g) = 1 - \frac{1}{q_o-q_i}\int_{q_o}^{q_i}\frac{H(q,q_o,v_o^2,c/g)}{1 - H}dq . \tag{35}$$

It is noted that the left side of Eq. (32) contains no drag coefficient c. It is the term G_x that is a function of H, which in turn is a function of the drag coefficient c. The numerator of the integral in Eq. (35) is H. Since $H_{c=0} = 0$ in Eq. (30), the range drag function at this condition is

$$G_{x(c=0)} = 1 . \tag{36}$$

A separate form of Eq. (32) can be written as

$$\frac{g(x_i-x_o)}{v_o^2} \frac{(1+q_o^2)}{(q_o-q_i)} = 1- \frac{1}{(q_o-q_i)} \int_{q_o}^{q_i} \frac{H}{1-H} dq . \tag{37}$$

VI. VARIATION OF THE NONDIMENSIONAL RANGE AND THE SLOPE FUNCTION. In order to obtain a first round hit of the target one of the conditions is that the variation of the range should be zero, i.e., from Eq. (33)

$$\delta(x_i-x_o) = 0 . \tag{38}$$

We take the perturbation for the nondimensional range from Eq. (33) as

$$\frac{\delta X}{X} = \frac{\delta(x_i-x_o)}{x_i-x_o} - \frac{2\delta v_o}{v_o}$$

$$\frac{\delta X}{X} = 0 - \frac{2\delta v_o}{v_o} . \tag{39}$$

The variation of the slope function Λ in Eq. (34) becomes

$$\frac{\delta\Lambda}{\Lambda} = - \frac{\delta q_i}{q_o-q_i} + \frac{\delta q_o}{q_o-q_i} - \frac{2q_o\delta q_o}{1+q_o^2} . \tag{40}$$

Next, taking the variation of Eq. (32) and using the expressions given in Eq. (39) and (40) we have

$$\frac{\delta X}{X} - \frac{\delta\Lambda}{\Lambda} = \frac{\delta G_x}{G_x} \tag{41}$$

or

$$- \frac{2\delta v_o}{v_o} + \frac{\delta q_i}{q_o-q_i} - \frac{\delta q_o}{q_o-q_i} + \frac{2q_o\delta q_o}{1+q_o^2} = \frac{\delta G_x}{G_x} . \tag{42}$$

It is noticed that in the absence of air damping the range drag function is unity from Eq. (36) and the variation $\delta G_x = 0$. Under this condition the solutions for Eq. (42) were given by the author in the paper entitled, "On the Sensitivity Coefficient of Exterior Ballistics and Its Potential Matching to Interior Ballistics Sensitivity". This paper was presented at the Second U.S. Army Symposium on Gun Dynamics, September 1978. With the velocity square damping the variation of G_x is not zero, i.e.,

$$\delta G_x \neq 0. \tag{43}$$

<u>VII. VARIATION OF THE RANGE DRAG INTEGRAND.</u> The range drag function in Eq. (35) can be written as

$$G_x(q_i,q_o,v_o{}^2,c/g) = 1-\frac{1}{q_o-q_i}\int_{q_o}^{q_i} F(q,q_o,v_o{}^2,c/g)dq \tag{44}$$

where the range integrand is

$$F(q,q_o,v_o{}^2,c/g) = \frac{H(q,q_o,v_o{}^2,c/g)}{1-H} . \tag{45}$$

The variation of G_x involves the initial velocity v_o, the initial slope q_o, and the impact slope q_i.

Eq. (44) has the parameters q_o and q_i in the denominator as well as in the integral of F. By chain rule we have

$$\delta G_x = -\frac{1}{q_o-q_i}\delta[\int_{q_o}^{q_i} F(q,q_o,v_o{}^2,c/g)dq]$$

$$-(-1)(q_o-q_i)^{-2}\delta(q_o-q_i)\int_{q_o}^{q_i} F(q,q_o,v_o{}^2,c/g)dq . \tag{46}$$

The variation of the integral of F is given in the next section.

<u>VIII. VARIATION OF THE RANGE DRAG INTEGRAL.</u> The parameters in the integral are q_o, q_i and v_o. The variation of the range drag integral follows the rules of differentiation under the integral sign.

$$\delta\left[\int_{q_o}^{q_i} F(q,q_o,v_o^2,c/g)dq\right]$$

$$= \left[\int_{q_o}^{q_i} \frac{\partial F}{\partial v_o} dq\right]\delta v_o + F(q=q_i,q_o,v_o^2,c/g)\delta q_i$$

$$+ \left[\int_{q_o}^{q_i} \frac{\partial F}{\partial q_o} dq\right]\delta q_o - F(q=q_o,q_o,v_o^2,c/g)\delta q_o \tag{47}$$

The last term of this equation is zero by virtue of Equations (29) and (45). Substituting Eq. (47) into Eq. (46) gives

$$\delta G_x = \left[-\frac{1}{q_o-q_i}\int_{q_o}^{q_i} \frac{\partial F}{\partial v_o} dq\right]\delta v_o$$

$$+ \left[-\frac{1}{q_o-q_i} F(q=q_i) - \frac{1}{(q_o-q_i)^2}\int_{q_o}^{q_i} F\, dq\right]\delta q_i$$

$$+ \left[-\frac{1}{q_o-q_i}\int_{q_o}^{q_i} \frac{\partial F}{\partial q_o} dq + \frac{1}{(q_o-q_i)^2}\int_{q_o}^{q_i} F\, dq\right]\delta q_o \tag{48}$$

IX. EVALUATION OF THE PARTIAL DERIVATIVES OF THE RANGE DRAG INTEGRAND. The partial derivatives of F with respect to v_o can be found by using Eqs. (45) and (28).

$$\frac{\partial F}{\partial v_o} = \frac{(1-H)\frac{\partial H}{\partial v_o} - H(-\frac{\partial H}{\partial v_o})}{(1-H)^2} = \frac{\frac{\partial H}{\partial v_o}}{(1-H)^2} \tag{49}$$

where

$$\frac{\partial H_o}{\partial v_o} = \frac{c}{g}\frac{2v_o}{1+q_o^2}[p(q)-p(q_o)] = \frac{2}{v_o} H \tag{50a}$$

Combining the above we have

$$\frac{\partial F}{\partial v_o} = \frac{2}{v_o}\frac{H}{(1-H)^2} \tag{50b}$$

Similarly the partial derivatives of F with respect to q_o is

$$\frac{\partial F}{\partial q_o} = \frac{\frac{\partial H}{\partial q_o}}{(1-H)^2} \tag{51}$$

where

$$\frac{\partial H}{\partial q_o} = \frac{c}{g} v_o^2(-1)(1+q_o^2)^{-2}2q_o[p(q)-p_o(q_o)]$$

$$+ \frac{c}{g} \frac{v_o^2}{1+q_o^2} (-1) \frac{dp_o}{dq_o}$$

$$= - \frac{2q_o}{1+q_o^2} H - \frac{c}{g} \frac{v_o^2}{1+q_o^2} \frac{dp_o}{dq_o} . \tag{52}$$

Substituting Eq. (52) into Eq. (51) one obtains

$$\frac{\partial F}{\partial q_o} = - \frac{2q_o}{1+q_o^2} \frac{H}{(1-H)^2} - \frac{c}{g} \frac{v_o^2}{(1+q_o^2)} \frac{dp_o}{dq_o} \frac{1}{(1-H)^2} . \tag{53}$$

X. <u>VARIATION OF THE RANGE DRAG FUNCTION.</u> By substituting Eqs. (45), (50) and (53) into Eq. (48), we have

$$\delta G_x = [- \frac{1}{q_o-q_i} \int_{q_o}^{q_i} \frac{H}{(1-H)^2} dq] \frac{2\delta v_o}{v_o}$$

$$+ [- \frac{H_{q=q_i}}{1-H_{q=q_i}} - \frac{1}{q_o-q_i} \int_{q_o}^{q_i} \frac{H}{1-H} dq] \frac{\delta q_i}{q_o-q_i}$$

$$+ [\frac{2q_o}{1+q_o^2} \int_{q_o}^{q_i} \frac{H}{(1-H)^2} dq + \frac{1}{q_o-q_i} \int_{q_o}^{q_i} \frac{H}{1-H} dq$$

$$+ \frac{c}{g} \frac{v_o^2}{1+q_o^2} \frac{dp_o}{dq_o} \int_{q_o}^{q_i} \frac{dq}{(1-H)^2}] \frac{\delta q_o}{q_o-q_i} . \tag{54}$$

It is noted that the difference of the end slope is not zero, i.e., $q_o-q_i \neq 0$. Therefore, the problem does not become singular. We have expressed the variation δG_x in terms of the variational parameter δv_o,

δq_i, and δq_o. However, the variational parameter δq_i at the impact point is not known explicitly and must be eliminated by using another variation in the direction of the elevation y.

XI. VARIATIONAL EQUATION FOR THE RANGE. By substituting Eq. (54) into Eq. (42) and grouping the coefficients for the variational terms, we have

$$I_v \frac{2\delta v_o}{v_o} + I_{qi} \frac{\delta q_i}{q_o - q_i} + I_{qo} \frac{\delta q_o}{q_o - q_i} = 0 \tag{55}$$

where

$$I_v = 1 - \frac{1}{G_x} \frac{1}{q_o - q_i} I_{12}(q_o, q_i) \tag{56}$$

$$I_{q_i} = -1 - \frac{1}{G_x} \frac{1}{q_o - q_i} I_{11}(q_o, q_i) - \frac{1}{G_x} \frac{H_i}{1 - H_i} \tag{57}$$

and

$$I_{q_o} = 1 + \frac{1}{G_x} \frac{1}{q_o - q_i} I_{11}(q_o, q_i) + \frac{1}{G_x} \left(\frac{2q_o}{1+q_o{}^2}\right) I_{12}(q_o, q_i)$$

$$+ \frac{1}{G_x} \frac{c}{g} \frac{v_o{}^2}{1+q^2} \frac{dp_o}{dq_o} I_{02}(q_o, q_i) - \frac{2q_o}{1+q_o{}^2} (q_o - q_i) . \tag{58}$$

In turn, the integral I_{11}, I_{12}, and I_{02}, and other terms are given as follows.

$$I_{11}(q_o, q_i) = \int_{q_o}^{q_i} \frac{H(q, q_o, v_o{}^2, c/g)}{1-H} dq , \tag{59}$$

$$I_{12}(q_o, q_i) = \int_{q_o}^{q_i} \frac{H}{(1-H)^2} dq , \tag{60}$$

$$I_{02}(q_o, q_i) = \int_{q_o}^{q_i} \frac{1}{(1-H)^2} dq , \tag{61}$$

$$H_i = H(q = q_i) \tag{62}$$

and

$$G_x = 1 - \frac{1}{q_o - q_i} I_{11} . \tag{63}$$

It is noted that δq_i in Eq. (55) has to be eliminated in solving the sensitivity problem. Similar variation equation may be obtained by considering the variation of the elevation.

XII. THE SOLUTION FOR ELEVATION. The differential equation for elevation was given in Eq. (22) and the solution for u is in Eq. (27).

Substituting Eq. (27) into Eq. (22) gives

$$dy = -\frac{v_o^2}{g(1+q_o^2)}\left[1 + \frac{H(q,q_o,v_o^2,c/g)}{1-H}\right]q\,dq \; . \tag{64}$$

Integrating the above one obtains

$$y_i - y_o = -\frac{v_o^2}{g(1+q_o^2)}\left[\frac{q_i^2-q_o^2}{2} + \int_{q_o}^{q_i}\frac{qH}{1-H}\,dq\right] \; . \tag{65}$$

Rearranging yields the relationship between the range Y, the end slope function Λ, and the elevation drag function G_y.

$$Y(y_i,y_o,v_o)/\Lambda(q_i,q_o) = \frac{1}{2}(q_o+q_i) + G_y(q_i,q_o,v_o^2,c/g) \tag{66}$$

where the nondimensional elevation is

$$Y(y_i,y_o,v_o) = \frac{g(y_i-y_o)}{v_o^2} \; . \tag{67}$$

Λ is given in Eq. (20) and the elevation drag function is

$$G_y(q_i,q_o,v_o^2,c/g) = -\frac{1}{q_o-q_i}\int_{q_o}^{q_i} qF(q,q_o,v_o^2,c/g)\,dq \; . \tag{68}$$

A separate form of Eq. (66) may be written as

$$\left[\frac{g(y_i-y_o)}{v_o^2}\right]/\left[\frac{q_o-q_i}{1+q_o^2}\right] = \frac{1}{2}(q_o+q_i) + \frac{1}{q_o-q_i}\int_{q_o}^{q_i} qF\,dq \; . \tag{69}$$

Note that the left side of Eq. (66) contains no drag coefficient c. It is the term G_y which is a function of drag coefficient c.

For $\qquad c = 0 \qquad G_y = 0 \; .$ (70)

XIII. TERRAIN SLOPE FROM LAUNCH POINT TO TARGET POINT. If Eq. (69) is divided by Eq. (37) with the aid of Eqs. (35) and (68), one obtains

$$\frac{y_i-y_o}{x_i-x_o} \overset{\Delta}{=} m = \frac{(1/2)(q_o+q_i)+G_y}{G_x} \qquad (71)$$

where m is the terrain slope from launch point to target point, a constant parameter. Therefore,

$$(1/2)(q_o+q_i) + G_y = mG_x \ . \qquad (72)$$

It is noted that for m = 0,

$$q_i = -q_o - 2G_y \ . \qquad (73)$$

From Equations (36), (70) and (72) we have for c = 0,

$$q_i + q_o = m \ . \qquad (74)$$

We use Equation (71) to find the variational equation for the elevation. Taking the variation of Eq. (72) for any given m, we have

$$(1/2)(\delta q_o+\delta q_i) + \delta G - mG_x\frac{\delta G_x}{G_x} = 0 \qquad (75)$$

where $\delta G_x/G_x$ is given in Equation (42) and

$$\delta G_y = - \frac{1}{q_o-q_i} \delta[\int_{q_o}^{q_i} qF\ dq]$$

$$- (-1)(q_o-q_i)^{-2}\delta(q_o-q_i)\int_{q_o}^{q_i} qF\ dq \qquad (76)$$

is obtained from Eq. (68).

It is noted that for c = 0, both δG_y and δG_x are zero in Eq. (75).

It can also be proved that the result for δG_y is

$$\delta G_y = [-\frac{1}{q_o-q_i}\int_{q_o}^{q_i}\frac{qH}{(1-H)^2}dq]\frac{2\delta v_o}{v_o}$$

$$+[-\frac{q_i H_{q=q_i}}{1-H_{q=q_i}}-\frac{1}{q_o-q_i}\int_{q_o}^{q_i}\frac{qH}{1-H}dq]\frac{\delta q_i}{q_o-q_i}$$

$$+[\frac{2q_o}{1+q_o^2}\int_{q_o}^{q_i}\frac{qH}{(1-H)^2}dq+\frac{1}{q_o-q_i}\int_{q_o}^{q_i}\frac{H}{1-H}dq$$

$$+\frac{c}{g}\frac{v_o^2}{1+q_o^2}\frac{dp_o}{dq_o}\int_{q_o}^{q_i}\frac{qdq}{(1-H)^2}]\frac{\delta q_o}{q_o-q_i}\ . \tag{77}$$

XIV. VARIATIONAL EQUATION FOR THE ELEVATION. By substituting Equations (77) and (42) into Equation (75) and grouping the coefficients for the variational terms, we have

$$J_v\frac{2\delta v_o}{v_o}+J_{q_i}\frac{\delta q_i}{q_o-q_i}+J_{q_o}\frac{\delta q_o}{q_o-q_i}=0 \tag{78}$$

where

$$J_v = -\frac{1}{q_o-q_i}J_{12}+mG_x \tag{79}$$

$$J_{q_i} = -\frac{1}{2}(q_o-q_i)-\frac{1}{q_o-q_i}J_{11}-\frac{q_iH_i}{1-H_i}-mG_x \tag{80}$$

and

$$J_{q_o} = \frac{1}{2}(q_o-q_i)+\frac{1}{q_o-q_i}J_{11}+\frac{2q_o}{1+q_o^2}J_{12}$$

$$+\frac{c}{g}\frac{v_o^2}{1+q_o^2}\frac{dp_o}{dq_o}J_{02}-mG_x[-1+\frac{2q_o(q_o-q_i)}{1+q_o^2}]\ . \tag{81}$$

In turn, the integral J_{11}, J_{12} and J_{02}, and other terms are given as follows.

$$J_{11} = \int_{q_o}^{q_i} qH(q,q_o,v_o^2,c/g)/(1-H)\,dq\ , \tag{82}$$

$$J_{12} = \int_{q_o}^{q_i} qH/(1-H)^2 \, dq \ , \tag{83}$$

$$J_{02} = \int_{q_o}^{q_i} q/(1-H)^2 \, dq \ , \tag{84}$$

and

$$G_x = 1 - \frac{1}{q_o - q_i} I_{11}. \tag{85}$$

XV. THE SENSITIVITY COEFFICIENT FOR EXTERIOR BALLISTICS. Eliminating $\delta q_i/(q_i - q_o)$ from Equations (55) and (78) we have

$$[(I_v/Iq_i) - (J_v/Jq_i)] \frac{2\delta v_o}{\delta v_o} + [(Iq_o/Iq_i) - (Jq_o/Jq_i)] \frac{\delta q_o}{q_o - q_i} = 0 \tag{86}$$

From which one obtains the sensitivity coefficient through the aid of Equation (19)

$$S = \frac{\delta\theta_o}{\delta v_o/v_o} = \frac{(I_v/Iq_i) - (J_v/Jq_i)}{(Iq_o/Iq_i) - (Jq_o/Jq_i)} \left[\frac{-2(q_o - q_i)}{1 + q_o^2}\right]. \tag{87}$$

It is noted that Equation (87) requires the evaluation of the integrals I and J, which are given in Equations (56) through (63) and Equations (78) through (85).

XVI. CONCLUSIONS. The following results are concluded in this paper:

1. The principal equation of exterior ballistics is derived with the trajectory slope as independent variables.

2. The closed form solution for the horizontal component of trajectory velocity is determined for the case of exterior ballistics with velocity square damping.

3. The nondimensional range is obtained in terms of an end slope function and a range drag function.

4. Variations of the nondimensional range are expressed as variations of launch velocity.

5. Variations of the range drag function are in terms of the variations of the range drag integral.

6. The range drag integral has parameters in the integrand as well as the upper and lower limits. The variations of this integral are found.

7. The partial derivatives of the range drag integrand are evaluated.

8. The variational equation for the range is in terms of elements involving three integrals as coefficients of three variational parameters.

9. The variational parameters are that of launch velocity, the launch elevation angle, and the impact elevation angle.

10. The average of the end slopes is equal to the terrain slope times the range drag function minus the elevation drag function.

11. Variations of the nondimensional elevation are expressed as variations of the end slopes and the variations of the drag function.

12. The variational equations for the elevation are determined similar to that for the range.

13. Eliminating the variations of impact slope, δq_i, from the set of two variational equations gives the ratio of the coefficients of $\delta v_o/v_o$ and $\delta q_o/(q_o-q_i)$.

14. The sensitivity $\delta\theta_o/(\delta v_o/v_o)$ may be obtained by dividing this ratio $\delta q_o/(\delta v_o/v_o)$ by the quantity $(1+q_o^2)$.

Numerical calculations of this problem will be carried out in the future.

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