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SECOND APPROXIMATION TO CONICAL FLOWS

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CORNELL UNIVERSITY

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Wright Air Development Center
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Foreword

This report is submitted as part of Contract AF33(038)9832, which was administered by the Aeronautical Research Laboratory, WADC, under RDO No. 465-1, Aerodynamics of Compressible Fluids, with Mr. Lee S. Wasserman acting as project engineer. The work reported was done at the Graduate School of Aeronautical Engineering at Cornell University during 1950.

The investigation was suggested by the thesis of Dr.F.K. Moore (Reference 2). It was desired to check Dr. Moore's analysis and to extend his calculations to several additional practical cases. The project was carried out under the immediate supervision of Professor W.R.Sears, who originally suggested the problem to Dr. Moore and directed his thesis research. Acknowledgment is also due to Mrs. Anne B. Kane, who worked faithfully and diligently on the extensive numerical calculations reported here.

Abstract

The method of expansion in powers of a thickness parameter ϵ is employed to calculate supersonic, inviscid flow about symmetrical arrow-head wings lying entirely within the tip Mach cone. In this method, the first-order terms in ϵ constitute the familiar, linearized Prandtl-Glauert approximation. The method is one of iteration, so that the i the approximation always depends on the (i^{-1})th, etc.

Here the second approximation, i.e., the terms in ϵ and ϵ^2, are computed and plotted for a family of arrow-head wings having various leading-edge angles and thicknesses, and flying at various Mach numbers. It is necessary to use Lighthill's method to determine the strength of the attached conical shock.

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Introduction

The problem to be considered is one in which dissipative phenomena appear only within the shock wave. Since this entropy change is of third order in thickness, it may be neglected in a second-order theory. Hence the entire flow field may be considered isentropic and irrotational, and the introduction of velocity potential is justified.

The exact flow over a conical body in an ideal, uniform supersonic flow field is conical. By expanding the solution in powers of a geometrical parameter ε, one sees that each term of the expansion must also be conical.

As disturbances may not propagate upstream, the flow over an "arrow head" wing is identical with that over an infinitely extended body.

That the solution to the differential equations of motion can be expanded in powers of thickness parameter can be justified by assuming an expansion in terms of diminishing order of magnitude and proceeding to satisfy the successive boundary conditions. For a flat airfoil, the first-order solution is formed to be of order ε (thickness parameter). This is dictated by the boundary condition on the normal velocity at the airfoil. On the airfoil, except at or near singular points of the first order solution (e.g., leading edges), the same boundary condition then requires that the next nontrivial approximation be of order $\mathcal{E}^2$, and so on. This statement may be checked against the boundary condition formulated in Eq. (29). Thus, the power expansion in $\mathcal{E}$ is the only expansion which will satisfy the boundary conditions on the body. Recent work by Lighthill indicates that, to second order, the power series expansion in $\mathcal{E}$ properly describes conditions at the Mach cone as well.

Further, it is expected that the solution, to each order will be analytic at some plane (e.g., a plane of symmetry) near the thin body; thus, the value of each approximation on the body can be expressed as a power series in $\mathcal{E}$, the leading term being its value on the plane of symmetry, say. As will be shown later, one may then formulate boundary conditions at the plane of symmetry to all orders.

Broderick and Lighthill have solved corresponding problems of the supersonic flow about bodies of revolution and, in order to satisfy boundary conditions at the surface, have found it necessary to introduce nonregular terms involving the logarithm of the thickness parameter. This is a consequence of the fact that the axis of the body (corresponding to the plane of symmetry in the present case) is a singularity of their solutions.

In the case of supersonic conical flows over slender bodies, an attached conical shock wave occurs in the vicinity of the leading Mach cone. In the linearized theory of such flows, it is customary to assume a zero-strength shock located at Mach cone. Lighthill has pointed out that while the assumption of zero shock strength is valid for the first (linearized) approximation, the true shock strength is is second order in disturbance velocities. He presents formulas that provide a quantitative second-order description of the flow near the shock wave in terms of results of the linearized solution. Lighthill's results will be used here to provide boundary conditions at the Mach cone.

The pressure coefficient is given by

$$C = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} = \varepsilon C^{(1)} + \varepsilon^2 C^{(2)} + \qquad (1)$$

Denoting the velocity vector by

$$\underline{i}'[U + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots] + \underline{j}'[\varepsilon v^{(1)} + \dots] + \underline{k}[\varepsilon w^{(1)} + \dots] \qquad (2)$$

where the unit vector $\underline{i}'$ lies in the streamwise direction, and

$$\beta \equiv \sqrt{M_\infty^2 - 1}, \qquad (3)$$

and applying the isentropic equations of motion, we have

$$C^{(1)} = -2(u^{(1)}/U) \qquad (4)$$

$$C^{(2)} = -[2(u^{(2)}/U) - \beta^2(u^{(1)}/U)^2 + (v^{(1)}/U)^2 + (w^{(1)}/U)^2] \qquad (5)$$

Thus, for pressure coefficient, one needs in addition to all three first-order cartesian velocity components, the second-order correction to the streamwise velocity component as well.

Notation

x, y, z = cartesian space coordinates

η = conical radial coordinate

s = Tschaplygin radial cooordinate

ω = space, conical, and Tschaplygin angular ordinate

ζ = complex coordinates in the Tschaplygin plane

P = pressure

ρ = density

C = pressure coefficient

U = free-stream velocity

M = Mach Number

β = a constant based on free-stream Mach Number

u, v, w = cartesian velocity components in the directions , , and respectively

ε, λ = angles - See Fig. 2

φ = velocity potential

a = velocity of sound

γ = ratio of specific heats

∇^2 = the Laplacian operator

l, r, θ = geometrical quantitites in the Tschaplygin plane - see Fig. 3

ψ, f, g, h = flow functions

A, D, K = constants

Subscripts:

∞ signifies evaluation in the free stream

b signifies evaluation on the body

o signifies evaluation on the plane of symmetry

p signifies particular integral

c signifies complemetary function.

The subscript notation for partial differentiation is used where convenient.

Superscripts:

(n) denotes the order of approximation.

Primes denote ordinary differentiation.

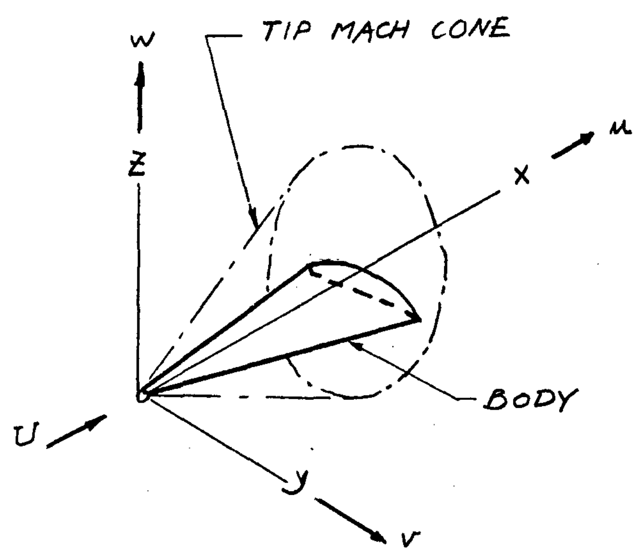


Fig. 1

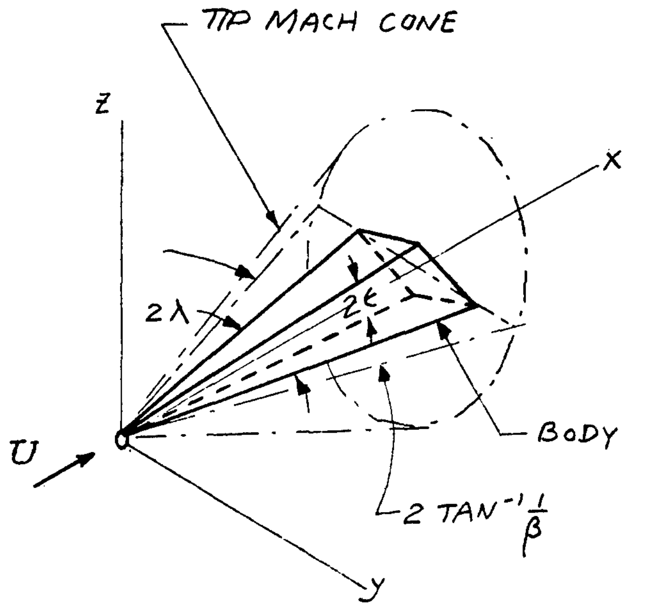

TIP MACH CONE
Z
X
2λ
2ϵ
U
BODY
$2\ TAN^{-1}\frac{1}{\beta}$
y

FIG. 2

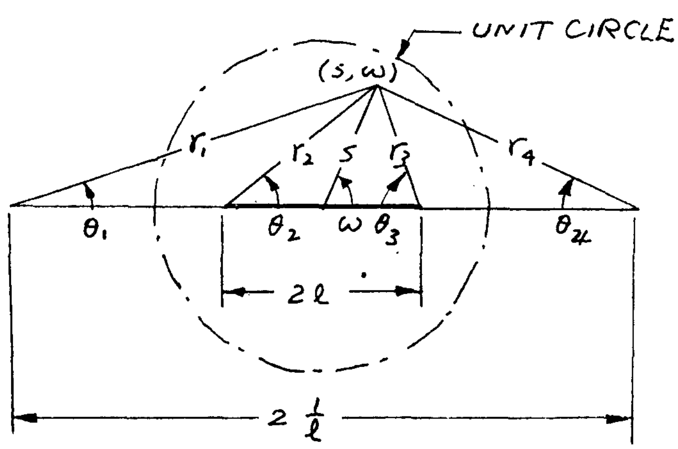

UNIT CIRCLE
(s, ω)
r_1
r_2
s
r_3
r_4
θ_1
θ_2
ω
θ_3
θ_4
2ℓ
$2\frac{1}{\ell}$

FIG. 3

PART I-THEORY*

A. The General Case

(1) Differential Equations for the Velocity Potential

For the scheme shown in Fig. 1, the isentropic equations of motion are

$$(a^2-\varphi_x^2)\varphi_{xx}+(a^2-\varphi_y^2)\varphi_{yy}+(a^2-\varphi_z^2)\varphi_{zz}$$
$$-2\varphi_x\varphi_y\varphi_{xy}-2\varphi_y\varphi_z\varphi_{yz}-2\varphi_x\varphi_z\varphi_{xz}=0 \qquad (6)$$

$$a^2=a_\infty^2+\frac{\gamma-1}{2}(U^2-\varphi_x^2-\varphi_y^2-\varphi_z^2) \qquad (7)$$

It is assumed that

$$\varphi=Ux+\varepsilon\varphi^{(1)}+\varepsilon^2\varphi^{(2)}+\cdots \qquad (8)$$

Introducing Eqs. (7) and (8) into Eq. (6) and collecting terms of order ε and ε^2, respectively, one finds

$$\beta^2\varphi_{xx}^{(1)}-\varphi_{yy}^{(1)}-\varphi_{zz}^{(1)}=0 \qquad (9)$$

$$\beta^2\varphi_{xx}^{(2)}-\varphi_{yy}^{(2)}-\varphi_{zz}^{(2)}$$
$$=-\frac{M_\infty}{a_\infty}\frac{\partial}{\partial x}\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\varphi_x^{(1)^2}+\varphi_y^{(1)^2}+\varphi_z^{(1)^2}\right] \qquad (10)$$

Eq. (9) is the equation of the linear-perturbation theory.

The solution for $\varphi^{(2)}$ willbe composed of a particular solution of Eq. (10) and a complementary solution satisfying the homogeneous counterpart of Eq. (10), which solutions, taken together, satisfy the boundary conditions appropriate to the particular case considered.

(2) Transformations and Particular Integrals

A transformation introduced by Tschaplygin is applicable to conical flows and changes the three-dimensional hyperbolic equations for velocity components into two-dimensional Laplace's equations. The transformation is introduced as follows:

$$\eta \equiv \beta \frac{\sqrt{y^2+z^2}}{x}; \qquad \omega \equiv \tan^{-1}\frac{z}{y}; \qquad s \equiv \frac{\eta}{1+\sqrt{1-\eta^2}} \tag{11}$$

Applying this transformation to Eqs. (9) and (10), it is found that

$$\nabla^2 u^{(1)}(s,\omega) = \nabla^2 v^{(1)} = \nabla^2 w^{(1)} = 0 \tag{12}$$

$$\nabla^2 u^{(2)} = \frac{2}{\beta}\frac{s}{(1-s^2)^3}\left[2h_s + s(1-s^2)h_{ss}\right] \tag{13}$$

$$\nabla^2 v^{(2)} = -\frac{1}{(1-s^2)^3}\left[(1+4s^2-s^4)\cos\omega\, h_s + s(1-s^4)\cos\omega\, h_{ss} - (1-s^2)^2\sin\omega\, h_{s\omega}\right] \tag{14}$$

$$\nabla^2 w^{(2)} = -\frac{1}{(1-s^2)^3}\left[(1+4s^2-s^4)\sin\omega\, h_s + s(1-s^4)\sin\omega\, h_{ss} + (1-s^2)^2\cos\omega\, h_{s\omega}\right] \tag{15}$$

where

$$h \equiv A^2\left[D^2 u^{(1)2} + v^{(1)2} + w^{(1)2}\right] \tag{16}$$

and

$$D^2 \equiv 1 + \frac{\gamma-1}{2}M_\infty^2; \qquad A^2 \equiv 2\frac{M_\infty^2}{U\beta} \tag{17}$$

The following transformation may now be introduced:

$$\zeta \equiv s e^{i\omega}; \qquad \bar{\zeta} \equiv s e^{-i\omega} \tag{18}$$

and one finds that

$$\nabla^2 = 4\,(\partial^2/\partial\varsigma\,\partial\bar{\varsigma}) \tag{19}$$

which makes it possible to define the following functions:

$$\left.\begin{aligned} f_1(\varsigma) + g_1(\bar{\varsigma}) &\equiv D A u^{(1)} \\ f_2(\varsigma) + g_2(\bar{\varsigma}) &\equiv A v^{(1)} \\ f_3(\varsigma) + g_3(\bar{\varsigma}) &\equiv A w^{(1)} \end{aligned}\right\} \tag{20}$$

and to write a particular integral for $u^{(2)}$, after numerous integrations by parts in Eq. (13), as follows:

$$\beta u_p^{(2)} = \frac{A^2}{2}\frac{\varsigma}{1-\varsigma^2}\frac{\partial}{\partial\varsigma}\left[D^2 u^{(1)2} + v^{(1)2} + w^{(1)2}\right] - \sum_{n=1}^{3}\left\{\int^{\varsigma} f_n'\,d\varsigma \int^{\bar{\varsigma}} \frac{(\bar{\varsigma}g_n')'\,d\bar{\varsigma}}{1-\varsigma\bar{\varsigma}} + \int^{\bar{\varsigma}} g_n'\,d\bar{\varsigma}\int^{\varsigma}\frac{(\varsigma f_n')'\,d\varsigma}{1-\varsigma\bar{\varsigma}}\right\} \tag{21}$$

Expressions for $v_p^{(2)}$ and $w_p^{(2)}$ similar to the above can be obtained from Eqs. (14) and (15).

It is clear at this point that the solution given in Eq. (21) will be infinite at the tip Mach cone $(\varsigma = 1)$ unless the derivative in the first right-hand term vanishes there. This is an example of singular behavior that concerned Lighthill in the investigation already mentioned. In the cases considered in detail here, in which the body lies entirely within the tip Mach cone, this derivative does in fact vanish, since the linearized theory leads to vanishing perturbation velocities at this cone. Consequently, the second-order solution is free of this particular difficulty, and one needs Lighthill's technique only to establish the upstream boundary condition.

(3) The Boundary Condition on the Body

One prescribes simply that the flow through the body surface vanish to the second order of approximation. It would be convenient to satisfy this boundary condition at some mean plane, as is customarily done in the linear theory. The assumption of analyticity of solutions at this plane enables one to do this. The details of this procedure will appear later in an illustrative example.

(4) The "Boundary Condition" at the Mach Cone

Since it is intended to deal herein with velocity components rather than the potential, it is necessary to formulate boundary conditions on these quantities at the Mach cone. As has already been mentioned, the formulation of correct boundary conditions to be satisfied by the perturbation velocity components at the tip Mach cone is based upon Lighthill's detailed analysis of the conditions near the shock wave. It may be of value to review briefly the salient features of his method.

The correct upstream "boundary conditions" are actually the shock-wave equations, applied at the shock-wave location, which is upstream of the tip Mach cone and is therefore outside the region of applicability of the method set forth above. To obviate this difficulty, Lighthill introduces a transformation of coordinates from η, ω to R, ω, which may be written in the form

$$\eta = R + \varepsilon r^{(1)}(\omega) + \varepsilon^2 r^{(2)}(\omega) + \cdots$$

where the successive functions $r^n(\omega)$ are to be determined as the calculation progresses. The method of approximations in successive powers of ε is then formulated in terms of the new independent variables R, ω. The singularity of the differential equations is thus shifted to the location $R=1$, instead of $\eta=1$. The choice of the functions $r^{(n)}(\omega)$ is made, however, in such a way as to avoid divergence of the method at $R=1$. It is then found that the shock wave occurs for $R<1$ - i.e., in the region of convergence.

Lighthill next introduces the Rankine-Hugoniot shock-wave equations to connect the conical flow to the flow upstream of the shock. This leads him to fictitious boundary values assumed by the velocity components at $R=1$. In particular, consider the perturbation velocity potential

$$\begin{aligned} \varphi &= Uxf(R,\omega) \\ &= Ux[\varepsilon f^{(1)}(R,\omega) + \varepsilon^2 f^{(2)}(R,\omega) + \cdots] \end{aligned}$$

According to the shock-wave equations, this quantity is continuous across the shock - i.e., it has the value zero just behind the shock in the present problem. Lighthill shows that the corresponding values to be applied at $R=1$ are

$$\left.\begin{aligned} f^{(1)}(1,\omega) &= f^{(2)}(1,\omega) = 0 \\ f_R^{(1)}(1,\omega) &= 0 \end{aligned}\right\} \qquad (22)$$

$$f_R^{(2)}(1,\omega) = \frac{\gamma+1}{2}\frac{M_\infty^4}{\beta^2}\left[\lim_{R\to 1}\frac{f_R^{(1)}}{\sqrt{1-R}}\right]^2 \Bigg\}$$

Furthermore, in any case where the body lies entirely within the tip Mach cone, it is found that $r^{(1)}(\omega)$ is identically zero. This means that, at any point in the flow field, η and R will differ only to second order in ε. Thus, away from the vicinity of the shock wave, the independent variable R in the functions $f^{(n)}(R,\omega)$ may be directly replaced by η without introducing any error greater than order ε^3.

It follows from the fact that $r^{(1)}(\omega)$ is zero that the differential equations satisfied by $f^{(1)}(R,\omega)$ and $f^{(2)}(R,\omega)$ are just the equations equivalent to Eqs. (9) and (10) and, hence, to Eqs. (12)-(15) (the only differences being those due to the definition of the variables). Finally, then, to the order of approximation contemplated in this paper, it will be correct to solve Eqs. (12)-(15) and to apply at $\eta = 1$ the boundary conditions (22). From the definition of $f^{(n)}(R,\omega)$ and the relation (11), it follows that these are equivalent to

$$\left.\begin{aligned} &\left[\frac{u^{(1)}}{U}, \frac{v^{(1)}}{U}, \frac{w^{(1)}}{U}\right]_{s=1} = 0 \\ &\left[\frac{u^{(2)}}{U}\right]_{s=1} = -(\gamma+1)\frac{M_\infty^4}{\beta^2}\left[\lim_{s\to 1}\frac{u^{(1)}/U}{1-s}\right]^2 \end{aligned}\right\} \tag{23}$$

(5) The Irrotationality Conditions

Applying the Tschaplygin transformation to the irrotationality conditions $u_z = w_x$; $v_x = u_y$; $v_z = w_y$ and evaluating them at the plane $\omega = 0$, π , one obtains the following useful relations:

$$\beta \left[u_\omega^{(n)} \right]_0 = -2 \frac{s}{1-s^2} \left[w_s^{(n)} \right]_0 \tag{24}$$

$$\beta \left[u_s^{(1)} \right]_0 = 2 \frac{1}{1-s^2} \left[w_s^{(1)} \right]_0 \tag{25}$$

$$\beta \left[u_s^{(2)} \right]_0 = 2 A^2 \frac{s^2}{(1-s^2)^2} \times$$
$$\left[\frac{\partial}{\partial s} \left(D^2 u^{(1)2} + v^{(1)2} + w^{(1)2} \right) \right]_0 + 2 \frac{1}{1-s^2} \left[w_\omega^{(2)} \right]_0 \tag{26}$$

$$\left[v_s^{(n)} \right]_0 = -\beta \frac{1+s^2}{2s} \left[u_s^{(n)} \right]_0 \tag{27}$$

$$\left[v_\omega^{(n)} \right]_0 = 2s \frac{1+s^2}{1-s^2} \left[w_s^{(n)} \right]_0 \tag{28}$$

B. Arrow-Head Inside Mach Cone

(1) Statement of Problem and Formulation of Boundary Conditions

Fig. 2 shows the configuration of the body - an "arrow-head" at zero angles of attack and yaw with respect to a free stream of velocity U and Mach Number M_∞. ϵ is the thickness parameter.

After certain geometrical maneuvers, the boundary condition at the body for $y > 0$ may be written as

$$[\varphi_z]_b = \epsilon [\varphi_x]_b - \epsilon \cot\Lambda [\varphi_y]_b \tag{29}$$

Assuming analyticity of solutions near the plane of symmetry, one writes

$$[\varphi_z^{(n)}(x,y,z)]_b = [\varphi_z^{(n)}]_0 + \epsilon \psi_1(x,y) + \epsilon^2 \psi_2(x,y) + \ldots \tag{30}$$

Substituting expressions (8) and (30) into Eq. (29) and collecting terms of like order in ϵ,

$$[\varphi_z^{(1)}]_0 = U \tag{31}$$

$$[\varphi_z^{(2)}]_0 = -\psi_1 + [\varphi_x^{(1)}]_0 - \cot\Lambda [\varphi_y^{(1)}]_0 \tag{32}$$

Eqs. (31) and (32) are applicable on the plane of symmetry within the leading edges.

Utilizing the Tschaplygin transformation, noting that $s = 1$ represents the Mach cone, and defining ℓ as the s-coordinate of the leading edges, Eqs. (31) and (32) may be written as

$$[w^{(1)}]_0 = U \qquad (\omega = 0,\ s < \ell) \tag{33}$$

$$[w^{(2)}]_0 = -\psi_1(s,\omega) + [u^{(1)}]_0 - \beta\frac{1+\ell^2}{2\ell}[v^{(1)}]_0 \qquad (\omega = 0,\ s < \ell) \tag{34}$$

By symmetry,

$$[w^{(1)}]_0 = [w^{(2)}]_0 = 0 \qquad (\omega = 0,\ \ell < s < 1) \tag{35}$$

$$\left.\begin{array}{l} u^{(n)} \text{ is even in both } z \text{ and } y \\ v^{(n)} \text{ is even in } z\text{, odd in } y \\ w^{(n)} \text{ is odd in } z\text{, even in } y \end{array}\right\} \tag{36}$$

On the circle $s = 1$,

$$\left.\begin{array}{l} [u^{(1)}, v^{(1)}, w^{(1)}]_{s=1} = 0 \\ \left[\dfrac{u^{(2)}}{U}\right]_{s=1} = -(\gamma+1)\dfrac{M_\infty^4}{\beta^2}\left[\lim_{s\to 1}\dfrac{u^{(1)}/U}{1-s}\right]^2 \end{array}\right\} \tag{37}$$

The differential equations applicable in the Tschaplygin plane are given by Eqs. (12)-(15). Use will be found for the irrotationality conditions (24) through (28).

Fig. 3 represents the Tschaplygin plane, in which the problem will be solved. Stated in its simplest terms, the plan of attack will be as follows:

(a) Find $u^{(1)}$, $v^{(1)}$, and $w^{(1)}$, which satisfy differential Eq. (12), and the irrotationality conditions, subject to the boundary conditions developed above.

(b) Find the particular integral for $u^{(2)}$, using the expression (21). To this will be added the complementary function required to ensure satisfaction of the boundary conditions, thus completing the solution for $u^{(2)}$.

(c) Using these results, apply Eqs. (4) and (5) to find the pressure coefficient.

(2) Singularities

Certain difficulties are encountered in carrying out the above procedure because of the presence of singularities in $u^{(1)}$ and $v^{(1)}$ and, hence, in all components in the second approximation, at the leading edges. This is due to the subsonic nature of the flow at these edges and the consequent stagnation in the component of velocity in the plane of symmetry and normal to the leading edge. A perturbation theory always fails to represent stagnation conditions and provides singularities in velocities instead.

One would expect that the singularities arising in this problem at the leading edtes would be of the same types as those appearing in the case of an infinite yawed wedge, when the angle of yaw is sufficient to provide subsonic normal flow at the leading edge. This normal component would contain the only singularities. Further, the subsonic two-dimensional flow over a wedge will have the same type of singularity as that of the infinite yawed wedge; the incompressible ($M_\infty \doteq 0$) two-dimensional flow over a wedge will afford the same singularities as the compressible subsonic case, inasmuch as the leading edge in the latter case represents a stagnation point in whose neighborhood the local Mach Number is nearly zero.

Examination of the incompressible, two-dimensional flow over a wedge shows singularities at the leading edge of the logarithmic type in the first approximation and the square of the logarithm and the logarithm in the second approximation. These singularities are all integrable. These, then, are the singularities to be expected in the "arrow-head" case.

Admission of such singularities raises the question of uniqueness. It is known that a potential problem has a unique solution when suitable boundary conditions are prescribed on a closed contour, provided the solution is to be regular everywhere inside. If singularities are admitted, an ambiguity may be expected - i.e., certain harmonic functions having singularities may be introduced without disturbing the boundary conditions. One therefore inquires what harmonic functions having singularities of the type $\ln|s-l|$ and/or $(\ln|s-l|)^2$ on the axis of symmetry may be added to the regular solution for $u^{(1)}$ or $u^{(2)}$ satisfying the appropriate boundary conditions, without disturbing these boundary conditions. Thus, in accordance with Eqs. (24), (36), and (37), it is required that such a singular function have zero angular derivative on the axis of symmetry, be even in both y and z, and vanish on the unit circle. It may be shown that the only singular function meeting these requirements is the following:

$$K_1^{(n)}\left[\ln(r_2)+\ln(r_3)-\ln(l r_1)-\ln(l r_4)\right] \tag{38}$$

Similar reasoning leads to the following ambiguous term for $v^{(1)}$ or $v^{(2)}$

$$K_2^{(n)}\left[\ln(r_2) - \ln(r_3) - \ln(\ell r_1) + \ln(\ell r_4)\right] \tag{39}$$

The constants $K_1^{(n)}$ and $K_2^{(n)}$ have superscripts indicating the order of approximation involved.

Since $w^{(1)}$ and $w^{(2)}$ have boundary conditions on value, they can have no ambiguity of this type. Therefore, knowing $w^{(1)}$ and $w^{(2)}$, one may find $K_1^{(1)}$, $K_1^{(2)}$, $K_2^{(1)}$, and $K_2^{(2)}$. Eqs. (25) and (26) are convenient for this purpose.

It appears that the lack of uniqueness discussed above arises because of the abandonment of the velocity potential in order to take advantage of the simplicity afforded by the Tschaplygin transformation.

(3) First-Order Solution

(a) Solution for $w^{(1)}$- The differential equation is given in Eq. (12), and the boundary conditions are given in Eqs. (33), (35), and (37). By analogy with incompressible fluid theory, the solution may be written down immediately, using the stream function for the incompressible source:

$$w^{(1)} = -(U/\pi)(\theta_1 + \theta_2 + \theta_3 + \theta_4 - \pi) \tag{40}$$

(b) Solution for $u^{(1)}$- The differential equation is given in Eq. (12) above. The boundary conditions are those given in Eqs. (25), (36), and (37), and from Eq. (24):

$$[u_\omega^{(1)}]_0 = 0 \qquad (s < \ell) \tag{41}$$

It is plain that the solution analytic inside the unit circle is identically zero, leaving only the constant $K_1^{(1)}$ to be found. Applying Eq. (25) to Eqs. (38) and (40), and, for convenience, evaluating terms near the leading edge, one may immediately write

$$u^{(1)} = \frac{U}{\beta\pi}\frac{2\ell}{1-\ell^2}\left[\ln(r_2) + \ln(r_3) - \ln(\ell r_1) - \ln(\ell r_4)\right] \tag{42}$$

(c) Solution for $v^{(1)}$. By a procedure similar to the one employed to find $u^{(1)}$, it is found that

$$v^{(1)} = \frac{U}{\pi}\frac{1+\ell^2}{1-\ell^2}\left[\ln(r_2) - \ln(r_3) - \ln(\ell r_1) + \ln(\ell r_4)\right] \tag{43}$$

(d) Information Obtained from the Above Solutions. - Expanding (40), one finds that

$$\psi_1 = - U(\beta/\pi)\left[(1+\ell^2)/\ell\right] \tag{44}$$

and from Eq. (42) is obtained

$$\lim_{s\to 1}\left[\frac{u^{(1)}/U}{1-s}\right] = -\frac{4}{\beta\pi}\left(\frac{1+\ell^2}{\ell} - \frac{4\ell}{1+\ell^2}\cos^2\omega\right)^{-1} \tag{45}$$

Eq. (45) may be substituted into the results of Lighthill for the strength and location of the shock wave.

For use in finding $u^{(2)}$, one puts $u^{(1)}$, $v^{(1)}$, and $w^{(1)}$ into complex form to conform to the expressions (20) above.

(4) Second-Order Solution

In philosophy, the solution for $u^{(2)}$ is obtained in the same way as was $u^{(1)}$ in the preceding paragraph. In detail, the procedure is as follows: One may write

$$\left.\begin{aligned} u^{(2)} &= u_p^{(2)} + u_c^{(2)} \\ w^{(2)} &= w_p^{(2)} + w_c^{(2)} \end{aligned}\right\} \tag{46}$$

where $u_p^{(2)}$ is obtained from Eq. (21), $w_p^{(2)}$ is obtained from a similar expression and $u_c^{(2)}$ and $w_c^{(2)}$ satisfy Laplace's equation in the Tschaplygin plane.

(a) Find $u_p^{(2)}$; note that the functions obtained from the integrations indicated in Eq. (21) may be complex. $u_p^{(2)}$ may be found in real form by adding to these complex functions any convenient functions of ζ or $\bar{\zeta}$ alone.

(b) Find $u_c^{(2)}$ such that $u_p^{(2)} + u_c^{(2)}$ satisfies the boundary conditions (24), (36), and (37) and has no singularities of order higher than $(\ln|\zeta - \ell|)^2$. Expression (38) is to be included in $u_c^{(2)}$.

(c) Find $w_p^{(2)}$.

(d) Find $w_c^{(2)}$ such that $w_p^{(2)} + w_c^{(2)}$ satisfies the boundary conditions (34), (35), and (36) and has no singularities of order higher than $(\ln|\zeta - \ell|)^2$. Note that no boundary condition on $w^{(2)}$ at the unit circle is applied.

(e) Apply Eq. (26) to find $K_1^{(2)}$. This step is simplified by considering only those terms that are singular at the leading edge. The equation will contain terms singular at $\zeta = \ell$, and one may require the equation to be satisfied upon arbitrarily close approach to the point $\zeta = \ell$. All terms nonsingular at this point may then be dropped, and all nonsingular factors in singular terms may be evaluated at $\zeta = \ell$. The term involving the unknown $K_1^{(2)}$ has a derivative with a simple-pole type of singularity. Thus, one needs only to formulate Eq. (26) for terms having simple poles in derivative. Finding $K_1^{(2)}$ in this way makes it possible to ignore any boundary condition on $w^{(2)}$ at the unit circle, since its application would result in a regular function that would make no contribution to the determination of $K_1^{(2)}$.

This procedure has been carried out and formulas have been obtained from which computations can be made (Part II).

PART II COMPUTATION AND RESULTS

A. Computation and Results

In the cases computed, the following values are used:

$$U = 1, \qquad \gamma = 1.4$$

$$M_\alpha = 1.2, \ \sqrt{2}, \ 2.$$

$$\ell = 0.1, \ 0.3, \ 0.78.$$

$$2\varepsilon = 2^\circ, \ 6^\circ, \ 10^\circ.$$

Introducing the notations

$$B^2 = 1 + \frac{\gamma - 1}{2} M_\alpha^2$$

$$L = \sqrt{\frac{2 M_\alpha^2}{U \beta}}$$

$${}_1\tau_2^{(1)} = -\frac{U\beta}{\pi} \frac{1+\ell^2}{\ell}$$

n	ε_1	ε_2	ε_3	ε_4
1	+1	+1	−1	−1
2	+1	−1	−1	+1
3	+1	−1	+1	−1

$$E_1 = -LB \frac{U}{\beta\pi} \frac{\ell}{1-\ell^2}, \qquad E_2 = -L \frac{U}{2\pi} \frac{1+\ell^2}{1-\ell^2}, \qquad E_3 = -L \frac{U}{2\pi i}$$

$$A_1 = \frac{1}{2} \sum_1^3 E_n^2, \qquad A_2 = \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \frac{1}{2} - \varepsilon_3 \frac{\ell^2}{1-\ell^2} + \varepsilon_4 \frac{\ell^2}{1+\ell^2}\right)$$

$$A_3 = \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \frac{1}{2} + \varepsilon_3 \frac{1}{1-\ell^2} + \varepsilon_4 \frac{1}{1+\ell^2}\right), \qquad A_4 = -\frac{1}{1-\ell^2} \sum_1^3 E_n^2 \varepsilon_3$$

$$A_5 = \frac{1}{1+\ell^2} \sum_1^3 E_n^2 \varepsilon_4 \qquad A_6 = \frac{1}{2} \sum_1^3 E_n^2 \varepsilon_2 = A_9$$

$$A_7 = -\frac{\ell^2}{(1-\ell^2)^2} \sum_1^3 E_n^2 \varepsilon_3 \qquad A_8 = \frac{2\ell^2}{(1+\ell^2)^2} \sum_1^3 E_n^2 \varepsilon_4$$

$$D_1 = -\frac{2}{\ell}(1-\ell^2) \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \frac{1}{2} - \varepsilon_3 \frac{\ell^2}{1-\ell^2} + \varepsilon_4 \frac{\ell^2}{1+\ell^2}\right)$$

$$D_2 = -\frac{2}{\ell} \sum_1^3 E_n^2 \varepsilon_3 \qquad D_3 = -\frac{2}{\ell} \frac{1-\ell^2}{1+\ell^2} \sum_1^3 E_n^2 \varepsilon_4$$

$$D_4 = 2 \frac{1-\ell^2}{\ell} \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \frac{1}{2} + \varepsilon_3 \frac{1}{1-\ell^2} + \varepsilon_4 \frac{1}{1+\ell^2}\right)$$

$$D_5 = -\frac{1-\ell^2}{\ell} \sum_1^3 E_n^2 \varepsilon_2 \qquad D_6 = (1-\ell^2)^2 \sum_1^3 E_n^2$$

$$D_7 = -\frac{2}{\ell}(1-\ell^2) \sum_1^3 E_n^2 \left[1 + \varepsilon_3 \left(\frac{1+\ell^2}{1-\ell^2}\right)^2 + \varepsilon_3 \left(\frac{2\ell^2}{(1+\ell^2)^2} + 1\right)\right]$$

$$D_8 = -2\ell \frac{1-\ell^2}{(1+\ell^2)^2} \sum_1^3 E_n^2 \varepsilon_4 \qquad D_9 = \frac{2}{\ell} \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \ell^2 + \varepsilon_3 \frac{1+3\ell^2}{1-\ell^2} + \varepsilon_4\right)$$

$$D_{10} = -2\ell \sum_1^3 E_n^2 \left(1 + \varepsilon_2 \frac{1}{\ell} - \varepsilon_3 \frac{3+\ell^2}{1-\ell^2} + \varepsilon_4\right)$$

and further putting $\mu^{(2)}$ in the form

$$\mu^{(2)} = \varphi_1 + \varphi_2 + \varphi_3$$

the procedure of computation can be summarized as follows:

a) Evaluate $(\varphi_1)_0$ for a series of values of s according to the formula (p. 144, Ref. 2):

$$
\begin{aligned}
\frac{\beta}{4}(\varphi_1)_0 = {} & n_1 \frac{s}{1-s^2}\left[\ln(\ell r_1) + \ln(\ell r_4) - \ln(r_2) - \ln(r_3)\right]\frac{\partial}{\partial s}\left[\ln(\ell r_1) + \ln(\ell r_4) - \ln(r_2) - \ln(r_3)\right] \\
& + n_2 \frac{s}{1-s^2}\left[\ln(\ell r_1) - \ln(\ell r_4)\right]\frac{\partial}{\partial s}\left[\ln(\ell r_1) - \ln(\ell r_4) - \ln(r_2) + \ln(r_3)\right] \\
& + s\left[\ln r_2 - \ln r_3\right]\left\{\left(\frac{n_2}{1-s^2} + n_3\right)\frac{\partial}{\partial s}(\ln r_2 - \ln r_3) + \left(\frac{n_4}{1-s^2} + n_5\right)\frac{\partial}{\partial s}\left[\ln(\ell r_1) - \ln(\ell r_4)\right]\right\} \\
& + n_6\left\{\left[\frac{\partial}{\partial s}\ln(\ell r_1)\right]\left[\frac{\partial}{\partial s}\ln(r_2)\right] + \left[\frac{\partial}{\partial s}\ln(\ell r_4)\right]\left[\frac{\partial}{\partial s}\ln r_3\right]\right\} \\
& + s\ln(r_2)\frac{\partial}{\partial s}\left[n_7\ln(\ell r_1) + n_8\ln(r_2) + n_9\ln(r_3) + n_{10}\ln(\ell r_4)\right] \\
& + s\ln(r_3)\frac{\partial}{\partial s}\left[n_{10}\ln(\ell r_1) + n_9\ln(r_2) + n_8\ln(r_3) + n_7\ln(\ell r_4)\right] \\
& + n_{11}\left\{\left[\ln r_2 - \ln(\ell r_1)\right]^2 + \left[\ln r_3 - \ln(\ell r_4)\right]^2\right\} \\
& + n_{12}\left[\ln r_2 - \ln(\ell r_4)\right]\left[\ln r_3 - \ln(\ell r_1)\right] \\
& + \left[\ln r_2 - \ln r_3\right]\left\{n_{13}\left[\ln(\ell r_1) - \ln(\ell r_4)\right] + n_{14}\left[\ln r_2 - \ln r_3\right] + n_{15}\left[\ln R_1 - \ln R_2\right]\right\} \\
& + n_{16}\left[\ln r_2 + \ln r_3\right]\left\{\ln(\ell r_1) + \ln(\ell r_4) + \tfrac{1}{2}\ln r_2 + \tfrac{1}{2}\ln r_3 - 2(\ln R_1 + \ln R_2)\right\} \\
& + \frac{C_1}{\ell}\left\{2 - s\frac{\partial}{\partial s}\left[\ln(\ell r_1) + \ln(\ell r_4) + \ln r_2 + \ln r_3\right]\right\} \\
& + n_{17}\left[\ln s\right]\left[\ln(\ell r_1) + \ln r_2 + \ln r_3 + \ln(\ell r_4) - 2\ln \ell - 2\left[\ln R_1 + \ln R_2\right]\right. \\
& + C_2\left[\ln R_1 + \ln R_2\right] \qquad + \tfrac{1}{2}(\psi)_0 \\
& + \frac{\beta}{4}K_1^{(2)}\left[\ln r_2 + \ln r_3 - \ln(\ell r_1) - \ln(\ell r_4)\right]
\end{aligned}
$$

where

$n_1 \equiv E_1^2$

$n_2 \equiv E_2^2$

$n_3 \equiv -\dfrac{E_3^2}{1-\ell^2}$

$n_4 \equiv -E_2^2$

$n_5 \equiv E_3^2 \dfrac{\ell^2}{1-\ell^2}$

$n_6 \equiv A_1$

$n_7 \equiv A_3$

$n_8 \equiv -A_4$

$n_9 \equiv A_5$

$n_{10} \equiv A_6$

$n_{11} \equiv A_7$

$n_{12} \equiv A_8$

$n_{13} \equiv -A_9 + \dfrac{\ell}{1-\ell^2}\left(E_3^2\dfrac{2\ell}{1-\ell^2} - \dfrac{E_1}{LB\pi}\right)$

$n_{14} \equiv \dfrac{1}{2}\dfrac{\ell}{1-\ell^2}\left(E_3^2\dfrac{2\ell}{1-\ell^2} - \dfrac{E_1}{LB\pi}\right)$

$n_{15} \equiv \dfrac{1+\ell^2}{1-\ell^2}\left(-E_3^2\dfrac{2\ell}{1-\ell^2} + \dfrac{E_1}{LB\pi}\right)$

$n_{16} \equiv -\dfrac{1}{2}\dfrac{E_2\beta}{L\pi}\dfrac{1+\ell^2}{1-\ell^2}$

$n_{17} \equiv -2n_{16}$

$$(\psi)_0 = 2\frac{\beta}{\pi}\frac{E_2}{L}\frac{1+\ell^2}{1-\ell^2}\left[\ln^2\ell + 2\sum_1^{\infty}(\ell^{2n}-2+\ell^{-2n})\frac{1}{(2n)^2}s^{2n}\right]$$

$$\begin{aligned}
C_1 &= -E_1^2\frac{\ell}{1-\ell^2}\left[\ln(1+\ell^2)+\ln(1-\ell^2)-\ln(2\ell)\right]\\
&\quad + E_2^2\frac{\ell}{1-\ell^2}\left[\ln(1+\ell^2)-\ln(1-\ell^2)-\ln(2\ell)\right]\\
&\quad - A_1\frac{\ell}{1-\ell^2} + A_5\ell\ln(2\ell) + E_3^2\frac{\ell}{1-\ell^2}\ln(2\ell)
\end{aligned}$$

$$\begin{aligned}
C_2 &= -E_3^2\, 2\ell\frac{1+\ell^2}{(1-\ell^2)^2}\left[\ln(1+\ell)-\ln(1-\ell)\right]\\
&\quad + \frac{1}{L\pi}\frac{1+\ell^2}{1-\ell^2}\left[\left(\frac{E_1}{B}-\beta E_2\right)\ln(1+\ell)-\left(\frac{E_1}{B}+\beta E_2\right)\ln(1-\ell)\right]
\end{aligned}$$

$$\begin{aligned}
K_1^{(2)}\frac{\beta}{4} &= A_3\frac{\ell^2}{1-\ell^2} + A_5\left[\ln(2\ell)-\tfrac{1}{2}\right] + A_4 - A_6\frac{2\ell^2}{1+\ell^2} + 2A_7\ln(1-\ell^2)\\
&\quad + A_8\left[\ln(1-\ell^2)-\ln(2\ell)\right] - A_9\left[\ln(1+\ell^2)-\ln(1-\ell^2)\right]\\
&\quad + \frac{2\ell^2}{(1-\ell^2)^2}E_3^2\left[\ln(1+\ell^2)-\ln(1-\ell^2)+\ln(2\ell)-\frac{1-\ell^2}{1+\ell^2}-\frac{1+\ell^2}{\ell}\ln\left(\frac{1+\ell}{1-\ell}\right)\right]\\
&\quad + \frac{1+\ell^2}{1-\ell^2}\left[\frac{E_1}{LB\pi}\ln\left(\frac{1+\ell}{1-\ell}\right) + \frac{E_2\beta}{L\pi}\ln(2)\right] - \frac{\ell}{1-\ell^2}\frac{1}{2\pi}\gamma_2^{(1)}\\
&\quad - \frac{2\ell^2}{(1-\ell^2)^2}\Big\{E_1^2\left[\ln(1+\ell^2)+\ln(1-\ell^2)-\ln(2\ell)\right]\\
&\qquad\qquad - E_2^2\left[\ln(1+\ell^2)-\ln(1-\ell^2)-\ln(2\ell)\right]\Big\}\\
&\quad + \frac{2\ell}{1-\ell^2}\left\{E_1^2\left[\frac{\ell}{1+\ell^2}-\frac{\ell}{1-\ell^2}-\frac{1}{2\ell}\right] - E_2^2\left[\frac{\ell}{1+\ell^2}+\frac{\ell}{1-\ell^2}-\frac{1}{2\ell}\right]\right\}\\
&\quad - \frac{1+\ell^2}{(1-\ell^2)^2}(E_1^2+E_2^2) - \frac{1}{2}D_1\frac{\ell^3}{(1-\ell^2)^2} + \frac{1}{4}D_3\frac{\ell}{1-\ell^2} + D_6\frac{\ell^2}{(1-\ell^2)^4}\\
&\quad + \frac{2\ell}{1-\ell^2}\frac{E_1}{LB\pi}\left[\ln(1-\ell^2)-\ln(2\ell)\right]
\end{aligned}$$

b) Tabulate $(\varphi_2)_0$ for series of values of ω from the formula (p. 148, Ref. 2):

$$
\begin{aligned}
-\frac{\beta}{4}(\varphi_2)_0 = {} & k_1\left\{\left[\frac{\partial}{\partial s}\ln r_2\right]^2+\left[\frac{\partial}{\partial s}\ln r_3\right]^2\right\} + k_2\left\{\theta_{2s}^{\,2}+\theta_{3s}^{\,2}\right\} + k_3\left[\theta_{2s}\right]\left[\theta_{3s}\right] \\
& + k_4\left[\frac{\partial}{\partial s}\ln r_2\right]\left[\frac{\partial}{\partial s}\ln r_3\right] + (A_3+A_6)(\ln r_2+\ln r_3) + (A_1+4E_1^2)\frac{\partial}{\partial s}(\ln r_2+\ln r_3) \\
& + (\omega-\theta_2)\frac{\partial}{\partial s}\left[k_6\theta_2+k_7\theta_3\right] + (\pi-\omega-\theta_3)\frac{\partial}{\partial s}\left[k_7\theta_2+k_6\theta_3\right] \\
& + \ln(r_2)\frac{\partial}{\partial s}\left[k_8\ln r_2+k_9\ln r_3\right] + \ln(r_3)\frac{\partial}{\partial s}\left[k_9\ln r_2+k_8\ln r_3\right] \\
& + k_{10}\left[\ln r_2-\ln r_3\right]^2 + k_{11}\left[\theta_2+\theta_3\right]^2 + k_{12}\left[\pi-\theta_2-\theta_3\right]\frac{\partial}{\partial s}\left[\theta_2+\theta_3\right] \\
& + k_{13}\left[\ln r_2-\ln r_3\right]\frac{\partial}{\partial s}\left[\ln r_2-\ln r_3\right] + k_{14}\left[\pi-\theta_2-\theta_3\right] + k_{15}\left[\ln r_2+\ln r_3\right]^2 \\
& + k_{16}\left[\theta_2-\theta_3\right]^2 + \ln(R_1)\left[(k_{17}+k_{18})\ln r_2 + (k_{18}-k_{17})\ln r_3\right] \\
& + \ln(R_2)\left[(k_{18}-k_{17})\ln r_2 + (k_{18}+k_{17})\ln r_3\right] \\
& + C_2\left[\ln R_1+\ln R_2\right] + k_{19}
\end{aligned}
$$

where

$$
\begin{aligned}
k_1 &\equiv -\left[2(E_1^2+E_2^2)+A_1\right] & k_{14} &\equiv -\pi\left[E_3^2\frac{\ell}{(1-\ell^2)^2}(1-\ell)^2-\frac{1}{2}\frac{E_1}{2B\pi}\frac{(1-\ell)^2}{1-\ell^2}\right] \\
k_2 &\equiv 2E_3^2+A_1 & k_{15} &\equiv -\frac{3}{4}\frac{E_2\beta}{2\pi}\frac{1+\ell^2}{1-\ell^2} \\
k_3 &\equiv 4E_3^2 & k_{16} &\equiv -\frac{1}{4}\frac{E_2\beta}{2\pi}\frac{1+\ell^2}{1-\ell^2} \\
k_4 &\equiv -4\left[E_1^2-E_2^2\right] & k_{17} &\equiv -\left[E_3^2\,2\ell\frac{1+\ell^2}{(1-\ell^2)^2}-\frac{E_1}{2B\pi}\frac{1+\ell^2}{1-\ell^2}\right] \\
k_6 &\equiv A_2+A_4\ell^2 & k_{18} &\equiv \frac{E_2\beta}{2\pi}\frac{1+\ell^2}{1-\ell^2} \\
k_7 &\equiv -(A_6+A_5\ell^2) & k_{19} &\equiv -2E_1^2 \\
k_8 &\equiv -(A_3+A_4) \\
k_9 &\equiv A_5-A_6 \\
k_{10} &\equiv -\left[A_8+A_9+E_3^2\frac{3\ell^2}{(1-\ell^2)^2}+\frac{3}{2}\frac{E_1}{2B\pi}\frac{\ell}{1-\ell^2}\right] \\
k_{11} &\equiv \left[E_3^2\frac{\ell^2}{(1-\ell^2)^2}-\frac{1}{2}\frac{E_1}{2B\pi}\frac{\ell}{1-\ell^2}\right] \\
k_{12} &\equiv -E_3^2 \equiv k_{13}
\end{aligned}
$$

Then since $(\varphi_2)_0$ is even about both vertical and horizontal axes, $(\varphi_2)_0$ may be expressed in a Fourier series

$$(\varphi_2)_0 = \frac{1}{2}a_0 + \sum_1^M a_{2n}\cos 2n\omega$$

where $$a_{2n} = \frac{1}{\pi}\int_0^{\pi}(\varphi_2)_0\cos 2n\omega\,d\omega = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}(\varphi_2)_0\cos 2n\omega\,d\omega$$

a_{2n} is evaluated by graphical integration using a planimeter. Thus the regular harmonic function associated with boundary value (1) is

$$\varphi_2 = \frac{1}{2}a_0 + \sum_1^\infty a_{2n} s^{2n} \cos 2n\omega$$

where $$(\varphi_2)_0 = \frac{1}{2}a_0 + \sum_1^\infty a_{2n} s^{2n}$$

$(\varphi_2)_0$ is then evaluated at the series of values s picked above.

c) A Fourier analysis of $(\varphi_3)_0$ is made for Lighthill's result on Mach cone (Appendix), which gives on $\omega = 0$,

$$(\varphi_3)_0 = \frac{1}{2}A_0 + \sum_1^\infty A_{2n} s^{2n}$$

where A_{2n} is obtained from formulas furnished in Appendix.

d) Results of a), b), and c) above are added to give $(u^{(2)})_0$.

e) $u^{(1)}$ and $u^{(2)}$ are computed from (42), (43).

f) $C_p^{(1)}$ and $C_p^{(2)}$ are computed from (4), (5).

In performing the calculations, resort has been made to various geometric relations furnished by Appendix A, Ref. 2.

The computation procedure is illustrated by tables for the case of $M_\infty = \sqrt{2}$, $\ell = 0.3$. The computed results for $u^{(1)}$, $u^{(2)}$ $C_p^{(1)}$, $C_p^{(2)}$ for various Mach numbers and values of λ are plotted in Curve 1 to Curve 20. Note that different scales have been adopted in plotting the curves. For $M_\infty = \sqrt{2}$, plots of u and C_p at $s = 0$ against ℓ are shown in Fig. 21. For $\ell = 0.3$, plots of u and C_p at $s = 0$ against M_∞ are given in Fig. 22. In both cases $2\varepsilon = 6°$ is used.

B. Concluding Remarks

The theory and computation formulas adopted in the present report are mainly taken from Ref. 2, by Dr. F. K. Moore. Those not to be found there are given in the Appendix to the present report.

The boundary condition on Mach cone given in Ref. 2 is incorrect. Prof. W. R. Sears pointed out that Lighthill's result in Ref. 3 should be introduced. Ref. 1 is the corrected version.

The computations have largely been carried out by Mrs. Anne Kane.

The effect of ε on the difference between the first and second approximation, as expected, grows with ε^2.

The reason that the discrepancy between first and second approximation changes radically with decreasing planform angle Λ is that the parameter ε is not, in fact, an ideal thickness parameter. A little reflection shows that $\varepsilon' = \varepsilon \csc \Lambda$ might be a better choice. Indeed, the boundary value on the singular line inside the unit circle of the conical plane is given by ε' rather than ε. Thus even for such small values of ε as might seem evidently to meet the requirement of linearization, the value of ε' can still cause breakdown, provided Λ be sufficiently small. It is suggested therefore that the present second approximation might be improved by expanding, instead, in powers of ε'.

It might also be of interest to note that, so far as computation results indicate, the present theory predicts a particular value of $\ell \ (\doteq 0.3)$, for which the first-order solution gives good approximation irrespective of Mach number. A further investigation to see whether this is a coincidence is also suggested.

APPENDIX

In order to utilize Lighthill's result, it should be noted that in his discussion, physical polar coordinates (r, ω) were used, while in the present report, conical coordinates (s, ω) are used. The problem can be stated in following manner. In the differential equation

$$\nabla^2 u^{(2)} = H(r, \omega) \tag{1}$$

a solution of the form

$$u^{(2)} = \varphi_1 + \varphi_2 + \varphi_3 \tag{2}$$

is sought, where

$$\nabla^2 \varphi_3 = 0 \tag{3}$$

and φ_3 satisfies the boundary conditions

$$(\varphi_3)_{\circledcirc} = - G(\omega) \tag{4}$$

$$(\varphi_3)_{\omega}(\omega = 0) = 0 \tag{5}$$

$G(\omega)$ is obtained from Lighthill's result on the study of conical shock. Thus, since φ_3 is analytic inside the unit circle $s = 1$, it takes on the form

$$\phi_3 = \sum_0^\infty A_{2n} s^{2n} \cos 2n\omega \tag{6}$$

Now using Lighthill's notation, Ref. 3, we have

$$u^{(2)} = U(f - r f_r) \tag{7}$$

$$u^{(2)}(r=1) = - U r f_r = - U f_{2R} \qquad (R = 1) \tag{8}$$

So $$\left(f_{2R}\right)_{R=1} = \frac{\gamma+1}{2}\frac{M^4}{\beta^2}A^2(\omega) = \frac{G(\omega)}{U} \quad \text{over } r = s = 1. \tag{9}$$

But as can easily be shown (Ref. 3)

$$A(\omega) = \left(\frac{\partial f_{1R}}{\partial\sqrt{1-R}}\right)_{R=1} \tag{10}$$

and since $f_R = \frac{1}{\beta U}(\phi_x\cos\omega + \phi_y\sin\omega) = \frac{1}{\beta U}(v\cos\omega + w\sin\omega)$ (11)

$$\left(\frac{dw}{d\sqrt{1-R}}\right)_{R=1} = \frac{4\sqrt{2}}{\pi}U\ell\sin\omega\,\frac{1+\ell^2}{(1+\ell^2)^2-4\ell^2\cos^2\omega} \tag{12}$$

$$\left(\frac{dv}{d\sqrt{1-R}}\right)_{R=1} = \frac{4\sqrt{2}}{\pi}U\ell\cos\omega\,\frac{1+\ell^2}{(1+\ell^2)^2-4\ell^2\cos^2\omega} \tag{13}$$

So $$\left(\frac{df_R}{d\sqrt{1-R}}\right)_{R=1} = A(\omega) = \frac{1}{\beta U}\left(\frac{dv}{d\sqrt{1-R}}\cos\omega + \frac{dw}{d\sqrt{1-R}}\sin\omega\right)$$

$$= \frac{4\sqrt{2}}{\pi\beta}\,\frac{1+\ell^2}{(1+\ell^2)^2-4\ell^2\cos^2\omega} \tag{14}$$

i.e. $$G(\omega) = \frac{\gamma+1}{2}\frac{M^4}{\beta^2}A^2(\omega) = \frac{(\gamma+1)16\ell^2M^4}{\pi^2\beta^4}\,\frac{(1+\ell^2)^2}{[(1+\ell^2)^2-4\ell^2\cos^2\omega]^2}$$

$$= \sum_0^\infty A_{2n}\cos 2n\omega \tag{15}$$

and finally we arrive at

$$(\phi_3)_0 = -U\sum_0^\infty A_{2n}s^{2n} \tag{16}$$

where $$\left.\begin{aligned}
A_0 &= \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4\frac{\ell^2}{(1-\ell^2)^2}\,\frac{1+\ell^4}{1-\ell^4}\\
A_2 &= \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4\frac{4\ell^4}{(1-\ell^2)^2(1-\ell^4)}\\
A_4 &= \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4\frac{2\ell^6(3-\ell^4)}{(1-\ell^2)^2(1-\ell^4)}\\
A_6 &= \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4\frac{4\ell^8(2-\ell^4)}{(1-\ell^2)^2(1-\ell^4)}\\
A_8 &= \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4\frac{2\ell^{10}(5-3\ell^4)}{(1-\ell^2)^2(1-\ell^4)}
\end{aligned}\right\} \tag{17}$$

$$A_{2n} = \frac{16}{\pi^2}(\gamma+1)\left(\frac{M}{\beta}\right)^4 \frac{\ell^2(1+\ell^2)^2}{(1+\ell^4)^2}\frac{1}{\pi}\int_0^{2\pi}\frac{\cos nu\,du}{(1-a\cos u)^2}$$

$$a = \frac{2\ell^2}{1+\ell^4} \qquad \sqrt{1-a^2} = \frac{1-\ell^4}{1+\ell^4} \tag{18}$$

putting $z = e^{i\theta}$, we have

$$\int_0^{2\pi}\frac{\cos m\theta\,d\theta}{(1-a\cos\theta)^2} = \oint \frac{\frac{1}{2}(z^m+z^{-m})(-\frac{i}{z})\,dz}{[1-\frac{a}{2}(z+z^{-1})]^2} = \oint\frac{(z^{2m+1}+1)(-\frac{i}{2z^{m+1}})\,dz}{(\frac{a}{2})^2(\frac{2}{a}z-1-z^2)^2\cdot\frac{1}{z^2}}$$

$$= -\frac{2i}{a^2}\oint\frac{(z^{2m+1}+1)\,dz}{z^{m-1}(z-\zeta)^2(z-\bar\zeta)^2} = -\frac{2i}{a^2}2\pi i\sum Re$$

$$= \frac{4\pi}{a^2}(Re_0 + Re_{\bar\zeta})$$

where

$$\zeta\bar\zeta = 1 \qquad \bar\zeta = \frac{1-\sqrt{1-a^2}}{a} \qquad \zeta = \frac{1+\sqrt{1-a^2}}{a} \tag{19}$$

$$\bar\zeta - \zeta = -\frac{2\sqrt{1-a^2}}{a} \qquad \bar\zeta+\zeta = \frac{2}{a}$$

$$Re_0 = \text{Coeff. of } z^{-1} \text{ in: } \frac{1}{z^{m-1}}\sum_k (k+1)\left(\frac{z}{\zeta}\right)^k\sum_\ell(\ell+1)\left(\frac{z}{\bar\zeta}\right)^\ell$$

$$= \sum_{n=1}^{[\frac{m}{2}]} n(m-n)(\zeta^{m-2n}+\bar\zeta^{m-2n})$$

$[\frac{m}{2}]$ denoting largest integer bounded by $\frac{m}{2}$. (20)

Now,

$$f'(z) = \frac{1}{2\pi i}\oint\frac{f(w)}{(w-z)^2}\,dw$$

i.e.

$$\oint\frac{f(w)}{(w-z)^2}\,dw = 2\pi i\,Re_z\,, \quad \text{so}$$

$$Re_{\bar\zeta} = \frac{d}{dz}\left\{\frac{z^{m+1}+z^{-m+1}}{(z-\zeta)^2}\right\}_{z=\bar\zeta}$$

$$= \frac{m(\bar\zeta^m-\zeta^m)+(\bar\zeta^m+\zeta^m)}{(\bar\zeta-\zeta)^2} - \frac{2\bar\zeta(\bar\zeta^m+\zeta^m)}{(\bar\zeta-\zeta)^3} \tag{21}$$

Thus

$$\int_0^{2\pi}\frac{\cos m\theta\,d\theta}{(1-a\cos\theta)^2} = \frac{4\pi}{a^2}\left[\sum_1^{[\frac{m}{2}]} n(m-n)(\zeta^{m-2n}+\bar\zeta^{m-2n}) + \frac{m(\bar\zeta^m-\zeta^m)+(\bar\zeta^m+\zeta^m)}{(\bar\zeta-\zeta)^2} - \frac{2\bar\zeta(\bar\zeta^m+\zeta^m)}{(\bar\zeta-\zeta)^3}\right] \tag{22}$$

<u>References</u>

1. Moore, F.K. Second Approximation to Supersonic Conical Flows. Journal of Aeronautical Scientist, Volume 17, No.6, pp 328-335, June 1950.

2. Moore, F.K. Second Approximation to Supersonic Conical Flows. Thesis for Ph.D., Cornell University, June 1949.

3. Lighthill, M.J. The Shock Strength in Supersonic "Conical Fields." Phil. Magazine. December 1949, pp. 1202-1223.

M	$\sqrt{2}$
β	1
ℓ	.3
υ	1
γ	1.4
$\cot\Lambda$	1.816667
2ϵ	$10^\circ, 6^\circ, 4^\circ, 2^\circ$

$1/1-\ell^2$	1.098901	$1/1+\ell^2$	.917431	$\ln(1+\ell^2)$	.0861777
$\ell/1-\ell^2$	.329670	$\ell/1+\ell^2$	.275229	$\ln(1-\ell^2)$	−.0943107
$\ell^2/1-\ell^2$	.0989011	$\ell^2/1+\ell^2$	.0825688	$\ln(2\ell)$	−.510826
$1/(1-\ell^2)^2$	1.207584	$1/(1+\ell^2)^2$	.841680	$\ln(\ell)$	−1.203973
$\ell^2/(1-\ell^2)^2$	.108683	$\ell^2/(1+\ell^2)^2$	.0757512	$\ln(1+\ell)$	.262364
$\ell^4/(1-\ell^2)^2$	.00978143	$\ell^4/(1+\ell^2)^2$	.00681761	$\ln(1-\ell)$	−.356675

$\ln(1+\ell^2)+\ln(1-\ell^2)-\ln(2\ell)$	.502693
$\ln(1+\ell^2)-\ln(1-\ell^2)-\ln(2\ell)$	.691314
$\ln(1+\ell)-\ln(1-\ell)$	.619039

L	2.	A_1	.0528568	n_3	.111342
B	1.183216	A_2	.1602971	n_4	−.145369
B^2	1.4	A_3	−.0545834	n_5	−.0100208
E_1	−.248327	A_4	.338853	n_6	.0528568
E_2	−.381272	A_5	.169746	n_7	−.0545833
E_3	$-\frac{1}{\pi}$ .318310	A_6	.0088095	n_8	−.338853
E_1^2	.0616663	A_7	.0335129	n_9	.169746
E_2^2	.145369	A_8	.0280315	n_{10}	.0088095
E_3^2	−.1013212	A_9	.0088095	n_{11}	.0335129
$E_1/LB\pi$	−.0334026	D_1	−.972469	n_{12}	.0280315
$E_2B/L\pi$	−.0606814	D_3	−1.029794	n_{13}	−.019822
$(1+\ell^2)/(1-\ell^2)$	1.197802	D_6	.0875415	n_{14}	−.0055059
$\frac{E_2\beta}{2L\pi}\frac{1+\ell^2}{1-\ell^2}$	−.0363421	n_1	.0616663	n_{15}	.0400097
$\Upsilon_2^{(1)}$	−1.156526	n_2	.145369	n_{16}	.0363422

n_{17}	$-.0726844$	k_5	$.299522$	k_{12}	$.1013212$
$\frac{1}{2}c_1$	$-.0115495$	k_5'	$-.0457739$	k_{13}	$.1013212$
c_2	$.0179125$	k_6	$.190794$	k_{14}	$.0282524$
$K_1^{(2)}$	$-.269640$	k_7	$-.0240867$	k_{15}	$.0545132$
k_1	$-.466927$	k_8	$-.284269$	k_{16}	$.0181711$
k_2	$-.149786$	k_9	$.160937$	k_{17}	$.0400097$
k_3	$-.405285$	k_{10}	$.0127110$	k_{18}	$-.0726843$
k_4	$.334809$	k_{11}	$-.00550592$	k_{19}	$-.123333$

S	0	.1	.2	.25	.275
$S/1-S^2$	0	.1010101	.208333	.266667	.297498
(ℓr_1)	1.	1.03	1.06	1.075	1.0825
(r_2)	.3	.4	.5	.55	.575
(r_3)	.3	.2	.1	.05	.025
(ℓr_4)	1.	.97	.94	.925	.9175
(R_1)	1.	1.1	1.2	1.25	1.275
(R_2)	1.	.9	.8	.75	.725
$\ln(\ell r_1)$	0	.0295588	.0582689	.0723207	.0792732
$\ln(r_2)$	-1.203973	-.916291	-.693147	-.597837	-.553385
$\ln(r_3)$	-1.203973	-1.60944	-2.30259	-2.99573	-3.68888
$\ln(\ell r_4)$	0	-.0304592	-.0618754	-.0779615	-.0861027
$\ln(R_1)$	0	.0953102	.182322	.223144	.242946
$\ln(R_2)$	0	-.1053605	-.223144	-.287682	-.321584
$\frac{\partial}{\partial S}\ln(\ell r_1)$	.3	.291262	.283019	.279070	.277136
$\frac{\partial}{\partial S}\ln(r_2)$	3.33333	2.5	2.	1.81818	1.73913
$\frac{\partial}{\partial S}\ln(r_3)$	-3.33333	-5.	-10.	-20.	-40.
$\frac{\partial}{\partial S}\ln(\ell r_4)$	-.3	-.309278	-.319149	-.324324	-.326975
S^{2n}					
$n=1$	0	.01	.04	.0625	.075625
$n=2$	0	.0001	.0016	.00390625	.00571914
$n=3$	0	.000001	.000064	.0002 4414	.00043251
$n=4$	0	.00000001	.0000 0256	.0000 1526	.0000327l

ω	0	$\pi/16$	$\pi/8$	$3\pi/16$	$\pi/4$	$5\pi/16$	$3\pi/8$	$7\pi/16$	$\pi/2$
$\cos\omega$	1	.980785	.923880	.831470	.707107	.555570	.382683	.195090	0
$\sin\omega$	0	.195090	.382683	.555570	.707107	.831470	.923880	.980785	1.
$(r_2)^2$	1.69	1.67847	1.644328	1.588882	1.514264	1.42334	1.319610	1.207054	1.09
$(r_3)^2$	.49	.501529	.535672	.591118	.665736	.756658	.860390	.972946	1.09
$(R_1)^2$	4.	3.96157	3.84776	3.66294	3.41421	3.11114	2.76537	2.39018	2.
$(R_2)^2$	0	.0384294	.152241	.337061	.585786	.888860	1.23463	1.609819	2.
θ_2	0	.151159	.303049	.456435	.612145	.771113	.934917	1.10332	1.27934
θ_3	3.14159	2.86251	2.59138	2.33403	2.09318	1.86900	1.66005	1.46424	1.27934
Θ_1	0	.0981748	.196350	.294524	.392699	.490874	.589049	.687223	.785398
$\ln(r_2)$	.262364	.25894	.248666	.231515	.207465	.176504	.138668	.0940914	.0430889
$\ln(r_3)$	-.356675	-.345047	-.312116	-.262870	-.203431	-.139422	-.0751848	-.0137134	.0430889
$\ln(R_1)$	+.693147	.688320	.673745	.649133	.613974	.567495	.508587	.435684	.346574
$\ln(R_2)$	$-\infty$	-1.62946	-.941145	-.543746	-.267400	-.0589080	.105387	.238061	.346574
$\frac{\partial}{\partial s}\ln(r_2)$	.769231	.771080	.776709	.786365	.8004760	.819670	.844799	.876951	.917431
$\frac{\partial}{\partial s}\ln(r_3)$	1.42857	1.40723	1.34940	1.26973	1.18345	1.101329	1.028830	.967652	.917431
$\frac{\partial}{\partial s}\theta_2$	0	.034869	.069819	.104848	.140084	.175250	.210035	.243763	.275229
$\frac{\partial}{\partial s}\theta_3$	0	.116697	.214320	.281959	.318643	.329661	.322138	.302417	.275229

$$\frac{1}{2}(\Psi)_0 = (-.1053395)+(-.145369)\left[-\sum_1^\infty \frac{(2-\ell^{2n})}{(2n)^2} S^{2n} + \sum_1^\infty \frac{1}{(2n)^2}\left(\frac{S}{\ell}\right)^{2n}\right]$$

S	0	.1	.2	.25	.275
$\sum_1^\infty \frac{2-\ell^{2n}}{(2n)^2} S^{2n}$	0	.004788	.019303	.030344	.036848
$\sum_1^\infty \frac{(S/\ell)^{2n}}{(2n)^2}$	0	.028590	.126759	.219961	.28977

Line \ S	0	.1	.2	.25	.275
1	0	.039034	.306133	1.070078	2.96906
2	0	-.006080	-.041472	-.123591	-.294197
3	0	.127688	.983686	3.42467	9.57990
4	.1057136	.120225	.198611	.369674	.716791
5	0	.157096	.331799	.602186	1.12570
6	0	-.344111	-1.72599	-5.32180	-14.0699
7	.097157	.113535	.187183	.300359	.448411
8	.040633	.040698	.041776	.044710	.049358
9	0	.002095	.008014	.010207	.006412
10	.105360	.114157	.154579	.218536	.303827
11	-.023099	-.026007	-.041662	-.075727	-.144778
12	0	-.016499	-.059631	-.107028	-.158005
13	0	-.000180	-.000731	-.001156	-.001409
14	-.105360	-.108820	-.120961	-.132904	-.142107
15	.162320	.170199	.201699	.241862	.285511
$\beta/4(\phi_1)_0$	.382725	.383030	.423036	.520072	.674576

Term \ ω	0	π/16	π/8	3π/16	π/4	5π/16	3π/8	7π/16	π/2
—1	1.22920	1.20227	1.131903	1.041516	.953149	.880056	.827476	.796293	.786005
—2	0	.002222	.007610	.013556	.018148	.020878	.022151	.022599	.022693
—3	0	.001649	.006064	.011987	.018091	.023415	.027422	.029877	.030701
—4	−.367922	−.363296	−.350910	−.334296	−.317173	−.302241	−.291001	−.284113	−.281802
—5	−.658290	−.652451	−.636816	−.615845	−.594231	−.575381	−.561193	−.552499	−.549582
—51	−.004317	−.003941	−.002904	−.001435	.000185	.001697	.002906	.003679	.003945
—6	0	−.000174	−.000732	−.001753	−.003301	−.005370	−.007874	−.010635	−.013373
—7	0	−.001773	−.006176	−.011203	−.015102	−.017066	−.017115	−.015699	−.013373
—8	−.002949	−.001885	.000902	.004443	.007695	.009842	.010341	.008803	.004875
—9	−.100690	−.095211	−.080711	−.061614	−.042231	−.025258	−.011767	−.001837	.004875
—10	−.004871	−.004637	−.003997	−.003107	−.002146	−.001269	−.000581	−.000148	0
—11	.054341	.050006	.046127	.042873	.040297	.038377	.037062	.036297	.036046
—12	0	−.001965	−.007116	−.013763	−.020277	−.025655	−.029501	−.031767	−.032511
—13	.041355	.038930	.032540	.024212	.015944	.009016	.003988	.000991	0
—14	0	−.003614	−.006983	−.009920	−.012325	−.014168	−.015457	−.016218	−.016469
—15	−.000485	−.000404	−.000219	−.000054	−.000001	−.000075	−.000220	−.000352	−.000405
—16	−.179341	−.133583	−.095152	−.064059	−.039858	−.021903	−.009568	−.002367	0
—17	−.021919	−.020941	−.018224	−.014319	−.009914	−.005644	−.002005	.000666	.002171
—18	———	−.029178	−.016776	−.009516	−.004474	−.000903	.001388	.002418	.002171
—19	−.012416	.016858	.004790	−.001888	−.006208	−.009110	−.010998	−.012068	−.012416
—20	.123333	.123333	.123333	.123333	.123333	.123333	.123333	.123333	.123333

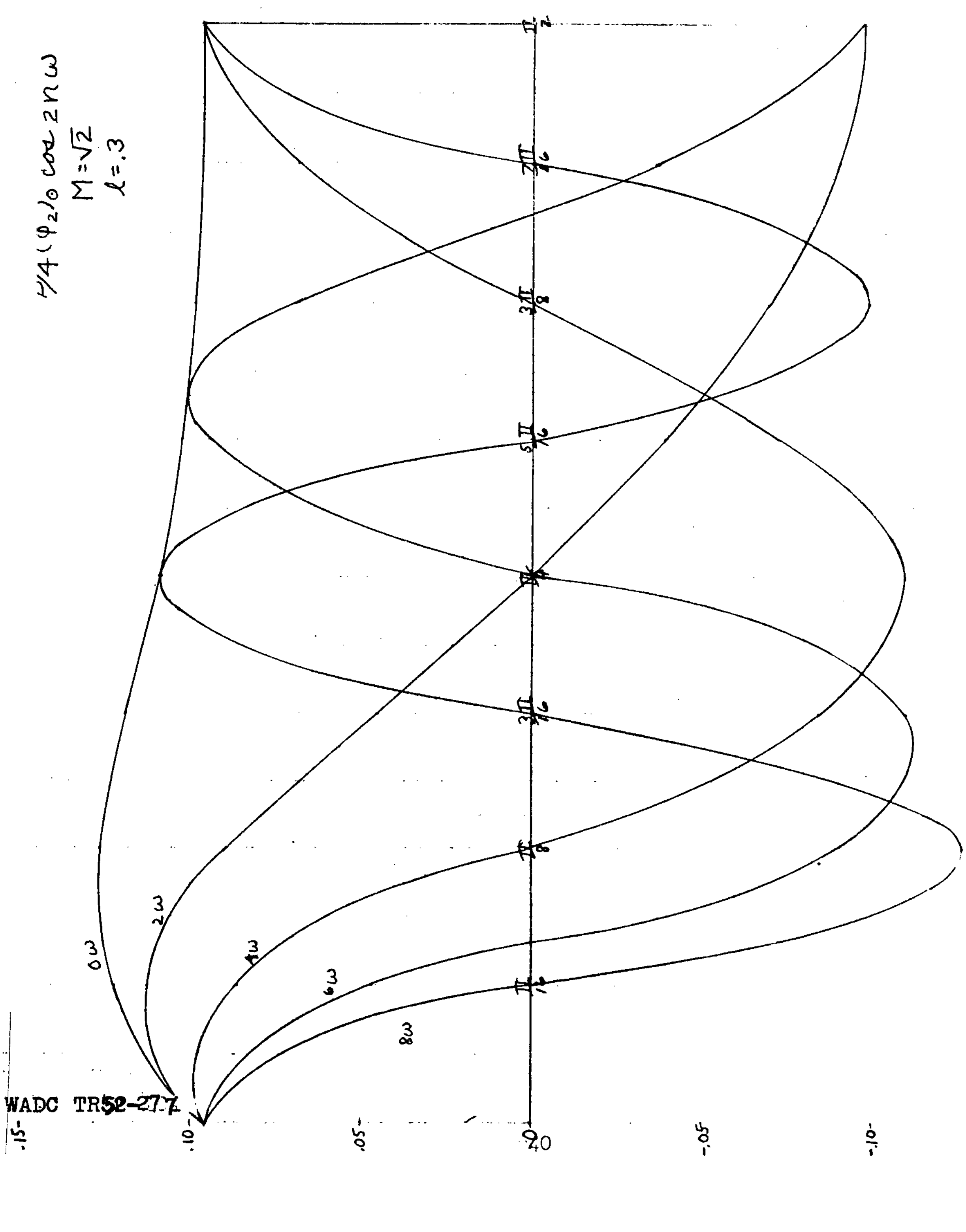
ω	0	$\pi/16$	$\pi/8$	$3\pi/16$	$\pi/4$	$5\pi/16$	$3\pi/8$	$7\pi/16$	$\pi/2$
$\beta/4\,(\phi_2)_\odot$	.095028	.122211	.126553	.119148	.109601	.102571	.098787	.097253	.096884
$\cos 2\omega$	1.	.923880	.707107	.382683	0	−.382683	−.707107	−.923880	−1.
$\cos 4\omega$	1.	.707107	0	−.707107	−1.	−.707107	0	.707107	1.
$\cos 6\omega$	1.	.382683	−.707107	−.923880	0	.923880	.707107	−.382683	−1.
$\cos 8\omega$	1.	0	−1.	0	1.	0	−1.	0	1.
$\frac{\beta}{4}(\phi_2)_\odot \cos 2\omega$	.095028	.112909	.089487	.045596	0	−.039252	−.069851	−.089851	−.096884
$\beta/4\,(\phi_2)_\odot \cos 4\omega$	.095028	.086417	0	−.084251	−.109601	−.072529	0	.068769	.096884
$\beta/4\,(\phi_2)_\odot \cos 6\omega$	.095028	.046768	−.089487	−.110079	0	.094763	.069851	−.037217	−.096884
$\beta/4\,(\phi_2)_\odot \cos 8\omega$	.095028	0	−.126554	0	.109601	0	−.098787	0	.096884

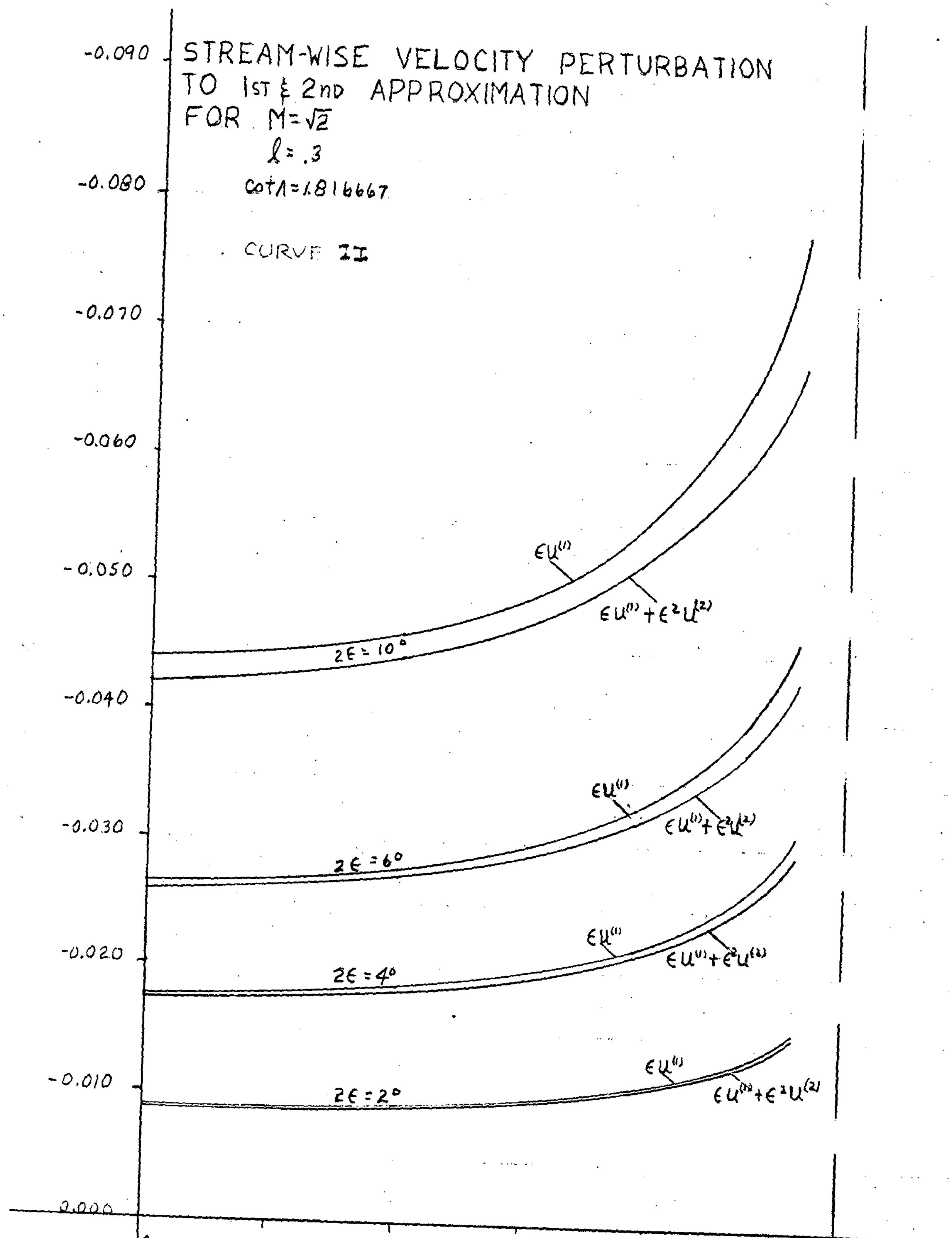
n	a_{2n}	$-A_{2n}$	b_{2n}
0	.8831	−1.7190	−1.2774
1	.0548	−.6138	−.5590
2	−.0060	−.0826	−.0886
3	−.0360	−.0099	−.0459
4	−.0160	−.0011	−.0171

S	0	$.1$	$.2$	$.25$	$.275$	$.300$
b_0	-1.2774	-1.2774	-1.2774	-1.2774	-1.2774	
$b_2\ s^2$	0	$-.0056$	$-.0224$	$-.0349$	$-.0423$	
$b_4\ s^4$	0	0	$-.0001$	$-.0003$	$-.0005$	
$b_6\ s^6$	0	0	0	0	0	
$b_8\ s^8$	0	0	0	0	0	
$(\phi_2)_0+(\phi_3)_0$	-1.2774	-1.2830	-1.2999	-1.3126	-1.3202	
$(\phi_1)_0$	1.5309	1.5321	1.6921	2.0803	2.6998	
$u^{(2)}$	$.2535$	$.2491$	$.3922$	$.7677$	1.3781	
$u^{(1)}$	$-.5054$	$-.5299$	$-.6280$	$-.7530$	$-.8997$	
$v^{(1)}$	0	$.2414$	$.5678$	$.8569$	1.1324	
$w^{(1)}$	$1.$	$1.$	$1.$	$1.$	$1.$	
$C_p^{(1)}$	1.011	1.060	1.256	1.506	1.773	
$C_p^{(2)}$	-1.252	-1.276	-1.712	-2.703	-4.243	
η	0	$.198$	$.385$	$.471$	$.511$	$.550$

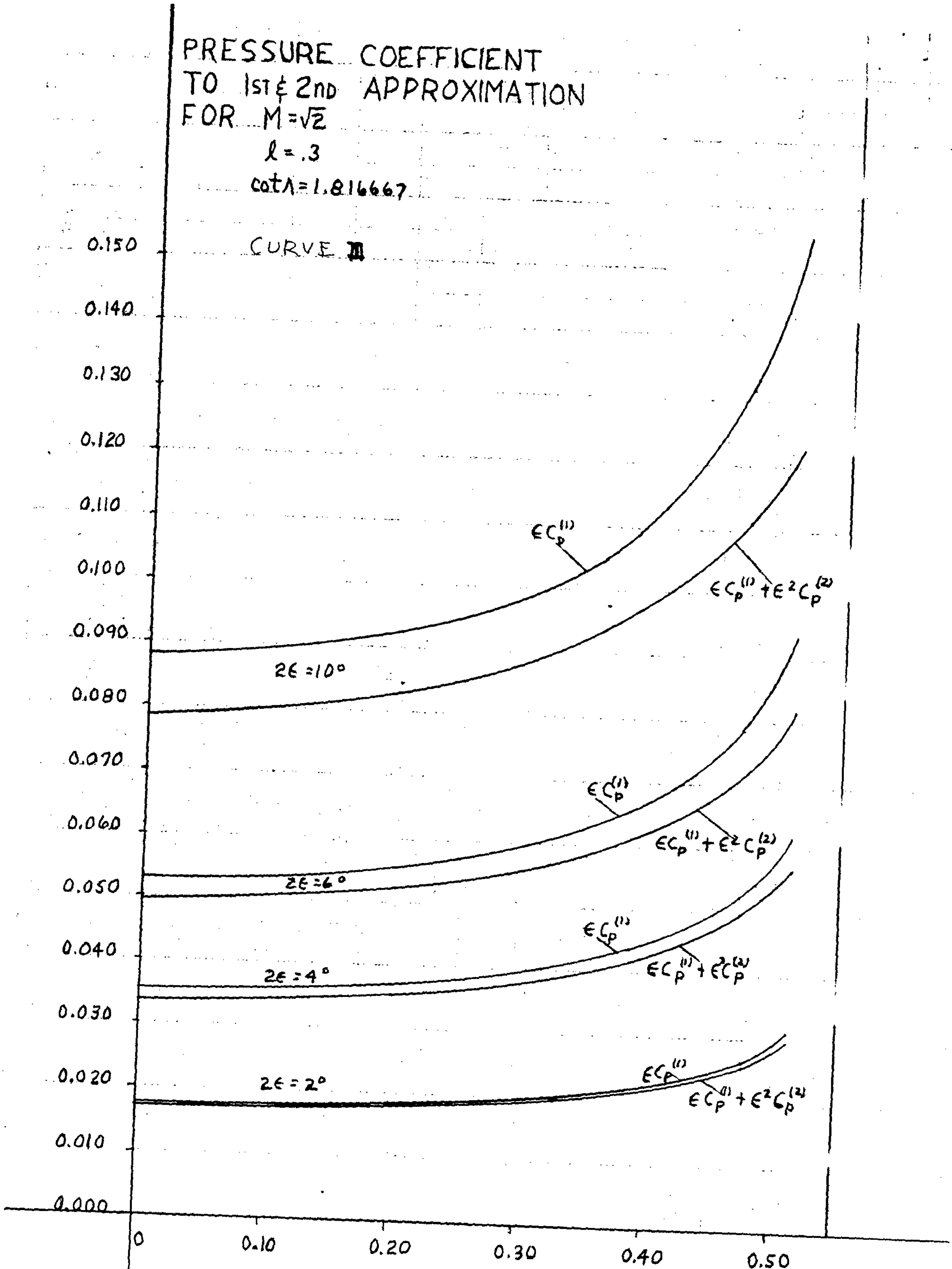
S	0	.1	.2	.25	.275
$2\epsilon = 2^\circ$	$\epsilon = .01745$	;	$\epsilon^2 = .000305$		
$\epsilon u^{(1)}$	$-.0088$	$-.0092$	$-.0110$	$-.0131$	$-.0155$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$	$-.0087$	$-.0092$	$-.0108$	$-.0129$	$-.0151$
$\epsilon C_p^{(1)}$	.0176	.0185	.0219	.0263	.0310
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$	.0173	.0181	.0214	.0255	.0297
$2\epsilon = 4^\circ$	$\epsilon = .03491$	;	$\epsilon^2 = .001219$		
$\epsilon u^{(1)}$	$-.0176$	$-.0185$	$-.0219$	$-.0263$	$-.0310$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$	$-.0173$	$-.0182$	$-.0214$	$-.0254$	$-.0294$
$\epsilon C_p^{(1)}$	.0353	.0370	.0438	.0526	.0621
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$	.0338	.0354	.0418	.0493	.0569
$2\epsilon = 6^\circ$	$\epsilon = .0524$	;	$\epsilon^2 = .00275$		
$\epsilon u^{(1)}$	$-.0265$	$-.0278$	$-.0329$	$-.0395$	$-.0466$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$	$-.0258$	$-.0271$	$-.0318$	$-.0374$	$-.0428$
$\epsilon C_p^{(1)}$	.0530	.0555	.0658	.0789	.0932
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$	.0496	.0520	.0611	.0715	.0815
$2\epsilon = 10^\circ$	$\epsilon = .0873$	;	$\epsilon^2 = .00762$		
$\epsilon u^{(1)}$	$-.0441$	$-.0462$	$-.0548$	$-.0657$	$-.0776$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$	$-.0422$	$-.0443$	$-.0518$	$-.0599$	$-.0671$
$\epsilon C_p^{(1)}$	.0882	.0925	.1096	.1314	.1552
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$	.0787	.0828	.0966	.1108	.1228

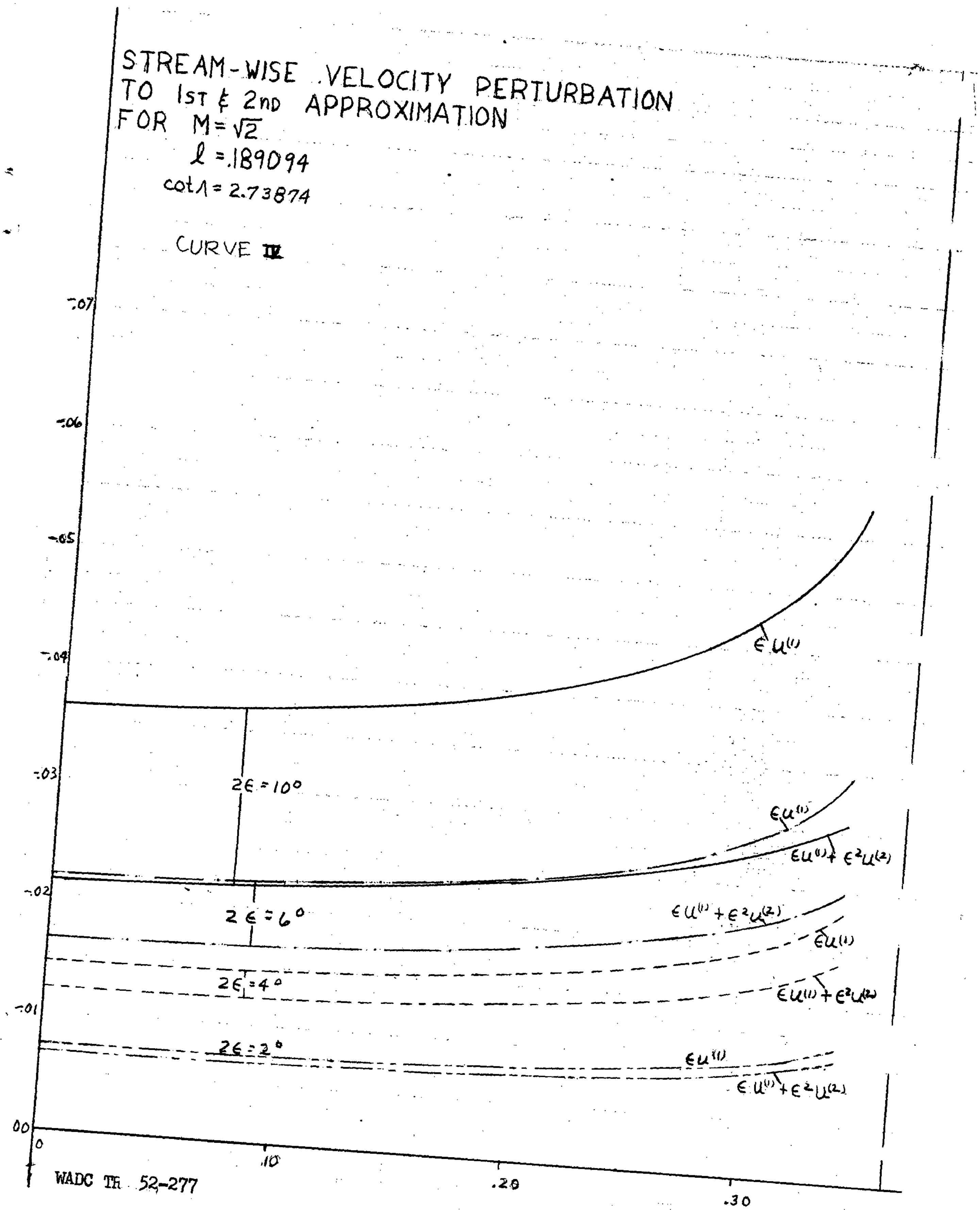
CURVE I

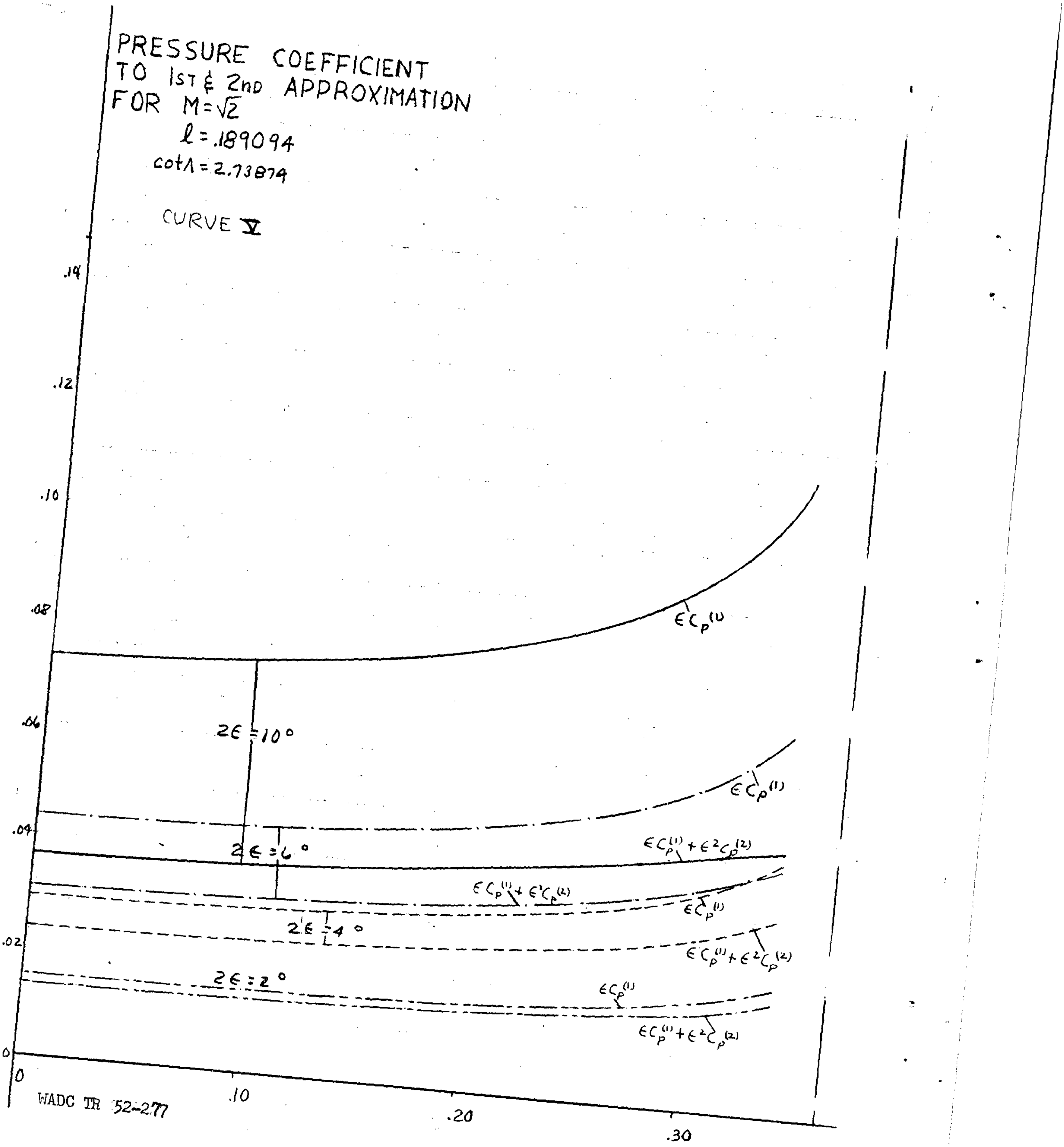


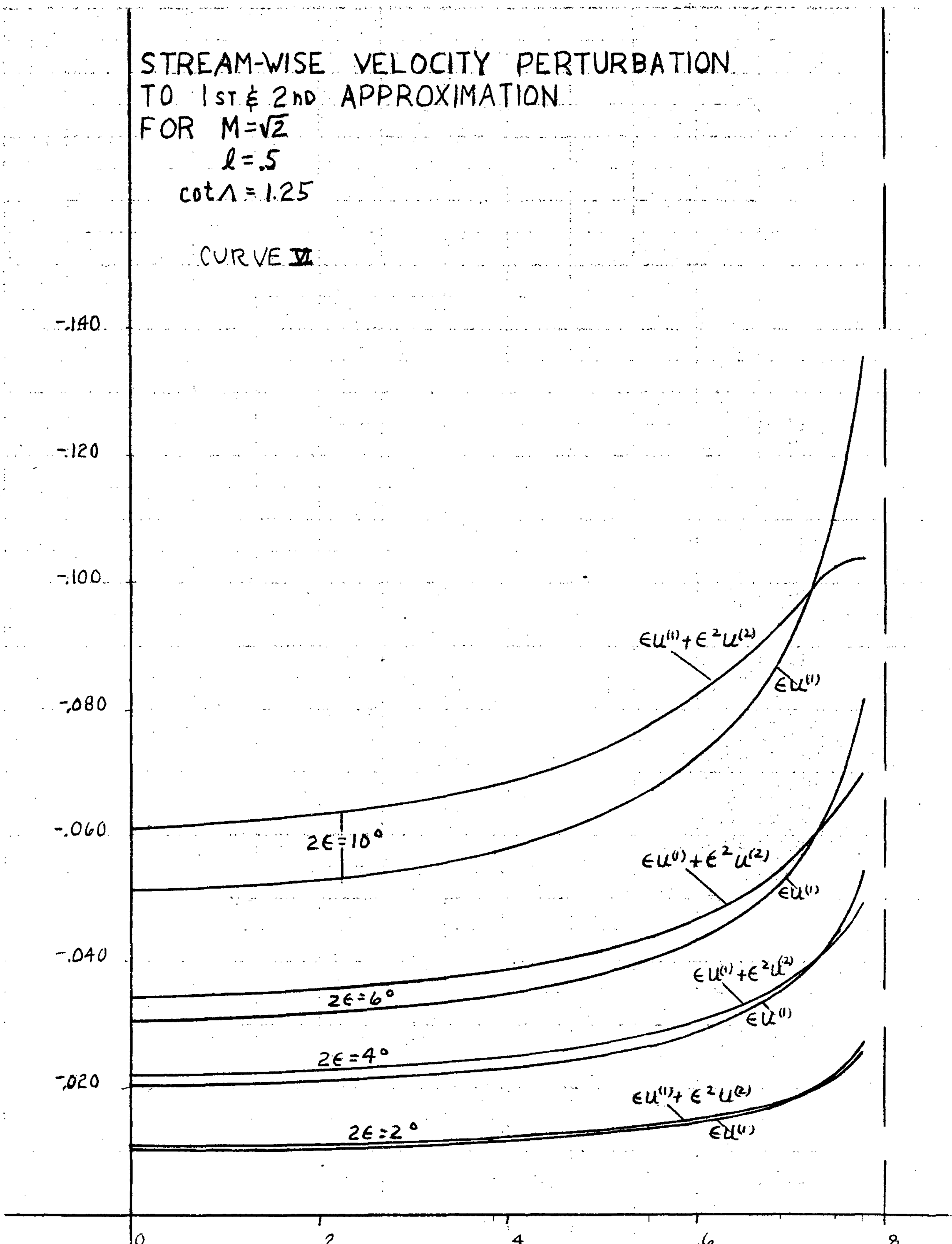


STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=√2
l= .3
cot Λ=1.816667
CURVE II
-0.090
-0.080
-0.070
-0.060
-0.050
-0.040
-0.030
-0.020
-0.010
0.000
0
0.10
0.20
0.30
0.40
0.50
2ε = 10°
2ε = 6°
2ε = 4°
2ε = 2°
εu(1)
εu(1)+ε²u(2)

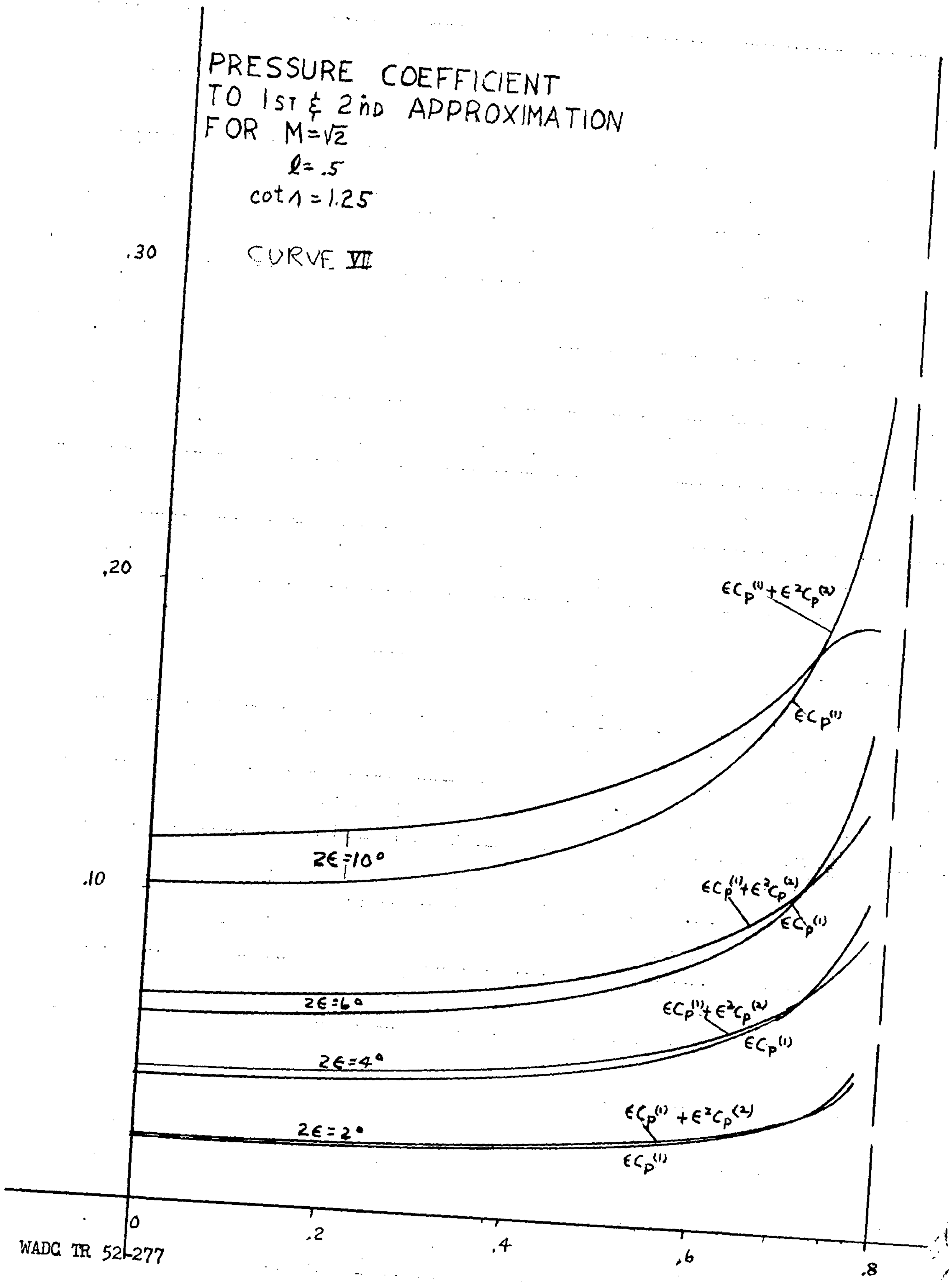

PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=√2
ℓ=.3
cotΛ=1.816667
CURVE III
ϵCp(1)
ϵCp(1)+ϵ²Cp(2)
2ϵ=10°
2ϵ=6°
2ϵ=4°
2ϵ=2°
0.150
0.140
0.130
0.120
0.110
0.100
0.090
0.080
0.070
0.060
0.050
0.040
0.030
0.020
0.010
0.000
0
0.10
0.20
0.30
0.40
0.50

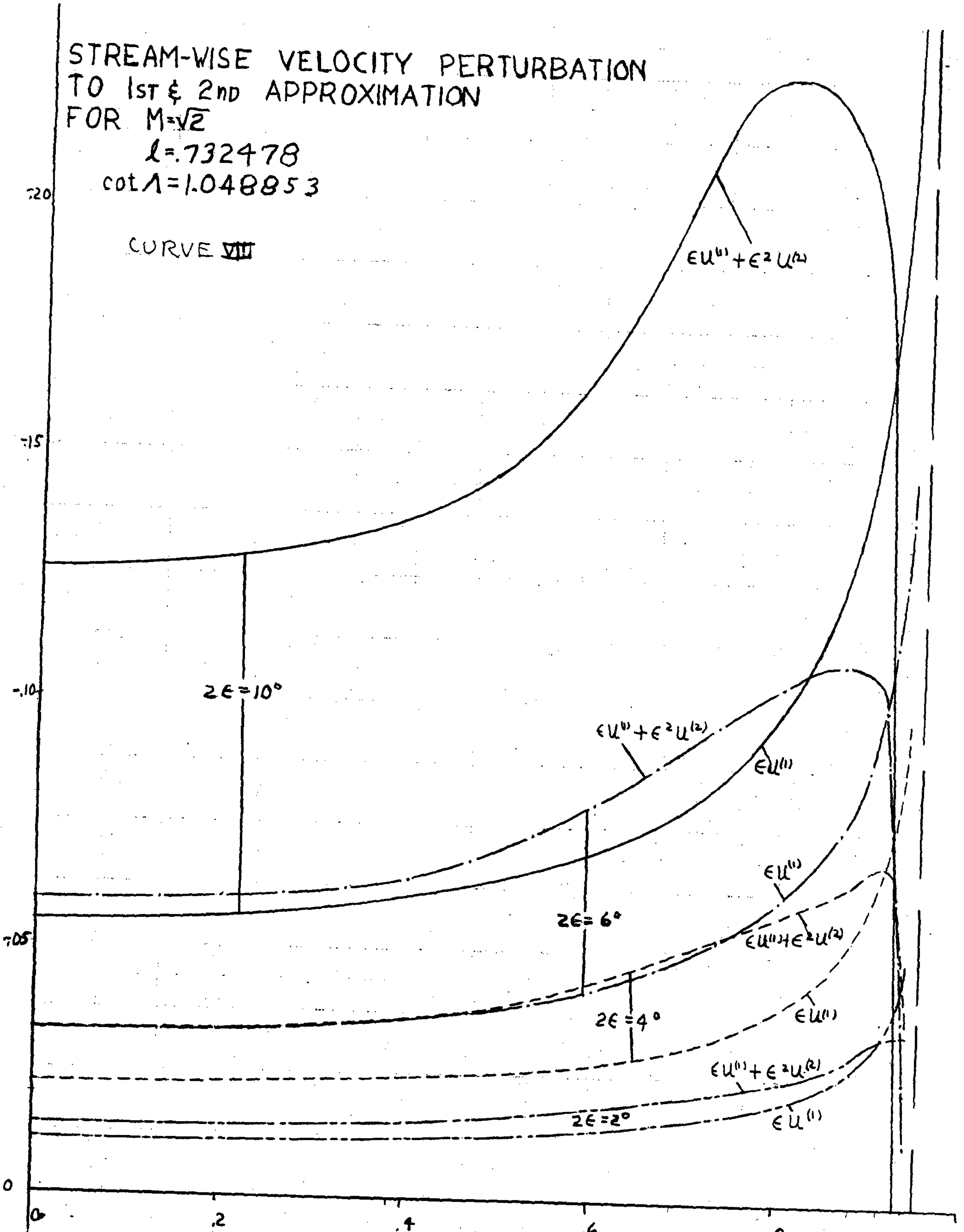

STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=√2
ℓ=.189094
cotΛ=2.73874
CURVE IV
ϵu(1)
ϵu(1)+ϵ²u(2)
2ϵ=10°
2ϵ=6°
2ϵ=4°
2ϵ=2°
.07
.06
.05
.04
.03
.02
.01
00
0
.10
.20
.30

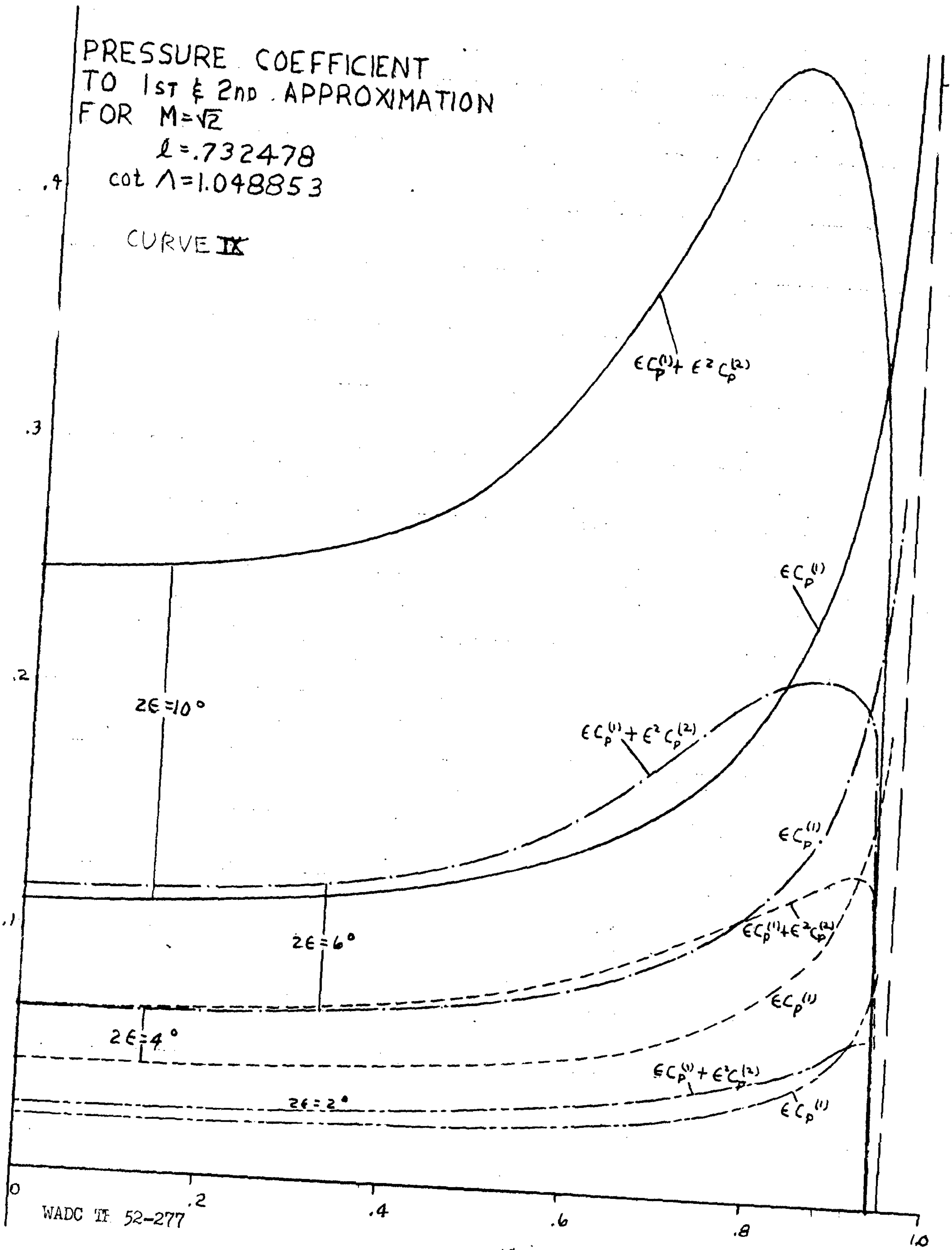

PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR $M=\sqrt{2}$
$\ell = .189094$
$\cot\Lambda = 2.73874$
CURVE V
.14
.12
.10
.08
.06
.04
.02
00
0
.10
.20
.30
$2\epsilon = 10°$
$2\epsilon = 6°$
$2\epsilon = 4°$
$2\epsilon = 2°$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$

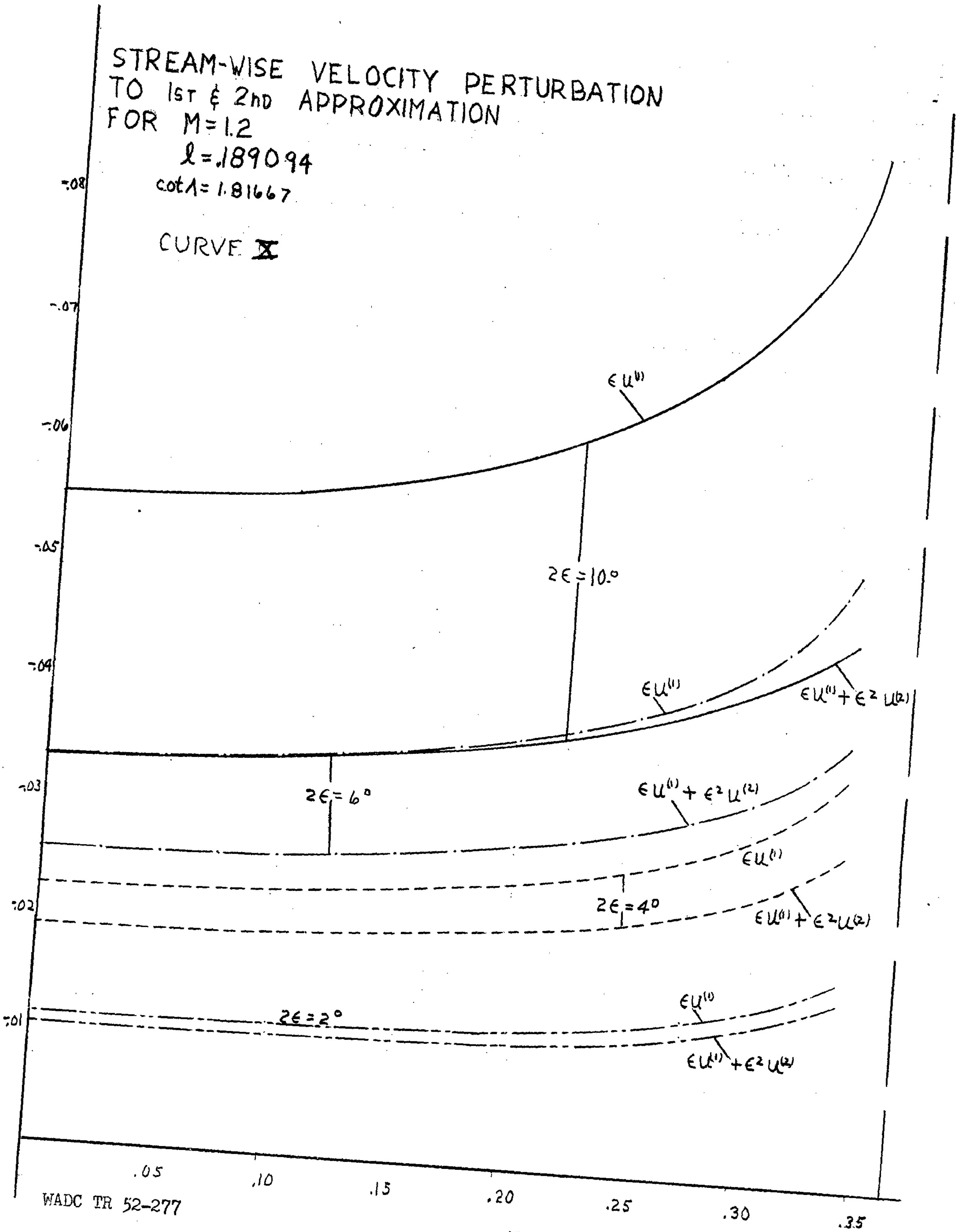


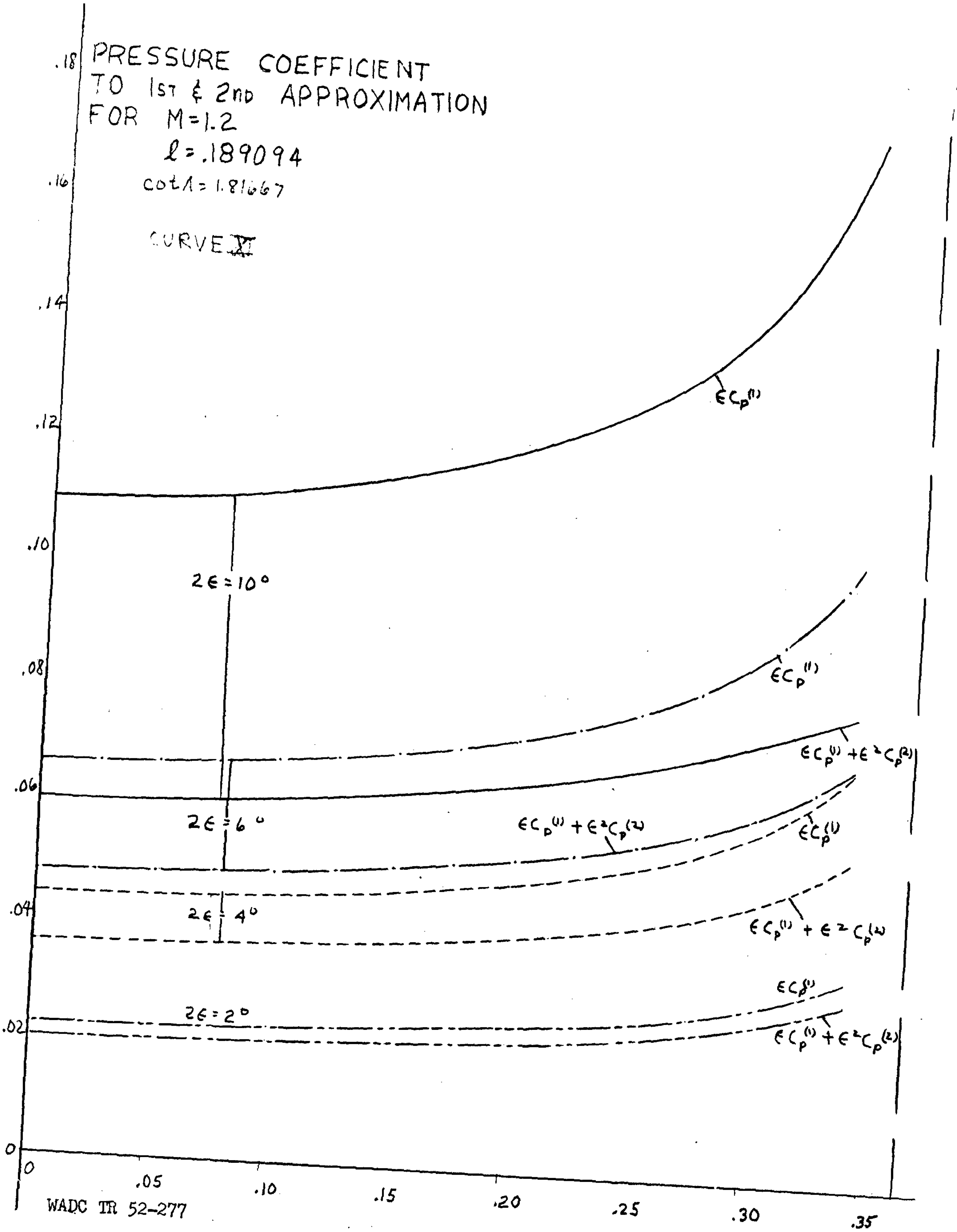
STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=√2
l=.5
cot Λ = 1.25
CURVE VI
-.140
-.120
-.100
-.080
-.060
-.040
-.020
0
.2
.4
.6
.8
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εu(1)+ε2u(2)
εu(1)

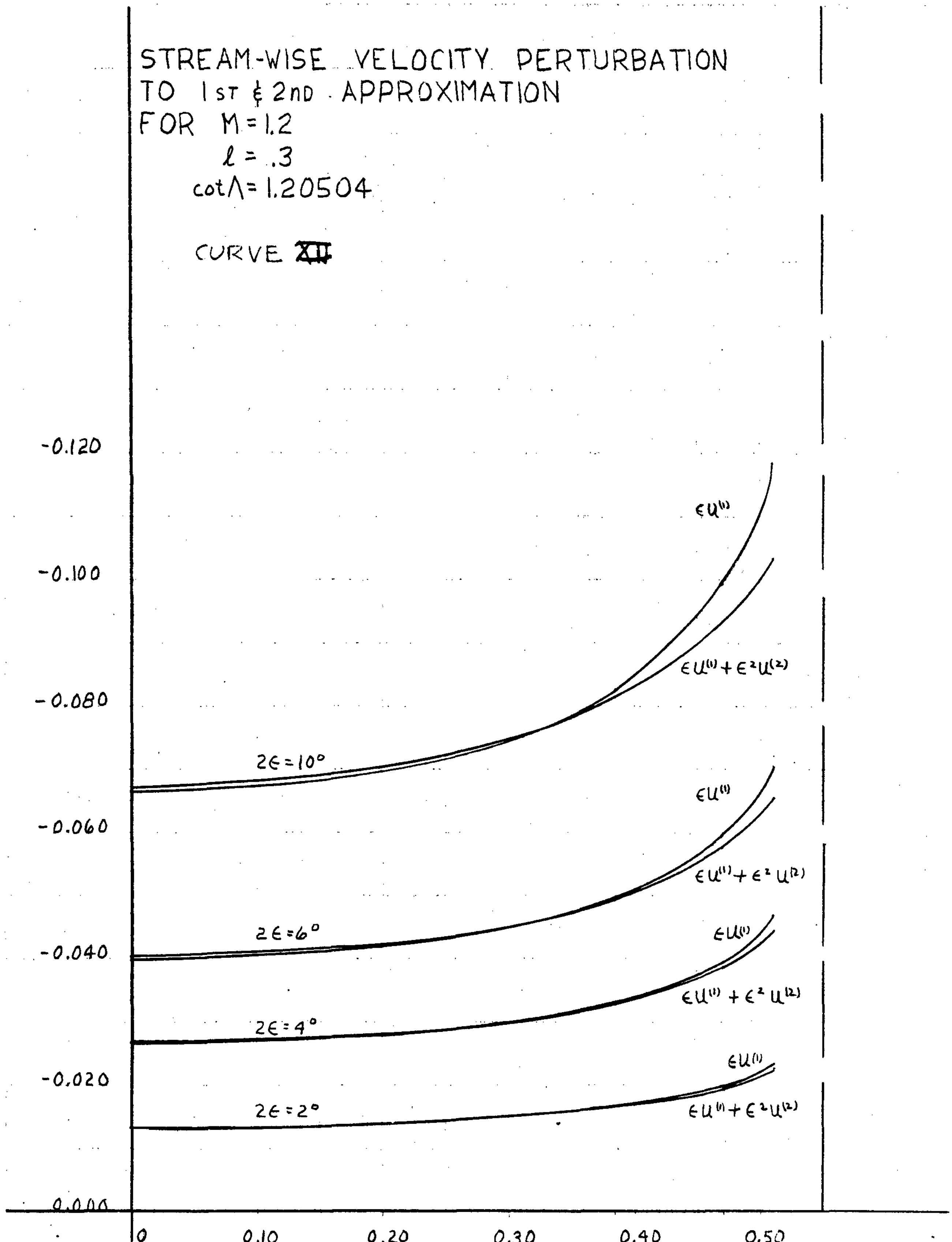

PRESSURE COEFFICIENT
TO 1st & 2nd APPROXIMATION
FOR M=√2
ℓ=.5
cot Λ=1.25
CURVE VII
.30
.20
.10
0
.2
.4
.6
.8
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εCp(1)+ε²Cp(2)
εCp(1)

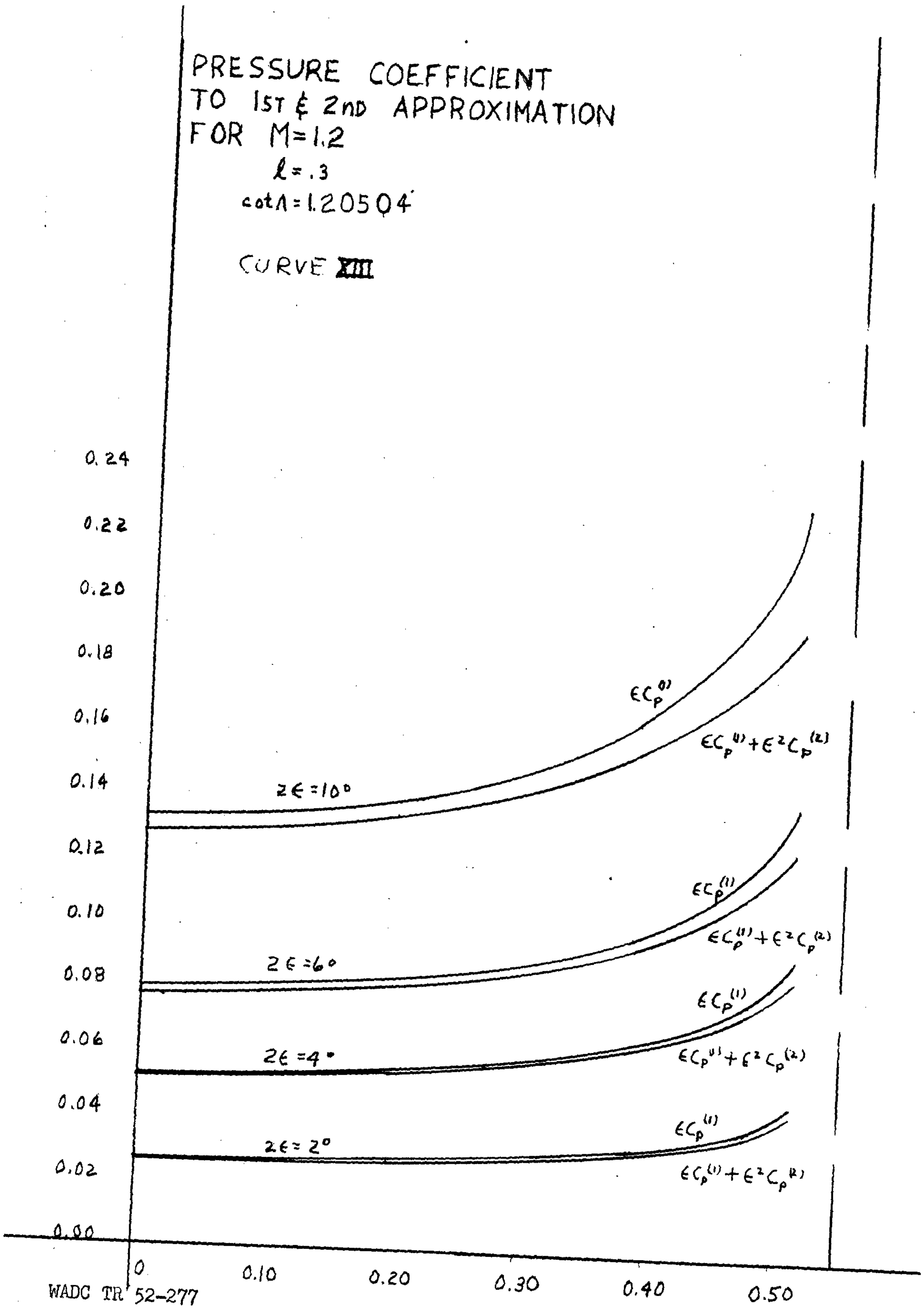

STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=√2
ℓ=.732478
cotΛ=1.048853
CURVE VIII
.20
.15
.10
.05
0
0
.2
.4
.6
.8
1.0
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$2\epsilon=10°$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$\epsilon u^{(1)}$
$\epsilon u^{(1)}$
$2\epsilon=6°$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$2\epsilon=4°$
$\epsilon u^{(1)}$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$2\epsilon=2°$
$\epsilon u^{(1)}$

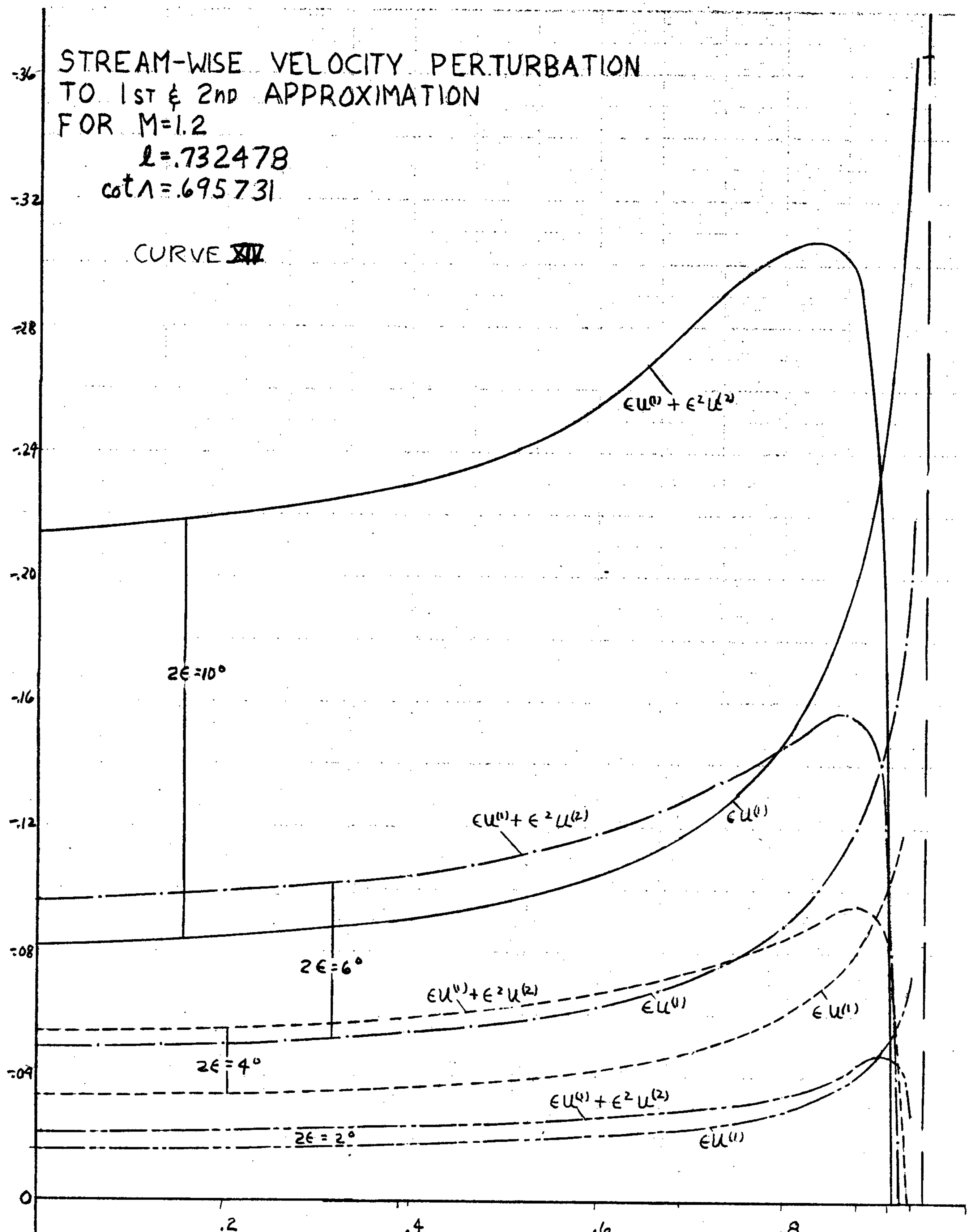

PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=√2
ℓ=.732478
cot Λ=1.048853
CURVE IX
2ε=10°
2ε=6°
2ε=4°
2ε=2°
.4
.3
.2
.1
0
.2
.4
.6
.8
1.0

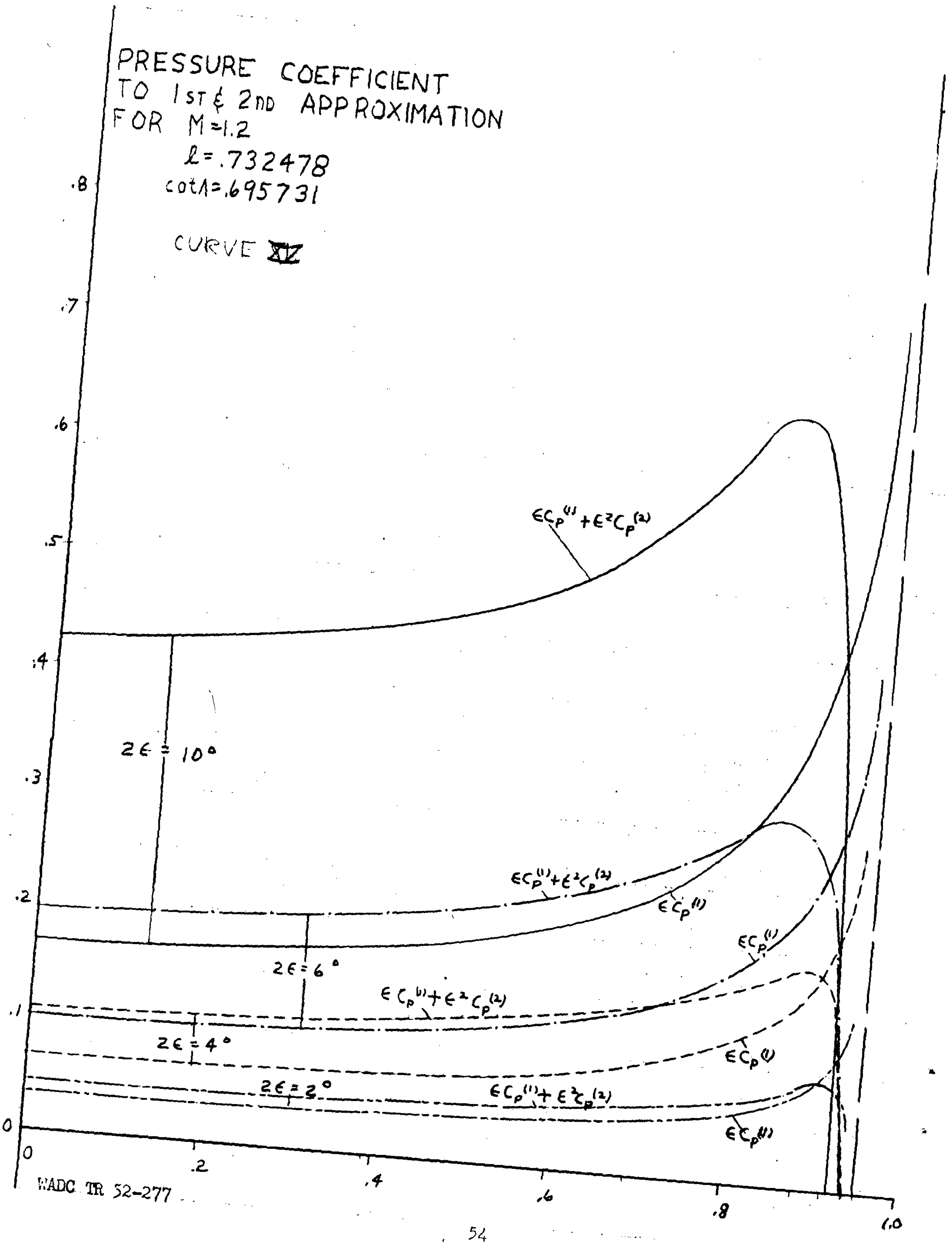

STREAM-WISE VELOCITY PERTURBATION
TO 1st & 2nd APPROXIMATION
FOR M=1.2
l=.189094
cotΛ= 1.81667
CURVE X
-.08
-.07
-.06
-.05
-.04
-.03
-.02
-.01
.05
.10
.15
.20
.25
.30
.35
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εu(1)
εu(1)+ε²u(2)

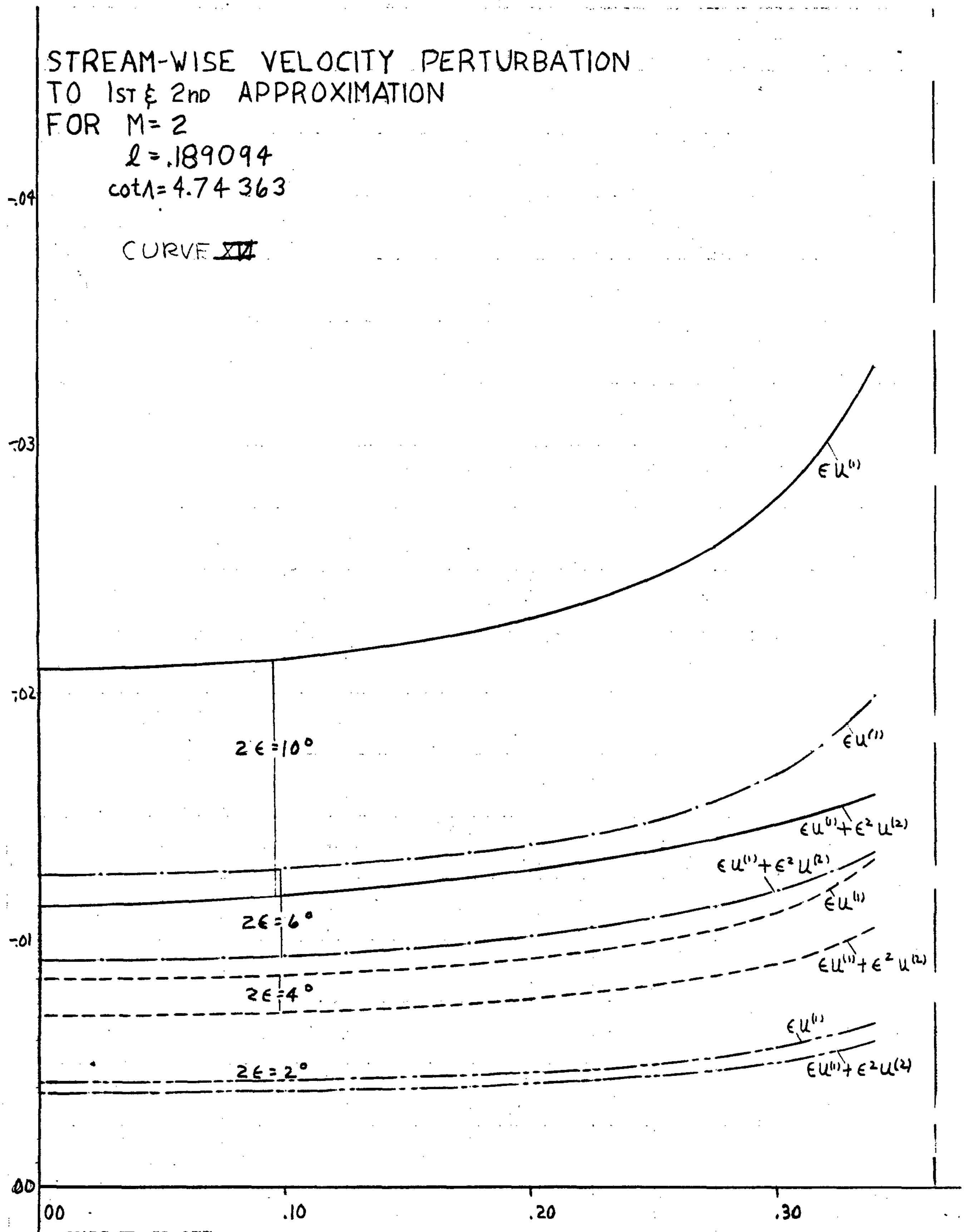

PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=1.2
ℓ=.189094
cotΛ=1.81667
CURVE XI
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εCp(1)
εCp(1)+ε2Cp(2)
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
.05
.10
.15
.20
.25
.30
.35

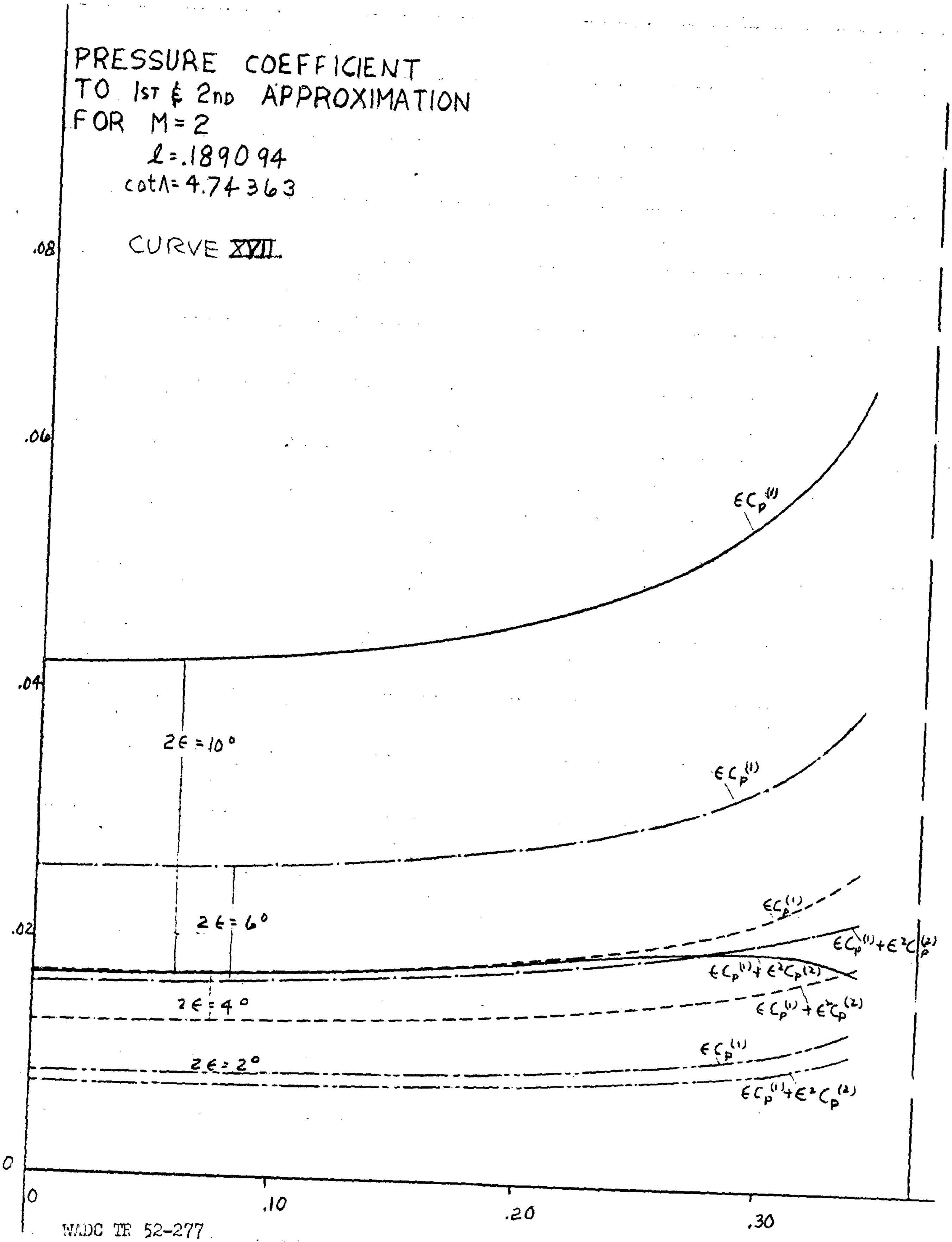

STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M = 1.2
ℓ = .3
cot Λ = 1.20504
CURVE XII
-0.120
-0.100
-0.080
-0.060
-0.040
-0.020
0.000
0
0.10
0.20
0.30
0.40
0.50
2ε = 10°
2ε = 6°
2ε = 4°
2ε = 2°
εu(1)
εu(1) + ε²u(2)


PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=1.2
ℓ=.3
cotΛ=1.20504
CURVE XIII
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εCp(1)
εCp(1)+ε²Cp(2)
0
0.10
0.20
0.30
0.40
0.50

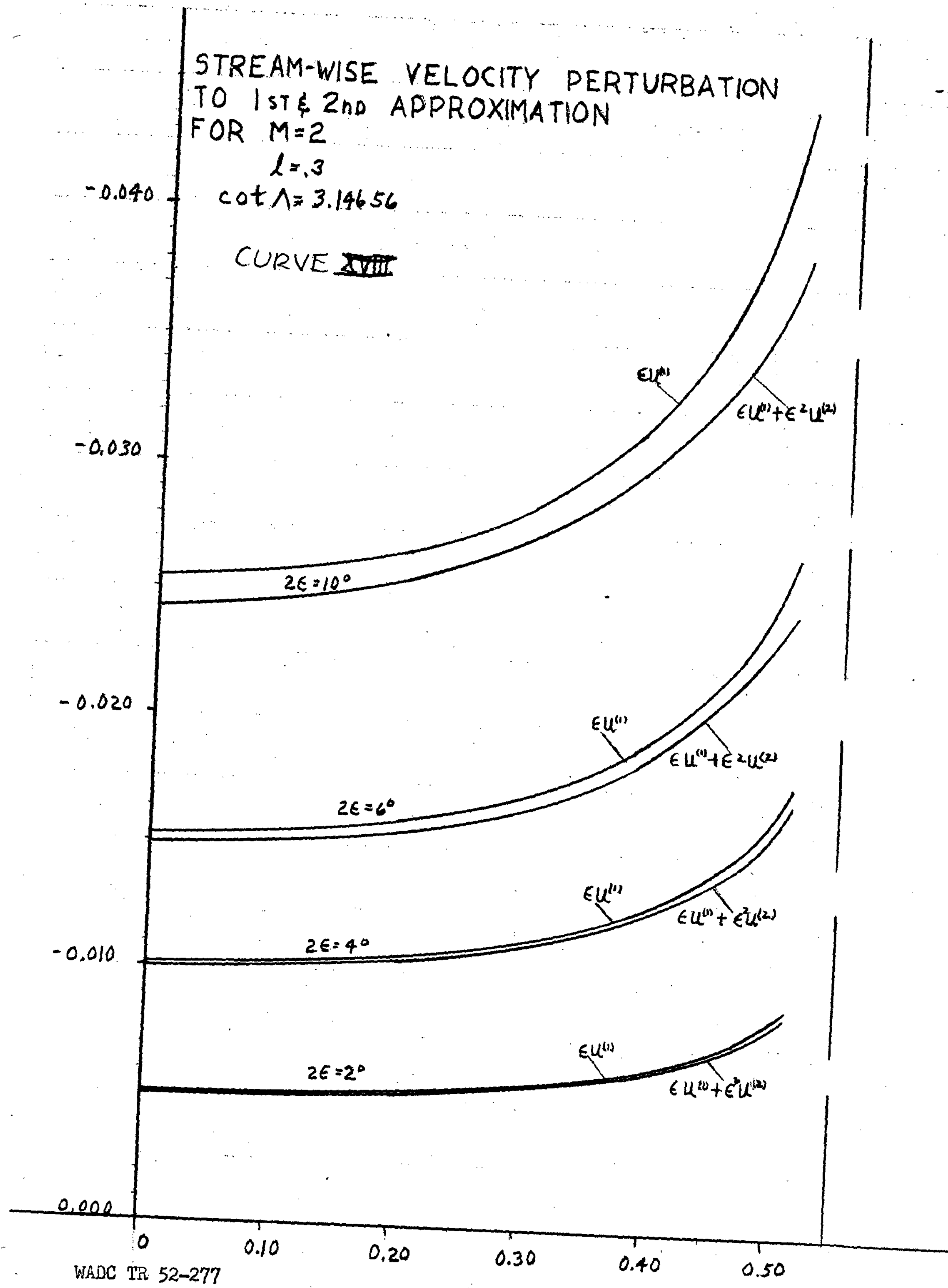

STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=1.2
ℓ=.732478
cot Λ=.695731
CURVE XIV
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$
$2\epsilon = 10°$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$
$\epsilon u^{(1)}$
$2\epsilon = 6°$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$
$\epsilon u^{(1)}$
$\epsilon u^{(1)}$
$2\epsilon = 4°$
$\epsilon u^{(1)} + \epsilon^2 u^{(2)}$
$2\epsilon = 2°$
$\epsilon u^{(1)}$
-.36
-.32
-.28
-.24
-.20
-.16
-.12
-.08
-.04
0
.2
.4
.6
.8

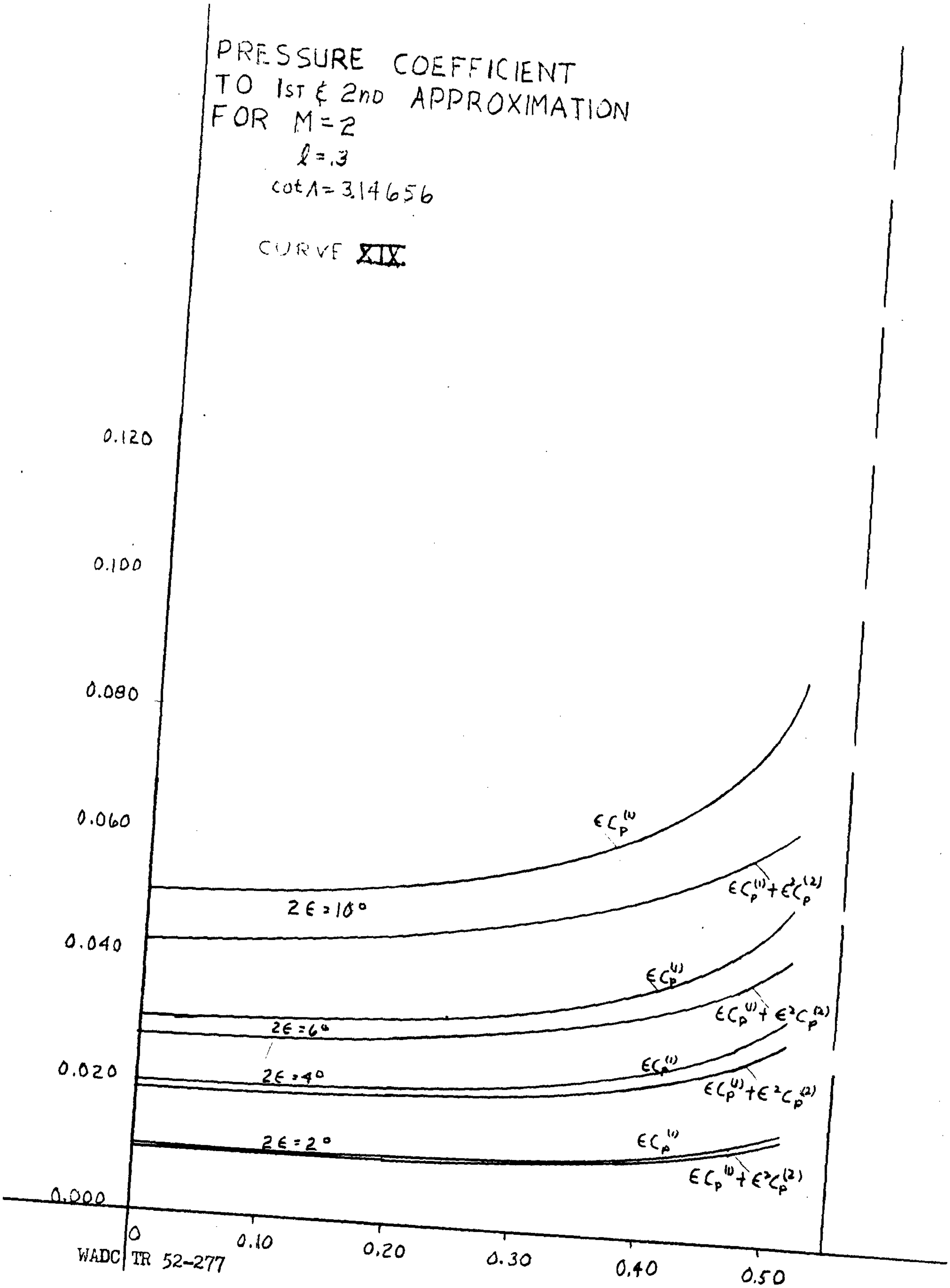

PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=1.2
ℓ=.732478
cotΛ=.695731
CURVE XV
$\epsilon C_p^{(1)}+\epsilon^2 C_p^{(2)}$
$\epsilon C_p^{(1)}$
$2\epsilon = 10°$
$2\epsilon = 6°$
$2\epsilon = 4°$
$2\epsilon = 2°$
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.2
.4
.6
.8
1.0


STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M = 2
l = .189094
cotΛ = 4.74363
CURVE XV
-.04
-.03
-.02
-.01
.00
00
.10
.20
.30
2ε = 10°
2ε = 6°
2ε = 4°
2ε = 2°
εu(1)
εu(1) + ε²u(2)

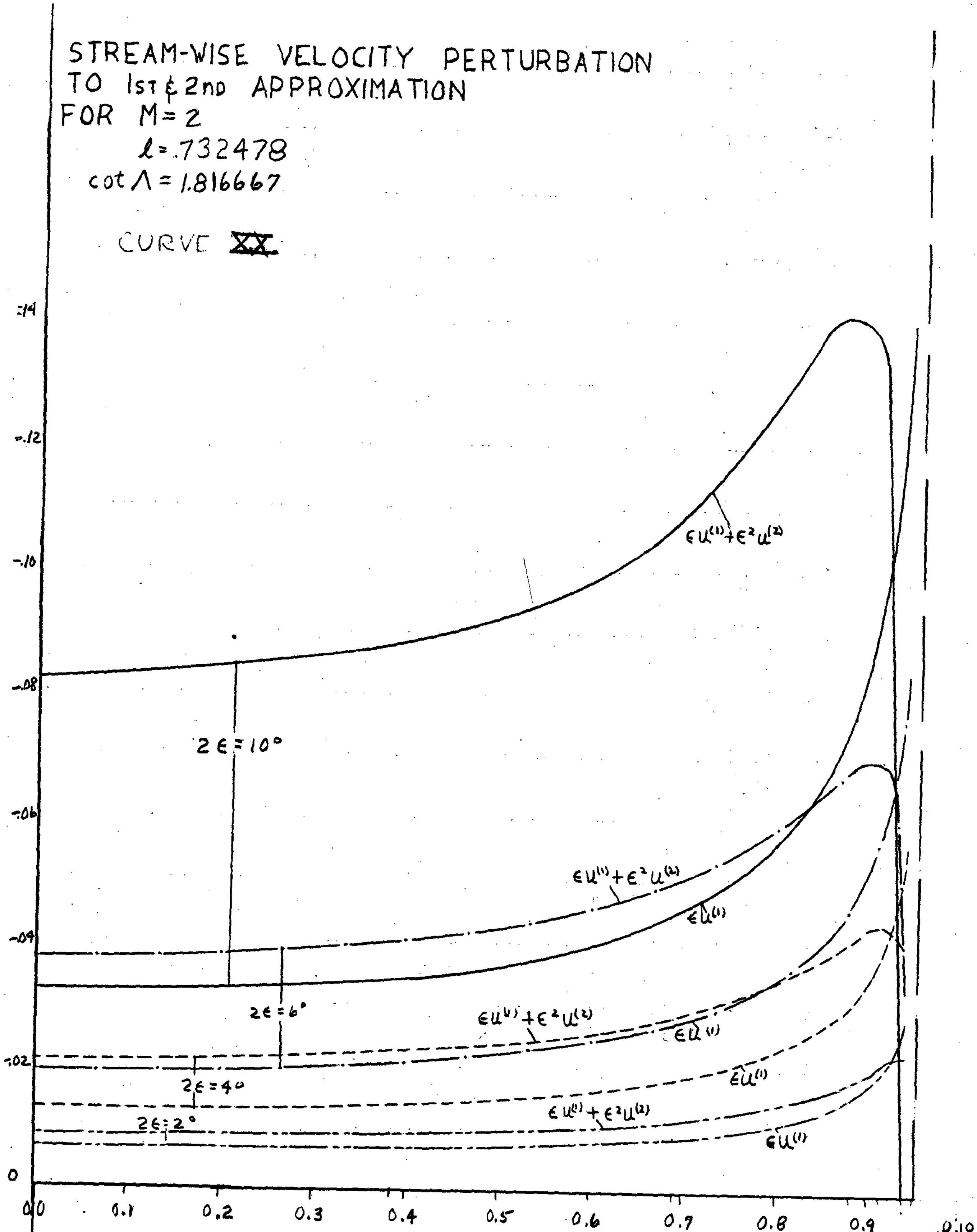


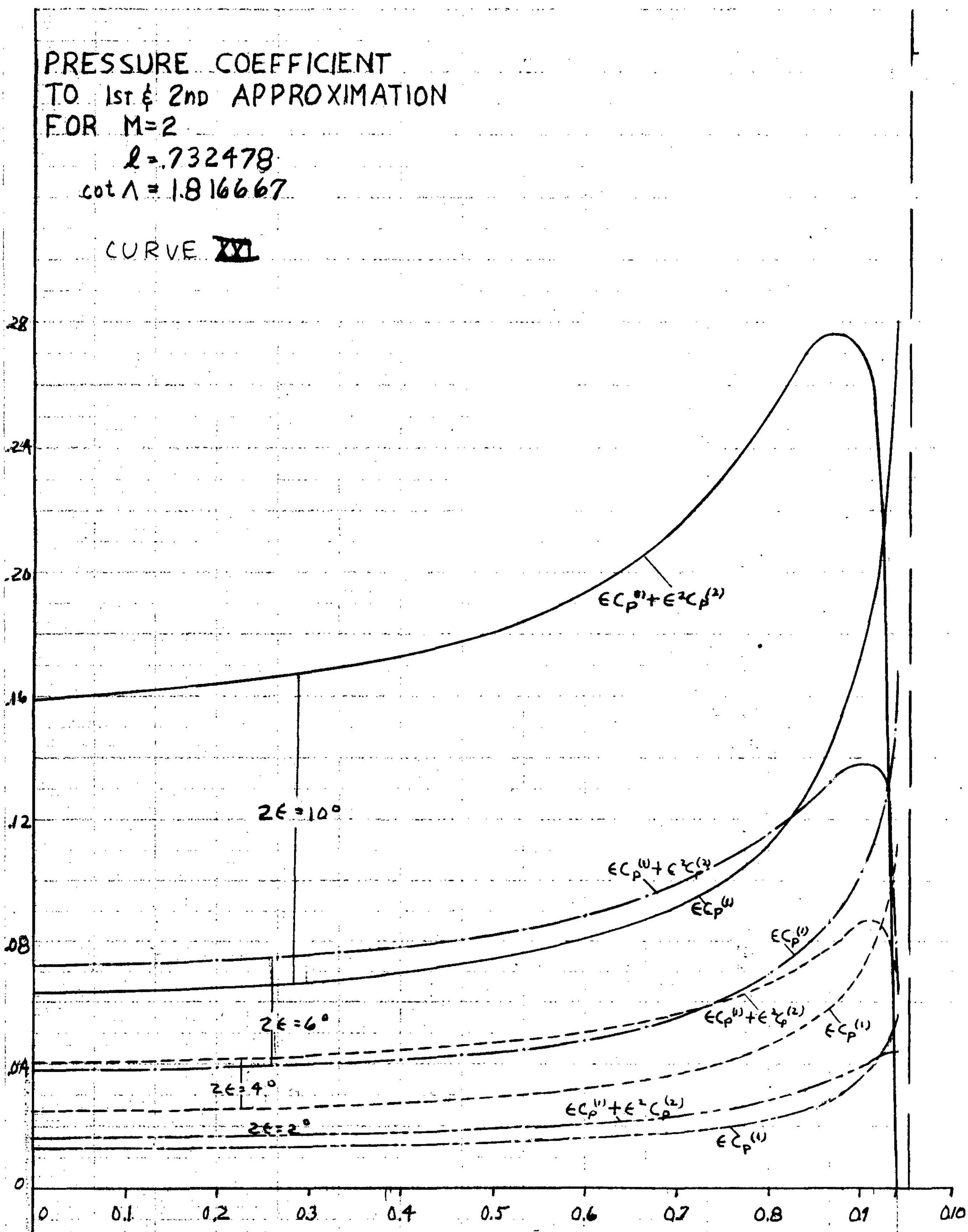
PRESSURE COEFFICIENT
TO 1ST & 2ND APPROXIMATION
FOR M=2
l=.18904
cotΛ=4.74363
CURVE XVII
.08
.06
.04
.02
0
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εCp(1)
εCp(1)+ε²Cp(2)
0
.10
.20
.30

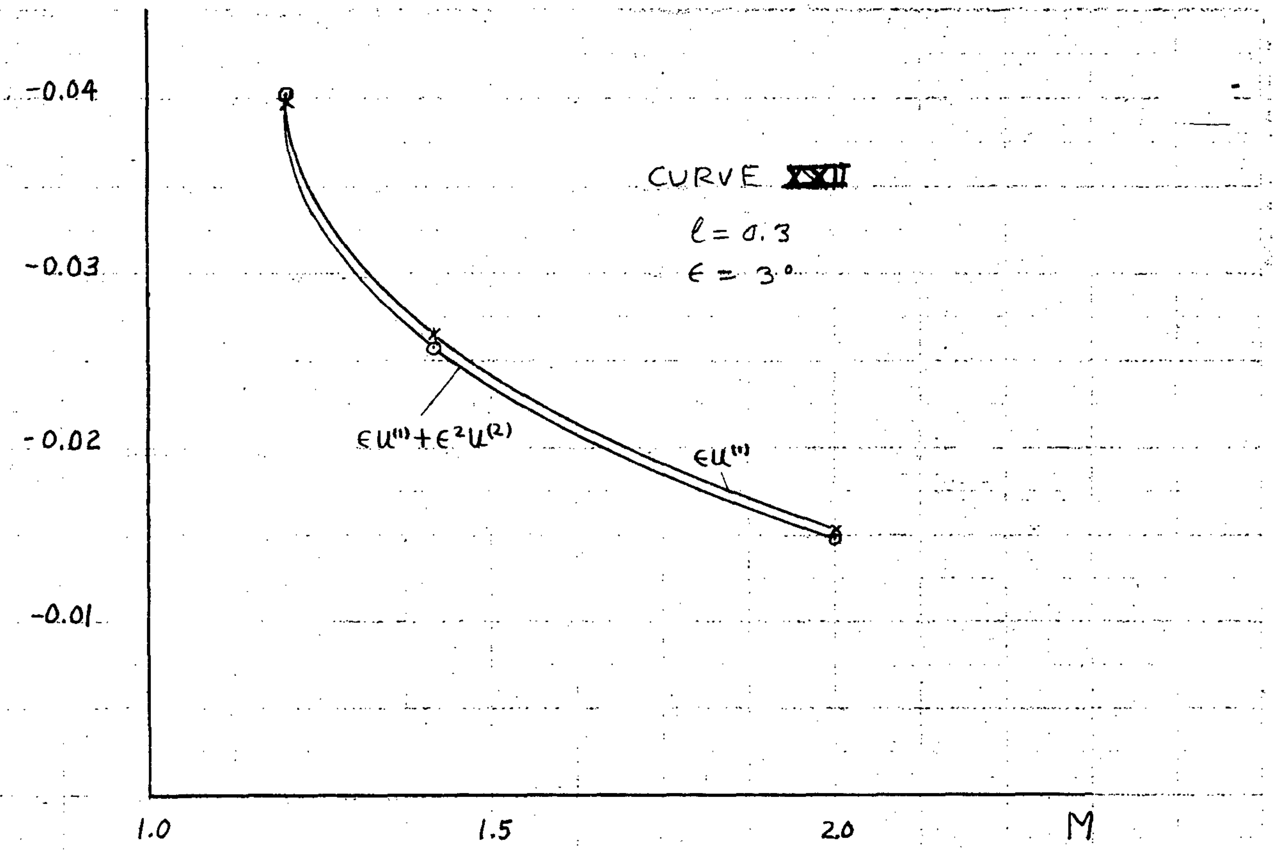

STREAM-WISE VELOCITY PERTURBATION
TO 1ST & 2ND APPROXIMATION
FOR M=2
l=.3
cot Λ= 3.14656
CURVE XVIII
-0.040
-0.030
-0.020
-0.010
0.000
0
0.10
0.20
0.30
0.40
0.50
2ε=10°
2ε=6°
2ε=4°
2ε=2°
εu(1)
εu(1)+ε²u(2)

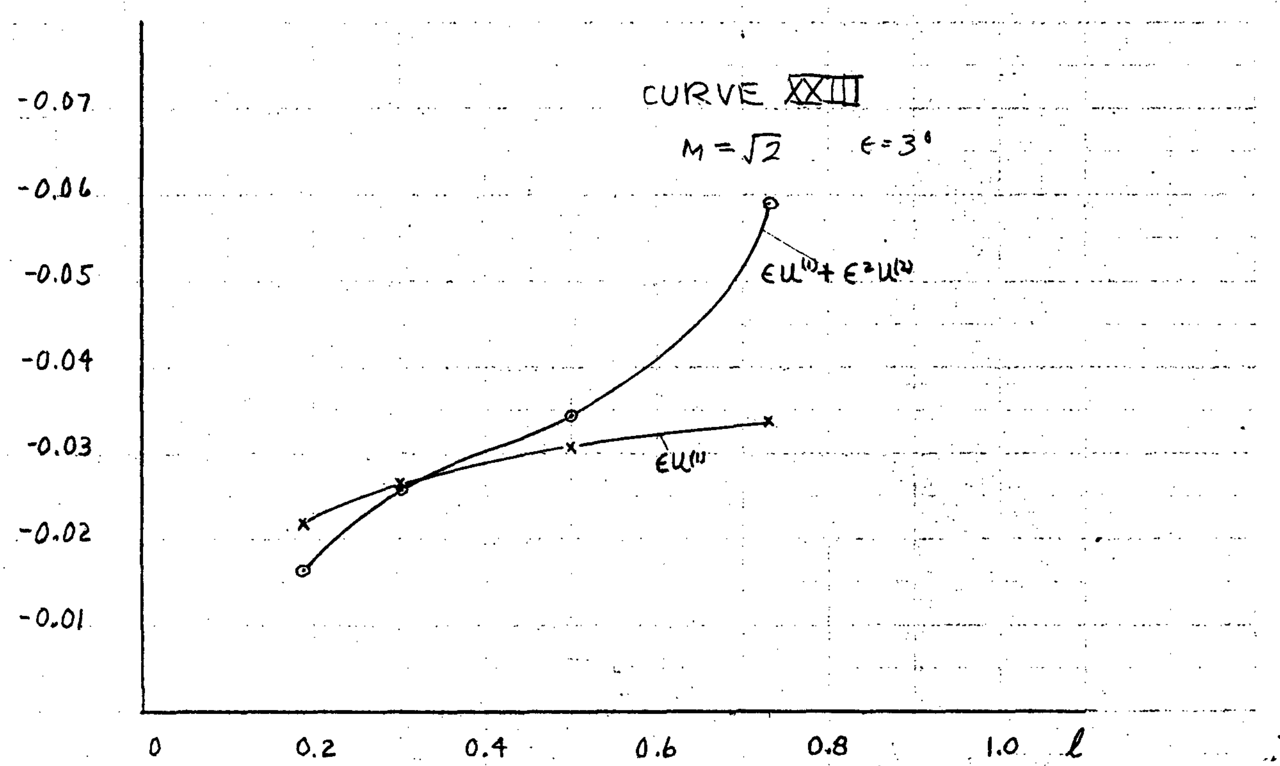


PRESSURE COEFFICIENT
TO 1st & 2nd APPROXIMATION
FOR M=2
$\ell = .3$
$\cot \Lambda = 3.14656$
CURVE XIX
0.120
0.100
0.080
0.060
0.040
0.020
0.000
0
0.10
0.20
0.30
0.40
0.50
$2\epsilon = 10°$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$2\epsilon = 6°$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$2\epsilon = 4°$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$
$2\epsilon = 2°$
$\epsilon C_p^{(1)}$
$\epsilon C_p^{(1)} + \epsilon^2 C_p^{(2)}$


STREAM-WISE VELOCITY PERTURBATION
TO 1st & 2nd APPROXIMATION
FOR M=2
ℓ=.732478
cot Λ=1.816667
CURVE XX
.14
-.12
-.10
-.08
-.06
-.04
-.02
0
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$2\epsilon=10°$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$\epsilon u^{(1)}$
$2\epsilon=6°$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$\epsilon u^{(1)}$
$2\epsilon=4°$
$\epsilon u^{(1)}$
$\epsilon u^{(1)}+\epsilon^2 u^{(2)}$
$2\epsilon=2°$
$\epsilon u^{(1)}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.10


PRESSURE COEFFICIENT
TO 1st & 2nd APPROXIMATION
FOR M=2
ℓ=.732478
cot Λ = 1.816667
CURVE XXI
.28
.24
.20
.16
.12
.08
.04
0
0.
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.10
$\epsilon C_p^{(1)}+\epsilon^2 C_p^{(2)}$
$2\epsilon=10°$
$\epsilon C_p^{(1)}$
$2\epsilon=6°$
$2\epsilon=4°$
$2\epsilon=2°$


-0.04
-0.03
-0.02
-0.01
1.0
1.5
2.0
M
CURVE XXII
l = 0.3
ϵ = 3°
ϵu(1) + ϵ2u(2)
ϵu(1)


-0.07
-0.06
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.2
0.4
0.6
0.8
1.0
l
CURVE XXIII
M = √2
ϵ = 3°
ϵu(1) + ϵ2u(2)
ϵu(1)