It is known that if $T: X \rightarrow X$ is completely continuous if there exists an $n > 0$ such that $T^n$ is completely continuous, then $T$ point dissipative implies $T$ is bounded dissipative and has a fixed point (see Billotti and LaSalle [1]). This result is used, for example, in studying retarded functional differential equations.

This result has been extended by Hale and Lopes [8]. They get the result that if $T$ is an $\alpha$-contraction and compact dissipative
then $T$ is bounded dissipative and has a fixed point. This applies, for example, to stable neutral functional differential equations and certain retarded functional differential equations of infinite delay. These results are contained in Hale [5].

The above result requires the stronger assumption of compact dissipative. The principle result of this paper will be to get similar results under the weaker assumption of point dissipative. We will need to add additional hypotheses on the space and the operator $T$. We will then show how these hypotheses are naturally satisfied for stable neutral functional differential equations.

The paper will be divided into four sections. The first will contain various definitions. The second will contain an abstract theorem relating point dissipative in one space to bounded dissipative in another, and to the existence of a fixed point. The third section will apply the result to stable neutral functional differential equations. The final section gives applications to retarded functional differential equations with infinite delay.
STABILITY AND FIXED POINTS OF
POINT DISSIPATIVE SYSTEMS

BY

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JUNE, 1979

*This research was supported by the Air Force Office of Scientific Research under AF-AFOSR 76-3092A. Approved for public release; distribution unlimited.
Abstract: It is known that if $T: X \rightarrow X$ is completely continuous or if there exists an $n_0 > 0$ such that $T^{n_0}$ is completely continuous, then $T$ point dissipative implies $T$ is bounded dissipative and has a fixed point (see Billotti and LaSalle [1]). This result is used, for example, in studying retarded functional differential equations.

This result has been extended by Hale and Lopes [8]. They get the result that if $T$ is an $\alpha$-contraction and compact dissipative then $T$ is bounded dissipative and has a fixed point. This applies, for example, to stable neutral functional differential equations and certain retarded functional differential equations of infinite delay. These results are contained in Hale [5].

The above result requires the stronger assumption of compact dissipative. The principle result of this paper will be to get similar results under the weaker assumption of point dissipative. We will need to add additional hypotheses on the space and the operator $T$. We will then show how these hypotheses are naturally satisfied for stable neutral functional differential equations and retarded functional differential equations of infinite delay.

The paper will be divided into four sections. The first will contain various definitions. The second will contain an abstract theorem relating point dissipative in one space to bounded
dissipative in another, and to the existence of a fixed point.
The third section will apply the result to stable neutral functional
differential equations. The final section gives applications to
retarded functional differential equations with infinite delay.

This paper is a part of my thesis at Brown University. I am
especially grateful to Jack K. Hale for his help and supervision
in the preparation of this paper.
1. **Definitions.**

In the following definitions, $X$ will denote a metric space, and $\mathcal{B}$ will denote the set of bounded subsets of $X$.

**Definition 1.1:** The $\alpha$-measure of noncompactness is a map $\alpha: \mathcal{B} \to [0, \infty)$ defined by $\alpha(B) = \inf \{d/\text{there is a finite cover of } B \text{ with sets in } X \text{ of diameter less than } d\}$.

**Definition 1.2:** $T: X \to X$ is an $\alpha$-contraction if there is a $k \in (0, 1)$ such that for all $B \in \mathcal{B}$ we have $\alpha(TB) \leq k\alpha(B)$.

**Definition 1.3:** $T: X \to X$ is $\alpha$-condensing if for all $B \in \mathcal{B}$ we have $\alpha(TB) < \alpha(B)$ with equality if and only if $\alpha(B) = 0$.

**Definition 1.4:** A measure of noncompactness on $X$ is a map $\beta: \mathcal{B} \to [0, \infty)$ with the two properties

(i) $\beta(B) = 0$ if and only if $\text{Cl}(B)$ is compact and

(ii) $\beta(B) \leq \beta(C)$ if $B \subseteq C$.

The definitions of a $\beta$-contraction and $\beta$-condensing map are analogous to those of an $\alpha$-contraction and $\alpha$-condensing map.

**Definition 1.5:** Let $T: X \to X$ and let $S$ be a collection of sets. A bounded set $B \subseteq X$ **dissipates** $S$-sets under $T$ if for
any \( C \in S \) there is an integer \( n_0(C) \) such that \( T^n(C) \subset B \) for \( n > n_0(C) \). If \( S = \{\{x\} / x \in X\} \), we say \( T \) is point dissipative. If \( S = \{J \subset X / J \) is compact\}, we say \( T \) is compact dissipative. If \( S \) contains a neighborhood of any point, we say \( T \) is local dissipative. If \( S \) contains a neighborhood of any compact set, we say \( T \) is local compact dissipative. If \( S \) contains all bounded subsets of \( X \), we say \( T \) is bounded dissipative or ultimately bounded.

Remark 1.1: Local dissipative and local compact dissipative are always equivalent. If \( T \) is continuous then local dissipative and compact dissipative are also equivalent.

Definition 1.6: The orbit of \( B, \gamma^+(B) \), for \( B \subset X \), is defined by
\[
\gamma^+(B) = \bigcup_{n=0}^{\infty} T^n(B)
\]

Definition 1.7: \( H: \emptyset \to \emptyset \) is a type 2 set operator if for any \( B \in \emptyset \), \( H(B) = \bigcup \{H(A) / A \) is a finite subset of \( B\} \).

Definition 1.8: Let \( B \in \emptyset \). The orbit of \( B \) under \( H \), \( \gamma^+_H(B) \), is defined
\[
\gamma^+_H(B) = \bigcup_{n=0}^{\infty} H^n(B)
\]

Definition 1.9: \( H: \emptyset \to \emptyset \) is asymptotically smooth if, for any \( B \in \emptyset \) with \( \gamma^+_H(B) \) bounded, there is a compact set \( J \) such that \( H^n(B) + J \) (i.e. for every \( \epsilon > 0 \) there exists an \( n_0 \) such that \( n > n_0 \) implies \( H^n(B) \subset J + B_\epsilon(0) \), where \( B_\epsilon(0) \) is the ball of radius \( \epsilon \) centered at \( 0 \)).

Definition 1.10: A set \( K \in \emptyset \) is stable under \( H \) if, for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, for any
Definition 1.11: A set \( K \in \mathcal{G} \) is uniformly asymptotically stable if it is stable and there is a \( \delta_0 > 0 \) such that for any \( \varepsilon > 0 \) there is an \( n_0 \) such that \( B \subset K + B_{\delta_0}(0) \) implies 
\[ H^n(B) \subset K + B_{\varepsilon}(0) \quad \text{for} \quad n \geq n_0. \]

Definition 1.12: A set \( J \in \mathcal{G} \) is invariant under \( T \) if \( TJ = J \).


Theorem 2.1: Let \( i: X_1 \subset X_2 \) be a compact imbedding where \( X_i \) are Banach spaces with norms \( \| \cdot \|_i \). Let \( T, C, \) and \( U \) be continuous operators mapping \( X_i \) into itself. Denote the topology of \( X_i \) by \( \mathcal{T}_i \). Let \( T = C + U \) with \( C \) a contraction in \( X \) and \( U: (X_1, \mathcal{T}_2) + (X_1, \mathcal{T}_1) \) mapping bounded sets to bounded sets. Let \( C(0) = 0 \). Let \( B_R^i = \{ x \in X_i \mid \| x \|_i < R \} \). Then the following conclusions holds:

1. If \( B \subset B^1_R \) and \( R > 0 \) then there exists a \( K = K(L, R) \) such that, for any \( n^* \) with \( 0 \leq n^* < \infty \), then \( \bigcup_{0 \leq m \leq n^*} T^m(B) \subset B^2_R \) implies \( \bigcup_{0 \leq m \leq n^*} T^m(B) \subset B^1_K \).

2. If \( T \) is point dissipative in \( X_2 \), then \( T \) is bounded dissipative in \( X_1 \).

Remark 2.1: It is not necessary to assume that \( U: (X_1, \mathcal{T}_2) + (X_1, \mathcal{T}_1) \) takes bounded sets into bounded sets. The conclusions are naturally weaker and hold only for those bounded sets whose
images under $U$ are bounded.

Proof of 1: Let $h: \mathbb{R}^+ \to \mathbb{R}^+$ be chosen so that $U(X_1 \cap B_R^2) \subseteq B_h^1(R)$. Suppose $B \subseteq B_L^1$ and $\bigcup_{0 \leq m \leq n^*} T^m(B) \subseteq B_R^2$. Let $\lambda \in (0,1)$ be the contraction constant for $C$. Let $K(L,R) = (1-\lambda)^{-1}(h(R) + L)$. If $x \in B$ we can easily show by induction $\bigcup_{0 \leq m \leq n^*} T^m(B) \subseteq B_R^1$. It is obviously true for $n = 0$. If it is assumed true for $n$, then

$$||T^{n+1}x||_1 \leq \lambda ||T^nx||_1 + h(R)$$

$$\leq \lambda (1-\lambda)^{-1}(h(R) + L) + h(R)$$

$$\leq \lambda (1-\lambda)^{-1}(h(R) + L) + (h(R) + L)$$

$$\leq (1-\lambda)^{-1}(h(R) + L) = K(L,R).$$

Hence, $\bigcup_{0 \leq m \leq n^*} T^m(B) \subseteq B_R^1$.

We shall need the following lemma before proceeding.

Lemma 2.1: Under the hypotheses of Theorem 2.1, there exists a number $n_1(L,R)$ and a number $Q(R) > 0$ such that if

$$\bigcup_{0 \leq m \leq n^*} T^m(B) \subseteq B_L^1 \cap B_R^2$$

then $\bigcup_{n_1(L,R) \leq m \leq n^*} T^m(B) \subseteq B_Q^1(R)$. Here we have $n_1$ independent of $n^*$ and $Q$ independent of both $n^*$ and $L$.

Proof of Lemma: Let $\delta_0 > 0$ and $Q(R) = (1-\lambda)^{-1}h(R) + \delta_0$.

If $B \subseteq X_1$ is bounded let $||B||_1 = \sup\{||x||_1/x \in B\}$. Also,
$K(0,R) = (1-\lambda)^{-1}h(R)$. If $L \leq K(0,R)$ we obtain the conclusion by letting $n_1 = 0$ and applying the same inductive argument used to prove part 1 of the theorem. If $L > K(0,R)$ choose $n_1$ large enough so that $\lambda^{n_1}(L-K(0,R)) < \delta_0$. Now we notice

$$||TB^1_L||_1 \leq \lambda L + h(R)$$

$$= \lambda(L+K(0,R)) + \lambda(1-\lambda)^{-1}h(R) + h(R)$$

$$= \lambda(L-K(0,R)) + (1-\lambda)^{-1}h(R)$$

$$= \lambda(L-K(0,R)) + K(0,R)$$

By induction we get for $n \leq n^*$

$$||T^{nB^1_L}||_1 \leq \lambda^n(L-K(0,R)) + K(0,R).$$

Hence, for $n_1 \leq n \leq n^*$ we have $||T^{nB^1_L}||_1 \leq K(0,R) + \delta_0 = Q(R)$. This completes the proof of the lemma.

**Proof of Part 2 of the Theorem:**

Let $B^2_k$ dissipate points in $X_1$. By part (1), it is clear that orbits of points are bounded in $X_1$. Since the orbit of any point is eventually dissipated by $B^2_R$, Lemma 2.1 implies that $B^1_{Q(R)}$ dissipates points in $X_1$.

We will now show $\gamma^+(B^1_{Q(R)})$ is bounded in $X_1$. Let $x \in B^1_{Q(R)}$ and $\gamma^+(x) \subset B^2_{C(x)}$. We may find a constant $C(x)$ since point dissipative in $X_2$ implies orbits of points are bounded.
in $X_2$. Then we have $\gamma^+(x) \subseteq B^1_{K(Q(R),C)}$ by part 1. Let $n_0(x)$ be chosen so that if $n > n_0(x)$ then $T^n x \in B^2_R$. Let $n_1(x) = n_1(K(Q(R),C(x)), R)$. Let $n^*(x) = n_0(x) + n_1(x)$. By continuity we may choose $\delta(x)$ so that

$$\bigcup_{0 \leq m \leq n^*(x)} T^m(B^2_\delta(x) \cap B^1_{Q(R)}) \subseteq B^2_C(x)$$

and Lemma 2.1 implies

$$T^n(x)(B^2_\delta(x) \cap B^1_{Q(R)}) \subseteq B^1_{Q(R)}.$$

Since $B^1_{Q(R)}$ is a compact set in $X_2$, the sets $B^1_{Q(R)} \cap B^2_\delta(x)$ form an open cover for which there is a finite subcover

$$\{B^1_{Q(R)} \cap B^2_\delta(x_1)\}_{i=1}^m.$$ Let $N = \max\{n^*(x_i)\}_{i=1}^m$. Then it is clear that $\gamma^+(B^1_{Q(R)}) = \bigcup_{0 \leq m \leq N} T^m(B^1_{Q(R)})$ since any point in $B^1_{Q(R)}$ returns to $B^1_{Q(R)}$ by the $N$th iteration.

To show any set $B^1_A$ is dissipated by $\gamma^+(B^1_{Q(R)})$ we use the same type of argument. We form an open cover of $B^1_A$ by neighborhoods $\{B^2_\delta(x) \cap B^1_A\}$ such that there is an $n^*(x)$ such that $T^n(x)(B^2_\delta(x) \cap B^1_A) \subseteq B^1_{Q(R)}$. Then we take a finite subcover

$$\{B^2_\delta(x_1) \cap B^1_A\}_{i=1}^m$$ and let $N = \max\{n^*(x_i)\}_{i=1}^m$. Clearly $n > N$ implies $T^n(B^1_A) \subseteq \gamma^+(B^1_{Q(R)})$. Since the argument is analogous I have only sketched the proof. This completes the proof.

**Corollary 2.1**: Suppose the assumptions of Theorem 2.1, $T$ is point dissipative in $X_2$ and $\beta$-condensing in $X_1$ with $\beta$ a
measure of noncompactness satisfying the property $\beta(A \cup B) = \beta(A)$ if $B$ is a finite set. Then there exists a maximal compact invariant set in $X_1$ which is uniformly asymptotically stable.

Proof: Theorem 2.1 implies that $T$ is bounded dissipative. Let $B^1_R$ dissipate bounded sets. Hence, orbits of bounded sets are bounded. Since $T$ is $\beta$-condensing, $T$ is asymptotically smooth (see Massatt [10]). For any bounded set $B$ there is a compact invariant set $\omega_B \subset B^1_R$ that attracts $B$. Let $\omega = \bigcup_{B \in \emptyset} \omega_B$. Then $\omega \subset B^1_R$ and $\omega$ is invariant. Since $T$ is $\beta$-condensing this implies $\omega$ is precompact. Clearly $\omega$ attracts all bounded sets. $\omega$ is stable since otherwise there would exist an $\varepsilon > 0$ and sequences $\{x_k\} \subset X_1, \{n_k\} \to \infty$ such that $d(x_k, \omega) \to 0$ and $d(T^n_{x_k} \omega) > \varepsilon$ for all $k$. But $\{x_k\}$ is a bounded set and so $\omega$ attracts $\{x_k\}$. This is clearly a contradiction. The fact that $\omega$ is stable and attracts bounded sets clearly implies that $\omega$ is uniformly asymptotically stable. Q.E.D.

Corollary 2.2: Under the hypotheses of Theorem 2.1, let $T$ be $\beta$-condensing in $X_1$ and point dissipative in $X_2$ with $\beta$-satisfying the following properties:

(i) $\beta(\overline{A}) = \beta(A)$ and

(ii) $\beta(A \cup B) = \max[\beta(A), \beta(B)]$. Then $T$ has a fixed point.

Proof. It is already known that if $T$ is $\beta$-condensing and compact dissipative it has a fixed point (see Massatt [10] for general measures of noncompactness and Hale and Lopes [8], Nussbaum [12] for $\alpha$-measures).
Since bounded dissipative implies compact dissipative, the corollary is proved.

3. Applications to Neutral Equations

In this section, we apply the results of Section 2 to show that a point dissipative periodic stable neutral functional differential equation has a periodic solution and that the period map is bounded dissipative in $W^\infty_1$. We will also get the existence of a maximal compact invariant set in $W^\infty_1$ which is uniformly asymptotically stable, with respect to the period map $T$.

Previously, the existence of a fixed point was only known under the assumption of compact dissipative. Under this assumption, the existence of a maximum compact invariant set in $C$ which is uniformly asymptotically stable is also known. (See Hale [5]).

For $r > 0$, let $C = C([-r,0],\mathbb{R}^n)$ be the space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ with the supremum norm. Let $W^\infty_1 = W^\infty_1([-r,0],\mathbb{R}^n)$ be the space of absolutely continuous functions with derivative essentially bounded. Let

$$
||\phi||_{W^\infty_1} = \sup_{-r,0]_{\infty}} |\phi(\theta)| + \text{ess sup}_{-r,0]} |\phi'(\theta)|
$$

If $x(\cdot): [-r,A) \rightarrow \mathbb{R}^n$, $A > 0$, let $x_t(\cdot): [-r,0] \rightarrow \mathbb{R}^n$ be defined by $x_t(\theta) = x(t+\theta)$ for $t \in [0,A)$, $-r < \theta < 0$. Suppose $D: C \rightarrow \mathbb{R}^n$ is linear and $D\phi = \phi(0) - L\phi$ where $L$ is nonatomic at zero. Suppose $f: \mathbb{R}^+ \times C \rightarrow \mathbb{R}^n$ is completely continuous. A neutral functional differential equation (NFDE) is a relation
\[ \frac{d}{dt} D_x t = f(t, x_t). \]  

(3.1)

A stable neutral functional differential equation (SNFDE) is a NFDE for which \( D \) is stable; that is, the zero solution of \( D x_t = 0 \) is uniformly asymptotically stable. We often use properties of these equations without proof. The details can be found in Hale [5].

Our first result deals with continuous dependence in \( W_1^{\infty} \).

**Theorem 3.1:** If the solutions of (3.1) are uniquely defined by the initial data then, for \( t > t_0 \), the solution map \( X_{D, f}(t, t_0) : W_1^{\infty} \to W_1^{\infty} \) given by \( X_{D, f}(t, t_0)\phi = x_t, x_{t_0} = \phi \), is continuous in \( \phi \). It is also continuous with respect to \( f \) with the topology of uniform convergence on bounded sets of \( \mathbb{R}^+ \times C \).

In general, there is no continuous dependence with respect to \( (t, t_0) \). This was the motivation for Melvin [11] to discuss continuous dependence in \( W_1^{\infty} \) in the weak-*-topology.

**Proof:** To prove the theorem we need only prove it for time \( t > 0 \) arbitrarily small, since for larger times, we may proceed by steps.

For \( t \) small enough we first prove \( X_{D, f}(t, 0) : W_1^{\infty} \to W_1^{\infty} \). The proof is based on the Schauder fixed point theorem. Let \( x(\cdot) : [-r, t] \to \mathbb{R}^n \) be the solution with initial condition \( \phi \in W_1^{\infty} \). Let \( z(\cdot) \in W_1^{\infty}[0, t] \). Let \( D\phi = \phi(0) + L\phi \) with \( L \) nonatomic. Let \( \phi^t : [-r, t] \to \mathbb{R}^n \) be defined by
\[
\phi^\tau(\theta) = \begin{cases} 
\phi(\theta), & \theta \in [-\tau,0] \\
\phi(0), & \theta > 0.
\end{cases}
\]

Let \( z_t: [-\tau,0] \to \mathbb{R}^n \) be defined by
\[
z_t(\theta) = \begin{cases} 
z(t+\theta), & \theta + t \geq 0 \\
0, & \theta + t < 0.
\end{cases}
\]

If
\[
x(t) = \begin{cases} 
\phi(t), & t < 0 \\
z(t) + \phi(0), & t \geq 0.
\end{cases}
\]

is a solution of (3.1) then \( z(\cdot) \) must be a fixed point of \( T \) where \((Tz)(t) = -Lz_t - L\Phi^\tau_t + \Phi + \int_0^t f(s,z_s)ds\). Since \( f \) is completely continuous and \( L \) is a contraction on \( z(\cdot) \) for small enough \( \tau \), there exists a \( \tau > 0 \), a closed, bounded, convex set \( B \subset W_1^{\infty} \) such that \( TB \subset B \). Since \( B \) is compact in \( C \) and \( T \) is continuous in \( C \) we have a fixed point by the Schauder fixed point theorem.

To prove continuous dependence in \( W_1^{\infty} \) is a similar argument. Given an initial function \( \phi_0 \in W_1^{\infty} \) and a solution \( x_0^0(t) \) defined on \([-\tau,\tau]\) with \( \tau > 0 \), \( x_0^0 = \phi_0 \), we show continuous dependence with respect to \( \phi_0 \) and \( f \). If \( x(t) = x_0^0(t) + z(t) + \phi(0) \) for \( t \in [0,\tau] \), \( x_0^0 = \phi_0 \), is the solution of (3.1) with initial condition \( \phi_0 + \phi \) then \( z(t) \) is a fixed point of
\[(Tz)(\theta) = -Lz_t - L\phi_t^\tau + L\phi + \int_0^t [f(s, x_0^0 + \phi_{s+s}^\tau + z_s) - f(s, x_0^0)]ds.\]

For \(\tau\) small enough, \(L\) is a contraction on \(z\) and there is an \(E \in \mathbb{R}^+\) such that \(TB^1_E \subset B^1_E\). Since \(B^1_E\) is compact in \(C\) we have a fixed point. Furthermore, as \(||\phi|||_{W^1_1} \to 0\) we may let \(E \to 0\). Since the fixed point must be in \(B^1_E\), the uniqueness implies the continuous dependence on \(f\). The proof for continuous dependence on \(f\) is analogous, and so the proof is omitted.

**Theorem 3.2:** A SNFDE given by (3.1) with \(f\) completely continuous, periodic in \(t\) of period \(\omega > 0\) and whose solution map \(X_D, f(\omega, 0) : C \to C\) is point dissipative, has a periodic solution of period \(\omega\).

**Proof:** We only prove the theorem under the hypothesis that the solution map takes bounded sets into bounded sets. The proof without this hypothesis is just a technical modification using the remark after Theorem 2.1.

Let \(X_1 = C[-r, 0]\) and \(X_2 = W^\omega[-r, 0]\). Let \(X_D, f(\omega) = X_D, f(\omega, 0)\) be the period map. Let \(T_{D, h(\omega)}\) be the solution map of the difference equation \(Dy_t = h(t)\). According to the theory in Hale [5] there are projection maps \(P\) and \(Q = I - P\), with \(Q\) finite dimensional and \(P : C \to Y_0\) where \(Y_0 = \{\phi \in C/D\phi = 0\}\). By the construction, the maps \(P\) and \(Q\) map \(X_1\) into itself also. If we let \(T_{D, 0} : Y_0 \to Y_0\) be the solution map of \(Dy_t = 0\) then, since \(D\) is stable, \(||T_{D, 0}^n(\omega)|| \leq K\lambda^n\) for some \(K > 0, \lambda \in [0,1]\).
We may now extend \( T_D^0 : Y_0 + Y_0 \) to \( T_D^0 : L^\infty + L^\infty \), and still get \( ||T_D^0(\omega)|| \leq K\lambda^n \). This is done by using the uniqueness of the \( X_D,0(\omega) \) map. Let \( z_k \in Y_0, y_0 \in L \) and \( \int_0^\theta z_k + \int_{-\theta}^0 y_0 \) in \( C \), with \( ||z_k||_C \leq ||y_0||_\infty \). Then \( \{X_D,0(\omega) \int_{-\theta}^0 z_k\} + X_D0(\omega) \int_{-\theta}^0 y_0 \)

in \( C \). But \( \frac{d}{d\theta} \int_{-\theta}^0 z_k \leq ||y_0||_\infty \) so for all \( k \) we get

\[
||\frac{d}{d\theta} X_D,0(\omega) \int_{-\theta}^0 z_k|| = ||T_D^n(\omega)z_k|| \leq K\lambda^n ||z_k|| \leq K\lambda^n ||y_0||_\infty.
\]

Hence, \( ||T_D^n(\omega)y_0|| \leq K\lambda^n ||y_0||_\infty \).

Since \( D \) is stable we know that \( X_D,0(\omega) \) is a stable operator in \( X_2 \). Also, from the theory in Hale \( \lim_{n \to \infty} X_D^n(\omega) \) exponentially approaches a finite dimensional subspace of \( C \) generated by the constant functions. We will call this map \( g(\cdot) : C \to C \). \( g \) may also be considered as a map \( g : X_1 \times X_1 \) which is continuous, finite dimensional and satisfies \( g : (X_1, \mathcal{F}_2) \to (X_1, \mathcal{F}_1) \) maps bounded sets to bounded sets.

Let \( C = X_D,0(\omega)P(\cdot) - g(P(\cdot)) \). Because of the exponential decay, there is an equivalent norm in \( X_1 \) where \( C \) is a contraction.

Now, let \( U = X_D,f(\omega) - C \). We need to show \( U : (X_1, \mathcal{F}_2) \to (X_1, \mathcal{F}_1) \) maps bounded sets to bounded sets and that

\( X_D,f(\omega) : X_1 \times X_1 \) is condensing. Notice \( U\phi = H\phi + g(P\phi) \) where \( H\phi \) is the solution at time \( \omega \) to the equation \( \frac{d}{dt} Dy_t = h_\phi(t) \)

with initial condition \( y_0 = Q\phi \) and \( h_\phi(t) = f(t, x_t) \) where \( x_t \) solves (3.1) with \( x_0 = \phi \). Recall that we are assuming
Let $X_{D,f}(\omega) : X_{2} \rightarrow X_{2}$ maps bounded sets to bounded sets. Let $B$ be a bounded set in $X_{2}$. Since $f$ is completely continuous, this insures $\{h_{\phi}(t)/\phi \in B\}$ are bounded uniformly on $[0,\omega]$. Since solutions to $Dy_{t} = h_{\phi}(t), \dot{y}_{0} = \frac{d}{dt} [Q\phi]$ are bounded we get $U : (X_{1}, \mathcal{T}_{2}) \rightarrow (X_{1}, \mathcal{T}_{1})$ maps bounded sets to bounded sets.

Next, we will show $U$ is completely continuous in $X_{1}$. Once we have this it is fairly easy to see $X_{D,f}(\omega) : X_{1} \rightarrow X_{1}$ is an $\alpha$-contraction.

For any $L > 0$ we must show $U(B_{L}^{1})$ is precompact. Since $\text{Cl} B_{L}^{1}$ is compact in $X_{2}$ we also get $\text{Cl}\{x_{t}/t \in [0,\omega], x_{0} \in B_{L}^{1}\}$ is compact in $X_{2}$. This implies $\{h_{\phi}(t)/\phi \in B_{L}^{1}\}$ lie in a compact subset of $C[0,\omega]$ since $h_{\phi}(t) = f(t,x_{t})$. By our continuous dependence theorem for $W_{1}^{\infty}$ this implies $\text{Cl} H(B_{L}^{1})$ is compact in $X_{1}$. Since $g$ also has finite dimensional range, we get $U : X_{1} \rightarrow X_{1}$ is completely continuous.

Hence, $X_{D,f}(\omega)$ is an $\alpha$-contraction in some equivalent norm in $X_{1}$. Since all the conditions of Theorem 2.1 are now satisfied we have a fixed point for $X_{D,f}(\omega)$, and hence a periodic solution of (3.1) of period $\omega$.

Remark 3.1: Under the assumption that $X_{D,f}(\omega) : X_{2} \rightarrow X_{2}$ takes bounded sets to bounded sets, the above proof also shows $X_{D,f}(\omega) : X_{1} \rightarrow X_{1}$ is bounded dissipative and there is a maximal compact invariant set in $X_{1}$ which is uniformly asymptotically stable.

Remark 3.2: The conclusions still hold if the operator $D$ in the NFDE is $\omega$-periodic in $t$. There are only minor technical changes.
in the proof for this case.

4. Applications to RFDE's with Infinite Delay.

In this section, we study the existence of periodic solutions of the equation

\[ \dot{x}(t) = f(t, x_t) \]  

where \( x_t(\mu) = x(t+\mu), \mu \in (-\infty, 0], \ \mathcal{F}: \mathbb{R}^+ \times X \rightarrow \mathbb{R}^n \) is completely continuous, \( X \) is a space of functions satisfying certain axioms to be specified below, and there is an \( \omega > 0 \) such that \( f(t, \phi) = f(t+\omega, \phi) \) for all \( t, \phi \).

Our objective is to show that point dissipative systems have an \( \omega \)-periodic solution. There are two notions of point dissipative for these systems. The usual one is the phase space \( X \) and the other is \( \mathbb{R}^n \). Let \( x(t, \phi) \) be the solution of \((\ast)\) with \( x(t, \phi) = \phi(t) \) for \( t \in (-\infty, 0] \). System \((\ast)\) is point dissipative in \( \mathbb{R}^n \) if there is a bounded set \( B_R \subseteq \mathbb{R}^n \) such that for any \( \phi \in X \) there is a \( t_0[\phi] > 0 \) such that \( x(t, \phi) \in B_R \) for \( t > t_0(\phi) \).

We will shortly give axioms which will show what phase spaces are permissible. From the axioms one can deduce that point dissipative in \( X \) implies point dissipative in \( \mathbb{R}^n \), but the converse need not hold. An example will be given later to illustrate this point. Our theorem will be proved for point dissipative in \( \mathbb{R}^n \).
We will first prove results for the space \( C_0^y = \{ y(\cdot): (-\infty, 0] \to \mathbb{R}^n/\|x\| = \sup_{\theta \in (-\infty, 0)} |e^{\gamma \theta} x(\theta)| < \infty, \lim_{\theta \to -\infty} |e^{\gamma \theta} x(\theta)| = 0 \} \) with \( \gamma > 0 \). This should help the reader understand some of the more general results which will be given later. Let

\[
W^{\infty, y'} = \{ x(\cdot): (-\infty, 0] \to \mathbb{R}^n/\|x\| = \sup_{\theta \in (-\infty, 0]} |e^{\gamma' \theta} x(\theta)| + \| e^{\gamma' \theta} x(\theta) \|_\infty < \infty \}
\]

**Theorem 4.1:** If the system (*) is point dissipative in \( \mathbb{R}^n \) and \( X = C_0^y \) then if \( \gamma' > y > 0 \) the period map is bounded dissipative in \( W^{\infty, y'} \) (if it maps bounded sets to bounded sets), there is a maximal compact invariant set which is uniformly asymptotically stable in \( W^{\infty, y'} \) and it has a fixed point.

**Proof:** Let \( X_2 = C_0^y, X_1 = W^{\infty, y'} \). We will show Theorem 2.1, Corollary 2.1, and Corollary 2.2 apply. Clearly \( i: X_1 \to X_2 \) is a compact imbedding. The solution map shall be denoted \( T_f(\omega) \). Clearly \( T_f(\omega): X_1 \to X_1 \) is continuous. We shall assume \( T_f(\omega) \) maps bounded sets to bounded sets (in \( C_0 \)) to avoid technical details. Let \( C\phi = \phi^\omega - \phi(0) \) where \( \phi(0) \) is the constant function. \( C \) is clearly a contraction. Since \( f \) is completely continuous, by arguments similar to the last theorem we get \( U: (X_1, E_2) \to (X_1, E_1) \) maps bounded sets to bounded sets and that \( U: X_1 \to X_1 \) is completely continuous. This also proves that \( T_f(\omega) \) is an \( \alpha \)-contraction. All the hypotheses are now
satisfied for applying Theorem 2.1, Corollary 2.1, and Corollary
2.2. Q.E.D.

Remark 4.1: There are many other spaces for which this theorem
could apply. For example, we could let $X_2 = L^2(g) \times \mathbb{R}^n = \{x(\cdot): (-\infty,0] \times \mathbb{R}^n/\|g(\cdot)x(\cdot)\|_2 + \|x(0)\| = \|x\| < \infty\}$ and

$X_1 = W^1_1(g') + \mathbb{R}^n = \{x(\cdot): (-\infty,0] + \mathbb{R}^n/\|x\| = \|g'(\cdot)x(\cdot)\|_2 + \|g'(\cdot)x(\cdot)\|_2 + \|x(0)\| < \infty\}$ with $0 < g(t) < g'(t) \leq K e^{at}$ for

some $K > 0$, $a > 0$, $g,g'$ monotonically increasing and

$$\lim_{t \to -\infty} \frac{g(t)}{g'(t)} = 0.$$

The local axioms on the phase space will be derived from
Schumacher [13]. The global axioms will come for the most past
from Hale and Kato [6].

Local Axioms: Let $X$ be a metrizable topological vector space
of functions $x: (-\infty,0] \rightarrow \mathbb{R}^n$ with Hausdorff equivalence $\sim$.
If $x,y \in X$ we say $x \sim y$ if there does not exist two open
disjoint sets $A$ and $B$ with $x \in A$ and $y \in B$. Let $X$ also
satisfy the following axioms:

1. For all $y \in X$, $\sigma \in \mathbb{R}^+$, the static continuation $y^\sigma$, 
defined by

$$y^\sigma(\mu) = \begin{cases} 
y(\sigma+\mu) & -\infty \leq \mu < -\sigma 
y(0) & -\sigma \leq \mu \leq 0 \end{cases}$$

belong to $X$.

2. $y \sim z$ implies $y^\sigma \sim z^\sigma$. 
(3) $\delta: X \rightarrow \mathbb{R}^n$ defined by $\delta x = x(0)$ is continuous.

(4) The map $\sigma \in \mathbb{R}^+ \rightarrow y^\sigma \in X$ is continuous for all $y \in X$.

(5) For all $\tau > 0$, the set $C_{\tau} = \{x(t): x \text{ is continuous with support in } [-\tau,0]\}$ belongs to $X$.

(6) If $C_{\tau}$ has the sup norm then $i: C_{\tau} \rightarrow X$, the inclusion map, is continuous for all $\tau > 0$.

Axiom 3 is stronger than Schumacher's stated assumption but the author feels Schumacher needs this assumption for his results.

With these assumptions and appropriate hypotheses on $f$ one can prove existence, uniqueness, continuous dependence, and continuation theorems. (See Hale and Kato [6] or Schumacher [13].) We shall always assume $f$ satisfies enough hypotheses to insure these properties.

Global Axioms:

(1) All continuous bounded functions are in $X$. The space of continuous bounded functions with the sup norm will be denoted $C_B^\sigma$.

(2) Let $B^\sigma = \{x^\sigma/x \in B\}$. If $B \subset X$ is bounded and $x \in B$ implies $\delta(x) = 0$ then $B^\sigma + 0$ as $\sigma \rightarrow \infty$.

(3) The map $i: C_B \rightarrow X$ is continuous.

Statement 2 may be stronger than one desires, and spaces have been suggested where (2) does not hold. A weaker version of (2) which suffices for our purposes is as follows.

(2') If $B \subset C_B$ is bounded in sup norm and $x \in B$ implies $\delta(x) = 0$ then $B^\sigma + 0$ in $X$. 
Examples.

(1) Let \( g(\cdot): (-\infty, 0] \to \mathbb{R}^+ \) be continuous, monotone increasing, and there exists \( K > 0, \alpha > 0 \) such that \( g(t) \leq Ke^{\alpha t} \). Let \( X = \{x(\cdot)/x(\cdot): (-\infty, 0] \to \mathbb{R}^n \) is continuous and \n\]
\[ \|x\| = \lim_{t \to (-\infty, 0)} |g(t)x(t)| < \infty. \]

For Axiom 2' we may replace the condition \( g(t) \leq Ke^{\alpha t} \) by \( \lim_{t \to -\infty} g(t) = 0. \)

(2) Let \( g \) be monotone increasing with \( \int_{-\infty}^{0} g < \infty \). Let \( r > 0 \) and for any locally measurable \( \phi: (-\infty, 0] \to \mathbb{R}^n \) whose restriction to \([-r, 0] \) is continuous let

\[ \|\phi\|_p = \left\{ \sup_{-r < \theta < 0} |\phi(\theta)|^p + \int_{-\infty}^{0} g(\theta)|\phi(\theta)|^p d\theta \right\}. \]

Let \( X \) denote the space of such functions with \( \|\phi\|_p < \infty \). This satisfies Axiom 2'. For Axiom 2 we may add the condition that there exists a \( K > 0, \alpha > 0 \) such that \( g(t) \leq Ke^{\alpha t} \).

(3) Let \( X \) be the space of continuous functions \( x(\cdot): (-\infty, 0] \to \mathbb{R}^n \) with the compact open topology.

Theorem 4.2: If the system (*) is point dissipative in \( \mathbb{R}^n \) and \( X \) is a Banach space satisfying global Axioms 1, 2, and 3 then there is an \( \omega \)-periodic solution of (*)}. Also, for some \( \gamma' > 0 \) the space \( W_1^\omega \gamma' \) may be compactly imbedded into \( X \) and is
bounded dissipative (if the solution map sends bounded sets to bounded sets) and there exists a compact invariant set which is uniformly asymptotically stable.

**Proof:** To prove this result we only need show there is a $\gamma > 0$ and an imbedding $i: C_0^\gamma \to X$ which is continuous. Then we may apply Theorem 4.1.

Let $B = \{ x \in C_B/ x(0) = 0, \ |x| < 1 \}$. Let $||B||_X = \lambda > 0$ where $||B||_X = \sup \{ |x|/ |x|/x \in B \}$. Let $\sigma > 0$ be chosen so that $||B^\sigma|| = \rho_0 \lambda$ for some $\rho_0 < 1$. Choose $\rho_1$ so that $\rho_0 < \rho_1 < 1$ and let $\gamma > 0$ be chosen so that $e^{-a\gamma} = \rho_1$. I claim $i: C_0^\gamma \to X$ is a continuous imbedding. Since $C_B$ is dense in $C_0^\gamma$ and any sequence of elements in $C_0^\gamma \cap C_B$ which is Cauchy in $C_0^\gamma$ is also Cauchy in $X$. We get that $i: C_0^\gamma \to X$ is a continuous imbedding. Q.E.D.

We now state the main result of this section.

**Theorem 4.3:** If system (*) is point dissipative in $\mathbb{R}^n$ then there is an $\omega$-periodic solution of (*). This theorem removes the restriction that $X$ be a Banach space and replaces the condition that $X$ satisfy global Axiom 2 with the weaker global Axiom 2'. The proof will use the following two results.

**Theorem 4.4** (Horn's Theorem [9]): Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space $X$ with $S_0, S_2$ compact and $S_1$ open in $S_2$. Let $T: S_2 \to X$ be a continuous mapping such that, for some integer $m > 0$, $T^j S_1 \subset S_2$ for $0 \leq j \leq m - 1$ and $T^j S_1 \subset S_0$
for \( j \geq m \). Then \( T \) has a fixed point.

The proof of the next theorem is identical to the proof of a
Theorem 4.1 in Massatt [10]. The proof will be given here, though.

**Theorem 4.5:** Let \( X_0 \) and \( X_1 \) be two Banach spaces with a con-
tinuous imbedding \( i: X_0 \hookrightarrow X_1 \). Let \( \mathcal{B}_1 \) be the collection of
bounded sets in \( X_1 \). Let \( H: \mathcal{B}_0 + \mathcal{B}_0 \) be a type 2 set operator.
Let \( \beta: \mathcal{B}_0 \to [0, \infty) \) be a map satisfying the following properties.

1. If \( B \in \mathcal{B}_0 \) then \( \beta(B) = 0 \) if and only if \( \text{Cl}_{X_1}(B) \) is
   compact in \( X_1 \).

2. Let \( A, B \in \mathcal{B}_0 \) and let \( B \) be a finite set, then
   \( \beta(A \cup B) = \beta(A) \). Under these conditions, if \( H \) is \( \beta \)-condensing
   then \( H \) restricted to \( \mathcal{B}_0 \) is asymptotically smooth in \( X_1 \).

**Proof of Theorem 4.4:** Let \( \gamma^+(B) \in \mathcal{B}_0 \). Let \( \mathcal{D}(B) = \{\{x^*_k, n^*_k\}/\{n_k\} + \infty, x^*_k \in H^k(B)\} \). Let \( P(\{x^*_k, n^*_k\}) = \{x^*_k\} \).
Let \( \eta = \sup\{\beta(Ph/h) \in \mathcal{D}(B)\} \). Note \( \eta < \infty \)
since \( \gamma^+(B) \in \mathcal{B}_0 \). We first show there is an \( h^* = \{x^*_k, n^*_k \} \in \mathcal{D}(B) \)
such that \( \beta(Ph^*) = \eta \). Let \( \{h_j\} \in \mathcal{D}(B) \) be a sequence
with \( \beta(Ph_j) \to \eta \). Let \( \hat{h}_j = \{(x^*_k, n^*_k) \in h_j/n_k > j\} \). Let
\( h^* = \bigcup_{j=1}^{\infty} h_j \) reordered in any way. Then we have \( h^* \in \mathcal{D}(B) \) and
so \( \eta \geq \beta(h^*) \geq \beta(\hat{h}_j) = \beta(h_j) + \eta \) as \( j \to \infty \). Hence, \( \beta(h^*) = \eta \).

Now for each \( (x^*_k, n^*_k) \in h^* \) there is a set \( \{x^*_{k,j}, n^*_{k,j} - 1\}_{j=1}^{n^*_k - 1} \in H^k(B) \times Z \) such that \( x^*_k \in H(\{x^*_{k,j}\}) \). Let
\[
g^* = \bigcup_{j=1}^{m_k} \{x^*_j, n^*_k - 1\}_{k=1}^\infty \in \mathcal{D}(B). \quad \text{Hence, } \eta \geq \beta(g^*) \geq \beta(Hg^*) \geq \beta(h^*) = \eta \quad \text{with equality if and only if } \beta(g^*) = 0. \quad \text{Hence, } \eta = 0.
\]

Now it is easy to see that there is a compact set \( J \subset X_1 \) such that \( H^n(B) \to J \) in the Hausdorff metric. One may use Lemma 3.1 in Massatt [10].

**Proof of Theorem 4.3:** Let \( ||x|| \) be the norm of \( x \) in \( X_1 \) or the distance from zero under some metric for \( X \). Let \( |x| \) be the sup norm for any \( x \in C_B \). Let \( B_N = \{ x \in C_B / x(0) = 0, |x| < N \} \).

If \( A \subset X \), let \( ||A|| = \sup \{ ||x|| : x \in A \} \). Pick \( \sigma_n \to \infty \) such that \( \sigma_1 = 0 \) and \( ||B_n^\sigma|| < \frac{1}{2^n} \) for \( n \geq 2 \). Let \( h(t) = \frac{(t+\sigma_{n+1})/(n(\sigma_{n+1} - \sigma_n) - (\sigma_n + t)/(n+1)(\sigma_{n+1} - \sigma_n))}{(n+1)(\sigma_{n+1} - \sigma_n)} \), i.e. \( h(-\sigma_n) = \frac{1}{n} \) and \( h(\cdot) \) connects these points with straight lines. Hence, we also have \( h(\cdot) \) strictly increasing. This allows us to define two Banach spaces.

\[
X_2 = \{ x \in C(\infty,0) / ||x||_2 = \sup_{\theta \in (-\infty,0)} h(x(\cdot)) < \infty, \\
\lim_{\theta \to -\infty} h(x(\cdot)) = 0 \}.
\]

\[
X_1 = \{ x \in W^{1,\infty}_{1,0}(\infty,0) / ||x||_1 = \sup_{\theta \in (-\infty,0)} h^{1/2}(\cdot)x(\cdot) + ||h^{1/2}(\cdot)x(\cdot)||_\infty < \infty \}.
\]

It is clear \( C_B \) is dense in \( X_2 \). Any Cauchy sequence of elements in \( C_B \) with \( ||\cdot||_2 \) is Cauchy in \( X \). Hence, we may consider \( X_2 \to X \) with a continuous imbedding. From Arzela-Ascoli's
theorem, we also get \( i: X_1 \rightarrow X_2 \) is a continuous, compact imbedding.

Under the general axioms imposed on \( X \) above, we cannot apply Theorem 2.1 directly, nor can we expect to get the solution map to be bounded dissipative in \( X_1 \). However, the proof will be in the same spirit.

As in Theorem 4.1, let \( T_f(\omega): X \rightarrow X \) be the solution map from initial time 0 to time \( \omega \), \( T_f(\omega)\phi = x_\omega(\cdot, \phi) \). Clearly, \( T_f(\omega): X_1 \rightarrow X_1 \). We assume \( T_f(\omega) \) maps bounded sets to bounded sets. The removal of this restriction involves only technical details. Let \( \phi = \phi - \phi(0) \) where \( \phi(0) \) is the constant function. Let \( U\phi = T_f(\omega)\phi - C\phi \). Since \( f \) is completely continuous, by arguments similar to the last theorem, we get \( U: (X_1, F_2) \rightarrow (X_1, F_1) \) maps bounded sets to bounded sets. One problem we immediately encounter is that \( C \) may not be a contraction under any equivalent norm.

The proof that \( U: X_1 \rightarrow X_1 \) is completely continuous is identical to the last theorem. Our goal now will be to show that in \( X_1 \) we have sets \( S_0, S_1, \) and \( S_2 \) satisfying all the hypotheses of Theorem 4.2.

Let \( A \subset X_1 \) be any bounded set and let \( A_r = \{x(\cdot): [-r,0] \rightarrow \mathbb{R}^n/x(\cdot) \text{ is the restriction on } [-r,0] \text{ of a function in } A\} \). Let \( r(A) = \sup \{r/C1 A_r \text{ is compact in } W_1^{\infty}[-r,0] \} \). Let \( \beta(A) = \frac{1}{1+r(A)} \). Hence \( \beta: \mathbb{R}_1 \rightarrow [0,\infty) \) is well-defined.

Let \( X_0 = \{x \in W_1^{10c}(-\infty,0)/||x||_0 = \sup h(\cdot)^{1/3}x(\cdot) + ||h(\cdot)^{1/3}x(\cdot)||_\infty < \infty \} \). The same arguments as before show
U: $X_0 \to X_0$ and $U: (X_0, \mathcal{F}_2) \to (X_0, \mathcal{F}_0)$ maps bounded sets to bounded sets.

Let $B^0_R$ dissipate points. Let $B^0_R \subset B^0_{Q(R)}$. If $x \in \text{Cl } B^0_{Q(R)}$ then $x$ is clearly dissipated by $\text{Cl } B^0_{Q(R)}$. Let $y \in \text{Cl } B^0_{Q(R)}$ be chosen so that for all $t \in (-\infty, 0]$ we have $h(t)\frac{1}{3}y(t) = q(r)$. Let $z(\cdot) = y(\cdot) - y(0)$. Then $z(\cdot) > 0$ since $h(\cdot)$ is strictly monotone increasing. This defines a set $A = \{x(\cdot)/|x(t)| < z(t) \text{ for all } t \in (-\beta, 0]\}$. Then any $x \in \text{Cl } (B^0_{Q(R)} + A)$ is also dissipated by $\text{Cl } B^0_{Q(R)}$. Now for every $x \in \text{Cl } (B^0_{Q(R)} + A)$ there exists an $n > 0$, $\epsilon > 0$ such that

$$T^n_f(\omega)\{(x) + B^2_{E}, \text{ Cl } (B^0_{Q(R)} + A)\} \subset \text{Cl } B^0_{Q(R)}.$$  

Because $\text{Cl } (B^0_{Q(R)} + A)$ is compact in $X_2$, there is a finite subcover of such sets and a maximum $N$ such that $n_i$ corresponding to the finite subcover satisfy $n_i \leq N$. Then, clearly $\gamma^+(\text{Cl } B^0_{Q(R)} + A) = \bigcup_{n=1}^{N} T^n_f(\omega)\{(x) + B^0_{Q(R)} + A\}$.

Now, we may replace $A$ by $A_N = \{x(\cdot)/|x(t)| < z(t) \text{ for all } t \in (-N, 0]\, \text{ and } x(t) = 0 \text{ for } t \in (-\infty, -N]\}$. Then we have

$$\gamma^+(\text{Cl } B^0_{Q(R)} + A_N) = \bigcap_{n=1}^{N} T^n_f(\omega)\{(x) + B^0_{Q(R)} + A_N\}. \quad \text{Let } D = \text{Cl } B^0_{Q(R)} + A_N.$$  

Define the set operator $H: \mathcal{A}_0 \to \mathcal{A}_0$ by $H(B) = \text{co}[T_f(\omega)\gamma^+(B \cap D)]$. The map $H$ is $\beta$-condensing and type 2. $H, X_0, X_1$, and $\beta$ satisfy all the properties of Theorem 4.4. Hence, $H$ is asymptotically smooth. Because of the continuity, $H: \mathcal{A}_0 \to \mathcal{A}_0$ defined by $H(B) = \text{Cl } \text{co}[T_f(\omega)\gamma^+(B \cap D)]$ is also asymptotically smooth. Let $E = \bigcap_{n=1}^{\infty} H^n(D)$. This is nonempty, bounded in $X_0$, compact in $X_1$, and...
convex. Now, let $S_0 = \text{Cl} \ B_0^Q(R) \cap E$, $S_1 = D \cap E$, and $S_2 = E$. Applying Theorem 4.3 we get a fixed point of $T_f(\omega)$. This implies a periodic solution to (*) of period $\omega$. This completes the proof.
REFERENCES


